A nonlocal sinusoidal shear deformation beam theory with application to bending, buckling, and vibration of nanobeams

Abstract

This paper presents a nonlocal sinusoidal shear deformation beam theory for the bending, buckling, and vibration of nanobeams. The present model is capable of capturing both small scale effect and transverse shear deformation effects of nanobeams, and does not require shear correction factors. Based on the nonlocal differential constitutive relations of Eringen, the equations of motion as well as the boundary conditions of the beam are derived using Hamilton's principle. Analytical solutions for the deflection, buckling load, and natural frequency are presented for a simply supported beam, and the obtained results are compared with those predicted by the nonlocal Timoshenko beam theory. The comparison firmly establishes that the present beam theory can accurately predict the bending, buckling, and vibration responses of short nanobeams where the small scale and transverse shear deformation effects are significant.

Keywords: Nonlocal theory; Sinusoidal theory; Bending; Buckling; Vibration; Nanobeam

1. Introduction

Nanostructures are widely used in micro- and nano-scale devices and systems such as biosensors, atomic force microscopes, micro-electro-mechanical systems (MEMS) and nano-electro-mechanical systems (NEMS) due to their superior mechanical, chemical, and electronic properties [1]. In such applications, small scale effects are often observed. These effects can be captured using size-dependent continuum mechanics such as strain gradient theory [2], modified couple stress theory [3], and nonlocal elasticity theory [4]. Among these theories, the nonlocal elasticity theory initiated by Eringen is the most commonly used theory. Unlike the local theories which assume that the stress at a point is a function of strain at that point, the nonlocal elasticity theory assumes that the stress at a point is a function of strains at all points in the continuum.

Based on the nonlocal constitutive relation of Eringen, a number of papers have been published attempting to develop nonlocal beam models for predicting the responses of carbon nanotubes. The nonlocal Euler-Bernoulli beam theory (EBT) and Timoshenko beam theory (TBT) first proposed by Peddieson et al. [5] and Wang [6], respectively, were adopted by many researchers to investigate bending [7-9], buckling [10-12], and vibration [13-15] responses of carbon nanotubes. A complete development of EBT and TBT was presented by Reddy and Pang [16] who provided the analytical solutions for the deflection, buckling load, and natural frequency of nanobeams with various boundary conditions. It should be noted that the EBT is only applicable for slender beams where the shear deformation effect is negligible and leads to underestimate deflection and overestimate buckling load as well as natural frequency for short beams. The TBT accounts for the shear deformation effect for short beams by assuming a constant shear strain through the height of the beam. Therefore, a shear correction factor is required to compensate for the difference between the actual stress state and the constant stress state. To avoid the use of shear correction factor, higher-order shear deformation theories were developed based on the assumption of the higher-order variation of axial displacement through the height of the beam, notable among them are the third-order theory of Reddy [17], generalized theory of Aydogdu [18], refined theory of Thai [19], and sinusoidal shear deformation theory of Touratier [20].

The sinusoidal shear deformation theory of Touratier [20] is based on the assumption that the transverse shear stress vanishes on the top and bottom surfaces of the beam and is nonzero elsewhere. Thus there is no need to use shear correction factors as in the case of TBT. This theory is also employed to predict the response of laminate plate [21] and functionally graded sandwich plates [22-24]. The aim of this paper is to propose a nonlocal sinusoidal theory which accounts for both small scale and shear deformation effects of nanobeams. The small scale effect is taken into account by using the nonlocal constitutive relations of Eringen, while the shear deformation effect is captured using the sinusoidal shear deformation theory [20]. The nonlocal equations of motion and boundary conditions are derived using Hamilton's principle. Analytical solutions for the deflection, buckling load, and natural frequency are presented for simply supported nanobeams, and the obtained results are compared with those predicted by the TBT to verify the accuracy of the present solution.

2. Equations of motion of the sinusoidal beam theory

Consider a beam length L and rectangular cross section $b \times h$, with b being the width and h being the height. The x-, y-, and z-coordinates are taken along the length, width, and height of the beam, respectively. Equations of motion are derived using Hamilton's principle. The principle can be stated in analytical form as [25]

$$0 = \int_0^T \left(¢U + ¢V - ¢K \right) dt \tag{1}$$

where $¢U$ is the variation of the strain energy; $¢V$ is the variation of the potential energy; and $¢K$ is the variation of the kinetic energy.

According to the sinusoidal theory, the displacement field is chosen based on the assumption that the transverse shear stress vanishes on the top and bottom surfaces of the beam and is nonzero elsewhere. The displacement field is given as [20]

$$\begin{aligned} u_1(x,z,t) &= u(x,t) - z\frac{dw}{dx} + f\theta \\ u_2(x,z,t) &= 0 \\ u_3(x,z,t) &= w(x,t) \end{aligned} \tag{2}$$

where $f = (h/\pi)\sin(\pi z/h)$, u and w are the axial and transverse displacements, respectively, of a point on the midplane of the beam and θ is the rotation of the cross section about the y-axis. The only nonzero strains are

$$\varepsilon_x = \frac{du}{dx} - z\frac{d^2w}{dx^2} + f\frac{d\theta}{dx} \tag{3a}$$

$$\gamma_{xz} = \cos\left(\frac{\pi z}{h}\right)\theta \tag{3b}$$

It can be observed from Eq. (3b) that the transverse shear strain γ_{xz} is zero at the top ($z = h/2$) and bottom ($z = -h/2$) surfaces of the beam thus satisfying the traction free conditions for σ_{xz}.

