POWER TRANSFER OPTIMISED
AUTOMATIC MATCHING
NETWORKS

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Abstract

Matching networks are widely used to enhance active power transfer when radio frequency generators drive complex loads. The tuning of the network for varying loads typically involves searching for optimum matching conditions.

However, improving the matching condition of the network does not necessarily indicate an increase in active power transfer. As an example, a π network with three adjustable elements may achieve comparable matches for a variety of elements’ settings, each matching triple exhibiting a different transferred active power.

Furthermore, the influence of the transmission lines used to connect the matching network to its source and load is rarely taken into account.

The purpose of the work is to optimise the power gain of a narrowband matching system in the frequency range of 1.8 – 30 MHz. The system consists of a source, a matching network, a load and two interconnecting lines whose characteristic impedance is complex. The optimisation process involves optimum choice of the transmission lines’ lengths and development of a matching strategy. Its objective is to ensure automatic and continuous adjustment of the matching network for optimum active power transfer to its load while matching the network’s input impedance to a resistive source.

The network topologies employed are limited to the most common π and T networks consisting of two variable capacitors and one central inductor. Losses are assumed to be mainly caused by the inductor. An appropriate simple and synthetic model of the losses is proposed — which is suitable for active power transfer optimisation. The model is validated against losses of inductors derived by different works.

After choosing a proper network parametrisation and exact inclusion of the losses during network design, the losses of a network terminated by a resistance and designed to match (exactly) a source resistance at its input are derived. Then its power gain is optimised by a proper choice of the network’s parameter and the impact of changing the purely resistive termination to impedances exhibiting capacitive or inductive imaginary parts is considered. An explicit solution is calculated for networks with a constant Q factor central inductor, its differences from the approximate solution (network elements designed as if the network would be lossless) are considered. Example diagrams are given illustrating those differences and power gain contour Smith charts are drawn for typical ranges of the L, π, and T networks’ elements. Combining the results of the different approaches yields an optimum matching strategy.

The losses of transmission lines connecting source and network of load and network are determined, where the lines’ complex characteristic impedance is taken into account. Those losses are included in a power gain optimisation of the complete matching system.

Finally, an experimental setup is designed under which the matching strategy of the network is tested and validated.
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Definitions

Notations

- Complex variable (or constant), e.g. \( V \) is the complex amplitude of a voltage

\( r \) Per-unit-length variable (or constant), e.g. \( R' \) is the per-unit-length resistance of a transmission line

* Complex conjugate, e.g. \( Z^* \) is the complex conjugate of \( Z \)

\(|\ldots|\) Absolute value of \( \ldots \)

\( \text{Im}\{\ldots\} \) Imaginary part of \( \ldots \)

\( \text{Max}\{\ldots\} \) Maximum of \( \ldots \)

\( \text{Min}\{\ldots\} \) Minimum of \( \ldots \)

\( \text{Re}\{\ldots\} \) Real part of \( \ldots \)

\( \varphi \ldots \) Phase of \( \ldots \)

(\#\#) The right hand side of the equation was derived by applying Equation (\#\#)

Indices

0 If used with an impedance/resistance/reacitance/admittance/conductance/susceptance, it denotes a characteristic value of a transmission line, if used with a voltage/current, it denotes the (complex) amplitude of an ideal voltage/current source

1 If used with an impedance/reflection coefficient/active power, it denotes a value at the input of a transmission line or a network as first index or the line number as second index

2 If used with an impedance/reflection coefficient/active power, it denotes a value at the load of a transmission line or a network as first index or the line number as second index

act Active power (index used only in conjunction with a power)

ax Axial

A Arnold

C Coil

i If used with an impedance/resistance/reacitance/admittance/conductance/susceptance/reflection coefficient, it denotes a value of a source impedance, if used with a voltage/current, it denotes an incident value in a transmission line

1,\ell Either index \( i \) or \( \ell \) is used in all variables with index \( 1,\ell \) throughout an equation

i,\ell,0 Either index \( i, \ell, \) or 0 is used in all variables with index \( 1,\ell,0 \) throughout an equation

inma network (exactly) matched at its input

\( \ell \) Load

\( \Pi \) Lossless

\( \text{Loaddes} \) Network designed as if all capacitors were lossless

\( \text{Loades} \) Network designed as if all elements were lossless

L Transmission line

M Medhurst

MB Matchbox (matching network)

\( \text{Outma} \) network (exactly) matched at its output

\( \Pi \) Parallel

\( r \) Reflected
DEFINITIONS

\[ r \text{ad} \] Radial
\[ s \] Series
\[ t \] T network (the lower case letter was chosen to avoid confusion of \( g_t \) (power gain of a T network), with \( g_T \) (transducer power gain))
\[ t_p \] Two-port
\[ T \] Transducer
\[ T \] Turn
\[ \bar{Z}_0 \] Reference impedance of a reflection coefficient, e.g. \( \bar{Z}_0 \)
\[ \pi \] \( \pi \) network

Latin letters

arctan() Arc tangens function
arsinh() Inverse hyperbolic sine function
\( B \) Susceptance, imaginary part of admittance \( Y \)
\( c_0 \) Speed of light in a vacuum
\( \cos() \) Cosine function
\( \cosh() \) Hyperbolic cosine function
\( C \) Capacitance
\( e \) Exponential constant (base of natural logarithms)
\( e^{(\cdots)} \) Exponential function of \( \cdots \)
\( f \) Frequency
\( g \) Power gain (named as usual, although it's a ratio of active powers (an active power gain))
\( g_T \) Transducer power gain
\( G \) Conductance, real part of admittance \( Y \)
\( G_p \) Parallel conductance
\( I \) Complex amplitude of a current
\( j \) Imaginary unit \( \sqrt{-1} \)
\( l \) Length of a transmission line
\( \log() \) Logarithm function to base 10
\( \ln() \) Natural logarithm function
\( L \) Inductance
\( \mathbb{R} \) Set of real numbers
\( \mathbb{R}_+^\mathbb{R} \) Set of real numbers greater or equal than zero
\( R \) Resistance, real part of impedance \( Z \)
\( R_s \) Series resistance
\( \sin() \) Sine function
\( \sinh() \) Hyperbolic sine function
\( \tanh() \) Hyperbolic tangens function
\( V \) Complex amplitude of a voltage
\( X \) Reactance, imaginary part of impedance \( Z \)
\( Y \) Admittance \( (Y = \frac{1}{Z}) \)
\( \mathbb{Z} \) Set of whole numbers (integers)
\( \bar{Z} \) Impedance

Greek letters

\( \alpha \) Attenuation constant of a transmission line
\( \beta \) Phase constant of a transmission line
\( \gamma \) Propagation constant of a transmission line
\( \mu \) Permeability
\( \mu_0 \) Permeability constant (in a vacuum), often approximated as \( 4\pi \times 10^{-7} \text{ H/m} \)
\pi \quad \text{Number } pi \\
\rho \quad \text{Resistivity} \\
\omega \quad \text{Angular frequency } (\omega = 2\pi f)

\textbf{Abbreviations}

(1.\#) \quad \text{Equation number of an equation in a chapter, e.g. equation number \# in Chapter 1} \\
(A.\#) \quad \text{Equation number of an equation in a chapter of the appendix, e.g. equation number \# in Appendix A} \\
[#] \quad \text{Citation number \#} \\
AC \quad \text{Alternating current} \\
DC \quad \text{Direct current} \\
MMIC \quad \text{Monolithic microwave integrated circuit}
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Declaration

I declare that the work contained in this thesis has not been submitted for any other award and that it is all my own work.

Name: Reinhard Jörg Glöckner (Reinhard Joerg Gloeckner)

Signature: [Signature]

Date: July 29th, 2008
Chapter 1

Introduction

1.1 Introduction

Matching networks are widely used to enhance active power transfer when radio frequency generators drive complex loads. The tuning of the network for varying loads typically involves searching for optimum matching conditions. A number of interesting algorithms and techniques were developed to search for optimum matching condition (see e.g. [1], [2], [3] for continuously variable elements or [4], [5], [6], [7], and [8] for binary switchable elements).

However, improving the matching condition of the network does not necessarily indicate an increase in active power transfer. As an example, a \(\pi\) network with three adjustable elements may achieve comparable matches for a variety of elements’ settings while each matching triple exhibits a different transferred active power.

Another important aspect in developing an optimum matching strategy is a proper choice of the network’s parametrisation. In effect, during the last decades, a couple of \(\pi\) and \(T\) network parametrisations were developed for lossless networks suit different needs. A more filter theoretical approach involving hyperbolic functions was summarised in [9e] and examined in more detail in [10]. The so called \(Q\)-based design was used in [11], [12] and further developed in [13], [14], [15], [16], [17], and [18]. The problem of exact loss inclusion in network design was introduced by [3]. However, the authors in the previous mentioned works assumed that the losses are low enough so that the same element values are used as in the lossless case whenever deriving analytical solutions. This obviously is not valid for the entire parameter range or for badly chosen element values. Additionally, although realistic loads are often complex loads, the element values were calculated for resistive loads only. In [13] and [18], absorption and resonance methods are used for complex load matching, but both works do not systematically describe the impact of complex loads on the chosen parametrisation. However, [19], [20], and [21] graphically visualise the complex load range for a given tunable range of the network’s components. In addition, the required minimum \(\pi\) network component values’ range to acquire a match for a given magnitude of the load reflection coefficient is calculated in [21], assuming a lossless network and using no explicit parametrisation (it’s done implicitly by variation of the components in their tunable range).
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Furthermore, in all the above work the losses are described by a constant $Q$ factor only.

On the other hand, it is important to realise that in practice, the source and the load are connected to the matching network via transmission lines which also have some influence on active power transfer. However, the influence of the transmission lines used to connect the matching network is rarely taken into account in various research works.

The purpose of the work is to optimise the power gain of a matching system in the frequency range of $1.8-30\,\text{MHz}$. The system consists of a source, a matching network, a load and two interconnecting lines whose characteristic impedance is complex. The optimisation process involves optimum choice of the transmission lines’ lengths and development of a matching strategy. Its objective is to ensure automatic and continuous adjustment of the matching network for optimum active power transfer to its load while matching the network’s input impedance to a resistive source.

Its application is restricted to carrier based applications, thus only narrowband matching is considered. The overall system can be applied to radio broadcast, ham radio, and industrial applications (as plasma deposition systems).

The network topology is limited to the most common $\pi$ and $T$ networks consisting of two variable capacitors and one central inductor, including the four possible $L$ networks derived by omitting one of the capacitors.

In the desired frequency range the total loss of such networks is mainly determined by the losses of the inductor while the capacitors’ losses can be neglected. Thus, a simple and synthetic model for the losses of the inductor is proposed. One of the main features of the proposed model is that it describes the general case of non constant $Q$ and includes the case of constant $Q$ as well as a special case. It applies to different variable inductor realisations, compared to losses derived in different works (examples for a solenoidal variable air-core inductor are [22] and [23] for a low number of turns, [24] for a high number of turns, and [25] for any number of turns).

Next, the losses due to the inductor are included exactly in the network’s parametrisation.

The choice of a proper parametrisation suiting the needs of active power transfer optimisation is especially important, because it determines the characteristic behaviour of the system. This in turn simplifies considerably the analysis of the system.

Within this work, the initial parametrisation of [13] (based on the graphical design method of [11]) is employed. Its extension to exact loss inclusion by [3] is used for $\pi$ and $T$ network design and the influence of complex loads on the parametrisation and the transferred active power is derived systematically.

Furthermore, additional losses of the feedlines between generator, matching network, and load are taken into account using the lines’ complex characteristic impedance, which results in a system description enabling active power transfer optimisation of the complete system.

Finally, the optimum matching strategy of the network is tested and validated experimentally by designing a prototype system.
1.2 The aims of the thesis

The main aims of the thesis are:

- To develop a narrowband matching strategy which is suited for automatic and continuous adjustment of L, π, and T matching networks for optimum active power transfer to their load while matching the network’s input impedance to a resistive source applicable in a frequency range of 1.8 – 30 MHz.

- To investigate the influence of the transmission lines’ lengths (which connect the matching network to source and load) on power gain of the matching system and derive optimum line lengths, where the transmission lines’ characteristic impedance is complex.

- To design a prototype system to test and validate the matching strategy of the network experimentally.

1.3 The objectives of the thesis

The main objectives of the proposed research work are:

- To propose a simple model for the losses in the central inductor of π and T networks in the given frequency range and validate it by comparison to existing loss models for different types of variable inductors used in matching networks.

- To choose a proper network parametrisation suitable for the needs of power gain optimisation.

- To derive optimum network settings with exact inclusion of the network’s central inductor’s losses for matching of a purely resistive load to a purely resistive source.

- To extend the method to complex loads.

- To solve explicitly for the central inductor’s loss resistance in case of a constant $Q$ factor inductor.

- To apply optimum network settings using the general loss model and appropriate results of the constant $Q$ factor inductor calculations to develop an optimum matching strategy for the network.

- To optimise the load impedance of a transmission line whose characteristic impedance is complex for optimum power gain and the according source impedance for maximum active power input.
• To examine the impact of common sub-optimal source impedance choices/matching network input impedance design on the power gain of the matching system for low-loss coaxial transmission lines.

• To investigate the influence of common sub-optimal source impedance choices on optimum matching strategy for low-loss coaxial transmission lines.

• To derive optimum line lengths in a matching system for lossless and lossy matching networks, thus optimising the power gain of the complete matching system.

• To test and validate the matching strategy of the network experimentally.

1.4 Scope of the thesis

The majority of research work in narrowband $\pi$ and T matching network design focus on constant $Q$ factor central network inductors and apply $Q$ based design to parameterise those networks. Exceptionally, the work in [3] exactly includes the variable inductor’s losses. On the other hand, whenever analytical solutions are given and parameter’s limits are derived, the networks are assumed to be designed as if all elements were lossless. Moreover, the impact of complex loads on the chosen parametrisation is usually not systematically described. In particular, the possibility that the resulting load related element of the equivalent circuit is an inductor instead of a capacitor is usually not considered.

Additionally, when losses in the connecting transmission lines are taken into account, their complex characteristic impedance is approximated by a real characteristic impedance, despite the fact that its complex propagation constant is used.

The present work uses a general loss model that is applicable to different types of variable inductors. The chosen parametrisation is particularly suited for power gain optimisation of $\pi$ and T matching networks because, unlike $Q$ based design, the power gain’s derivative with respect to the parameter exhibits a unique sign over the entire parameter range for purely resistive and capacitive complex loads if the general loss model is applied. The losses of the central inductor are exactly included in network design and the design formulae are systematically extended to complex loads. Analytical solutions and parameter limits are given for $\pi$ and T matching networks with a constant $Q$ factor central inductor and any types of complex loads. The results obtained from the general loss model and those from the constant $Q$ factor inductor are combined to derive a general optimum matching strategy for L, $\pi$, and T networks.

The losses in the connecting transmission lines are taken into account by their complex characteristic impedance and attenuation.
1.5 Contributions

The contributions resulting from this work are as follows:

- A simple and synthetic general loss model applicable to different types of variable inductors used in matching networks is derived.

- Applying the loss model to L, π, and T networks, an analytically proven matching strategy which optimises the networks' power gain if purely resistive or capacitive complex loads are matched to a purely resistive source is given. In addition, an adaptation to match purely resistive/capacitive complex sources to purely resistive loads is presented.

- An analytical solution for the L, π, and T networks' central inductor's loss resistance as a function of the chosen parametrisation and the appropriate parameter's limits if the inductor's Q factor is constant is provided.

- An optimum matching strategy suitable for automatic and continuous adjustment of L, π, and T matching networks obtained by combining general loss model and constant Q factor results is derived.

- An optimisation of the complete matching system including optimum splitting of the connecting lines' total length is achieved.

- Finally, an experimental setup is designed under which the matching strategy of the network is tested and validated.

1.6 Thesis structure

The thesis is broken down in seven chapters. Chapter 1 gives a brief introduction to matching network system design, the scope of the thesis, and summaries of each chapter.

Chapter 2 describes basic transmission line properties and gives a motivation as to why the complex characteristic impedance of coaxial transmission lines has to be considered although its imaginary part is much lower than its real part. Furthermore, it includes definitions, basic equations used in subsequent chapters, and a derivation of the complete matching system's power gain.

Chapter 3 considers the losses in π and T matching networks, proposes a simple loss model suitable for different types of the networks' central inductor, and validates the model by comparing to existing loss models for different types of variable inductors. A proper network parametrisation is chosen suitting the needs of power gain optimisation. Application of the general loss model to π and T networks designed including the losses exactly yields the optimum parameter's (and load resistance's) value for maximum active power transfer in case of resistive sources and loads.

Since realistic loads are usually complex, Chapter 4 extends the method presented in Chapter 3 to complex loads, thereby obtaining the first part of the optimum
matching strategy which applies to purely resistive or capacitive complex loads. To implement a matching strategy for inductive complex loads, an explicit solution is calculated for networks with a constant $Q$ factor central inductor, its differences from the approximate solution (network elements designed as if the network would be lossless) are considered. Example diagrams are given illustrating those differences and power gain contour Smith charts are drawn for typical ranges of the $L$, $\pi$, and $T$ networks' elements. Combining the results of the different approaches yields an optimum matching strategy for $L$, $\pi$, and $T$ matching networks suitable for any types of loads.

Chapter 5 determines the losses of transmission lines connecting source and network or load and network. The impact of the complex characteristic impedance on both lines is considered. Splitting of the total line length is optimised for highest power gain, and a power gain optimisation of the complete matching system is derived.

Chapter 6 describes the experimental setup designed to test the network's matching strategy. The performed measurements are explained and results are given validating the matching strategy for optimum power gain of the network.

Chapter 7 presents final conclusions, discusses optimum matching strategies, and suggests possible directions for further work.

Note: In order to facilitate the readability of Chapters 2–5, lengthy calculations or proofs were moved towards the appropriate appendices.
Chapter 2

RF Basics

2.1 Introduction

In this chapter, we derive all the necessary equations that describe the active power transfer in a system consisting of a matching network embedded in two lossy transmission lines — described by their attenuation and complex characteristic impedance — between source and load.

In effect, the calculation and optimisation of active power transfer in matching systems using matching networks involves lossy transmission lines. This is due to the fact that generally the matching network is embedded in two lines connecting it to source and load. Both lines introduce impedance transformation and losses. Although lossy transmission lines are usually described by their attenuation and an approximately real characteristic impedance, the small imaginary part of the characteristic impedance of the mismatched line between matching network and load needs to be taken into account.

We start by recalling some basic definitions and notations that are used throughout the thesis.