The variation of the strain energy of the beam can be stated as

$$\delta U = \int_0^L \int_A (\sigma_x \delta\varepsilon_x + \sigma_{xz}\delta\gamma_{xz})\, dA dx = \int_0^L \left(N\frac{d\delta u}{dx} - M\frac{d^2\delta w}{dx^2} + P\frac{d\delta\theta}{dx} + Q\delta\theta \right) dx \tag{4}$$

where N, M, P, and Q are the stress resultants defined as

$$(N, M, P) = \int_A (1, z, f)\sigma_x dA \quad \text{and} \quad Q = \int_A \cos(\pi z/h)\sigma_{xz} dA \tag{5}$$

The variation of the potential energy of the applied loads can be expressed as

$$\delta V = -\int_0^L q\delta w dx - \int_0^L N_0 \frac{dw}{dx}\frac{d\delta w}{dx} dx \tag{6}$$

where q and N_0 are the transverse and axial loads, respectively.

The variation of the kinetic energy is obtained as

$$
\begin{aligned}
¢K &= \int_0^L \int_A ° \left(\dot{u}_1 ¢\dot{u}_1 + \dot{u}_3 ¢\dot{u}_3 \right) dA dz \\
&= \int_0^L \left[m_0 \left(\dot{u} ¢\dot{u} + \dot{w} ¢\dot{w} \right) + m_2 \frac{d\dot{w}}{dx} \frac{d¢\dot{w}}{dx} - \frac{24 m_2}{®^3} \left(\frac{d\dot{w}}{dx} ¢¨ + ¨ \frac{d¢\dot{w}}{dx} \right) + \frac{6 m_2}{®^2} ¨ ¢¨ \right] dx
\end{aligned}
\tag{7}
$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $°$ is the mass density; and $\left(m_0, m_2\right)$ are mass inertias defined as

$$
\left(m_0, m_2\right) = \int_A \left(1, z^2\right) ° \, dA \tag{8}
$$

Substituting the expressions for $¢U$, $¢V$, and $¢K$ from Eqs. (4), (6), and (7) into Eq. (1) and integrating by parts, and collecting the coefficients of $¢u$, $¢w$, and $¢¨$, the following equations of motion of the beam are obtained

$$
¢u: \; \frac{dN}{dx} = m_0 \ddot{u} \tag{9a}
$$

$$
¢¨: \; \frac{dP}{dx} - Q = \frac{6 m_2}{®^2} \ddot{¨} - \frac{24 m_2}{®^3} \frac{d\ddot{w}}{dx} \tag{9b}
$$

$$
¢w: \; \frac{d^2 M}{dx^2} + q - N_0 \frac{d^2 w}{dx^2} = \frac{24 m_2}{®^3} \frac{d\ddot{¨}}{dx} + m_0 \ddot{w} - m_2 \frac{d^2 \ddot{w}}{dx^2} \tag{9c}
$$

The boundary conditions of the present theory are of the form

$$
\begin{array}{llll}
\text{specify} & u & \text{or} & N \\
\text{specify} & ¨ & \text{or} & P \\
\text{specify} & w & \text{or} & \dfrac{dM}{dx} - N_0 \dfrac{dw}{dx} + m_2 \dfrac{d\ddot{w}}{dx} - \dfrac{24 m_2}{®^3} \ddot{¨} \\
\text{specify} & \dfrac{dw}{dx} & \text{or} & M
\end{array}
\tag{10}
$$

3. Nonlocal theory

3.1. Constitutive relations

Unlike the local theory, the nonlocal theory assumes that the stress at a point depends not only on the strain at that point but also on strains at all other points of the body.

According to Eringen [26], the nonlocal stress tensor $\pm$ at point x is expressed as

$$\pm - \text{«}\nabla^2 \pm = {}^{2} \tag{11}$$

where ^{2} is classical stress tensor at a point x related to the strain by the Hooke's law; $\text{«} = (e_0 a)^2$ is the nonlocal parameter which incorporates the small scale effect, a is the internal characteristic length and e_0 is a constant appropriate to each material. The nonlocal parameter depends on the boundary conditions, chirality, mode shapes, number of walls, and type of motion [27]. So far, there is no rigorous study made on estimating the value of the nonlocal parameter. It is suggested that the value of nonlocal parameter can be determined by experiment or by conducting a comparison of dispersion curves from the nonlocal continuum mechanics and molecular dynamics simulation [6, 28]. In general, a conservative estimate of the nonlocal parameter is $e_0 a < 2.0\,\mathrm{nm}$ for a single wall carbon nanotube [29].

3.2. Stress resultants

For an isotropic material in a one-dimensional case, the nonlocal constitutive relation in Eq. (11) takes the following forms

$$\begin{aligned} \pm_x - \text{«}\frac{d^2 \pm_x}{dx^2} &= E\text{£}_x \\ \pm_{xz} - \text{«}\frac{d^2 \pm_{xz}}{dx^2} &= G\text{¥}_{xz} \end{aligned} \tag{12}$$

where E and G are the elastic and shear modulus of the beam, respectively. By substituting Eq. (3) into Eq. (12) and the subsequent results into Eq. (5), the stress resultants are obtained as

$$N - \text{«}\frac{d^2 N}{dx^2} = EA\frac{du}{dx} \tag{13a}$$

$$M - \text{«} \frac{d^2 M}{dx^2} = -EI \frac{d^2 w}{dx^2} + \frac{24EI}{\text{®}^3} \frac{d \text{¨}}{dx} \tag{13b}$$

$$P - \text{«} \frac{d^2 P}{dx^2} = -\frac{24EI}{\text{®}^3} \frac{d^2 w}{dx^2} + \frac{6EI}{\text{®}^2} \frac{d \text{¨}}{dx} \tag{13c}$$

$$Q - \text{«} \frac{d^2 Q}{dx^2} = \frac{GA}{2} \text{¨} \tag{13d}$$

where

$$\left(A, I\right) = \int_A \left(1, z^2\right) dA \tag{14}$$

4. Equations of motion in terms of displacements

The nonlocal equations of motion of the proposed beam theory can be expressed in terms of displacements ($u, w, \text{¨}$) by substituting stress resultants in Eq. (13) into Eq. (9) as

$$EA \frac{d^2 u}{dx^2} = m_0 \left(\ddot{u} - \text{«} \frac{d^2 \ddot{u}}{dx^2} \right) \tag{15a}$$

$$-\frac{24EI}{\text{®}^3} \frac{d^3 w}{dx^3} + \frac{6EI}{\text{®}^2} \frac{d^2 \text{¨}}{dx^2} - \frac{GA}{2} \text{¨} = \frac{6m_2}{\text{®}^2} \left(\ddot{\text{¨}} - \text{«} \frac{d^2 \ddot{\text{¨}}}{dx^2} \right) - \frac{24m_2}{\text{®}^3} \left(\frac{d\ddot{w}}{dx} - \text{«} \frac{d^3 \ddot{w}}{dx^3} \right) \tag{15b}$$

$$\begin{aligned} &-EI \frac{d^4 w}{dx^4} + \frac{24EI}{\text{®}^3} \frac{d^3 \text{¨}}{dx^3} + q - \text{«} \frac{d^2 q}{dx^2} - N_0 \left(\frac{d^2 w}{dx^2} - \text{«} \frac{d^4 w}{dx^4} \right) \\ &= \frac{24m_2}{\text{®}^3} \left(\frac{d \ddot{\text{¨}}}{dx} - \text{«} \frac{d^3 \ddot{\text{¨}}}{dx^3} \right) + m_0 \left(\ddot{w} - \text{«} \frac{d^2 \ddot{w}}{dx^2} \right) - m_2 \left(\frac{d^2 \ddot{w}}{dx^2} - \text{«} \frac{d^4 \ddot{w}}{dx^4} \right) \end{aligned} \tag{15c}$$

The equations of motion of local beam theory can be recovered from Eq. (15) by setting the nonlocal parameter « equal to zero. Also, the equations of motion of the nonlocal EBT can be obtained from Eq. (15) by setting the rotation ¨ equal to zero. It can be seen from Eq. (15) that the axial deformation u is uncoupled from transverse deformations ($\text{¨}, w$). Thus, the equations of motion for the transverse response of the beam are reduced to Eqs. (15b) and (15c).

5. Analytical solution of simply supported beams

Consider a simply supported beam with length L subjected to transverse load q and axial load N_0. The simply supported boundary conditions of the beam are

$$w = M = 0 \quad \text{at} \quad x = 0, L \tag{16}$$

The following expansions of the generalized displacements ($\text{¨}, w$) satisfy the boundary conditions in Eq. (16)

$$\begin{aligned} \text{¨}\left(x,t\right) &= \sum_{n=1}^{\infty} \text{¤}_n e^{i\text{μ}t} \cos \text{~} x \\ w\left(x,t\right) &= \sum_{n=1}^{\infty} W_n e^{i\text{μ}t} \sin \text{~} x \end{aligned} \tag{17}$$

where $i = \sqrt{-1}$, $\text{~} = n\text{®} / L$, $\left(\text{¤}_n, W_n\right)$ are coefficients, and μ is the natural frequency. The transverse load q is also expanded in the Fourier sine series as

$$q\left(x\right) = \sum_{n=1}^{\infty} Q_n \sin \text{~} x \tag{18}$$

where

$$Q_n = \frac{2}{L}\int_0^L q(x) \sin \text{~} x dx = \begin{cases} q_0 \; (n=1) & \text{for sinusoidal load } q_0 \\ \dfrac{4q_0}{n\text{®}} (n = 1, 3, 5, ...) & \text{for uniform load } q_0 \\ \dfrac{2}{\mathrm{L}} Q_0 \sin \dfrac{n\text{®}}{2} (n = 1, 2, 3, ...) & \text{for point load } Q_0 \text{ at the center} \end{cases} \tag{19}$$