2.2 Definitions/notations

From this chapter on, the following definitions/notations will be used:

Latin letters

- $a$ : Attenuation of a transmission line
- $a_{dB/100\,m}$ : Attenuation of a transmission line in $\,dB/100\,m$
- $C'$ : Per-unit-length capacitance of a transmission line
- $g_{tot}$ : Power gain of the complete matching system
- $g_p$ : Power gain of a two-port
- $g_{p,\phi}(\xi_{tp}, Z_{0p})$ : Power gain of a two-port as a function of reflection coefficient $\xi_{tp}, Z_{0p}$
- $g_L$ : Power gain of a transmission line
- $g_L_1$ : Power gain of a transmission line 1
- $g_{L_1}(\xi_{tp}, Z_{0_1})$ : Power gain of a transmission line 1 as a function of reflection coefficient $\xi_{tp}, Z_{0_1}$
CHAPTER 2. RF BASICS

\( g_{L2} \)  
Power gain of a transmission line 2

\( g_{L2} \left( \Gamma, \overline{Z}_{02} \right) \)  
Power gain of a transmission line 2 as a function of reflection coefficient \( \Gamma \) to \( \overline{Z}_{02} \)

\( g_{L1} \)  
Transducer power gain of a transmission line 1

\( g_{L1} \left( \Gamma, \overline{Z}_{01} \right) \)  
Transducer power gain of a transmission line 1 as a function of the reflection coefficients \( \Gamma \) to \( \overline{Z}_{01} \) and \( \Gamma \) to \( \overline{Z}_{01} \)

\( g_{Ttot} \)  
Transducer power gain of the complete matching system

\( l_i \)  
Length of transmission line 1

\( l_2 \)  
Length of transmission line 2

\( P_{act1} \)  
Active power transferred to the input of a transmission line

\( P_{act2} \)  
Active power transferred to the load of a transmission line

\( P_{act1} \)  
Active power transferred to the input of a transmission line 1

\( P_{act2} \)  
Active power transferred to the input of a transmission line 2

\( P_{act1p} \)  
Active power transferred to the input of a two-port

\( P_{act2} \)  
Active power transferred to the load of transmission line 1

\( P_{act2} \)  
Active power transferred to the load of transmission line 2

\( P_{act} \)  
Active power transferred to the load impedance \( Z_\ell \)

\( P_{act1} \)  
Active power transferred to the load of transmission line 1

\( P_{act2} \)  
Active power transferred to the load of transmission line 2

\( P_{act1p} \)  
Active power transferred to the load of a two-port

\( P_{actmaxi} \)  
Available power of source with source impedance \( Z_i \)

\( \tau \)  
(Complex) reflection coefficient (without reference index referred to \( Z_0 \))

\( \tau(\overline{\Gamma}) \)  
Reflection coefficient as a function of reflection coefficient \( \Gamma \) (both reflection coefficients referred to \( Z_0 \))

\( \tau(Z) \)  
Reflection coefficient of \( Z \) (without reference index referred to \( Z_0 \))

\( \tau_1, Z_{01} \)  
Reflection coefficient at the beginning of a transmission line (at its input, referred to \( Z_{01} \))

\( \tau_{12}, Z_{02} \)  
Reflection coefficient at the beginning of transmission line 2 (at its input) referred to \( Z_{02} \)

\( \tau_{1p}, Z_{op1} \)  
(Complex) reflection coefficient \( \tau_{1p}, Z_{op1} \) as a function of reflection coefficient \( \tau_{op2}, Z_{op2} \)

\( \tau_2 \)  
Reflection coefficient at the end of a transmission line (of its load, referred to \( Z_0 \))

\( \tau_{21}, Z_{01} \)  
Reflection coefficient at the end of transmission line 1 (of its load) referred to \( Z_{01} \)

\( \tau_{22}, Z_{02} \)  
Reflection coefficient at the end of transmission line 2 (of its load) referred to \( Z_{02} \)

\( \tau_1 \)  
Reflection coefficient of source impedance \( Z_i \) (referred to \( Z_0 \))

\( \tau_{i1}, Z_{01} \)  
Reflection coefficient of source impedance \( Z_i \) referred to \( Z_{01} \)

\( \tau_{i2}, Z_{02} \)  
Reflection coefficient of the source impedance of transmission line 1 referred to \( Z_{01} \)

\( \tau_{ip}, Z_{op1} \)  
Reflection coefficient of the source impedance of a two-port referred to \( Z_{op1} \)

\( \tau_{ip}, Z_{op2} \)  
Reflection coefficient of the source impedance of a two-port referred to \( Z_{op2} \)

\( \tau_{ip}, Z_{op1} \)  
Reflection coefficient of load impedance \( Z_\ell \) (referred to \( Z_0 \))

\( \tau_{ip}, Z_{op2} \)  
Reflection coefficient of the load impedance of \( Z_\ell \) referred to \( Z_{02} \)

\( \tau_{ip}, Z_{op2} \)  
Reflection coefficient of the load impedance of a two-port referred to \( Z_{op2} \)

\( \tau_{ip}, Z_{op1} \)  
Reflection coefficient of the load impedance of a two-port referred to \( Z_{op1} \)

\( \tau_\ell \)  
Reflection coefficient of load impedance \( Z_\ell \) (referred to \( Z_0 \))

\( \tau_\ell, Z_{02} \)  
Reflection coefficient of the load impedance of \( Z_\ell \) referred to \( Z_{02} \)

\( \tau_\ell, Z_{op2} \)  
Reflection coefficient of the load impedance of a two-port referred to \( Z_{op2} \)

\( \tau_\ell, Z_{op1} \)  
Reflection coefficient of the load impedance of a two-port referred to \( Z_{op1} \)

\( \tau_\ell, Z_{01} \)  
Reflection coefficient of the load impedance of \( Z_\ell \) referred to \( Z_{01} \)

\( \tau_\ell, Z_{02} \)  
Reflection coefficient of the load impedance of \( Z_\ell \) referred to \( Z_{02} \)

\( \text{Re}(\rho) \)  
Per-unit-length resistance of a transmission line

\( R_0 \)  
Real part of \( Z_0 \), the complex characteristic impedance of a transmission line

\( R_{01} \)  
Real part of \( Z_{01} \), the complex characteristic impedance of transmission line 1

\( R_{02} \)  
Real part of \( Z_{02} \), the complex characteristic impedance of transmission line 2

\( \text{Im}(\rho) \)  
Complex amplitude of an ideal voltage source

\( X_0 \)  
Imaginary part of \( Z_0 \), the complex characteristic impedance of a transmission line

\( X_{01} \)  
Imaginary part of \( Z_{01} \), the complex characteristic impedance of transmission line 1

\( X_{02} \)  
Imaginary part of \( Z_{02} \), the complex characteristic impedance of transmission line 2
CHAPTER 2. RF BASICS

$Z_0$ Complex characteristic impedance of a transmission line
$Z_{01}$ Complex characteristic impedance of transmission line 1
$Z_{02}$ Complex characteristic impedance of transmission line 2
$Z_{0\|}$ Real characteristic impedance of a lossless transmission line, nominal (real) characteristic impedance of a transmission line
$Z_{0\rho1}$ Reference impedance of the reflection coefficients at a two-port's input
$Z_{0\rho2}$ Reference impedance of the reflection coefficients at a two-port's output
$Z_s$ Source impedance
$Z_L$ Load impedance

Greek letters

$\alpha_1$ Attenuation constant of transmission line 1
$\alpha_2$ Attenuation constant of transmission line 2
$\beta_1$ Phase constant of transmission line 1
$\beta_2$ Phase constant of transmission line 2

2.3 Basic definitions

- The active power transferred to an impedance is the real part of the product of the voltage at the impedance times the complex conjugate of the current flowing through the impedance.
- As usual, a power gain is the ratio of active powers, although “active power gain” would be a more precise terminology.
- The $Q$ factor of an impedance is the ratio of its reactance to its resistance.

2.4 Properties of lossy (coaxial) transmission lines

In the system under consideration, lossy coaxial transmission lines connect the source and the load to the matching network. These lines exhibit a nominal characteristic impedance of

$$Z_{0\|} = 50 \Omega,$$

(2.1)
a per-unit-length capacitance $C'$ and an attenuation $\alpha$ at the frequency of interest, usually specified as $\alpha_{dB/100\,m}$ in dB/100 m.

In this work, matching network and connecting lines will be operated within the short wave frequency range of about 1.8 – 30 MHz. It is well known that the coaxial line’s losses in this frequency range are dominated by its series resistance characterised by the per-unit-length resistance $R'$ ([26a], [26b], [27a], [27b], and [27c]). Thus certain approximations apply, in particular for the calculation of the attenuation constant $\alpha^\dagger$, the phase constant $\beta^\dagger$, and the line’s characteristic impedance $Z_0$

$^\dagger$Despite their names, $\alpha$, $\beta$, and thus $\gamma$ are not constant, but depend on the operating frequency.
(26h), (26b), (26c) or (27a)). Additionally the ratio of attenuation to phase constant may be assumed to be much less than unity, $\alpha / \beta \ll 1$, and $\alpha$ is approximately proportional to the square root of the operating frequency $f$ or angular frequency $\omega = 2\pi f$, respectively; i.e. $\alpha \sim \sqrt{\omega}$.

More specifically,

\[ \alpha = \frac{1}{2} \left( \frac{\text{/100 m}}{10 \text{ dB/100 m}} \right) \ln 10 \left[ \approx \frac{R'}{2Z_{0ll}} \right], \]  
(2.2)

\[ \beta \approx \omega Z_{0ll} C', \]  
(2.3)

\[ Z_0 \approx Z_{0ll} \left( 1 - j \frac{\alpha}{\beta} \right) \]  
(2.4)

Since $\alpha \ll \beta$, the worse approximation

\[ Z_0 \approx Z_{0ll} \]  
(2.5)

is usually employed.

The error introduced by this approximation is low for transmission lines terminated by its characteristic impedance. Conversely, it may be quite high for terminations strongly deviating from its characteristic impedance. This issue is discussed in Section 5.4.1 in detail.

Thus in the following sections all formulae will be derived for a complex characteristic impedance. To use the real characteristic impedance instead, we just have to put $X_0 = \ln \{Z_0\} = 0$.

As usual, the transmission line’s propagation constant \[^\dagger\]

\[ \gamma = \alpha + j\beta \]  
(2.6)

is always assumed to be complex.

### 2.5 Rewriting of some circuit parameters as functions of their reflection coefficients

All equations derived in the following sections will depend solely on reflection coefficients, S-parameters, and power gains. Thus, as a first step, some of the original circuit parameters have to be rewritten as functions of their reflection coefficients; namely, the real and imaginary part of the impedance or admittance, the change of the reference characteristic impedance of a reflection coefficient, and the reflection coefficient of the conjugate complex impedance.

In this section, the “reflection coefficient” is used as a conformal mapping with parameter $Z_0$. Its physical relevance for transmission lines is delayed to Section 2.6.
The “reflection coefficient” \( r \) of a complex impedance
\[
Z := R + jX
\]
referred to a reference impedance
\[
Z_0 := R_0 + jX_0,
\]
is defined by [26e]:
\[
r := \frac{Z - Z_0}{Z + Z_0}.
\]
Conversely, the complex impedance \( Z \) may be calculated as a function of \( r \) as
\[
Z = Z_0 \frac{1 + r}{1 - r}.
\]
Solving for the real part \( R \) of the impedance yields
\[
R = \text{Re} \left\{ \frac{Z_0 \frac{1 + r}{1 - r}}{1 + \frac{r}{1 - r}} \right\} = \frac{1}{2} \left( \frac{Z_0 \frac{1 + r}{1 - r} + Z_0^* \frac{1 + r^*}{1 - r^*}}{1 - r^2} \right),
\]
or equivalently,
\[
R = \frac{1}{2} \frac{Z_0 (1 + r - r^* - |r|^2) + Z_0^* (1 - r + r^* - |r|^2)}{|1 - r|^2}.
\]
By applying (2.8), it can be shown that
\[
R = \frac{R_0 (1 - |r|^2)}{1 - 2 \text{Re}\{r\} + |r|^2} = \text{Re}\{Z\}.
\]
Similar derivation of the imaginary part \( X \) of the impedance gives
\[
X = \text{Im} \left\{ \frac{Z_0 \frac{1 + r}{1 - r}}{1 + \frac{r}{1 - r}} \right\} = \frac{1}{2j} \left( \frac{Z_0 \frac{1 + r}{1 - r} - Z_0^* \frac{1 + r^*}{1 - r^*}}{1 - r^2} \right)
\]
or
\[
X = \frac{2R_0 \text{Im}\{r\} + X_0 (1 - |r|^2)}{1 - 2 \text{Re}\{r\} + |r|^2} = \text{Im}\{Z\}.
\]
In some cases we will use the admittance
\[
Y := \frac{1}{Z} = G + jB
\]
rather than the impedance \( Z \).
Rearranging (2.9) as \(-r = (\frac{1}{2} - \frac{1}{Z_0})/(\frac{1}{2} + \frac{1}{Z_0})\) and applying \((-r)^* = -r^*\), one can solve, in a similar fashion as above, for the real part \( G \) and the imaginary part \( B \) of the admittance:
\[
G = \frac{R_0 (1 - |r|^2)}{|Z_0|^2 (1 + 2 \text{Re}\{r\} + |r|^2)} = \text{Re} \left\{ \frac{1}{Z} \right\},
\]
\[
B = \frac{2R_0 \text{Im}\{r\} + X_0 (1 - |r|^2)}{|Z_0|^2 (1 + 2 \text{Re}\{r\} + |r|^2)} = \text{Im} \left\{ \frac{1}{Z} \right\}.
\]
Sometimes we may need to change the reference characteristic impedance from a given reflection coefficient $\tau$ referred to $Z_0$ to a reflection coefficient $\tau\tilde{Z}_0$ referred to $\tilde{Z}_0$, which is equivalent to a transformation of the coordinates from parameter $Z_0$ to parameter $\tilde{Z}_0$.

Both reflection coefficients describe the same impedance, hence combining of (2.9) and (2.10) yields

$$\tau\tilde{Z}_0 = \frac{(1 + \tau) Z_0 - (1 - \tau) \tilde{Z}_0}{(1 + \tau) Z_0 + (1 - \tau) \tilde{Z}_0} = \frac{\tau - \tau\tilde{Z}_0(\tilde{Z}_0)}{1 - \tau\tilde{Z}_0(\tilde{Z}_0)} = \frac{\tau + \tau\tilde{Z}_0(Z_0)}{1 + \tau\tilde{Z}_0(Z_0)} \tag{2.16}$$

where

$$\tau\tilde{Z}_0(\tilde{Z}_0) = \frac{\tilde{Z}_0 - Z_0}{\tilde{Z}_0 + Z_0} = \frac{Z_0 - \tilde{Z}_0}{Z_0 + \tilde{Z}_0} = -\tau\tilde{Z}_0(Z_0). \tag{2.17}$$

In what follows the reflection coefficient $\tau(Z^*)$ of the impedance’s $Z$ conjugate complex $Z^*$ will be derived.

Starting from the reflection coefficient’s definition in (2.9)

$$\tau(Z^*) = \frac{Z^* - Z_0}{Z^* + Z_0}$$

and applying (2.10) yields

$$\tau(Z^*) = \frac{Z^*_0 \frac{1 + \tau^*}{1 - \tau^*} - Z_0}{Z^*_0 \frac{1 + \tau^*}{1 - \tau^*} + Z_0} = \frac{Z^*_0(1 + \tau^*) - Z_0(1 - \tau^*)}{Z^*_0(1 + \tau^*) + Z_0(1 - \tau^*)}$$

In other words:

$$\tau(Z^*) = \frac{(Z_0 + Z^*_0) \tau^* - (Z_0 - Z^*_0)}{Z_0 + Z^*_0 - (Z_0 - Z^*_0) \tau^*} = \frac{Z_0 \tau^* - jX_0}{R_0 - jX_0 \tau^*} \tag{2.18}$$

We may conclude from (2.18) that $\tau(Z^*)$ equals $\tau^*$ for any $\tau$ for a real characteristic impedance ($X_0 = 0$) only⁴.

If $Z = Z_0$, thus $Z^* = Z^*_0$, (2.18) becomes

$$\tau(Z^*_0) = -j \frac{X_0}{R_0}. \tag{2.19}$$

⁴However, two $\tau(Z)$ exist where $\tau(Z^*) = \tau(Z) = \tau^*(Z)$, in particular $\tau(Z) = \pm 1$. They describe an impedance of $Z \to \infty$ and $Z = 0$, respectively, which are idealised impedance limits.
2.6 Active power in a transmission line and its transducer and power gain

![Figure 2.1: Basic transmission line with source and load](image)

In Figure 2.1, a transmission line with input reflection coefficient $r_1$ connected to a source $V_0$ and load with impedance $Z_\ell$ is depicted.

In what follows we are going to derive the transmission line’s power gain $g_L$.

All the necessary formulae that are given below are presented in detail in Appendix A.2 and Appendix A.3.

From (A.3), the reflection coefficient $r_1$ at the input of the line is obtained as

$$r_1 = r_\ell e^{-2\alpha l} e^{-j2\beta l}.$$  (2.20)

The transducer power gain $g_{T,L}$ of the line is given in (A.10) by

$$g_{T,L} := \frac{P_{act}}{P_{act_{maxi}}} = e^{-2\alpha l} \frac{|R_0 (1 - |r_\ell|^2) - 2X_0 \text{Im}\{r_\ell\}| |R_0 (1 - |r_\ell|^2) - 2X_0 \text{Im}\{r_\ell\}|}{|Z_0|^2 |1 - r_1 r_\ell e^{-2\alpha l} e^{-j2\beta l}|^2},$$  (2.21)

where $P_{act_{maxi}}$ is the available power, defined in (A.1) as

$$P_{act_{maxi}} = \frac{|V_0|^2}{8 \text{Re}\{Z_1\}}.$$  (2.22)

The power gain $g_L$ of the line is described in (A.12) as

$$g_L := \frac{P_{act}}{P_{act1}} = e^{-2\alpha l} \frac{R_0 (1 - |r_\ell|^2) - 2X_0 \text{Im}\{r_\ell\}}{R_0 (1 - e^{-4\alpha l} |r_\ell|^2) - 2X_0 e^{-2\alpha l} \text{Im}\{r_\ell e^{-j2\beta l}\}}.$$  (2.23)

A similar formula is derived in [28] and [29].
CHAPTER 2. RF BASICS

To illustrate the magnitude of the error in mismatched transmission lines, an example for (2.23) was calculated in Figure 2.2. Numerical values employed are:

\[ Z_0 \| = 50 \Omega, \quad C' = 101 \text{pF/m}, \quad a_{dB/100m} = 3.7 \text{dB/100m}, \quad \alpha = 0.004260 \frac{1}{\text{m}}, \quad \beta = 0.9519 \frac{1}{\text{m}}, \]

\[ R_0 = \text{Re} \{Z_0\}, \quad X_0 = \text{Im} \{Z_0\}, \]

and the line’s length \( l \). The transmission line was terminated by 5Ω in series to 1.06μH (equal to 5Ω + j200Ω).

Although \( \frac{g}{Z} = 0.004475 \ll 1 \), using (2.5) instead of (2.4) causes a deviation of up to 33% for the line’s power gain!

Despite the impressive result, we should keep in mind that the error depends on line length and termination impedance. Additionally, even the “more accurate” calculation using (2.4) relies on cable data which might be erroneous due to changes of its characteristic impedance caused by bending, connectors, etc.

However, regarding the system under consideration, we might conclude that calculations should always be done using the complex characteristic impedance (2.4), enabling calculation of its influence on the optimisation’s results and estimation of the errors introduced by using (2.5).

2.7 Transferred active power in a source, line 1, two-port, line 2, and load system

![Diagram of a system consisting of a source, transmission line 1, two-port tp, transmission line 2, and load](image)

Figure 2.3: Definitions and setup of a system consisting of a source, transmission line 1, two-port tp, transmission line 2, and load
As shown in Figure 2.3, the input of a two-port (tp) is connected to a source (source impedance $Z_{s}$) using transmission line 1 of length $l_{1}$. Its output is connected to a load $Z_{l}$ using transmission line 2 of length $l_{2}$. The complex characteristic impedances of the lines are $Z_{01,2}$, their attenuation constants $\alpha_{1,2}$, and their phase constants $\beta_{1,2}$.

The two-port will be characterised by its power gain

$$g_{tp} = \frac{P_{act \text{tp}}}{P_{act \text{tp}}}$$

(2.24)

and its input impedance, which depend on its load impedance.