Substituting the expansions of ¨, w, and q from Eqs. (17) and (18) into the equations of motion Eq. (15), the closed-form solutions can be obtained from the following equations

$$\left(\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} - k \end{bmatrix} - \text{ª}\text{μ}^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \right) \begin{Bmatrix} \text{¤}_n \\ W_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \text{ª} Q_n \end{Bmatrix} \tag{20}$$

where

$$s_{11} = \frac{GA}{2} + \frac{6EI}{®^2}\sim^2,\ s_{12} = -\frac{24EI}{®^3}\sim^3,\ s_{22} = EI\sim^4$$
$$m_{11} = \frac{6m_2}{®^2},\ m_{12} = -\frac{24m_2}{®^3},\ m_{22} = m_0 + m_2\sim^2,\ ª = 1 + «\sim^2,\ k = ªN_0\sim^2 \tag{21}$$

The static deflection is obtained from Eq. (20) by setting N_0 and µ equal to zero

$$w(x) = \sum_{n=1}^{\infty} \frac{ªQ_n}{s_{22} - s_{12}^2 / s_{11}} \sin\sim x \tag{22}$$

The buckling load is obtained from Eq. (20) by setting q and µ equal to zero

$$N_0 = \frac{s_{22} - s_{12}^2 / s_{11}}{ª\sim^2} \tag{23}$$

By setting q and N_0 in Eq. (20) equal to zero, the natural frequency can be obtained from the following equation

$$\left(m_{11}m_{22} - m_{12}^2\right)ª^2µ^4 - \left(m_{11}s_{22} + m_{22}s_{11} - 2s_{12}m_{12}\right)ªµ^2 + \left(s_{11}s_{22} - s_{12}^2\right) = 0 \tag{24}$$

The closed-form solutions of the EBT can be obtained from Eq. (20) by setting $¤_n$ equal to zero. Thus, the deflection w, buckling load N_0, and frequency µ of the EBT are expressed as

$$w(x) = \sum_{n=1}^{\infty} \frac{ªQ_n}{s_{22}} \sin\sim x = \sum_{n=1}^{\infty} \frac{ªQ_n}{EI\sim^4} \sin\sim x \tag{25}$$

$$N_0 = \frac{s_{22}}{ª\sim^2} = \frac{EI\sim^2}{ª} \tag{26}$$

$$µ = \sqrt{\frac{s_{22}}{ªm_{22}}} = \sqrt{\frac{EI\sim^4}{ª\left(m_0 + m_2\sim^2\right)}} \tag{27}$$

6. Results and discussions

Table 1 shows the nondimensional deflection, critical buckling load, and fundamental frequency of a simply supported beam subjected to uniform load. The obtained results are compared with those reported by Thai [19] based on nonlocal TBT for a wide range of small scale coefficient and length-to-depth ratio L/h. Shear correction factor and Poisson's ratio are taken as 5/6 and 0.3, respectively. The side of nanobeam L is assumed to be 10 nm. Nondimensional deflection, buckling load, and natural frequency are defined as

$$\bar{w} = \frac{100EI}{q_0 L^4}, \quad \bar{N} = \frac{N_{cr} L^2}{EI}, \quad \bar{\mu} = \mu L^2 \sqrt{\frac{m_0}{EI}} \tag{28}$$

It can be seen that the results of present theory are in excellent agreement with those predicted by TBT for all values of small scale coefficient and length-to-depth ratio even for short beams at the higher vibration modes where the effects of transverse shear deformation and rotary inertia are significant (see Table 2). It is worth noting that the TBT requires a shear correction factor to satisfy the free transverse shear stress conditions on the top and bottom surfaces of the beam, whereas the present theory satisfies the free transverse shear stress conditions on the top and bottom surfaces of the beam without using any shear correction factors.

The effect of shear deformation on the bending, buckling, and vibration responses of nanobeams is shown in Fig. 1 for a simply supported beam with $e_0 a = 1$ nm. In this figure, the deflection, buckling load, and frequency ratios are defined as the ratios of those by present theory to the correspondences by EBT where the shear deformation effect is omitted. It is observed that, the effect of shear deformation is to increase the deflections and decrease the buckling loads and natural frequencies, and this effect is significant for short beams at higher vibration modes (see Fig. 2). This indicates that the shear deformation effect results in a reduction of the beam stiffness. Another illustration of the shear deformation effect is also shown in Table 3 for a simply supported single-walled carbon nanotube (SWCNT) with diameter $d = 0.678$ nm and effective tube thickness $t = 0.066$ nm [30]. The material properties of SWCNT are [13]: $E = 5.5$ TPa, ¬ $= 0.19$, and ° $= 2.3$ g/cm^3. The results of present theory obtained from Eq. (24) are compared with those of EBT from Eq. (27) for various values of length-to-diameter ratio L/d. It can be seen from Table 3 that the frequencies obtained by present theory are smaller than those given by EBT. The difference between present theory and EBT is significant for higher modes and small L/d. For example, for the fundamental mode $n = 1$, the differences are only 5.23% and 0.45% for short tube ($L/d = 5$) and long tube ($L/d = 20$), respectively. However, for the higher mode, say $n = 5$, the differences are 43.20% and 7.77% for short tube ($L/d = 5$) and long tube ($L/d = 20$), respectively. So the present beam should be used to predict the responses of short beams at higher modes where the shear deformation effect is significant.

To illustrate the small scale effect on the responses of nanobeams, Fig. 3 plots the deflection, buckling load, and frequency ratios with respect to the small scale coefficient e_0a for a simply supported beam with $L/h = 10$. It is noted that the values of e_0a should be smaller than 2.0 nm for a SWCNT as pointed out by Wang and Wang [29]. The deflection, buckling load, and frequency ratios are defined as the ratios of those by the nonlocal theory to the correspondences by the local theory (i.e., e_0a =0). It can be seen that the deflection ratio is greater than unity, whereas the buckling load and frequency ratios are smaller than unity. It means that the local theory underestimates the deflections and overestimates the buckling loads as well as natural frequencies of the nanobeams compared to the nonlocal one. This is due to the fact that the local theory is unable to capture the small scale effect of the nanobeams. The difference between the local and nonlocal theories is especially significant for the higher modes (see Fig. 4). In general, the effects of small scale and shear deformation are similar, and the inclusion of the small scale and shear deformation effects will reduce the stiffness of the beam, and consequently, leads to an increase in the deflections and a reduction of the buckling loads and natural frequencies.

7. Conclusions

A nonlocal sinusoidal beam theory is developed for the bending, buckling, and vibration of nanobeams. The present model is capable of capturing both small scale and shear deformation effects of nanobeams, and does not require shear correction factors. Based on the nonlocal differential constitutive relations of Eringen, the equations of motion as well as the boundary conditions of the beam are derived using Hamilton's principle. Analytical solutions for deflection, buckling load, and natural frequency are presented for a simply supported beam, and the obtained results are compared well with those predicted by the TBT. It is observed that the inclusion of the small scale and shear deformation effects lead to an increase in the deflections and a reduction of the buckling loads and natural frequencies of nanobeams.

References

[1] X. Li, B. Bhushan, K. Takashima, C.W. Baek, Y.K. Kim, Mechanical characterization of micro/nanoscale structures for MEMS/NEMS applications using nanoindentation techniques, Ultramicroscopy, 97 (2003) 481-494.

[2] W.D. Nix, H. Gao, Indentation size effects in crystalline materials: A law for strain gradient plasticity, Journal of the Mechanics and Physics of Solids, 46 (1998) 411-425.

[3] H.M. Ma, X.L. Gao, J.N. Reddy, A microstructure-dependent Timoshenko beam

model based on a modified couple stress theory, Journal of the Mechanics and Physics of Solids, 56 (2008) 3379-3391.

[4] A.C. Eringen, Nonlocal polar elastic continua, International Journal of Engineering Science, 10 (1972) 1-16.

[5] J. Peddieson, G.R. Buchanan, R.P. McNitt, Application of nonlocal continuum models to nanotechnology, International Journal of Engineering Science, 41 (2003) 305-312.

[6] Q. Wang, Wave propagation in carbon nanotubes via nonlocal continuum mechanics, Journal of Applied Physics, 98 (2005) 124301.

[7] Q. Wang, K.M. Liew, Application of nonlocal continuum mechanics to static analysis of micro-and nano-structures, Physics Letters A, 363 (2007) 236-242.

[8] C.M. Wang, S. Kitipornchai, C.W. Lim, M. Eisenberger, Beam bending solutions based on nonlocal Timoshenko beam theory, Journal of Engineering Mechanics, 134 (2008) 475-481.

[9] O. Civalek, C. Demir, Bending analysis of microtubules using nonlocal Euler-Bernoulli beam theory, Applied Mathematical Modelling, 35 (2011) 2053-2067.

[10] C.M. Wang, Y.Y. Zhang, S.S. Ramesh, S. Kitipornchai, Buckling analysis of micro- and nano-rods/tubes based on nonlocal Timoshenko beam theory, Journal of Physics D: Applied Physics, 39 (2006) 3904-3909.

[11] Q. Wang, V.K. Varadan, S.T. Quek, Small scale effect on elastic buckling of carbon nanotubes with nonlocal continuum models, Physics Letters A, 357 (2006) 130-135.

[12] T. Murmu, S.C. Pradhan, Buckling analysis of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity and Timoshenko beam theory and using DQM, Physica E: Low-dimensional Systems and Nanostructures, 41

(2009) 1232-1239.

[13] C.M. Wang, Y.Y. Zhang, X.Q. He, Vibration of nonlocal Timoshenko beams, Nanotechnology, 18 (2007) 105401.

[14] Q. Wang, V.K. Varadan, Vibration of carbon nanotubes studied using nonlocal continuum mechanics, Smart Materials and Structures, 15 (2006) 659-666.

[15] Y.Q. Zhang, G.R. Liu, X.Y. Xie, Free transverse vibrations of double-walled carbon nanotubes using a theory of nonlocal elasticity, Physical Review B, 71 (2005) 195404.

[16] J.N. Reddy, S.D. Pang, Nonlocal continuum theories of beams for the analysis of carbon nanotubes, Journal of Applied Physics, 103 (2008) 023511.