However, for our purpose it’s more convenient to replace the two-port’s input impedance with its reflection coefficient $\tau_{1tp, Z_{\text{opt1}}}$, which is referred to $Z_{\text{opt1}}$, and the two-port’s load with its reflection coefficient $\tau_{2tp, Z_{\text{opt2}}}$, which is referred to $Z_{\text{opt2}}$.

Choosing $Z_{\text{opt1}} = Z_{01}$ and $Z_{\text{opt2}} = Z_{02}$ simplifies calculations, but such choice might be impossible if measured values are used. For example, the network analyser measurement results will typically be referred to $Z_{\text{opt1}} = Z_{\text{opt2}} = 50 \, \Omega$. However, $Z_{01,2}$ might be complex.

If $Z_{\text{opt1}} \neq Z_{01}$ or $Z_{\text{opt2}} \neq Z_{02}$, then $\tau_{1tp, Z_{\text{opt1}}}$ has to be rewritten in $\tau_{1tp, Z_{01}} = \tau_{21, Z_{01}}$ or $\tau_{2tp, Z_{02}} = \tau_{12, Z_{02}} = \tau_{t, Z_{02}} e^{-2\alpha_{2}l_{2}} e^{-i2\beta_{2}l_{2}}$ in $\tau_{2tp, Z_{\text{opt2}}}$ by changing the reflection coefficient’s reference characteristic impedance using (2.16).

Finally, the active power delivered to the load $Z_{l}$ in the system of Figure 2.3 is given by

$$P_{act} = P_{act \text{maxi}} g_{T\text{tot}},$$

(2.25)

where $g_{T\text{tot}}$ is the transducer power gain of the complete system.

The system’s transducer power gain $g_{T\text{tot}}$ might be computed using (2.21) and (2.23):

$$g_{T\text{tot}} := \frac{P_{act}}{P_{act \text{maxi}}} = g_{T, L1}(\tau_{t, Z_{01}}, \tau_{1tp, Z_{01}}) g_{tp}(\tau_{2tp, Z_{\text{opt2}}}) g_{L2}(\tau_{t, Z_{02}})$$
which after simplification yields:

\[
g_{\text{Tot}} = e^{-2(\alpha_{1l_1} + \alpha_{2l_2})} \left[ \frac{R_{01} \text{Im}\{r_{l_1, Z_{01}}\}}{|Z_{01}|^2} \frac{R_{01} (1 - |r_{\text{1tp}, Z_{01}}|^2) - 2X_{01} \text{Im}\{r_{\text{1tp}, Z_{01}}\}}{1 - r_{l_1, Z_{01}} r_{\text{1tp}, Z_{01}} e^{-2\alpha_{l_1} l_1} e^{-j2\beta_{l_1} l_1}} \frac{R_{02} (1 - |r_{l, Z_{02}}|^2) - 2X_{02} \text{Im}\{r_{l, Z_{02}}\}}{R_{02} (1 - e^{-4\alpha_{l_2} l_2} |r_{l, Z_{02}}|^2) - 2X_{02} e^{-2\alpha_{l_2} l_2} \text{Im}\{r_{l, Z_{02}} e^{-j2\beta_{l_2}}\}} \right].
\]

(2.26)

To minimise the losses of the complete system, its power gain \(g_{\text{tot}}\) is used instead, which is given by

\[
g_{\text{tot}} := \frac{P_{\text{act}}}{P_{\text{act}11}} = g_{l_1}(r_{\text{1tp}, Z_{01}}) g_{\text{1tp}}(r_{\text{1tp}, Z_{01}}) g_{l_2}(r_{l, Z_{02}}),
\]

which after simplification yields:

\[
g_{\text{tot}} = e^{-2(\alpha_{1l_1} + \alpha_{2l_2})} \frac{R_{01} (1 - |r_{\text{1tp}, Z_{01}}|^2) - 2X_{01} \text{Im}\{r_{\text{1tp}, Z_{01}}\}}{R_{01} (1 - e^{-4\alpha_{l_1} l_1} |r_{\text{1tp}, Z_{01}}|^2) - 2X_{01} e^{-2\alpha_{l_1} l_1} \text{Im}\{r_{\text{1tp}, Z_{01}} e^{-j2\beta_{l_1}}\}} \frac{R_{02} (1 - |r_{l, Z_{02}}|^2) - 2X_{02} \text{Im}\{r_{l, Z_{02}}\}}{R_{02} (1 - e^{-4\alpha_{l_2} l_2} |r_{l, Z_{02}}|^2) - 2X_{02} e^{-2\alpha_{l_2} l_2} \text{Im}\{r_{l, Z_{02}} e^{-j2\beta_{l_2}}\}}.
\]

(2.27)

### 2.8 Summary

In this chapter we have recalled some definitions and shown how different variables can be written in terms of reflection coefficients. The power gain of a single transmission line and of the complete matching system is given. Several notations used throughout this work are also presented.
Chapter 3

Lossy L, π, and T Matching Networks

3.1 Introduction

The following chapter addresses the losses in L, π, and T matching networks. Firstly, we propose a simple and synthetical loss model, which will be validated using the results of [22], [24], and [25]. Next, after choosing a proper parametrisation, we consider the networks’ behaviour if source and load are purely resistive. Application of the model yields the optimum parameter’s (and load resistance’s) value for maximum active power transfer.

3.2 Definitions/notations

From this chapter on, the following definitions/notations will be used:

Latin letters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>Real coefficient of the series loss model</td>
</tr>
<tr>
<td>( a_{0d} \Phi_M )</td>
<td>Constant linearisation coefficient of ( \frac{dC}{dC} \Phi_M )</td>
</tr>
<tr>
<td>( a_{0r} )</td>
<td>Constant linearisation coefficient of ( r_{PS} )</td>
</tr>
<tr>
<td>( a_{0L} )</td>
<td>Constant linearisation coefficient of series inductance ( L_s )</td>
</tr>
<tr>
<td>( a_{0N} \Phi_A )</td>
<td>Constant linearisation coefficient of ( N_T \Phi_A )</td>
</tr>
<tr>
<td>( a_{0R} )</td>
<td>Constant linearisation coefficient of series resistance ( R_s )</td>
</tr>
<tr>
<td>( a_{0\Phi_A} )</td>
<td>Constant linearisation coefficient of ( \Phi_A )</td>
</tr>
<tr>
<td>( a_{0\Phi_M} )</td>
<td>Constant linearisation coefficient of ( \Phi_M )</td>
</tr>
<tr>
<td>( a_{1d} \Phi_M )</td>
<td>Linear linearisation coefficient of ( \frac{dC}{dC} \Phi_M )</td>
</tr>
<tr>
<td>( a_{1r} )</td>
<td>Linear linearisation coefficient of ( r_{PS} )</td>
</tr>
<tr>
<td>( a_{1L} )</td>
<td>Linear linearisation coefficient of series inductance ( L_s )</td>
</tr>
<tr>
<td>( a_{1N} \Phi_A )</td>
<td>Linear linearisation coefficient of ( N_T \Phi_A )</td>
</tr>
<tr>
<td>( a_{1R} )</td>
<td>Linear linearisation coefficient of series resistance ( R_s )</td>
</tr>
<tr>
<td>( a_{1\Phi_A} )</td>
<td>Linear linearisation coefficient of ( \Phi_A )</td>
</tr>
<tr>
<td>( a_{1\Phi_M} )</td>
<td>Linear linearisation coefficient of ( \Phi_M )</td>
</tr>
<tr>
<td>( a_{2R} )</td>
<td>Quadratic linearisation coefficient of series resistance ( R_s )</td>
</tr>
</tbody>
</table>
$a_n$  
$n$-th real coefficient of the series loss model

$\delta_n$  
$n$-th real coefficient of the parallel loss model

$b_n$  
$n$-th real exponent of the series loss model

$\delta_b_n$  
$n$-th real exponent of the parallel loss model

$b_{N}$  
Real exponent of the series loss model with the highest index, thus the highest real exponent of the model

$C_{m1}$  
Capacitance calculated from imaginary part of parallel-to-series transformed $G_1$ and $C_1$ in a $\pi$ network

$C_{m1}$  
Capacitance calculated from imaginary part of series-to-parallel transformed $R_1$ and $C_1$ in a $T$ network

$C_{m2}$  
Capacitance calculated from imaginary part of parallel-to-series transformed $G_{\ell}$ and $C_2$ in a $\pi$ network

$C_{m2}$  
Capacitance calculated from imaginary part of series-to-parallel transformed $R_{\ell}$ and $C_2$ in a $T$ network

$C_p$  
Parallel capacitance

$C_s$  
Series capacitance

$d$  
Diameter of round wire used to make the windings of a solenoidal coil

$d_C$  
Mean diameter of a solenoidal coil

$D$  
Distance of wire centres (winding centres) in a solenoidal coil (pitch)

$g_T$  
Power gain of a $T$ network

$g_{\pi,nma}$  
Power gain of a $\pi$ network which is (exactly) matched at its input

$g_{\pi}$  
Power gain of a $\pi$ network which is (exactly) matched at its input

$G_l$  
Source conductance

$G_{\ell}$  
Load conductance — “special” definition in matching networks: in case of a $\pi$ network, load conductance (real part of load admittance) of that network, in case of a $T$ network, $G_{\ell} = \frac{1}{R_{\ell}}$, where $R_{\ell}$ is the load resistance (real part of load impedance)

$G_m$  
“Virtual” conductance, the parameter describing the degree of freedom in a $T$ network

$G_{m1}$  
Real part of series-to-parallel transformed $R_1$ and $C_1$ in a $T$ network

$G_{m2}$  
Real part of series-to-parallel transformed $R_{\ell}$ and $C_2$ in a $T$ network

$G_{mlim}$  
Limit of $G_m$ derived by solving (3.34) — may yield several solutions — without additional index the highest valid solution of (3.34) ($T$ network exactly matched at its input) or (B.16) ($T$ network exactly matched at its output)

$G_p$  
Parallel conductance of parallel inductance $L_p$

$G_{p0}$  
Constant conductance term of the parallel loss model

$G_{p0}$  
Constant parallel conductance of constant parallel inductance $L_{p0}$

$L_0$  
Complex amplitude of an ideal current source

$K$  
(Tabulated) factor to derive the inductance of an air core single-layer solenoidal coil

$K(\ldots)$  
(Tabulated) factor to derive the inductance of an air core single-layer solenoidal coil as a function of . . .

$l_{ax}$  
Axial length of rectangular wire used to make the windings of a solenoidal coil

$l_{rd}$  
Radial length of rectangular wire used to make the windings of a solenoidal coil

$l_C$  
Mean length of a solenoidal coil

$L_p$  
Parallel inductance

$L_{p0}$  
Constant parallel inductance

$L_{p1}$  
Parallel inductance of the left $L$ network if a $T$ network is separated in two $L$ networks

$L_{p2}$  
Parallel inductance of the right $L$ network if a $T$ network is separated in two $L$ networks

$L_s$  
Series inductance

$L_{s0}$  
Constant series inductance

$L_{s1}$  
Series inductance of the left $L$ network if a $\pi$ network is separated in two $L$ networks

$L_{s2}$  
Series inductance of the right $L$ network if a $\pi$ network is separated in two $L$ networks

$L_{snc}$  
One series inductance of switchable series connected inductors (sorted in ascending order of inductances)

$L_{\nu\nu_n}$  
Last series inductance of switchable series connected inductors (highest inductance)

$n$  
Natural number
CHAPTER 3. LOSSY L, π, AND T MATCHING NETWORKS

\[ n_C \] Number of one switchable series connected inductor

\[ n_T \] Partial number of turns

\[ N \] Natural number

\[ N_C \] Total number of switchable series connected inductors

\[ N_T \] Total number of turns

\[ P_{act} \] Active power transferred to the parallel conductance \( G_p \)

\[ P_{trans} \] Active power transferred to the series resistance \( R_s \)

\[ Q \] \( Q \) factor of the central inductor of a \( \pi \) or \( T \) network

\[ r_{PS} \] Ratio of the per-unit-length resistance including proximity effect to the per-unit-length-resistance calculated for a pure skin effect

\[ r'_{PS}(\ldots,\ldots) \] Ratio of the per-unit-length resistance including proximity effect to the per-unit-length-resistance calculated for a pure skin effect as a function of \( \ldots \) and \( \ldots \)

\[ R_l \] Source resistance

\[ R_t \] Load resistance — “special” definition in matching networks: in case of a \( T \) network, load resistance (real part of load impedance) of that network, in case of a \( \pi \) network, \( R_l = \frac{1}{G_t} \), where \( G_t \) is the load conductance (real part of load admittance)

\[ R_m \] “Virtual” resistance, the parameter describing the degree of freedom in a \( \pi \) network

\[ R_m1 \] Real part of parallel-to-series transformed \( G_t \) and \( C_t \) in a \( \pi \) network

\[ R_m2 \] Real part of parallel-to-series transformed \( G_t \) and \( C_t \) in a \( \pi \) network

\[ R_{lim} \] Limit of \( R_m \) derived by solving (3.21) — may yield several solutions — without additional index the highest valid solution of (3.21) (\( \pi \) network exactly matched at its input) or (B.7) (\( \pi \) network exactly matched at its output)

\[ R_s \] Series resistance of series inductance \( L_s \)

\[ R_{00} \] Constant resistance term of the series loss model

\[ R_{0c} \] Constant series resistance of constant series inductance \( L_{0c} \)

\[ R_{nc} \] Series resistance of one of the series inductances of switchable series connected inductors

\[ R_{ncC} \] Series resistance of the last series inductance of switchable series connected inductors

\[ S_{nc} \] Series resistance of one of the series inductances of switchable series connected inductors

\[ S_{ncC} \] Switch shorting one of the series inductances of switchable series connected inductors

\[ S_{t} \] Switch shorting the last series inductance of switchable series connected inductors

\[ S_{\pi} \] Abbreviation for an inductance section (3.5.3.2)

\[ Y_{11} \] Admittance of source and first L network as defined in Figure 3.15

\[ Y_{21} \] Admittance of source, first L network, and parallel conductance as defined in Figure 3.15

\[ Y_{1t} \] Admittance of load, second L network, and parallel conductance as defined in Figure 3.15

\[ Y_{2t} \] Admittance of load and second L network as defined in Figure 3.15

\[ Z_{11} \] Impedance of source and first L network as defined in Figure 3.14

\[ Z_{1t} \] Impedance of load, second L network, and series resistance as defined in Figure 3.14

\[ Z_{21} \] Impedance of load and second L network as defined in Figure 3.14

\[ Z_{2t} \] Impedance of load and second L network as defined in Figure 3.14

Greek letters

\[ \Lambda \] An abbreviation for the reactance of the parallel-to-series transformed lossy inductor of a \( T \) network \( \Lambda = \frac{1}{\omega C_p + j2L_2} \)

\[ \Phi_A \] Ratio of an inductor’s AC resistance to a straight wire’s AC resistance at \( \omega \rightarrow \infty \) derived by [25] (straight wire is uncoiled windings’ wire)

\[ \Phi_A(\ldots,\ldots) \] Ratio of an inductor’s AC resistance to a straight wire’s AC resistance at \( \omega \rightarrow \infty \) derived by [25] as a function of \( \ldots \), \( \ldots \), and \( \ldots \) (straight wire is uncoiled windings’ wire)

\[ \Phi_M \] Ratio of an inductor’s AC resistance to a straight wire’s AC resistance at \( \omega \rightarrow \infty \) derived by [24] (straight wire is uncoiled windings’ wire)

\[ \Phi_M(\ldots,\ldots) \] Ratio of an inductor’s AC resistance to a straight wire’s AC resistance at \( \omega \rightarrow \infty \) derived by [24] as a function of \( \ldots \) and \( \ldots \) (straight wire is uncoiled windings’ wire)

\[ \psi_i \] Implicit function of parallel conductance \( G_p \) in a \( T \) network

\[ \psi_{\pi} \] Implicit function of series resistance \( R_s \) in a \( \pi \) network
3.3 Considered types of networks, losses involved

If a matching network is used where radio frequency generators drive complex loads, it is usually a L, π, or T network with a “central” inductor as indicated in Figure 3.1. The four possible L networks are just limits of π or T networks if one of the two variable capacitances is set to zero or tends to infinity, respectively.

Thus consideration of π and T networks includes all networks we are interested in.

Variable capacitors and inductors used in those networks are not ideal. They exhibit a couple of parasitic elements as indicated in [30b] or [9b], for example. This is also true for any connection made between them. However, some parasitics may be neglected.

In the frequency range of 1.8–30 MHz variable capacitors are usually high-Q air or vacuum variable types, especially in high power applications. Both exhibit very low (dielectric) losses which are small and negligible compared to the losses of the variable inductor. Additionally (high power) variable capacitors are usually employed for connections of high cross-sectional area. Thus parasitic inductances and resistances of those connections may also be neglected.

These assumptions hold even for switchable automatic antenna matching units, developed in [4], [5], [6], [7], and [8]. Their tuning procedure is enhanced in [1] and “quiet” tuning is considered in [31], [2]. Useful element values are proposed in [32], [3]. For example, a realisation is given in [33] for a frequency range of 1.8–30 MHz. Such automatic antenna matching units use switchable capacitors, whose dielectric losses are higher than those of air or vacuum variable capacitors, but still low compared to the losses of the switchable inductors due to the relatively low frequency. However, care has to be taken because the nonzero minimum inductance and capacitance steps may prevent exact matching.

Compared to variable capacitors, the variable inductor involves relatively high losses of different origins (which are indicated in detail in the following section). Thus the number of variable inductors is usually reduced to a minimum — as in the networks of Figure 3.1 which use only one.
The losses of the inductor may be described by a series resistance, but variable inductors additionally have a parasitic parallel capacitance which may be estimated according to [34b], [35], [36], [37], and [38]. An approach more focused on air core single-layer solenoidal coils is given in [24], [39], and [40]. It limits the useful frequency range of the coil. However, for a useful variable inductor the parallel capacitance should be negligible in the frequency range it’s used.

Parasitic inductances and resistances of the variable inductor’s connections may simply be added to its inductance and loss resistance.

![Simplified equivalent circuit of a) a π and b) a T network](image)

Figure 3.2: Simplified equivalent circuit of a) a π and b) a T network

Combining all preceding considerations yield the π and T network’s simplified equivalent circuits of Figure 3.2, which are quite common in literature (see e.g. [9c] and [18]).

### 3.4 Losses of a variable inductor

#### 3.4.1 Modeling losses of a variable inductor

Losses in inductors have different origins (refer to [30c] for a thorough review\(^1\)). Losses are introduced by finite conductance of the wire, skin effect, proximity effect, or core losses (whenever a core is used); all of which may be described by series resistors, and a parallel resistor describing losses by finite resistance of the wire’s insulation. The latter may be neglected for variable inductors used in matching networks, because usually there is a certain gap between adjacent wires to ensure a low parasitic parallel capacitance.

In our application the bandwidth of the generated signal is low compared to its mean frequency and we are interested in narrow band matching only — we simply intend to match at some given angular frequency \(\omega\). We therefore need to know the

---

\(^1\)However, all formulae used in [30c] cannot be applied unless their references are checked to be valid within the frequency range of interest. For example, the formula describing eddy current losses assumes that the current flows through the complete cross-section of the wire, which would require much lower frequencies than those in our application or moderately lower frequencies and the use of litz wire.
losses' dependence on the number of turns, hence on the inductance, rather than its
dependence on frequency. Due to the fixed matching frequency, we also may use the
inductor’s reactance \( \omega L_s \) instead of its inductance \( L_s \).

The equivalent resistance of all types of losses indicated above is entirely positive
and increases with an increasing number of turns. Thus the simplest model would
consist of a sum of functions of the type \( a (\omega L_s)^b \) where \( a > 0 \) (positive losses) and
\( b > 0 \) (increase with increasing number of turns or inductance). Each term (or a
sum of several terms) would then describe the different types of losses.

Then the total equivalent series resistance is given by

\[
R_s = R_{s0} + \sum_{n=1}^{N} a_n (\omega L_s)^{b_n}, \text{ where } R_{s0}, a_n \geq 0, \ b_N > b_{N-1} > \ldots > b_1 > 0. \tag{3.1}
\]

\( R_{s0}, a_n, \) and \( b_n \) are real and may all be frequency dependent.

Theoretically the model given by (3.1) is mainly restricted by the assumption
\( R_{s0}, a_n \geq 0 \), in particular if we describe the losses by several terms of a Taylor
series (however, since real coefficients \( b_n \) are used (3.1), it generally differs from a
Taylor series employing integer exponents). Although \( R_s \) has to be entirely positive
in the range of \( \omega L_s \) the series applies, \( R_{s0} \) or some \( a_n \) could be negative provided
the sum of all terms is still positive. However, the proofs in the following sections
require \( R_{s0}, a_n \geq 0 \), and at least we may use real \( b_n > 0 \) to further improve the series
approximation.

Of course the restriction to \( R_{s0}, a_n \geq 0 \) is only useful if the losses of the inductor
may be described by the model, which will be proved in the following sections for
derifferent types of inductors used in matching networks.

The model also allows piecewise description of the losses, provided each line segment
can be described by (3.1) and \( R_s \) exhibits no steps at the segments’ limits (Section 4.6
addresses the impact of such steps).

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3.4.2 Application and validation of the proposed model to different types of inductors used in matching networks

3.4.2.1 Variable air inductors

![Diagram of variable air inductor with labels: \( d \), \( d_C \), \( l_{rad} \), \( l_{ax} \), \( l_C = N_T D \), \( R_s \), \( L_s \)]

Figure 3.3: a) Air core single-layer solenoidal coil of round wire b) Simplified equivalent circuit of the inductor according to Section 3.3

In Figure 3.3 a), an air core single-layer solenoidal coil is shown. Its windings consist of round wire whose diameter is \( d \), or of rectangular wire, whose radial length is \( l_{rad} \) and axial length is \( l_{ax} \). As an example, the figure is drawn for a number of turns \( N_T = 3 \). \( D \) denotes the distance between centres of adjacent wires, \( d_C \) the mean diameter of the coil, and \( l_C = N_T D \) the mean length of the coil.

Variable air core inductors usually look like a single-layer solenoid as illustrated in Figure 3.3. Additionally, they have a rotating contact which slides along the windings and provides a tap for the required amount of inductance.

As indicated in the previous section, we need to derive the coils's series loss resistance \( R_s \) as a function of its inductance \( L_s \) or its reactance \( \omega L_s \). However, \( R_s \) is usually calculated as a function of \( N_T \), the number of coil’s turns. Thus at first we have to derive the coil’s inductance as a function of its number of turns.

---

\(^1\)Care has to be taken regarding the definition of the coil’s mean length, because it might differ if inductance or loss resistance are calculated. As in [41c], the inductance is usually calculated using \( l_C = N_T D \). However, to calculate its series loss resistance, the helical coil is often approximated by parallel circular rings, whose dimensions are similar to those of Figure 3.3 (except \( l_C \)), but the coil’s mean length is then defined by \( (N_T - 1) D \).

Fortunately we do not have to distinguish between both definitions in the following sections, because [23] and [25] use the number of turns (and never its length) in all calculations and [24] defines the coil’s length similar to Figure 3.3, although it examines coils of many windings, where \( N_T \approx N_T - 1 \) would hold.
The inductance of air core single-layer solenoidal coils is given by [41b]

\[
L_s = \mu_0 \frac{\pi}{4} \frac{d_C}{D} \left( \frac{d_C}{D} \right)^2 \frac{l_C}{d_C} K \left( \frac{l_C}{d_C} \right),
\]

(3.2)

where \( \mu_0 \approx 4\pi 10^{-7} \text{ H/m} \) and \( l_C = N_T D \). The factor \( K \) is a function of the shape ratio \( \frac{l_C}{d_C} \). Note that an estimate of \( K \) is given in [30a] as \( K \approx \frac{1}{1 + 0.45 \frac{d_C}{l_C}} \) for \( \frac{l_C}{d_C} \geq \frac{1}{3} \) with a \(< 1\% \) error\(^2\).

In order to cater for different wire topologies, such as wires of circular or rectangular cross-section, some correction in (3.2) has to be made (see [41d], [41e], and [41c]). However, those corrections are usually small. Additionally, [41f] introduces a frequency correction, but it’s also small and negligible.

Hence, using the basic inductance value of (3.2) provides sufficient accuracy. To gain insight in the inductance’s dependence on the number of windings, we simply have to draw \( \frac{l_C}{d_C} K \left( \frac{l_C}{d_C} \right) \) as a function of \( \frac{l_C}{d_C} \) because all other factors do not depend on the number of turns and \( \frac{l_C}{d_C} \sim N_T \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure_3.4}
\caption{\( \frac{l_C}{d_C} K \left( \frac{l_C}{d_C} \right) \) as a function of \( \frac{l_C}{d_C} \) if a) \( \frac{l_C}{d_C} \in [0, 2] \) or b) \( \frac{l_C}{d_C} \in [0, 100] \)
\end{figure}

Inspection of Figure 3.4 reveals a linear function if \( \frac{l_C}{d_C} \gg 1 \), but a nonlinear function if \( \frac{l_C}{d_C} \ll 1 \).

Depending on dimensions and maximum number of turns of the used variable air core inductor we may assume a linear function in the range of interest, provided its accuracy suffices. If the approximation’s accuracy is too poor, it’s better to perform a piecewise linearisation which also may include all correcting terms mentioned above.

Performing the linearisation yields a functional description of the inductance \( L_s \) depending on the inductor’s number of turns \( N_T \),

\[
L_s \approx a_{0L} + a_{1L} N_T,
\]

(3.3)

from which we obtain by solving for the number of turns

\[
N_T \approx \frac{L_s - a_{0L}}{a_{1L}}.
\]

\(^2\)[30a] indicates \( \frac{l_C}{d_C} \geq \frac{1}{3} \), but compares the approximation to an inductance formula given in a different reference instead of comparing it to the tabulated values of [41b].
By analysing Figure 3.4 it can easily be observed that $a_{0L} \leq 0$ and $a_{1L} \geq 0$.

In the next step we calculate the loss itself.

In air-core inductors several different types of losses occur. As indicated in [34a], the current in the wire causes a current crowding in that wire towards its surface. This is the skin effect and is also observed in straight wires. The currents in adjacent wires lead to additional current crowding — the proximity effect. Both effects raise the inductor’s AC resistance above its DC resistance.

Although losses caused by a pure skin effect may be calculated quite easily, description of the combined losses including the proximity effect is quite complicated and may be performed for round wires only.

Besides early approaches in [42] and [43] (corrected in [44]), the basis of the most common loss formulae are given in [45], [46], [47], and [48] — which derive the series loss resistance for short coils ($l_C \ll d_C$) or coils with a large number of turns. These works are summarised in [49] and [9a]. In addition, [24] checks the formula for coils with a large number of turns by measuring the losses of solenoidal air-core inductors of 30–50 turns and uses the results to correct that formula empirically. In [25], the formulae for short coils and coils of many turns given in [47] are combined and verified. The resulting (unrestricted) single formula is compared to several measurements of different works including [24]. In [39] a summary of all different existing formulae is given and the limiting conditions for their applicability is indicated. Note that in particular [47] does not state those limits explicitly. Furthermore, [39] confirms the results of [25] by additional measurements (in the range of 10 kHz to 23 MHz, all measurements are included in [50]) and derives a transmission line model of the coil incorporating parasitic capacitances. Finally, in [40] the preceding results are used to design optimum $Q$ factor coils, which is described a little bit more detailed in [51a].

Additionally, [22] and [23] derive the series loss resistance of short coils using a different method involving fourier series, apply the results to multiturn loop antennas and check them by appropriate measurements. However, [22] and [23] assume the skin depth to be much lower than the wire’s radius, but this assumption also holds for typical wire diameters of several millimeters used in solenoidal air-core inductors if the frequency is in the range of 1.8 – 30 MHz.

Regarding the publication dates of the preceding citations it should be noted that they apparently quite completely solve the problem of deriving losses of air-core inductors. “Modern” authors seem to focus on objectives applicable to integrated circuits (in particular losses of spiral inductors in MMICs, e.g. described in [52], [53], and [54]) or to coils in switching power supplies, e.g. considered in [55], [56], [57] — compared in [58] — and [59]. The former is not applicable due to the different shape of the inductor, the latter due to the assumption that the entire magnetic field is parallel to the coil’s axis. Although there are also new techniques to calculate parasitic capacitances as those derived in [60], this work assumes that adjacent wires touch, but this winding style is not used for solenoidal air-core inductors to keep parasitic capacitances low and to ensure a high self resonance frequency.
Now the main question is:

*If we compute all applicable loss formulae indicated above, can \( R_s \) be described by (3.1) including all proposed restrictions?*

First, we consider short single-layer solenoidal air-core inductors. In [23], \( r'_{\text{PS}} \), the ratio of the per-unit-length resistance including proximity effect to the per-unit-length-resistance calculated for a pure skin effect, is defined. Assuming \( d < D, \)
\( d \ll d_C, \)
\( \frac{1}{\frac{d}{D} \left( \frac{\mu_0}{\pi} \right)^2} \ll 1 \), \( N_T^2 D^2 \ll d_C^2 \) (which in our application limits the maximum number of turns the following formula applies to), and \( N_T \frac{c_0}{d} \ll 1 \), where \( c_0 \) is the speed of light in vacuum. Then the series loss resistance \( R_s \) is obtained in [23] as

\[
R_s = \frac{d_C}{d} \sqrt{\frac{\omega \mu_0}{2}} N_T \left( 1 + r'_{\text{PS}} \left( N_T, \frac{D}{d} \right) \right).
\]  

(3.5)

\( r'_{\text{PS}} \) is tabulated in [22] as a function of \( N_T \) and \( \frac{D}{d} \) for \( N_T \in \{2, \ldots, 8\} \) and \( \frac{D}{d} \in [1, 4] \).

![Graphs](image)

*Figure 3.5: a) \( r'_{\text{PS}} \) and b) \( N_T (1 + r'_{\text{PS}}) \) as functions of \( N_T \) for \( \frac{D}{d} \in \{1.05, 1.1, 1.15, 1.4, 2.2, 4\} \).*

The resulting \( r'_{\text{PS}} \) is shown in Figure 3.5 a). \( r'_{\text{PS}} = 0 \) was added for \( N_T \in [0, 1] \) (approximately assuming a pure skin effect). It’s interesting to note that the deviation from the formula for short coils given in [47] is < 5% in its applicable range (\( \frac{d}{D} \leq 0.6 \) or \( \frac{D}{d} \geq 1.7 \)).

Performing a piecewise linearisation of \( r'_{\text{PS}} \),

\[
r'_{\text{PS}} \approx a_{0r} + a_{1r} N_T,
\]

(3.6)

we obtain by combining with (3.5)

\[
R_s \approx \frac{d_C}{d} \sqrt{\frac{\omega \mu_0}{2}} (1 + a_{0r}) N_T + a_{1r} N_T^2.
\]

(3.7)
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Figure 3.6: a) $1 + a_{0r}$ and b) $a_{1r}$ for $\frac{D}{d} \in \{1.05, 1.1, 1.15, 1.4, 2.2, 4\}$. (Dots at $N_T = 0, 1, \ldots, 7$ are calculated values of a piecewise linearisation within the interval $[N_T, N_T + 1)$). Dashed lines illustrate the tendency of values.

Inspection of Figure 3.6 yields $1 + a_{0r} > 0$ and $a_{1r} \geq 0$ if $\frac{D}{d} \geq 1.1$. Taking into account the tendency of the linearisation coefficients, we would expect those relations to hold if $N_T \geq 8$. However, we cannot predict the behaviour for $1 \leq \frac{D}{d} < 1.1$ due to lack of data, but $\frac{D}{d}$ of useful single-layer solenoidal air-core inductors are never within this range because it would involve high parasitic capacitances.

Next, we consider single-layer solenoidal air-core inductors with a large number of turns.

In [24], $\Phi_M$ is defined as the ratio of the inductor’s AC resistance to a straight wire’s AC resistance at $\omega \rightarrow \infty$, where the straight wire’s length equals the length of the coiled wire. Assuming $\frac{d}{D} \sqrt{\frac{\omega \mu \rho}{\epsilon}} \gg 1$, the series loss resistance $R_s$ is obtained in [24] as

$$R_s = \frac{dC}{d} \sqrt{\frac{\omega \mu \rho}{2}} N_T \Phi_M \left( \frac{N_T D}{dC}, \frac{d}{D} \right) = \frac{dC}{d} \sqrt{\frac{\omega \mu \rho}{2}} \frac{dC}{dC} \frac{lC}{lC} \Phi_M \left( \frac{lC}{dC}, \frac{d}{D} \right). \quad (3.8)$$

In [24], $\Phi_M$ is tabulated as a function of $\frac{lC}{dC}$ and $\frac{d}{D}$ for $\frac{lC}{dC} \in [0, \infty)$ and $\frac{d}{D} \in [0.1, 1]$. An indication for the range of $\frac{d}{D}$ and $\frac{lC}{dC}$ the formula is valid in is given in [39].

Figure 3.7: a) $\Phi_M$ and b) $\frac{lC}{dC} \Phi_M$ as functions of $\frac{lC}{dC}$ for $\frac{d}{D} \in \{0.1, 0.4, 0.6, 0.7, 0.8, 0.9, 1\}$.
If we perform a piecewise linearisation of $R_s$ as follows:

$$\begin{align*}
R_s &\approx \begin{cases} 
\frac{dc}{d} \sqrt{\frac{\omega \mu_0}{2}} \frac{dc}{dC} \left( a_0 \Phi_M + a_1 \Phi_M \frac{dc}{dC} \right) & \text{if } \frac{d \Phi_M}{d \frac{dc}{dC}} \geq 0 \\
\frac{dc}{d} \sqrt{\frac{\omega \mu_0}{2}} \frac{dc}{dC} \left( a_{0\text{dd}} \Phi_M + a_{1\text{dd}} \Phi_M \frac{dc}{dC} \right) & \text{if } \frac{d \Phi_M}{d \frac{dc}{dC}} < 0
\end{cases} 
\end{align*}$$

we obtain $a_0 \Phi_M$, $a_1 \Phi_M$, $a_{0\text{dd}} \Phi_M$, $a_{1\text{dd}} \Phi_M$ $\geq 0$ (as indicated in Table 3.1 and Table 3.2). If $\frac{dc}{dC} \geq 10$, the first type of linearisation applies, because $\Phi_M$ moderately increases in this case (check [24]). It's interesting to note that the practically relevant case $\frac{d}{D} \leq 0.6$ also involves only the first type of linearisation.

Applying $l_C = N_T D$ yields the required functional description. (In [24], an additional frequency correction would be performed, but usually the skin depth is much lower than the wire's diameter in variable single-layer solenoidal air-core inductors in our frequency range of interest, thus it's small and negligible.)

Table 3.1: Calculated values of $a_0 \Phi_M$ or $a_{0\text{dd}} \Phi_M$ (in boldface) as a function of $l_C \frac{df}{dc}$ with parameter $\frac{d}{D}$ after a piecewise linearisation as given in (3.9). The values of each column are valid between $l_C \frac{df}{dc}$ of that column and $l_C \frac{df}{dc}$ of the next column to the right.

<table>
<thead>
<tr>
<th>$\frac{d}{D}$</th>
<th>$\frac{l_C \frac{df}{dc}}{0}$</th>
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<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0 \Phi_M$</td>
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<td>5.25</td>
<td>5.35</td>
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<td>2.24</td>
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<td>0.86</td>
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</tr>
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<td>2.12</td>
<td>2.08</td>
<td>2.20</td>
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<td>0.20</td>
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Table 3.2: Calculated values of $a_{1 \Phi_M}$ or $a_{1ldd \Phi_M}$ (in boldface) as a function of $\frac{d}{d_C}$ with parameter $\frac{d}{D}$ after a piecewise linearisation as given in (3.9). The values of each column are valid between $\frac{d}{d_C}$ of that column and $\frac{d}{d_C}$ of the next column to the right.

<table>
<thead>
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<th>$a_{1 \Phi_M}$, $a_{1ldd \Phi_M}$</th>
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<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
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<td>0.75</td>
<td>0.00</td>
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<td>2.98</td>
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<td>0.30</td>
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<td>0.70</td>
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<td>0.30</td>
<td>0.15</td>
<td>2.17</td>
<td>2.220</td>
<td>0.010</td>
<td>0.025</td>
<td>0.015</td>
<td>(na)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.15</td>
<td>0.30</td>
<td>0.30</td>
<td>0.15</td>
<td>0.10</td>
<td>0.04</td>
<td>0.015</td>
<td>0.010</td>
<td>0.025</td>
<td>0.010</td>
<td>(na)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.20</td>
<td>0.30</td>
<td>0.30</td>
<td>0.20</td>
<td>0.15</td>
<td>0.07</td>
<td>0.020</td>
<td>0.010</td>
<td>0.005</td>
<td>0.010</td>
<td>(na)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.15</td>
<td>0.20</td>
<td>0.25</td>
<td>0.20</td>
<td>0.15</td>
<td>0.05</td>
<td>0.020</td>
<td>0.010</td>
<td>0.005</td>
<td>0.005</td>
<td>(na)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.15</td>
<td>0.10</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
<td>0.020</td>
<td>0.010</td>
<td>0.005</td>
<td>0.005</td>
<td>(na)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.05</td>
<td>0.00</td>
<td>0.10</td>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
<td>0.010</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
<td>(na)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.00</td>
<td>0.05</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>(na)</td>
</tr>
</tbody>
</table>

Finally, we consider single-layer solenoidal air-core inductors with any number of turns.

In [25], an (unrestricted) formula was derived based on [47] and a large amount of measurements including those of [24].

The resulting formula describes the ratio of the inductor’s AC resistance to a straight wire’s AC resistance, where the straight wire’s length equals the length of the coiled wire. The formula depends on $\frac{d}{d_C}$, $\frac{D}{d_C}$, $\frac{d}{D}$, and $N_T$.

To reduce the number of parameters, we define the function $\Phi_A$ as the limit of the formula from [25] mentioned above for $\omega \to \infty$. In order to keep the values of $\Phi_A$ close to the formula’s values in our frequency range of interest, $d \geq 2$ mm has to hold (ensuring the value of the function $\phi$ defined in [25] is less than 5% lower compared to its value for $\omega \to \infty$ which is used with powers up to two in the formula, hence the error is estimated to stay within 10%). which is likely to be satisfied by the “high power” inductors under consideration. Then the losses are given by

$$R_a = \frac{d_C}{d} \sqrt{\frac{\omega \mu_\varphi}{2}} N_T \Phi_A \left( \frac{D}{d_C}, \frac{d}{D}, N_T \right).$$

(3.10)

Compared to the previous loss descriptions, the resulting $\Phi_A\left( \frac{D}{d_C}, \frac{d}{D}, N_T \right)$ still involves one additional parameter, $N_T$. Thus its diagrams cannot be drawn as simple as those of the previous loss descriptions and the parameters have to be limited to useful ranges.