[17] J.N. Reddy, Nonlocal theories for bending, buckling and vibration of beams, International Journal of Engineering Science, 45 (2007) 288-307.

[18] M. Aydogdu, A general nonlocal beam theory: its application to nanobeam bending, buckling and vibration, Physica E: Low-dimensional Systems and Nanostructures, 41 (2009) 1651-1655.

[19] H.T. Thai, A nonlocal beam theory for bending, buckling, and vibration of nanobeams, International Journal of Engineering Science, (2011) doi:10.1016/j.ijengsci.2011.1011.1011.

[20] M. Touratier, An efficient standard plate theory, International Journal of Engineering Science, 29 (1991) 901-916.

[21] A.M. Zenkour, Analytical solution for bending of cross-ply laminated plates under thermo-mechanical loading, Composite Structures, 65 (2004) 367-379.

[22] A.M. Zenkour, A comprehensive analysis of functionally graded sandwich plates: Part 1-Deflection and stresses, International Journal of Solids and Structures, 42 (2005) 5224-5242.

[23] A.M. Zenkour, A comprehensive analysis of functionally graded sandwich plates: Part 2-Buckling and free vibration, International Journal of Solids and Structures, 42 (2005) 5243-5258.

[24] A.M. Zenkour, Generalized shear deformation theory for bending analysis of functionally graded plates, Applied Mathematical Modelling, 30 (2006) 67-84.

[25] J.N. Reddy, Energy principles and variational methods in applied mechanics, John Wiley & Sons Inc, 2002.

[26] A.C. Eringen, On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves, Journal of Applied Physics, 54 (1983) 4703-4710.

[27] B. Arash, Q. Wang, A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes, Computational Materials Science, 51 (2012) 303-313.

[28] B. Arash, R. Ansari, Evaluation of nonlocal parameter in the vibrations of single-walled carbon nanotubes with initial strain, Physica E: Low-dimensional Systems and Nanostructures, 42 (2010) 2058-2064.

[29] Q. Wang, C. Wang, The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes, Nanotechnology, 18 (2007) 075702.

[30] B.I. Yakobson, C.J. Brabec, J. Bernholc, Nanomechanics of carbon tubes: instabilities beyond linear response, Physical Review Letters, 76 (1996) 2511-2514.

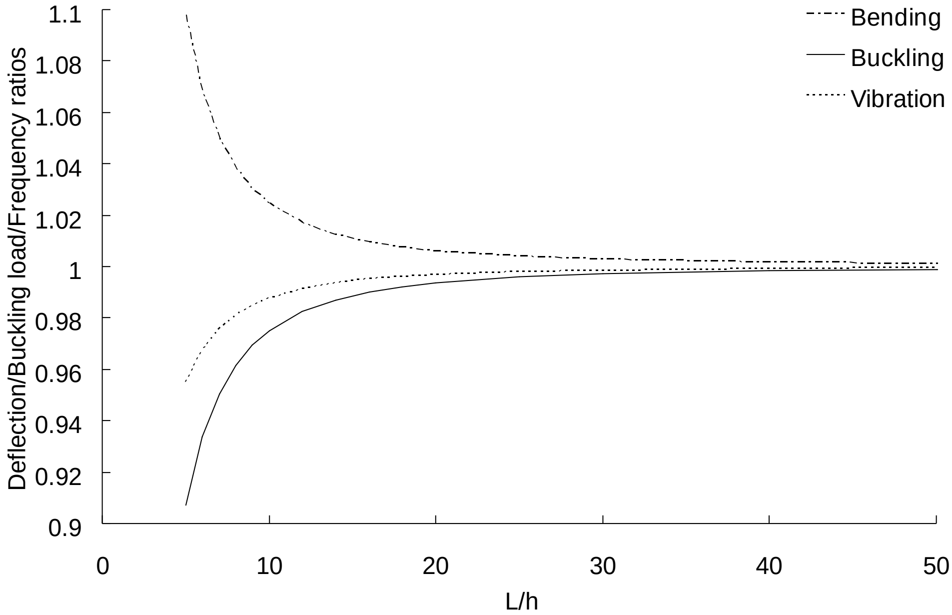

Fig. 1. Effect of transverse shear deformation on the deflection, critical buckling load, and fundamental frequency ratios for a simply supported beam with $e_0 a = 1\,\mathrm{nm}$