Firstly, typical variable “high power” inductors do not have more than 30–40 turns. Thus $N_T \leq 50$ is assumed. Secondly, $0.1 \leq \frac{d}{D} \leq 1$ is considered as [24] did (although it’s typically close to 0.5). Finally, the diameter of typical variable “high power” inductors does not exceed 200 mm. Since $d \geq 2$ mm and $\frac{d}{D} \leq 1$ was assumed,
0.01 \leq \frac{d}{d_{c}} \leq 0.5 is chosen — at the lower limit, \(d_c = 200\) mm if \(d = 2\) mm and \(\frac{d}{d_{c}} = 1\), the upper limit is introduced by one of the parameters’ limits in the tables of [25] (in our notation, [25] defines \(\xi\) as \(\frac{d_{c}}{d}\), whose range is \(0 \leq \xi \leq 1\) in the tables).

![Graphs](image)

Figure 3.8: a) \(\Phi_A\) and b) \(N_T \Phi_A\) as functions of \(N_T\) for \(\frac{d}{d_{c}} \in \{0.1, 0.2, \ldots, 0.9, 1\}\) and \(\frac{d}{d_{c}} = \frac{1}{20}\).

An example for the obtained \(\Phi_A\) is shown in Figure 3.8 a) for \(\frac{d}{d_{c}} = \frac{1}{20}\) (a variable inductor with \(d = 2\) mm and \(\frac{d}{d_{c}} = 0.5\) would have a \(d_c = 80\) mm). \(\Phi_A = 1\) was added for \(N_T \in [0, 1]\) (approximately assuming a pure skin effect).

If we perform a piecewise linearisation as follows:

\[
R_s \approx \begin{cases} 
\frac{d_{c}}{d} \sqrt{\frac{\omega \mu \rho}{2}} N_T \left(a_{0}\Phi_A + a_{1}\Phi_A N_T\right) & \text{if } \frac{d \Phi_A}{d N_T} \geq 0 \\
\frac{d_{c}}{d} \sqrt{\frac{\omega \mu \rho}{2}} \left(a_{0N}\Phi_A + a_{1N}\Phi_A N_T\right) & \text{if } \frac{d \Phi_A}{d N_T} < 0
\end{cases}
\]

we obtain \(a_{0}\Phi_A, a_{1}\Phi_A, a_{0N}\Phi_A, a_{1N}\Phi_A \geq 0\) within the limits given above\(^3\).

![Graphs](image)

Figure 3.9: a) \(a_{0}\Phi_A\) and b) \(a_{1}\Phi_A\) as functions of \(N_T\) for \(\frac{d}{d_{c}} \in \{0.1, 0.2, \ldots, 0.9, 1\}\) and \(\frac{d}{d_{c}} = \frac{1}{20}\). (Dots at \(N_T = 0, 1, \ldots, 49\) are calculated values of a piecewise linearisation within the interval \([N_T, N_T + 1]\). Dashed lines illustrate the tendency of values.)

\(^3\)Calculations were performed with \(\frac{d}{d_{c}} \in \{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}, \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \ldots, \frac{1}{100}\}\). Since \(\Phi_A\) may be quite flat in some regions, care has to be taken if it is intended to get “smooth” linearisation coefficients, involving usage of the tabulated values of \(u_1\) and \(u_2\) instead of the approximate formulae, calculation of additional \(u_n\), \(v_n\), and \(w_n\) from the formulae given in [47], and application of the interpolation proposed in [47] for \(u_n, v_n,\) and \(w_n\). Furthermore, \(u_2\) may be computed directly as a function and it seems advantageous to interpolate \(u_1\) uniformly against \(\frac{1}{N_T}\). (\(u_1, u_2, u_n, v_n,\) and \(w_n\) are defined in [25].)
Figure 3.10: a) $a_{0N\Phi_A}$ and b) $a_{1N\Phi_A}$ as functions of $N_T$ for $\frac{D}{D} \in \{0.8, 0.9\}$ and $\frac{D}{D} = \frac{1}{20}$. (Dashed lines illustrate the tendency of values.)

Figure 3.9 and Figure 3.10 are examples for such a linearisation performed for $\Phi_A$ of Figure 3.8 a). Dots at $N_T = 0, 1, \ldots, 49$ are calculated values of a piecewise linearisation within the interval $[N_T, N_T + 1]$. The type of linearisation changes for $N_T \geq 33$ in case $\frac{D}{D} \in \{0.8, 0.9\}$, which is clearly visible in Figure 3.9 a).

Summarising all preceding calculations, we may conclude that performing a piecewise linearisation of single-layer solenoidal air-core inductors’ series loss resistance $R_s$, which was computed according to [22] for a small number of turns, [24] for a large number of turns, or [25] for any number of turns, yielded functions of the kind

$$ R_s \approx a_{0R} + a_{1R} N_T + a_{2R} N_T^2, $$

(3.12)

where $a_{0R}, a_{1R}, a_{2R} \geq 0$.

Applying (3.4) gives

$$ R_s \approx a_{0R} + \frac{-a_{0L}}{a_{1L}} \left( a_{1R} + \frac{-a_{0L}}{a_{1L}} a_{2R} \right) + \frac{1}{a_{1L}} \left( a_{1R} + 2 \frac{-a_{0L}}{a_{1L}} a_{2R} \right) L_s + \frac{a_{2R}}{a_{1L}^2} L_s^2, $$

or, using new abbreviations,

$$ R_s \approx R_{s0} + a_1 \omega L_s + a_2 (\omega L_s)^2, $$

(3.13)

where

$$ R_{s0} = a_{0R} + \frac{-a_{0L}}{a_{1L}} \left( a_{1R} + \frac{-a_{0L}}{a_{1L}} a_{2R} \right), $$

$$ a_1 = \frac{1}{\omega a_{1L}} \left( a_{1R} + 2 \frac{-a_{0L}}{a_{1L}} a_{2R} \right), $$

$$ a_2 = \frac{a_{2R}}{\omega^2 a_{1L}^2}. $$

As indicated above, $R_{s0}, a_1, a_2 \geq 0$, hence (3.1) is a valid model for the (piecewise linearised) series loss resistance of (variable) single-layer solenoidal air-core inductors.

Alternatively we may perform a series expansion of $R_s$ as a function of $\omega L_s$ (without the temporary use of $N_T$), which also may be piecewise linearised.
3.4.2.2 Switchable inductors

Modern automatic antenna matching units (developed in [4], [5], [6], [7], and [8], a realisation given in [33] for a frequency range of 1.8–30 MHz) use switchable inductors (and capacitors).

![Simplified equivalent circuit of series connected switchable inductors](image)

Figure 3.11: Simplified equivalent circuit of series connected switchable inductors

Switchable inductors in such matching networks are built as illustrated in Figure 3.11, wherein the inductances are sorted in ascending order. The first inductor \((n_C = 1)\) has the lowest inductance \(L_{s1}\) and the \(N_C\)-th inductor (the last one, \(n_C = N_C\)) has the highest inductance \(L_{sN_C}\). Unused inductors are shorted by closing the appropriate switches.

Usually the inductances increase by powers of two, hence the inductance of the \(n_C\)-th inductor may be computed from the first (and lowest) inductance as

\[
L_{s_{nC}} = L_{s1} 2^{n_C - 1}.
\]

Then the total inductance is changeable in inductance steps of \(L_{s1}\) and stays constant (likewise the losses) between two adjacent values — inductance and series loss resistance are piecewise constant,

\[
L_s = \sum_{\text{all } n_C \text{ used}} L_{s_{nC}} = L_{s0}, \quad \text{(3.14)}
\]

\[
R_s = \sum_{\text{all } n_C \text{ used}} R_{s_{nC}} = R_{s0}. \quad \text{(3.15)}
\]

The resulting piecewise constant series loss resistance can be described by (3.1) (application to \(T\) networks is similar to those described at the end of Section 3.4.2.3), but it involves steps at the line segments’ limits (Section 4.6 addresses the impact of such steps).

Due to nonzero minimum inductance and capacitance step size we usually cannot tune the network for exact input matching. Then the transferred active power has to be computed using different formulae than those given in Sections 3.5.2.1 and 3.5.3.1. However, if the minimum inductance and capacitance step is sufficiently small, we may approximately use the inductance and capacitances required for exact input matching and the series loss resistance of the closest (switchable) inductance value.
3.4.2.3 Inductor in series/parallel to a variable capacitor

Another possible realisation of a variable inductor is a variable capacitor in series to a (fixed) inductor for usage in a π network or a variable capacitor in parallel to a (fixed) inductor for usage in a T network.

![Circuit Diagram]

Figure 3.12: Simplified equivalent circuit of a) a variable capacitor and a serial (fixed) inductor (for use in a π network) and b) a variable capacitor and a parallel (fixed) inductor (for use in a T network).

Inspection of its simplified equivalent circuit in Figure 3.12 reveals its advantage — if the elements' values are chosen properly, the total inductance at matching frequency may be reduced to zero in a) or increased to infinity in b), which is difficult to achieve by "real" variable inductors if the matching frequency is high — and disadvantage — the series loss resistance in a) and the parallel loss conductance in b) are constant for any total inductance. Thus even low total inductances involve the "high" losses of the maximum inductance value. However, the losses may be reduced by using a switchable tapped inductor as proposed in [3].

The series loss resistance in a) is just a constant loss resistance $R_{s0}$, hence included in (3.1). The constant loss conductance in parallel to a variable inductance in b) is fortunately one of the exceptions mentioned in Footnote 7 (at the end of Section 3.5.3.1), where π and T network are exactly equivalent (this issue is also addressed in Section 3.5.3.4). Thus all relations derived in the following sections for π networks whose central inductor's series loss resistance is constant apply to T networks whose central inductor's parallel loss conductance is constant. Hence the model proposed in (3.1) is suitable to describe an inductor in series/parallel to a variable capacitor used as central variable inductor of a π or T network, respectively.

3.5 Matching of a resistive load to a resistive source (matching at the network's input)

3.5.1 Definition of the objectives and suitable parametrisations for π and T networks

The basic objective of the following sections is to match a (real) load resistance $R_L = \frac{1}{\omega L} > 0$ to a (real) source resistance $R_i = \frac{1}{\omega C} > 0$ at a given frequency using a π or T network as indicated in Figure 3.13.
The T network contains an inductance $L_p$ in parallel to a conductance $G_p$ to simplify its parametrisation and the matched network's element values' equations. The element values of the "original" network, $L_s$ and $R_s$, are then computed by applying a parallel-to-series transformation according to (A.20) and (A.19).

![Diagram](image)

Figure 3.13: Considered lossy a) π and b) T network matching a (real) load resistance $R_L = \frac{1}{\omega L}$ to a (real) source resistance $R_i = \frac{1}{\omega C_i}$.

π and T networks consist of three (independent) elements, which have to be tuned to transform the load resistance to the source resistance, hence achieving a (conjugate complex) match of the load to the source at the network's input. Conjugate complex matching involves two independent equations to be solved, one for the real part and one for the imaginary part of the network's input impedance, yielding two conditions for the network's element settings. Since three independently adjustable elements are available, one may be chosen arbitrarily.

However, if a lossless L network is considered, two conditions yield unique settings of its elements, because it consists just of two independently adjustable elements. In a lossy L network, there may exist more than one distinct settings depending on the loss model applied (for example consider the capacitive solution in the right half of Figure 4.8 b)), although usually only one solution remains valid, in particular if network elements' limits are taken into account.

The degree of freedom in π and T networks is described by a parameter of those networks. To simplify calculations and optimisation proofs, it's especially important to choose a proper parametrisation suitting the needs of active power transfer optimisation.

During the last decades, a couple of π and T network parametrisations were developed for lossless networks suitting different needs. A more filter theoretical approach involving hyperbolic functions was summarised in [9c] and examined in more detail in [10]. Q-based design was used in [11], [12] and further developed in [13], [14], [15] (corrected/extended in [16]), [17], and [18].

Common to all preceding parametrisations is the difficulty to include the series loss resistance. Thus calculations of the losses of the inductor are usually performed using lossless network elements' values (assuming those values are not changed significantly by low losses), e.g. in [9c] or [18]. Then the power losses involved may be computed exactly, but the network's input is never matched to the source — the higher the losses, the worser the match.

It's interesting to note the loss calculation method of [61] using Tellegen's theorem [62] — it includes the losses of all network elements, but considers constant $Q$ factors.
only, assumes elements of the same type having similar $Q$ factors, and designs the network using lossless network elements' values.

However, the parametrisation derived in [11] and [13] is extendable to exact loss inclusion (introduced by [3]), e.g. if the π network’s equations for $C_{1,2}$ and $L_s$ are rewritten in resistances and the loss resistance is taken into account when choosing the parameters of the two L networks the π network was split in. For a single L network, the exact loss inclusion is given in [3] (and rewritten for $Q$-based design).

Then the network’s input is exactly matched within the entire parameter range. After deriving the network’s power gain, it’s optimised by applying (3.1) and the implicit function theorem.

To gain more insight in the network’s behaviour, its power gain is derived in a little bit more complicated manner. The method described in [51b] is simpler and especially useful if the losses of the capacitors have to be included, too. It’s used in Appendix B.3 where design and power gain of π and T networks are considered including losses of all network’s elements.

### 3.5.2 π network

**3.5.2.1 Elements’ values and losses involved**

![Figure 3.14: Resulting (transformed) π network’s equivalent circuit for matching of a (real) load conductance $G_L = \frac{1}{R_L}$ to a (real) source conductance $G_i = \frac{1}{R_i}$](image)

To derive the elements’ values for matching of a (real) load conductance $G_L > 0$ to a (real) source conductance $G_i > 0$ if a π network with a lossy “central” inductor $L_s$ is used as shown in Figure 3.13 a), we initially split $L_s$ in two parts, $L_{s1}$ and $L_{s2}$. We then may consider the π network to be composed of two lossless L networks ($C_1$, $L_{s1}$ or $C_2$, $L_{s2}$, respectively), and $R_s$, which takes the (total) losses into account.

Applying Thevenin’s theorem [63] to the voltage source of (complex) amplitude $V_0$ and (real) source resistance $R_i = \frac{1}{G_i}$ in parallel to the capacitance $C_1$ yields
CHAPTER 3. LOSSY L, π, AND T MATCHING NETWORKS

a Thevenin equivalent voltage source of (complex) amplitude $\tilde{V}_0$ and a (complex)
source impedance consisting of a resistor $R_{m1}$ in series to a capacitor $C_{m1}$.

Finally $G_\ell = \frac{1}{j R_\ell}$ in parallel to $C_2$ is parallel-to-series transformed in $R_{m2}$ in series to $C_{m2}$ using (A.19) and (A.20).

We then obtain Figure 3.14 from Figure 3.13, where

$$\tilde{V}_0 = V_0 \frac{1}{\frac{1}{C_1} + \frac{1}{j \omega C_1}} = V_0 \frac{G_1}{G_1 + j \omega C_1} = \frac{V_0}{1 + j \omega R_\ell C_1}. \tag{3.16}$$

(The formulae obtained for $R_{m1,2}$ and $C_{m1,2}$ will not be needed later on and thus are not shown here.)

Which conditions will cause matching in the circuit of Figure 3.14?

Firstly, capacitive and inductive reactances have to cancel at the matching frequency.
If an approach similar to [13] (derivation for a lossless $\pi$ network) is used, reactive
cancellation applies to each of the two L networks — $C_{m1}$ cancels $L_{s1}$ and $C_{m2}$
cancels $L_{s2}$ at the matching frequency.

Secondly, the remaining resistances have to be equal. For a lossless inductor, choosing of $R_{m1} = R_{m2} = R_m$ would yield $Z_{11} = Z_{1\ell} = Z_{21} = Z_{2\ell} = R_m$ at the matching frequency [$R_m$ is the independent parameter referred to in Section 3.5.1, denoted by
“virtual resistance” $R$ in [13]), thus providing a (conjugate complex) match of the
source resistance to the load resistance and vice versa.

However, a lossy inductor involves an “asymmetry” $R_s$, which urges us to decide
whether we want to match the load to the source or the source to the load. In the
first case, we may choose $R_{m2} = R_m$ (as before, $R_m$ is the independent parameter).
At matching frequency, $Z_{2\ell} = R_m$ holds and thus $Z_{1\ell} = Z_{2\ell} + R_s = R_m + R_s$.

To achieve a (conjugate complex) match of the load to the source, $Z_{1\ell}$ must be
equal to $R_{m1}$, hence $R_{m1} = R_m + R_s$ (for a L network, where $C_1 = 0$ and thus
$R_{m1} = R_\ell$; this condition was given in [3]). Conversely, $Z_{11} = R_{m1} = R_m + R_s$ and
$Z_{21} = Z_{11} + R_s = R_m + 2R_s$ at matching frequency, which is not equal to $R_{m2} = R_m$
as it would be for a conjugate complex match of the source to the load.

Different choices of $R_{m1,2}$ are also possible, but those shown above are ideally suited
for the extension to complex loads.

For our purpose, matching of the load to the source is needed. However, matching
of the source to the load is quite similar and derived in Appendix B.4.1.

Using the preceding choice for $R_{m1,2}$, we would obtain $C_{12}$ (and $C_{m1,2}$) from parallel-
to-series transformation equations for $R_{m1,2}$. Applying reactive cancellation would
then yield $L_{s1,2}$ (and $L_s$). The calculations are not shown here because we alterna-
tively may use the formulae from [13] if they are rewritten in our resistance notation
and if we use $R_{m1} = R_m + R_s$ for the left L network and $R_{m2} = R_m$ for the right
L network ([13] derives the lossless case, where $R_{m1} = R_{m2} = R_m$).

Since $R_{m1,2}$ was obtained from parallel-to-series transformation of passive compo-
nents (source resistance $R_\ell > 0$ and load resistance $R_s > 0$), $R_{m1,2} \geq 0$ holds. Then
the calculations described above yield the capacitances \( C_{1,2} \) and the inductance \( L_s \) of the original \( \pi \) network in Figure 3.13 a), where

\begin{align}
C_1 &= \frac{1}{\omega R_1} \sqrt{\frac{R_1 - R_m - R_s}{R_m + R_s}}, \quad (3.17) \\
C_2 &= \frac{1}{\omega R_\ell} \sqrt{\frac{R_\ell - R_m}{R_m}}, \quad (3.18) \\
L_s &= \frac{1}{\omega} \sqrt{\frac{(R_m + R_s) (R_\ell - R_m - R_s)}{R_m (R_\ell - R_m)}}. \quad (3.19)
\end{align}

(The negative signs in parentheses/square brackets would apply if we would use (lossless) variable inductors \( L_{1,2} = \frac{1}{\omega^2 C_{1,2}} \). Then a negative \( L_s \) might occur, which would be just equal to a variable capacitor \( C_s = \frac{1}{\omega^2 L_m} \). As indicated by using parentheses/square brackets, reactive cancellation causes a negative sign of the first square root in \( L_s \), if \( L_1 \) is used instead of \( C_1 \), or a negative sign of the second square root in \( L_s \), if \( L_2 \) is used instead of \( C_2 \), respectively. We will need these changes during the extension to inductive loads.)

In order to get valid solutions, all square roots must be real (if not, the \( \pi \) network of Figure 3.13 a) wouldn’t be able to match). Thus, certain restrictions to the parameter \( R_m \) must apply. Since \( R_\ell > 0 \), only \( 0 \leq R_m \leq R_\ell \) yields \( R_m (R_\ell - R_m) \geq 0 \). Similarly, \( R_s \geq 0 \) and \( R_1 > 0 \) imply \( 0 \leq R_m + R_s \leq R_1 \). Hence, for valid \( R_m \) the designability conditions are

\[
0 \leq R_m \leq R_\ell \quad \text{and} \quad 0 \leq R_m + R_s \leq R_1. \quad (3.20)
\]

As \( R_s \geq R_{s0} \) (from (3.1)), we immediately obtain that \( R_{s0} \) is limited by \( R_{s0} \leq R_1 \). If \( R_{s0} \) would be higher, the \( \pi \) network would not be able to match any load.

However, both inequalities mainly describe the upper limit of the parameter \( R_m \). This issue is best understood if we initially compute its limit for the lossless case (\( R_s = 0 \)). Since both inequalities apply, the upper limit of \( R_m \) is given by \( R_\ell \) if \( R_\ell \leq R_t \) or by \( R_t \) if \( R_\ell > R_t \), respectively. The change in the upper limit occurs at \( R_\ell = R_t \), where a match would not need any additional components (\( L_s = 0 \) and \( C_{1,2} = 0 \) would match).

The lossy network behaves quite similarly. The upper limit of \( R_m \) is given by \( R_\ell \) if \( R_\ell \leq R_t - R_{s0} \) or by \( R_{m\text{lim}} \) if \( R_\ell > R_t - R_{s0} \), respectively. \( R_{m\text{lim}} \) is the highest parameter \( R_m \) which solves

\[
R_{m\text{lim}} = R_t - R_s (R_{m\text{lim}}) \quad (3.21)
\]

for \( 0 \leq R_{m\text{lim}} < R_\ell \) and involves valid network elements. The change in the upper limit occurs at \( R_\ell = R_t - R_{s0} \), where, as before, a match would not need any additional components (\( L_s = 0 \) and \( C_{1,2} = 0 \) would match). It’s interesting to note that (for a resistive load) there is no other solution where \( L_s = 0 \).  

\footnote{Theoretically \( L_s = 0 \) might additionally occur for \( R_m = 0 \) if \( R_{s0}(R_m = 0) = 0 \), which would be possible for \( R_{s0} = 0 \). However, since \( L_s = 0 \), the load resistor would not be transformed. Hence matching would be impossible unless \( R_\ell = R_t - R_{s0} = R_t = R_m = 0 \) — this solution is already described by (a limit of) \( R_{m\text{lim}} \).}
Depending on the particular \( R_s \) used (given as a function of \( \omega L_s \) or \( R_m \)), additional restrictions to \( R_m \) might apply, limiting \( R_m \) even further or involving intervals within the limits derived above where matching is impossible. However, (3.20) holds throughout the valid range of \( R_m \).

Additionally, the upper limits in (3.20) separate \( \pi \) and L networks — \( C_1 = 0 \) if \( R_m = R_{\text{min}} \) or \( C_2 = 0 \) if \( R_m = R_{\ell} \), respectively.

After designing the \( \pi \) network's input impedance to match the source resistance \( R_l \), we compute the active power \( P_{\text{act}} \) delivered to its load resistance \( R_\ell \). Since the capacitive and inductive reactances in Figure 3.14 cancel at matching frequency, we obtain

\[
P_{\text{act}} = \frac{1}{2} \frac{R_m}{2} \frac{\left| \tilde{V}_0 \right|^2}{(R_{\text{min}} + R_s + R_{\text{m2}})^2} \overset{(3.16)}{=} \frac{1}{2} \frac{R_m}{4} \frac{|V_0|^2}{(R_m + R_s)^2} \overset{(3.17)}{=} \frac{1}{2} \frac{R_m}{8} \frac{|V_0|^2}{R_m + R_s} \overset{(2.22)}{=} P_{\text{act maxi}} \frac{1}{1 + \frac{R_s}{R_m}}.
\]

(3.22)

Conversely, due to input matching the active power \( P_{\text{act}} \) delivered to the loss resistance \( R_s \) is given by

\[
P_{\text{act}} = P_{\text{act maxi}} - P_{\text{act}} = P_{\text{act maxi}} \frac{1}{1 + \frac{R_s}{R_m}}.
\]

(3.23)

For optimisation, we will need the power gain \( g_{\pi, \text{inna}} \) of the matched \( \pi \) network. Since its input impedance is matched to \( R_l \), the active power delivered to the network’s input is \( P_{\text{act}} = P_{\text{act maxi}} \) (refer to Figure A.4). Thus we obtain

\[
g_{\pi, \text{inna}} = \frac{P_{\text{act}}}{P_{\text{act1}}} = \frac{P_{\text{act}}}{P_{\text{act maxi}}} \overset{(3.22)}{=} \frac{R_m}{R_m + R_s} \underset{(3.24)}{=} \frac{1}{1 + \frac{R_s}{R_m}}.
\]

(3.24)

(Alternatively we could derive \( g_{\pi, \text{inna}} \) from (A.15) if we put \( C_{\text{2tot}} = C_2 \). However, we gain more insight in the circuit’s behaviour if it is computed as shown above.)

3.5.2.2 Optimum value of parameter \( R_m \) for maximum power gain

Minimising “losses of the \( \pi \) network” is achieved by transferring the highest portion of the active power delivered to the network’s input to its load \( R_\ell = \frac{1}{G_l} \). Thus neither \( P_{\text{act}} \) has to be minimised nor \( P_{\text{act}} \) to be maximised, but we have to maximise \( g_{\pi, \text{inna}} \). However, \( P_{\text{act maxi}} \) is independent of \( R_m \) and \( P_{\text{act}} = g_{\pi, \text{inna}} P_{\text{act maxi}} \) holds (input matching), hence the highest \( g_{\pi, \text{inna}} \) occurs for maximum \( P_{\text{act}} \).

To derive the value of \( R_m \) for maximum \( g_{\pi, \text{inna}} \), we compute its derivative with respect to \( R_m \).

Using (3.24) yields

\[
\frac{dg_{\pi, \text{inna}}}{dR_m} = \frac{R_m}{(R_m + R_s)^2} \left( \frac{R_s}{R_m} - \frac{dR_s}{dR_m} \right).
\]

(3.25)
Since \( R_m, R_s \geq 0 \), the first term is always \( \geq 0 \). Thus we need to check the sign of the second term only.

As \( R_s \) is a function of \( L_s \), which itself is a function of \( R_m \) and implicitly\(^4\) of \( R_s \) according to (3.19), we define the implicit function \( \psi_x \equiv 0 \) from (3.1),

\[
\psi_x := R_{s0} + \left( \sum_{n=1}^{N} a_n (\omega L_s)^n \right) - R_s \equiv 0.
\]

Then we compute the implicit function’s partial derivatives with respect to \( R_m \) and \( R_s \),

\[
\frac{\partial \psi_x}{\partial R_m} = \sum_{n=1}^{N} a_n b_n (\omega L_s)^{n-1} \frac{\partial (\omega L_s)}{\partial R_m},
\]

\[
\frac{\partial \psi_x}{\partial R_s} = \left( \sum_{n=1}^{N} a_n b_n (\omega L_s)^{n-1} \frac{\partial (\omega L_s)}{\partial R_s} \right) - 1.
\]

Applying the implicit function theorem \([64]\) gives

\[
\frac{dR_s}{dR_m} = -\frac{\frac{\partial \psi_x}{\partial R_m}}{\frac{\partial \psi_x}{\partial R_s}} = \frac{\sum_{n=1}^{N} a_n b_n (\omega L_s)^{n-1} \frac{\partial (\omega L_s)}{\partial R_m}}{1 - \sum_{n=1}^{N} a_n b_n (\omega L_s)^{n-1} \frac{\partial (\omega L_s)}{\partial R_s}}.
\]

If we multiply numerator and denominator by \( \omega L_s \), put the result in the second term of (3.25), and rearrange it, we obtain

\[
\frac{R_s}{R_m} - \frac{dR_s}{dR_m} = \frac{R_s \omega L_s - \sum_{n=1}^{N} a_n b_n (\omega L_s)^n \left( R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} \right)}{R_m \left( \omega L_s - \sum_{n=1}^{N} a_n b_n (\omega L_s)^n \frac{\partial (\omega L_s)}{\partial R_s} \right)}.
\]  

(3.26)

Replacing of \( \omega L_s \) with (3.19) yields the missing partial derivatives,

\[
\frac{\partial (\omega L_s)}{\partial R_m} = \frac{R_1 - 2 (R_m + R_s)}{2 \sqrt{(R_m + R_s) (R_1 - R_m - R_s)}} + \frac{R_t - 2 R_m}{2 \sqrt{R_m (R_t - R_m)}},
\]

(3.27)

\[
\frac{\partial (\omega L_s)}{\partial R_s} = \frac{R_1 - 2 (R_m + R_s)}{2 \sqrt{(R_m + R_s) (R_1 - R_m - R_s)}}.
\]

(3.28)

To check which value of \( R_m \) yields the highest \( g_{\pi, \text{inw}} \), we need to know the sign of \( \frac{R_s}{R_m} - \frac{dR_s}{dR_m} \), the second term in (3.26), throughout the valid range of \( R_m \) given by (3.20).

---

\(^4\)In particular, \( R_s \) and \( L_s \) are both functions of \( R_m \) only. However, we usually may not be able to solve for \( R_s \) explicitly after applying (3.19) to (3.1).
CHAPTER 3. LOSSY L, π, AND T MATCHING NETWORKS

For ease of understanding, an overview of the following proof is given below. The proof itself is quite lengthy and thus performed in Appendix B.2.1.1.

The idea is that \( \frac{R_s}{R_m} - \frac{\frac{dR_s}{dR_m}}{dR_m} \) remains \( \geq 0 \) for all valid \( R_m \). Then the highest possible value of \( R_m \) would yield the highest \( g_{\pi, \text{inma}} \).

To prove it, we will derive the highest \( b_N \) which still yields a numerator \( \geq 0 \) and simultaneously a denominator \( \geq 0 \). This approach involves more restrictions than necessary, because the ratio of a negative numerator to a negative denominator would also be positive, but the proof is then quite "easy" and straightforward. However, we should keep in mind that due to the additional restrictions on \( \frac{R_s}{R_m} - \frac{\frac{dR_s}{dR_m}}{dR_m} \) might remain positive even for a higher \( b_N \) than those derived.

If we abbreviate \( S_\pi = \sum_{n=1}^{N} a_n b_n (\omega L_a)^{b_n} \), we obtain from (3.26)

\[
\frac{R_s}{R_m} - \frac{dR_s}{dR_m} = \frac{R_s \omega L_a - S_\pi \left( R_s \frac{\partial(\omega L_a)}{\partial R_s} + R_m \frac{\partial(\omega L_a)}{\partial R_m} \right)}{R_m \left( \omega L_a - S_\pi \frac{\partial(\omega L_a)}{\partial R_s} \right)}.
\]

To prove that numerator and denominator are \( \geq 0 \) for certain \( b_N \), we firstly assume \( S_\pi = 0 \) or \( R_s = R_{s0} \) and calculate \( \frac{R_s}{R_m} - \frac{dR_s}{dR_m} \), which will be \( \geq 0 \) throughout the valid range of \( R_m \).

Secondly, we assume \( S_\pi \neq 0 \), hence \( S_\pi > 0 \) (according to (3.1)), and check numerator and denominator separately:

1. Since \( \omega L_a, R_m, R_s \geq 0 \), and \( S_\pi > 0 \), the numerator will always be \( \geq 0 \) for \( R_s \frac{\partial(\omega L_a)}{\partial R_s} + R_m \frac{\partial(\omega L_a)}{\partial R_m} \leq 0 \). Conversely, if \( R_s \frac{\partial(\omega L_a)}{\partial R_s} + R_m \frac{\partial(\omega L_a)}{\partial R_m} > 0 \), we have to use certain inequalities to prove that (at least) for \( b_N \leq 2 \) the numerator will be \( \geq 0 \).

2. Since \( \omega L_a, R_m \geq 0 \), and \( S_\pi > 0 \), the denominator will always be \( \geq 0 \) for \( \frac{\partial(\omega L_a)}{\partial R_s} \leq 0 \). Conversely, if \( \frac{\partial(\omega L_a)}{\partial R_s} > 0 \), we will have to use certain inequalities to prove that (at least) for \( b_N \leq 2 \) the denominator will be \( \geq 0 \).

We then know that for all possible values of \( R_{s0}, a_n, \) and \( b_n \) according to (3.1), \( \frac{\partial g_{\pi, \text{inma}}}{\partial R_m} \geq 0 \) if (at least) \( b_N \leq 2 \).

Hence the highest \( g_{\pi, \text{inma}} \) is obtained for the highest valid value of \( R_m \), which is \( R_e \) if \( R_e \leq R_i - R_{s0} \) or \( R_{\text{lim}} \) defined in (3.21) if \( R_e > R_i - R_{s0} \), respectively. If we choose \( R_m \) accordingly, the particular type of network involved is one of the two L networks where \( C_2 = 0 \) or \( C_1 = 0 \).

However, we should keep in mind that the restriction \( 0 < b_N \leq 2 \) aroused after applying inequalities to get upper estimates for certain terms. If \( b_N > 2 \), the method used in the preceding proof gives no indication of the network's behaviour.\(^5\)

\(^5\)It's not obvious that by using the method applied within the proof \( 0 < b_N \leq 2 \) is the utmost obtainable range for \( b_N \). However, it can be achieved by performing it in a slightly different manner. Unfortunately the proof is then much more complicated and harder to understand.
It’s interesting to note that usually the highest $g_{\pi,\,\text{inma}}$ does not coincide with the lowest $R_s$ (or, according to (3.1), the lowest $L_s$). This issue may be illustrated by comparison of Figure 4.11 to Figure 4.15.

3.5.2.3 Optimum value of load resistance $R_\ell$ for maximum power gain

In Section 3.5.2.2 we derived which value of $R_m$ maximises $g_{\pi,\,\text{inma}}$. However, the highest $g_{\pi,\,\text{inma}}$ still depends on $R_\ell$ ($R_i$ is fixed in our application of matching the load to the source).

Thus further investigations are required to determine which $R_\ell$ yields the highest $g_{\pi,\,\text{inma}}$ if the parameter $R_m$ was chosen for maximum $g_{\pi,\,\text{inma}}$ according to Section 3.5.2.2.

$R_\ell$ is variable because the connecting line between matching network and load transforms the load depending on the line’s length. As transformation usually involves additional susceptances even for a purely resistive load, optimisation of $g_{\pi,\,\text{inma}}$ with respect to $R_\ell$ is only part of the solution, “complete” optimisation needs application of Sections 4.3.1.1 and 4.3.2.1, too.

The derivation of the optimum value of $R_\ell$ is quite similar to Section 3.5.2.2, but lengthy and thus performed in Appendix B.2.1.2.

Summarising Appendix B.2.1.2, we may conclude that if $0 < b_N \leq 2$ and $R_m$ is chosen according to Section 3.5.2.2, $R_\ell = R_i - R_{s0}$ would yield the highest $g_{\pi,\,\text{inma}}$. $g_{\pi,\,\text{inma}}$ decreases for higher or lower $R_\ell$ with exception of the constant $R_s$ case (all $a_n = 0$, hence $R_s = R_{s0}$), where $g_{\pi,\,\text{inma}}$ stays constant for higher $R_\ell$.

Unfortunately the optimisation gives no indication for which of the two load resistors $R_{\ell 1} > R_i - R_{s0}$, $R_{\ell 2} = \frac{(R_i - R_{s0})^2}{R_{\ell 1}} < R_i - R_{s0}$, exhibiting similar reflection coefficients referred to $R_i - R_{s0}$, we would achieve the higher $g_{\pi,\,\text{inma}}$ (reference $R_i - R_{s0}$ is not related to the interconnecting transmission lines’ characteristic impedances, but was used to obtain equal “transformation distances” from optimum value $R_i - R_{s0}$).

As mentioned before, “complete” optimisation involves taking into account of Sections 4.3.1.1 and 4.3.2.1. However, at least we may conclude that minimising losses of the matching network will always require consideration (and optimisation) of the length of the transmission line connecting it to the load.
3.5.3 T network

3.5.3.1 Elements’ values and losses involved

![T Network Diagram]

Figure 3.15: Resulting (transformed) T network’s equivalent circuit for matching of a (real) load resistance \( R_\ell = \frac{1}{G_\ell} \) to a (real) source resistance \( R_i = \frac{1}{G_i} \)

To derive the elements’ values for matching of a (real) load resistance \( R_\ell > 0 \) to a (real) source resistance \( R_i > 0 \) if a T network with a lossy “central” inductor \( L_p \) is used as shown in Figure 3.13 b), we initially split \( \frac{1}{L_p} \) in two parts, \( \frac{1}{L_{p1}} \) and \( \frac{1}{L_{p2}} \). We then may consider the T network to be composed of two lossless L networks \((C_1, L_{p1} \text{ or } C_2, L_{p2}, \text{ respectively})\), and \( G_p \), which takes the (total) losses into account.

Applying Norton’s theorem [63] to the voltage source of (complex) amplitude \( V_0 \) and (real) source resistance \( R_i = \frac{1}{G_i} \) in series to the capacitance \( C_1 \) yields a Norton equivalent current source of (complex) amplitude \( \tilde{I}_0 \) and a (complex) source admittance consisting of a conductor \( G_{m1} \) in parallel to a capacitor \( C_{m1} \).

Finally \( R_\ell = \frac{1}{G_\ell} \) in series to \( C_2 \) is series-to-parallel transformed in \( G_{m2} \) in parallel to \( C_{m2} \) using (A.16) and (A.17) (or [34c]).

We then obtain Figure 3.15 from Figure 3.13, where

\[
\tilde{I}_0 = \frac{V_0}{R_i + \frac{1}{j\omega C_i}} = V_0 \frac{j\omega C_1}{1 + j\omega R_i C_1} = V_0 \frac{j\omega G_i C_1}{G_i + j\omega C_1}.
\]

(3.29)

(The formulae obtained for \( R_{m1,2} \) and \( C_{m1,2} \) will not be needed later on and thus are not shown here.)

Which conditions will cause matching in the circuit of Figure 3.15?

Firstly, capacitive and inductive susceptances have to cancel at the matching frequency. If an approach similar to [13] (derivation for a lossless T network) is used, susceptive cancellation applies to each of the two L networks — \( C_{m1} \) cancels \( L_{p1} \) and \( C_{m2} \) cancels \( L_{p2} \) at the matching frequency.
Secondly, the remaining conductances have to be equal. For a lossless inductor, choosing of \(G_{m1} = G_{m2} = G_m\) would yield \(Y_{11} = Y_{1\ell} = Y_{2i} = Y_{2\ell} = G_m\) at the matching frequency \((G_m)\) is the independent parameter referred to in Section 3.5.1, which would be equivalent to a “virtual conductance” \(G\) — however, \([13]\) uses a “virtual resistance” instead), thus providing a (conjugate complex) match of the source resistance to the load resistance and vice versa.

However, a lossy inductor involves an “asymmetry” \(G_p\), which urges us to decide whether we want to match the load to the source or the source to the load. In the first case, we may choose \(G_{m2} = G_m\) (as before, \(G_m\) is the independent parameter). At matching frequency, \(Y_{2\ell} = G_m\) holds and thus \(Y_{1\ell} = Y_{2\ell} + G_p = G_m + G_p\). To achieve a (conjugate complex) match of the load to the source, \(Y_{1\ell}\) must be equal to \(G_{m1}\), hence \(G_{m1} = G_m + G_p\). Conversely, \(Y_{1\ell} = G_{m1} = G_m + G_p\) and \(Y_{2\ell} = Y_{2\ell} + G_p = G_m + 2G_p\) at matching frequency, which is not equal to \(G_{m2} = G_m\), as it would be for a conjugate complex match of the source to the load.

Different choices of \(G_{m1,2}\) are also possible, but those shown above are ideally suited for the extension to complex loads.

For our purpose, matching of the load to the source is needed. However, matching of the source to the load is quite similar and derived in Appendix B.4.2.