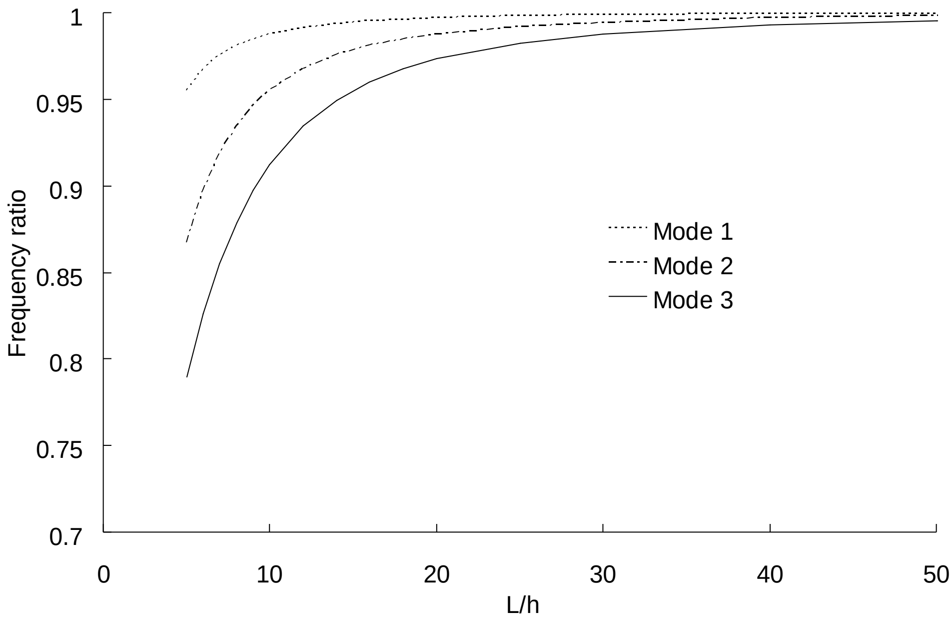

Fig. 2. Effect of transverse shear deformation on higher frequency ratios for a simply supported beam with $e_0 a = 1\,\mathrm{nm}$

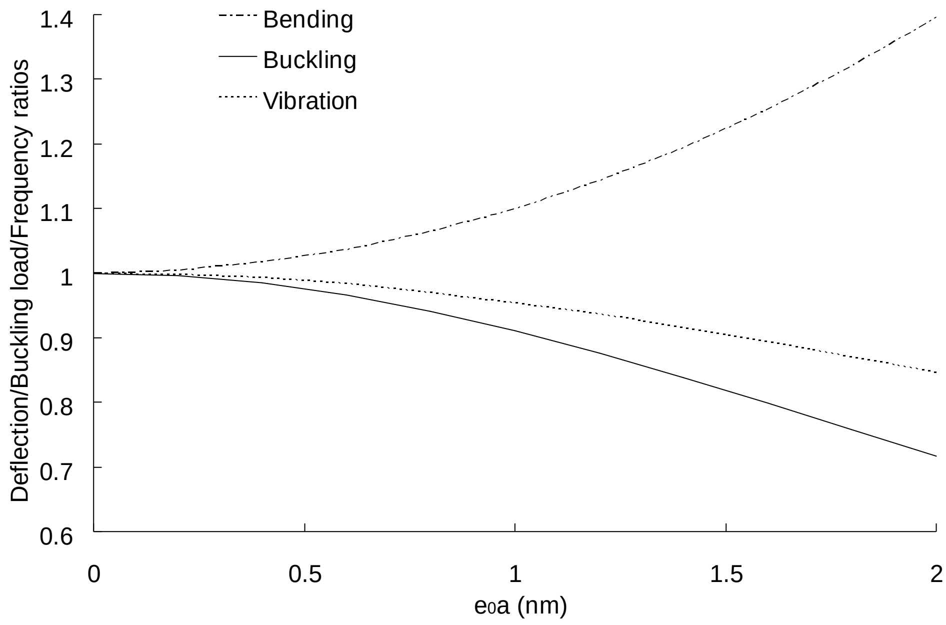

Fig. 3. Effect of small scale on the deflection, critical buckling load, and fundamental frequency ratios for a simply supported beam with $L/h = 10$

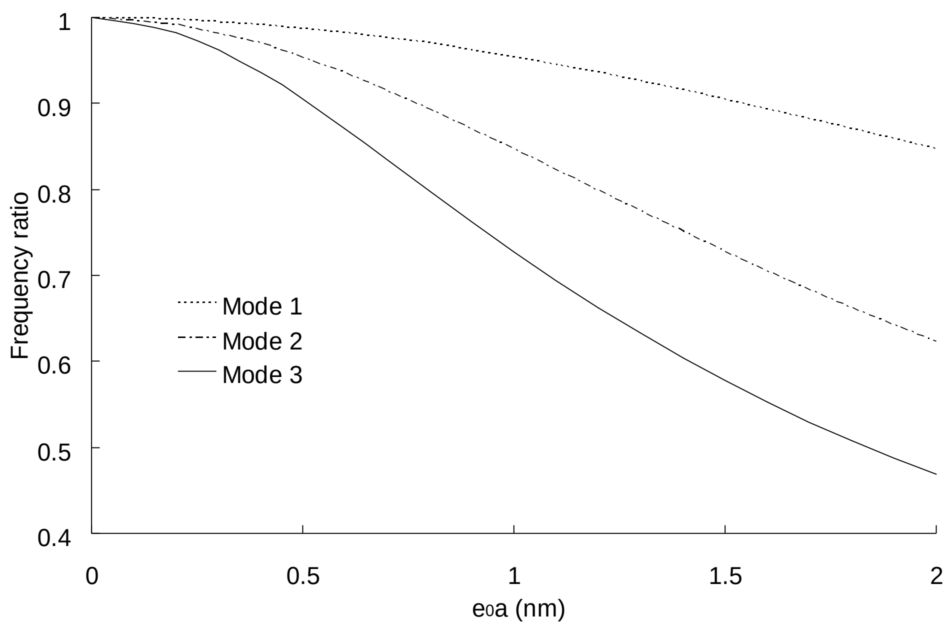

Fig. 4. Effect of small scale on higher frequency ratios for a simply supported beam with $L/h = 10$