Using the preceding choice for \(G_{m1,2}\), we would obtain \(C_{1,2}\) (and \(C_{m1,2}\)) from series-to-parallel transformation equations for \(G_{m1,2}\). Applying susceptive cancellation would then yield \(L_{p1,2}\) (and \(L_p\)). The calculations are not shown here because we alternatively may use the formulae from \([13]\) if they are rewritten in our conductance notation and if we use \(G_{m1} = G_m + G_p\) for the left L network and \(G_{m2} = G_m\) for the right L network (\([13]\) derives the lossless case in resistive notation, where \(G_{m1} = G_{m2} = G_m\) and \(1/G_m\) is denoted by \(R\)).

Since \(G_{m1,2}\) was obtained from series-to-parallel transformation of passive components (source conductance \(G_1 > 0\) and load conductance \(G_\ell > 0\)), \(G_{m1,2} \geq 0\) holds. Then the calculations described above yield the capacitances \(C_{1,2}\) and the inductance \(L_p\) of the original T network in Figure 3.13 b), where

\[
C_1 = \left(\frac{G_i}{\omega}\right) \sqrt{\frac{G_m + G_p}{G_i - G_m - G_p}}, \quad (3.30) \quad C_2 = \left(\frac{G_\ell}{\omega}\right) \sqrt{\frac{G_m}{G_\ell - G_m}}, \quad (3.31)
\]

\[
L_p = \frac{1}{\omega \left(\frac{1}{\omega} \sqrt{(G_m + G_p) (G_i - G_m - G_p)} \right) \sqrt{G_m (G_\ell - G_m)}} \quad (3.32)
\]

(The negative signs in parentheses/square brackets would apply if we would use (lossless) variable inductors \(L_{1,2} = -\omega^2 G_{1,2}^{-1}\). Then a negative \(L_p\) might occur, which would be just equal to a variable capacitor: \(C_p = -\omega^2 L_p\). As indicated by using parentheses/square brackets, susceptive cancellation causes a negative sign of the first square root in \(L_p\), if \(L_1\) is used instead of \(C_1\), or a negative sign of the second
square root in $L_p$, if $L_2$ is used instead of $C_2$, respectively. We will need these changes during the extension to inductive loads.)

In order to get valid solutions, all square roots must be real (if not, the T network of Figure 3.13 b) wouldn’t be able to match). Thus, certain restrictions to the parameter $G_m$ must apply. Since $G_\ell > 0$, only $0 \leq G_m \leq G_\ell$ yields $G_m(G_\ell - G_m) \geq 0$. Similarly, $G_p \geq 0$ and $G_i \geq 0$ imply $0 \leq G_m + G_p \leq G_i$.

Hence, for valid $G_m$ the designability conditions are

$$0 \leq G_m \leq G_\ell \quad \text{and} \quad 0 \leq G_m + G_p \leq G_i.$$  \hspace{1cm} (3.33)

Both inequalities mainly describe the upper limit of the parameter $G_m$. This issue is best understood if we initially compute its limit for the lossless case ($R_a = 0$ and thus $G_p = 0$). Since both inequalities apply, the upper limit of $G_m$ is given by $G_\ell$ if $G_\ell \leq G_i$ or by $G_i$ if $G_\ell > G_i$, respectively\(^6\). The change in the upper limit occurs at $G_\ell = G_i$, where a match would not need any additional components ($L_a \to \infty$, hence $L_p \to \infty$ (from (A.18)), and $C_{1,2} \to \infty$ would match).

The lossy network behaves quite similarly. The upper limit of $G_m$ is given by $G_\ell$ if $G_\ell \leq G_i$ or by $G_{\text{mlim}}$ if $G_\ell > G_i$, respectively. $G_{\text{mlim}}$ is the highest parameter $G_m$ which solves

$$G_{\text{mlim}} = G_i - G_p(G_{\text{mlim}})$$  \hspace{1cm} (3.34)

for $0 \leq G_{\text{mlim}} < G_\ell$ and involves valid network elements. The change in the upper limit occurs at $G_\ell = G_i$, where, as before, a match would not need any additional components ($L_a \to \infty$, hence $L_p \to \infty$, and $C_{1,2} \to \infty$ would match). It’s interesting to note that (for a resistive load) there is no other solution where $L_{a,p} \to \infty$ or $\frac{1}{L_p} = 0$\(^\dagger\).

---

\(^6\) Usually there is no restriction to $R_{a0}$ similar to those in Section 3.5.2.1. The reason is that $G_p$ is usually not equal to $\frac{1}{R_a} + \ldots$ due to series-to-parallel transformation of $R_a$ and $L_a$, where $R_a$ is given by (3.1). This issue is also addressed by the last two paragraphs of this section dealing with the differences of $T$ and $\pi$ network and by the exceptions mentioned in Footnotes\(^7\) and \(^8\).

\(^\dagger\) Theoretically $\frac{1}{L_p} = 0$ might additionally occur for $G_m = 0$ if $G_p(G_m = 0) = 0$. However, since $L_{a,p} \to \infty$ ($\frac{1}{L_p} = 0$ and $G_p = 0$ is case (ii) in (A.18)), the load conductor would not be transformed. Hence matching would be impossible unless $G_\ell = G_i = G_m = 0$ — this solution is already described by (a limit of) $G_{\text{mlim}}$.

Unique to the T network, case (i) b) in (A.18) describes a second point where $\frac{1}{L_p} = 0$ or $L_p \to \infty$, but $L_a = 0$, if $G_i \geq \frac{1}{R_{a0}}$ for $R_{a0} > 0$ and $G_m = G_{\text{mlim}} = G_i - \frac{1}{R_{a0}} = G_\ell$, hence $G_p = \frac{1}{R_{a0}}$ and $C_{1,2} \to \infty$. Compared to $L_a \to \infty$, which was lossless, $L_a = 0$ involves losses in $R_{a0}$.
Depending on the particular $R_s$ used, hence the particular $G_p$ used (given as a function of $\omega L_p$ or $G_m$), additional restrictions to $G_m$ might apply, limiting $G_m$ even further or involving intervals within the limits derived above where matching is impossible. However, (3.33) holds throughout the valid range of $G_m$.

Additionally, the upper limits in (3.33) separate T and L networks — $C_1 \to \infty$ if $G_m = G_{m\text{lim}}$ or $C_2 \to \infty$ if $G_m = G_t$, respectively.

After designing the T network’s input impedance to match the source conductance $G_t$, we compute the active power $P_{actt}$ delivered to its load conductance $G_t$. Since the capacitive and inductive susceptances in Figure 3.15 cancel at matching frequency, we obtain

$$P_{actt} = \frac{1}{2} \frac{G_m}{(G_m + G_p) G_m^2} |\overline{V}_0|^2 \tag{3.29}$$
$$P_{actt} = \frac{1}{2} \frac{1}{G_m} \frac{|\overline{V}_0|^2 G_m}{G_m + G_p} \stackrel{(2.22)}{=} \frac{1}{1 + \frac{G_p}{G_m}}. \tag{3.35}$$

Conversely, due to input matching the active power $P_{actp}$ delivered to the loss conductance $G_p$ is given by

$$P_{actp} = P_{actmaxi} - P_{actt} \stackrel{(3.35)}{=} P_{actmaxi} \frac{1}{1 + \frac{G_p}{G_m}}. \tag{3.36}$$

For optimisation, we will need the power gain $g_{t,\text{inma}}$ of the matched T network. Since its input impedance is matched to $G_t$, the active power delivered to the network’s input is $P_{act1} = P_{actmaxi}$ (refer to Figure A.6). Thus we obtain

$$g_{t,\text{inma}} = \frac{P_{actt}}{P_{act1}} = \frac{P_{actt}}{P_{actmaxi}} \stackrel{(3.35)}{=} \frac{G_m}{G_m + G_p} = \frac{1}{1 + \frac{G_p}{G_m}}. \tag{3.37}$$

(Alternatively we could derive $g_{t,\text{inma}}$ from (A.23) if we put $C_{2\text{cot}} = C_2$. However, we gain more insight in the circuit’s behaviour if it is computed as shown above.)

Comparing the preceding formulae to those derived for the $\pi$ network, we may conclude that $\pi$ and T network would be equivalent if we would use an inductor $L_p$ with a parallel loss conductance $G_p$ which could be described as a function of $\frac{1}{\omega L_p}$ similar to the series loss resistance’s $R_s$ function of $\omega L_s$ for the $\pi$ network.
Unfortunately, the same inductor is used in both networks, thus exhibiting the same \( R_s \) as a function of \( \omega L_s \). If it is rewritten in \( G_p \) as a function of \( \frac{1}{\omega L_p} \), its functional description usually changes\(^7\).

Thus the properties of lossy \( \pi \) and T networks with a lossy “central” inductor are usually not equivalent. Hence the following derivation of the optimum value of the parameter \( G_m \) or \( G_t \), respectively, differ from those for the \( \pi \) network, involving additional restrictions within the proofs.

3.5.3.2 Optimum value of parameter \( G_m \) for maximum power gain

Minimising “losses of the T network” is achieved by transferring the highest portion of the active power delivered to the network’s input to its load \( G_t = \frac{1}{R_t} \). Thus neither \( P_{actp} \) has to be minimised nor \( P_{actT} \) to be maximised, but we have to maximise \( g_{t,\text{inna}} \). However, \( P_{act\text{maxi}} \) is independent of \( G_m \) and \( P_{actT} = g_{t,\text{inna}} P_{act\text{maxi}} \) holds (input matching), hence the highest \( g_{t,\text{inna}} \) occurs for maximum \( P_{actT} \).

To derive the value of \( G_m \) for maximum \( g_{t,\text{inna}} \), we compute its derivative with respect to \( G_m \).

Using (3.37) yields

\[
\frac{dg_{t,\text{inna}}}{dG_m} = \frac{G_m}{(G_m + G_p)^2} \left( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \right). \tag{3.38}
\]

Since \( G_m, \ G_p \geq 0 \), the first term is always \( \geq 0 \). Thus we need to check the sign of the second term only.

As \( R_s \) is a function of \( L_s \), and both are functions of \( G_p \) and \( L_p \) according to (A.19), (A.20), and \( G_m \) according to (3.32), where \( G_p \) itself is a function of \( G_m \) and implicitly\(^8\) of \( G_p \), we define the implicit function \( \psi_t \equiv 0 \) from (3.1), (A.19), and (A.20),

\[
\psi_t := R_{s0} + \left( \sum_{n=1}^{N} a_n \Lambda^\Lambda \right) - \frac{G_p}{G_p^2 + \frac{1}{\omega^2 L_p^2}} \equiv 0, \quad \text{where} \quad \Lambda = \frac{1}{\omega^2 L_p^2}.
\]

\(^7\) One exception is the case of a coil with a constant \( Q \) factor, where \( R_s = \frac{1}{Q} \omega L_s \) can be rewritten in \( G_p = \frac{1}{Q} \frac{1}{\omega L_p} \).

There is another equivalence if the “variable inductor” consists of a fixed inductor \( L_{s0} = \text{const.} \) in series to a variable capacitance for the \( \pi \) network or in parallel to a variable capacitance for the T network. Neglecting the losses in the variable capacitor yields \( R_s = R_{s0} \) for the \( \pi \) network and \( G_p = \frac{b_{s0}}{G_p^0 + \omega L_s} = G_{p0} \) for the T network. Similar to the \( \pi \) network in Section 3.5.2.1, matching in the T network requires \( G_{p0} \leq G_t \).

\(^8\) In particular, \( R_s, L_s, G_p, \) and \( L_p \) are all functions of \( G_m \) only. However, we usually may not be able to solve for \( G_p \) explicitly after applying (A.19), (A.20), and (3.32) to (3.1).
Then we compute the implicit function's partial derivatives with respect to $G_m$ and $G_p$,

$$\frac{\partial \psi_t}{\partial G_m} = \left(2 G_p \frac{1}{\omega L_p} - \left(\frac{1}{\omega^2 L_p^2} - G_p^2\right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) \frac{\partial \left(\frac{1}{\omega L_p}\right)}{\partial G_m} \left(G_p^2 + \frac{1}{\omega^2 L_p^2}\right)^2,$$

$$\frac{\partial \psi_t}{\partial G_p} = \frac{1}{\omega^2 L_p^2} - G_p^2 + 2 G_p \frac{1}{\omega L_p} \left(\sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) - \left(2 G_p \frac{1}{\omega L_p} - \left(\frac{1}{\omega^2 L_p^2} - G_p^2\right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) \frac{\partial \left(\frac{1}{\omega L_p}\right)}{\partial G_p} \left(G_p^2 + \frac{1}{\omega^2 L_p^2}\right)^2.$$

Applying the implicit function theorem [64] gives

$$\frac{d G_p}{d G_m} = -\frac{\frac{\partial \psi_t}{\partial G_m}}{\frac{\partial \psi_t}{\partial G_p}} = \frac{\left(2 G_p \frac{1}{\omega L_p} - \left(\frac{1}{\omega^2 L_p^2} - G_p^2\right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) \frac{\partial \left(\frac{1}{\omega L_p}\right)}{\partial G_m}}{\frac{1}{\omega^2 L_p^2} - G_p^2 + 2 G_p \frac{1}{\omega L_p} \left(\sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) - \left(2 G_p \frac{1}{\omega L_p} - \left(\frac{1}{\omega^2 L_p^2} - G_p^2\right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) \frac{\partial \left(\frac{1}{\omega L_p}\right)}{\partial G_p}}.$$

Then we obtain for the second term in (3.38)

$$\frac{G_p}{G_m} \frac{d G_p}{d G_m} = \frac{G_p \left(\frac{1}{\omega^2 L_p^2} - G_p^2 + 2 G_p \frac{1}{\omega L_p} \left(\sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) - \left(2 G_p \frac{1}{\omega L_p} - \left(\frac{1}{\omega^2 L_p^2} - G_p^2\right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) \left(G_p \frac{\partial \left(\frac{1}{\omega L_p}\right)}{\partial G_p} + G_m \frac{\partial \left(\frac{1}{\omega L_p}\right)}{\partial G_m}\right)\right)}{G_p \left(\frac{1}{\omega^2 L_p^2} - G_p^2 + 2 G_p \frac{1}{\omega L_p} \left(\sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) - \left(2 G_p \frac{1}{\omega L_p} - \left(\frac{1}{\omega^2 L_p^2} - G_p^2\right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1}\right) \frac{\partial \left(\frac{1}{\omega L_p}\right)}{\partial G_p}\right)} \left(\Lambda = \frac{1}{\omega L_p}\right). (3.39)$$
CHAPTER 3. LOSSY \( L, \pi, \) AND \( T \) MATCHING NETWORKS

Replacing of \( \frac{1}{\omega L_p} \) with (3.32) yields the missing partial derivatives,

\[
\frac{\partial (\frac{1}{\omega L_p})}{\partial G_m} = \frac{G_i - 2(G_m + G_p)}{2 \sqrt{(G_m + G_p)(G_i - G_m - G_p)}} + \frac{G_t - 2 G_m}{2 \sqrt{G_m(G_t - G_m)}}, \tag{3.40}
\]

\[
\frac{\partial (\frac{1}{\omega L_p})}{\partial G_p} = \frac{G_i - 2(G_m + G_p)}{2 \sqrt{(G_m + G_p)(G_i - G_m - G_p)}}. \tag{3.41}
\]

To check which value of \( G_m \) yields the highest \( g_{t, inna} \), we need to know the sign of \( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \), the second term in (3.39), throughout the valid range of \( G_m \) given by (3.33).

For ease of understanding, an overview of the following proof is given below. The proof itself is quite lengthy and thus performed in Appendix B.2.2.1.

The idea is that \( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \) remains \( \geq 0 \) for all valid \( G_m \). Then the highest possible value of \( G_m \) would yield the highest \( g_{t, inna} \).

To prove it, we will derive the highest \( b_N \) which still yields a numerator \( \geq 0 \) and simultaneously a denominator \( \geq 0 \). This approach involves more restrictions than necessary, because the ratio of a negative numerator to a negative denominator would also be positive, but the proof is then quite "easy" and straightforward. During the proof, we will limit the inductor’s \( Q \) factor to \( Q \geq 1 \), which just indicates that \( \omega L_a \geq R_a \) or \( \frac{1}{\omega L_p} \geq G_p \) and is likely to assume for useful variable inductors.

However, we should keep in mind that due to the additional restrictions \( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \) might remain positive even for a higher \( b_N \) or lower \( Q \) than those derived.

If we abbreviate \( S_t = \sum_{n=1}^{N} a_n b_n \Lambda^{n-1} \), we obtain from (3.39)

\[
\frac{G_p}{G_m} - \frac{dG_p}{dG_m} = \frac{G_p \left( \frac{1}{\omega L_p} - G_p^2 + 2 G_p \omega_L \frac{1}{\omega L_p} S_t \right) - \left( 2 G_p \frac{1}{\omega L_p} - \frac{1}{\omega L_p} G_p^2 S_t \right)}{G_m \left( \frac{1}{\omega L_p} - G_p^2 + 2 G_p \omega_L \frac{1}{\omega L_p} S_t - \left( 2 G_p \frac{1}{\omega L_p} - \frac{1}{\omega L_p} G_p^2 \right) S_t \right) \frac{d \frac{1}{\omega L_p}}{G_m}}.
\]

To prove that numerator and denominator are \( \geq 0 \) for certain \( b_N \) and \( Q \), we firstly assume \( S_t = 0 \) or \( R_a = R_{a0} \) and calculate \( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \), which will be \( \geq 0 \) throughout the valid range of \( G_m \).

Secondly, we assume \( S_t \neq 0 \), hence \( S_t > 0 \) (according to (3.1)). We then prove that \( 2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega L_p} G_p^2 \right) S_t \geq 0 \) if \( 0 < b_N \leq 2 \). If we further limit the variable inductor’s \( Q \) factor to \( Q \geq 1 \), hence \( \frac{1}{\omega L_p} \geq G_p \), it is obvious that \( \frac{1}{\omega L_p} - G_p^2 \geq 0 \), thus \( 2 G_p \frac{1}{\omega L_p} \geq 2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega L_p} G_p^2 \right) S_t \geq 0 \) and \( \frac{1}{\omega L_p} - G_p^2 + 2 G_p \frac{1}{\omega L_p} S_t \geq 0 \).

We then check numerator and denominator separately:

1. Since \( \omega L_p, G_m, G_p \) are \( \geq 0 \), \( S_t > 0 \), and the preceding estimates apply, the numerator will always be \( \geq 0 \) for \( G_p \frac{d \frac{1}{\omega L_p}}{G_m} + G_m \frac{d \frac{1}{\omega L_p}}{G_m} \leq 0 \). Conversely, if
\( G_p \frac{\partial L_p}{\partial G_p} + G_m \frac{\partial L_m}{\partial G_m} > 0 \), we have to use the preceding estimates and certain inequalities to prove that then the numerator will be \( \geq 0 \).

2. Since \( \omega L_p, G_m, G_p \) are \( \geq 0, S_t > 0 \), and the preceding estimates apply, the denominator will always be \( \geq 0 \) for \( \frac{\partial (L_p)}{\partial G_p} \leq 0 \). Conversely, if \( \frac{\partial (L_p)}{\partial G_p} > 0 \), we will have to use the preceding estimates and certain inequalities to prove that then the denominator will be \( \geq 0 \).

We then know that for all possible values of \( R_s, a_n, \) and \( b_n \) according to (3.1), \( \frac{d g_{t, inna}}{d G_m} \geq 0 \) if (at least) \( b_N \leq 2 \) and \( Q \geq 1 \) (hence \( \omega L_s \geq R_s \) or \( \omega L_p \geq G_p \)).