Table 1. Nondimensional deflection $\bar{w}$, critical buckling load $\bar{N}$, and fundamental frequency $\bar{\mu}$ of simply supported beams

L/h	« (nm^2)	Deflection $\bar{w}$		Buckling load $\bar{N}$		Frequency $\bar{\mu}$	
		TBT [19]	Present	TBT [19]	Present	TBT [19]	Present
5	0	1.4321	1.4317	8.9509	8.9533	9.2740	9.2752
	1	1.5674	1.5671	8.1468	8.1490	8.8477	8.8488
	2	1.7028	1.7025	7.4753	7.4773	8.4752	8.4763
	3	1.8381	1.8379	6.9061	6.9079	8.1461	8.1472
	4	1.9734	1.9733	6.4174	6.4191	7.8526	7.8536
10	0	1.3346	1.3345	9.6227	9.6231	9.7075	9.7077
	1	1.4622	1.4621	8.7583	8.7587	9.2612	9.2614
	2	1.5898	1.5897	8.0364	8.0367	8.8713	8.8715
	3	1.7173	1.7173	7.4244	7.4247	8.5269	8.5271
	4	1.8449	1.8449	6.8990	6.8994	8.2196	8.2198
20	0	1.3102	1.3102	9.8067	9.8068	9.8281	9.8282
	1	1.4359	1.4358	8.9258	8.9258	9.3763	9.3764
	2	1.5615	1.5615	8.1900	8.1901	8.9816	8.9816
	3	1.6871	1.6871	7.5664	7.5665	8.6328	8.6329
	4	1.8128	1.8128	7.0310	7.0310	8.3218	8.3218
100	0	1.3024	1.3024	9.8671	9.8671	9.8679	9.8679
	1	1.4274	1.4274	8.9807	8.9807	9.4143	9.4143
	2	1.5525	1.5525	8.2405	8.2405	9.0180	9.0180
	3	1.6775	1.6775	7.6130	7.6130	8.6678	8.6678
	4	1.8025	1.8025	7.0743	7.0743	8.3555	8.3555

Table 2. First three nondimensional frequencies $\bar{\mu}$ of simply supported beams

L/h	« (nm^2)	$\bar{\mu}_1$		$\bar{\mu}_2$		$\bar{\mu}_3$	
		TBT [19]	Present	TBT [19]	Present	TBT [19]	Present
5	0	9.2740	9.2752	32.1665	32.1948	61.4581	61.6192
	1	8.8477	8.8488	27.2364	27.2604	44.7247	44.8420
	2	8.4752	8.4763	24.0453	24.0664	36.8831	36.9798
	3	8.1461	8.1472	21.7642	21.7833	32.1036	32.1878
	4	7.8526	7.8536	20.0293	20.0470	28.8023	28.8778
10	0	9.7075	9.7077	37.0962	37.1009	78.1547	78.1855
	1	9.2612	9.2614	31.4105	31.4146	56.8753	56.8977
	2	8.8713	8.8715	27.7303	27.7339	46.9034	46.9219
	3	8.5269	8.5271	25.0996	25.1029	40.8254	40.8415
	4	8.2196	8.2198	23.0989	23.1019	36.6272	36.6416
20	0	9.8281	9.8282	38.8299	38.8308	85.6619	85.6671
	1	9.3763	9.3764	32.8786	32.8793	62.3385	62.3422
	2	8.9816	8.9816	29.0263	29.0270	51.4087	51.4118
	3	8.6328	8.6329	26.2727	26.2733	44.7469	44.7496
	4	8.3218	8.3218	24.1785	24.1790	40.1454	40.1478
100	0	9.8679	9.8679	39.4517	39.4517	88.6914	88.6915
	1	9.4143	9.4143	33.4051	33.4051	64.5431	64.5432
	2	9.0180	9.0180	29.4911	29.4912	53.2268	53.2269
	3	8.6678	8.6678	26.6934	26.6934	46.3294	46.3295
	4	8.3555	8.3555	24.5657	24.5657	41.5652	41.5653

Table 3. First five frequencies (THz) of a simply supported single-walled carbon nanotube with $e_0 a = 1.0\,\mathrm{nm}$

Modes	$L/d=5$		$L/d=10$		$L/d=20$		$L/d=100$	
	EBT	Present	EBT	Present	EBT	Present	EBT	Present
1	1.0459	0.9939	0.3283	0.3237	0.0885	0.0881	0.0036	0.0036
2	2.5622	2.2006	1.0459	0.9939	0.3283	0.3237	0.0145	0.0145
3	3.7911	2.9689	1.8294	1.6550	0.6643	0.6445	0.0324	0.0324
4	4.7180	3.4516	2.5622	2.2006	1.0459	0.9939	0.0572	0.0570
5	5.4019	3.7723	3.2171	2.6310	1.4402	1.3364	0.0885	0.0881