Hence the highest \( g_{t, inna} \) is obtained for the highest valid value of \( G_m \), which is \( G_\ell \) if \( G_\ell \leq G_i \) or \( G_{}\text{lim} \) defined in (3.34) if \( G_\ell > G_i \), respectively. If we choose \( G_m \) accordingly, the particular type of network involved is one of the two \( L \) networks where \( C_2 \to \infty \) or \( C_1 \to \infty \).

However, we should keep in mind that the restrictions \( 0 < b_N \leq 2 \) and \( Q \geq 1 \) (hence \( \omega L_s \geq R_s \) or \( \omega L_p \geq G_p \)) aroused after applying inequalities to get upper estimates for certain terms. If \( b_N > 2 \) or \( 0 \leq Q < 1 \), the method used in the preceding proof gives no indication of the network’s behaviour. To illustrate this, it should be noted that if all \( a_n = 0 \), hence \( R_s = R_{e0} \), the proof involved no restrictions to the \( Q \) factor of the variable inductor\(^9\).

It’s interesting to note that usually the highest \( g_{t, inna} \) does not coincide with the lowest \( R_s \) (or, according to (3.1), the lowest \( L_s \)), where \( R_s, L_s \) are the elements yielded by parallel-to-series transformation of \( G_p \) and \( L_p \).

### 3.5.3.3 Optimum value of load conductance \( G_\ell \) for maximum power gain

In Section 3.5.3.2 we derived which value of \( G_m \) maximises \( g_{t, inna} \). However, the highest \( g_{t, inna} \) still depends on \( G_\ell \) (\( G_i \) is fixed in our application of matching the load to the source).

Thus further investigations are required to determine which \( G_\ell \) yields the highest \( g_{t, inna} \) if the parameter \( G_m \) was chosen for maximum \( g_{t, inna} \) according to Section 3.5.3.2.

\( G_\ell \) is variable because the connecting line between matching network and load transforms the load depending on the line’s length. As transformation usually involves additional reactances even for a purely resistive load, optimisation of \( g_{t, inna} \) with

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\(^9\)Similar proofs without any restrictions to the \( Q \) factor of the variable inductor could also be performed for different single loss terms, e.g. \( R_s = a_1 (\omega L_s)^3 \), \( R_s = \frac{1}{2} \omega L_s \), and \( R_s = a_2 (\omega L_s)^2 \), which also holds for the two cases equivalent to the \( \pi \) network, \( G_p = G_{p0} \) and \( G_p = \frac{1}{Q} \omega L_p \) (described in Footnote \(^7\)). However, each term generates a different solution for \( G_p \), and usually the sum of the single loss terms does not generate a \( G_p \) equal to the sum of the single terms’ \( G_p \). Thus these proofs do not help to extend the proof for (3.1).
respect to \( G_\ell \) is only part of the solution, “complete” optimisation needs application of Sections 4.3.1.2 and 4.3.2.2, too.

The derivation of the optimum value of \( G_\ell \) is quite similar to Section 3.5.3.2, but lengthy and thus performed in Appendix B.2.2.2.

Summarising Appendix B.2.2.2, we may conclude that if \( 0 < b_N \leq 2 \), \( Q \geq 1 \) (hence \( \omega L_p \geq R_s \) or \( \frac{1}{\omega L_p} \geq G_p \)), and \( G_m \) is chosen according to Section 3.5.3.2, \( G_\ell = G_i \) would yield the highest \( g_{\ell, \text{inma}} \). \( g_{\ell, \text{inma}} \) decreases for higher or lower \( G_\ell \).

Unfortunately the optimisation gives no indication for which of the two load conductors \( G_{\ell_1} > G_i \), \( G_{\ell_2} = \frac{G_i^2}{G_{\ell_1}} < G_i \), exhibiting similar reflection coefficients referred to \( R_i = \frac{1}{G_i^2} \), we would achieve the higher \( g_{\ell, \text{inma}} \) (reference \( R_i = \frac{1}{G_i^2} \) is not related to the interconnecting transmission lines’ characteristic impedances, but was used to obtain equal “transformation distances” from optimum value \( R_i = \frac{1}{G_i^2} \)).

As mentioned before, “complete” optimisation involves taking into account of Sections 4.3.1.2 and 4.3.2.2. However, at least we may conclude that minimising losses of the matching network will always require consideration (and optimisation) of the length of the transmission line connecting it to the load.

### 3.5.3.4 Approximate equivalence of \( \pi \) and \( T \) network for a high-\( Q \) central inductor \( L_s \)

Usually the \( T \) network behaves different than the \( \pi \) network because the losses of the inductor are of a serial type in both cases (some remarks at the end of Section 3.5.3.1 already addressed this issue). Thus, if we perform a series to parallel transformation of \( L_s \) and \( R_s \) in \( L_p \) and \( G_p \) as shown in Figure 3.13 b), we may rewrite (3.1), the functional description of the losses, using (A.19) and (A.20), obtaining

\[
\frac{G_p}{G_p^2 + \left( \frac{1}{\omega L_p} \right)^2} = R_{s0} + \sum_{n=1}^{N} a_n \Lambda^{b_n},
\]

or, slightly rearranged,

\[
G_p = R_{s0} \left( G_p^2 + \left( \frac{1}{\omega L_p} \right)^2 \right) + \sum_{n=1}^{N} a_n \left( \frac{1}{\omega L_p} \right)^{b_n} \left( G_p^2 + \left( \frac{1}{\omega L_p} \right)^2 \right)^{b_n-1},
\]

which, if solved for \( G_p \), is usually not equal to \( G_p = G_{p0} + \sum_{n=0}^{N} \tilde{a}_n \left( \frac{1}{\omega L_p} \right)^{b_n} \) (with some exceptions indicated in Footnote 7).

Although the proofs for the \( T \) network did not need any approximation, we might get some hint why they also involved a constraint of \( b_N \leq 2 \) if we consider an inductor
with $\omega L_n \gg R_n$ (or $Q \gg 1$). Hence $\frac{1}{\omega L_p} \gg G_p$ holds, allowing us to approximate

$$G_p \approx R_{s0} \left( \frac{1}{\omega L_p} \right)^2 + \sum_{n=1}^{N} a_n \left( \frac{1}{\omega L_p} \right)^{2-b_n}.$$ 

Assuming $0 < b_N \leq 2$, the powers are simply "mirrored" at $b_n = 1$. If we define $\bar{a}_n := a_n$, $\bar{b}_n := 2 - b_n$, we get an equation in $G_p$ and $\frac{1}{\omega L_p}$ similar to those in $R_s$ and $\omega L_s$. If $b_N = 2$ for the $R_s$ used, a $G_{p0} = a_N$ would occur, involving additional restrictions similar to those involved by $R_{s0}$ for the $\pi$ network.

Thus, if $Q \gg 1$ for the central inductor $L_s$, the $T$ network is (approximately) equivalent to the $\pi$ network.
Chapter 4

Extension of the Method to Complex Loads

4.1 Introduction

In the previous chapter, the optimum parameter’s (and load resistance’s) value for maximum active power transfer was derived for resistive sources and loads.

Since realistic loads are usually complex, the method presented in the previous chapter is extended to complex loads in the following chapter. Then exact analytical solutions for a constant $Q$ factor central inductor are derived. Thereby the power gain of different types of networks may be compared and its increase introduced by a suitable matching strategy may be estimated. Important characteristics of the exact solution help to derive an optimum matching strategy for any types of loads. Finally its applicability to switchable matching networks is considered.

4.2 Definitions/notations

From this chapter on, the following definitions/notations will be used:

**Latin letters**

- $B_{2tot}$: Total load susceptance (including $C_2$ of the network and the load’s (equivalent) inductance)
- $C_{2tot}$: Total load capacitance (including $C_2$ of the network and the load’s (equivalent) capacitance or inductance)
- $C_L$: Load capacitance in parallel to load conductance $G_L$
- $C_{ta}$: Load capacitance in series to load resistance $R_L$
- $g_{r,inma,maxind+}$: Maximum $g_{r,inma}$ (of solution $G_{pind,+}$ if $G_L < C_L$)
- $g_{r,inma,maxind+}$: Maximum $g_{r,inma}$ (of solution $G_{pind,+}$ if $R_L < R_L$)
- $g_{p,ideal}$: Power gain of a π network designed as if it would be lossless
- $G_{climo}G_{lim}$: Load conductance for which the maximum of $C_L$ tends to infinity (solution $G_{pcap+}$)
- $G_m/l_p0$: $G_m$ which yields $1/l_p = 0$ in the case of an inductively loaded T network (equivalent square roots in $L_p$ formula)
- $G_mC_{ra}$: $G_m$ which yields $C_{2tot} = C_{ta}$
CHAPTER 4. EXTENSION OF THE METHOD TO COMPLEX LOADS

\( G_{\text{maxind}+} \quad G_m \) which yields the maximum \( g_{t, \text{lima}} \) (solution \( G_{\text{pind}+} \))

\( G_{\text{lim}1} \quad C_1 \) Lower pole of \( C_1 \) (solution \( G_{\text{cap}+} \))

\( G_{\text{lim}2} \quad C_1 \) Upper pole of \( C_1 \) (solution \( G_{\text{cap}+} \))

\( G_{m, t} \quad C_m \) which yields \( L_{2\text{tot}} = L_\delta \)

\( G_{\text{mov}01} \quad C_1 \) Lower zero of \( u_t \)

\( G_{\text{mov}02} \quad C_1 \) Upper zero of \( u_t \)

\( G_{\text{mov}01} \quad u_t \) Lower zero of \( v_t \)

\( G_{\text{mov}02} \quad u_t \) Upper zero of \( v_t \)

\( G_{\text{pind}+} \quad u_t \) Exact solutions of \( G_p \) for a constant \( Q \) factor central inductor and capacitive/resistive/(over-)compensated inductive loads, distinguished by the sign in front of \( \sqrt{u_t} \)

\( G_{\text{cap}+} \quad v_t \) Exact solutions of \( G_p \) for a constant \( Q \) factor central inductor and undercompensated inductive loads, distinguished by the sign in front of \( \sqrt{v_t} \)

\( L_{2\text{tot}} \quad \delta \) Total load inductance (including \( C_2 \) of the network and the load's (equivalent) inductance)

\( L_{\text{tp}} \quad \delta \) Load inductance in parallel to load conductance \( G_{\delta} \)

\( L_{\text{tp0}} \quad \delta \) Load inductance in series to load resistance \( R_{\delta} \) (T network)

\( L_{\text{pind}+} \quad \delta \) Parallel inductance derived from the exact solution \( G_{\text{pind}+} \) for a constant \( Q \) factor central inductor and undercompensated inductive loads

\( L_{\text{ind}+} \quad \delta \) Series inductance derived from the exact solution \( R_{\text{ind}+} \) for a constant \( Q \) factor central inductor and undercompensated inductive loads \( (\pi \text{ network}) \)

\( L_{\text{ind}+} \quad \delta \) Series inductance yielded by parallel-to-series transformation of \( G_{\text{pind}+} \) and \( L_{\text{pind}+} \) \( (\pi \text{ network}) \)

\( R_{\text{limodRlim}} \quad \delta \) Load resistance for which the minimum of \( C_1 \) equals zero (solution \( R_{\text{cap}+} \))

\( R_{\text{lim}0} \quad \delta \) \( R_m \) which yields \( C_{2\text{tot}} = C_{\delta} \)

\( R_{\text{maxind}+} \quad \delta \) \( R_m \) which yields the maximum \( g_{t, \text{lima}} \) (constant \( Q \) factor central inductor and undercompensated inductive load)

\( R_{\text{lim}0} \quad \delta \) \( R_m \) which yields \( L_{2\text{tot}} = L_\delta \)

\( R_{\text{lim}2} \quad \delta \) \( R_m \) which yields \( \delta = 0 \) in the case of an inductively loaded T network (equal square roots in \( L_\delta \) formula)

\( R_{\text{mov}01} \quad \delta \) Lower zero of \( u_\pi \)

\( R_{\text{mov}02} \quad \delta \) Upper zero of \( u_\pi \)

\( R_{\text{mov}01} \quad \delta \) Lower zero of \( v_\pi \)

\( R_{\text{mov}02} \quad \delta \) Upper zero of \( v_\pi \)

\( R_{\text{cap}+} \quad \delta \) Exact solutions of \( R_\pi \) for a constant \( Q \) factor central inductor and capacitive/resistive/(over-)compensated inductive loads, distinguished by the sign in front of \( \sqrt{v_\pi} \)

\( R_{\text{ind}+} \quad \delta \) Exact solutions of \( R_\pi \) for a constant \( Q \) factor central inductor and undercompensated inductive loads, distinguished by the sign in front of \( \sqrt{v_\pi} \)

\( \delta \quad \pi \) Argument of "large" square root in \( G_{\text{cap}+} \)

\( \delta \quad \pi \) Argument of "large" square root in \( R_{\text{cap}+} \)

\( \delta \quad \pi \) Argument of "large" square root in \( G_{\text{pind}+} \)

\( \delta \quad \pi \) Argument of "large" square root in \( R_{\text{ind}+} \)

\( X_{2\text{tot}} \quad \pi \) Total load reactance (including \( C_2 \) of the network and the load's (equivalent) inductance)

**Greek letters**

\( \Delta G_m \quad \pi \) Step in \( G_m \) accompanied by a resistance step due to switching to next inductor value in a T network with switchable elements

\( \Delta R_m \quad \pi \) Step in \( R_m \) accompanied by a resistance step due to switching to next inductor value in a \( \pi \) network with switchable elements
4.3 Matching of a complex load to a resistive source (matching at the network’s input)

4.3.1 Complex load exhibiting a capacitive imaginary part

4.3.1.1 π network

![Circuit Diagram]

Figure 4.1: Resulting π network’s equivalent circuit for matching of a (complex) load admittance consisting of a conductance \( G_\ell \) in parallel to a capacitance \( C_{tp} \) to a (real) source conductance \( G_i \)

If the load impedance \( Z_\ell \), hence the load admittance \( \frac{1}{Z_\ell} \), exhibits a capacitive imaginary part, thus \( \text{Im}\{Z_\ell\} < 0 \) and \( \text{Im}\left\{\frac{1}{Z_\ell}\right\} > 0 \), we may describe the load admittance at matching frequency by a conductance \( G_\ell \) in parallel to a capacitance \( C_{tp} \), where

\[
G_\ell = \text{Re}\left\{\frac{1}{Z_\ell}\right\}, \quad (4.1)
\]

\[
C_{tp} = \frac{1}{\omega} \text{Im}\left\{\frac{1}{Z_\ell}\right\}. \quad (4.2)
\]

If we combine the π network’s capacitance \( C_2 \) and the load capacitance \( C_{tp} \) (refer to Figure 4.1), and denote the total capacitance by \( C_{2\text{tot}} \), we obtain

\[
C_{2\text{tot}} = C_2 + C_{tp}.
\]

The resulting equivalent circuit’s components are similar to those of the π network in Section 3.5.2.1. Thus all results obtained in Sections 3.5.2.1 and 3.5.2.2 are applicable if we put \( R_i = \frac{1}{G_i} \), \( R_\ell = \frac{1}{G_\ell} \), and replace \( C_2 \) with \( C_{2\text{tot}} \) in all formulae derived\(^1\), in particular in (3.18). Then the remaining capacitance \( C_2 \) of the π network is given by

\[
C_2 = C_{2\text{tot}} - C_{tp} = \frac{1}{\omega R_\ell} \sqrt{\frac{R_\ell - R_m}{R_m}} - C_{tp}. \quad (4.3)
\]

However, \( C_2 \) is a variable capacitance, thus the lowest value of \( C_{2\text{tot}} \) is \( C_{tp} \), which differs from the case of a resistive load, where the minimum value was (theoretically) zero (but a "real world" variable capacitor's minimum value wouldn't be zero).

To illustrate the impact of the preceding restriction to the minimum \( C_{2\text{tot}} \), we compute the derivative of \( C_{2\text{tot}} \) with respect to \( R_m \) from (3.18), \[ \frac{dC_{2\text{tot}}}{dR_m} = \]

\(^1\text{We should note that there is a slight inconsistency in the definition of } R_\ell \text{ because } R_\ell \neq \text{Re}\left\{\frac{1}{G_{\ell+}+j\omega C_m}\right\} \text{ as we would expect it to be according to the definitions in (2.7) and (2.13).} \]
which is \( < 0 \) for the (utmost) valid range of \( R_m \) indicated in (3.20). Thus \( R_m \) has to be lower than or equal to

\[
R_m C_{tp} = \frac{R_\ell}{1 + (\omega R_\ell C_{tp})^2} \left( \frac{\text{Re}\left\{\frac{1}{Z_\ell}\right\}}{\frac{1}{Z_\ell}} \right) = \text{Re}\{Z_\ell\},
\]

(4.4)

ensuring \( C_{2tot} \geq C_{tp} \). For AC excitation \( \omega > 0 \), hence \( R_m C_{tp} < R_\ell \) holds (which is easy to understand because \( C_{2tot} = 0 \) if \( R_m = R_\ell \), but \( C_{2tot} \geq C_{tp} > 0 \) due to \( C_{tp} > 0 \) for a complex load exhibiting a capacitive imaginary part).

Differing from a purely resistive load, the (utmost) limits of the parameter \( R_m \) are given by \( 0 \leq R_m \leq R_m C_{tp} \) if \( R_\ell \leq R_i - R_{\delta 0} \) or by \( 0 \leq R_m \leq \min\{R_m C_{tp}, R_{\text{mlim}}\} \) if \( R_\ell > R_i - R_{\delta 0} \).

Similar to Section 3.5.2.2, \( g_{r,\text{inma}} \) increases (or stays constant) with increasing \( R_m \). Thus the highest valid \( R_m \) yields the maximum \( g_{r,\text{inma}} \), hence \( R_m = R_m C_{tp} \) if \( R_\ell \leq R_i - R_{\delta 0} \) or \( R_m = \min\{R_m C_{tp}, R_{\text{mlim}}\} \) if \( R_\ell > R_i - R_{\delta 0} \), respectively. In both cases, the resulting matching network is a \( L \) network, because \( C_{2tot} = C_{tp} \) and \( C_2 = 0 \) for \( R_m = R_m C_{tp} \), or \( C_1 = 0 \) for \( R_m = R_{\text{mlim}} \). In particular, for \( R_m = R_m C_{tp} \) the resulting circuit is a \( \pi \) network, but the type of the \textit{external} matching network is a \( L \) network\(^2\).

### 4.3.1.2 T network

![T network diagram](image)

Figure 4.2: Resulting T network's equivalent circuit for matching of a (complex) load reactance consisting of a resistance \( R_\ell \) in series to a capacitance \( C_{lbs} \) to a (real) source resistance \( R_i \)

If the load impedance \( Z_\ell \) exhibits a capacitive imaginary part, thus \( \text{Im}\{Z_\ell\} < 0 \), we may describe the load impedance at matching frequency by a resistance \( R_\ell \) in series to a capacitance \( C_{lbs} \), where

\[
R_\ell = \text{Re}\{Z_\ell\}, \quad (4.5)
\]

\[
C_{lbs} = -\frac{1}{\omega \text{Im}\{Z_\ell\}}, \quad (4.6)
\]

\(^2\) However, due to parasitic capacitances, a "real world" matching network would always be a \( \pi \) network and could not be set to \( R_m = R_{\text{mlim}} \), although these parasitic capacitances might be remarkably small, e.g. for switchable capacitors in a network similar to [33]. But within this section we investigate the networks' principal behaviour, the impact of minimum and maximum values of \( C_1, L_{\text{m}}, \text{ and } C_2 \) will be considered in Section 4.5.1.
If we combine the T network's capacitance $C_2$ and the load capacitance $C_{fs}$ (refer to Figure 4.2), and denote the total capacitance by $C_{2tot}$, we obtain

$$C_{2tot} = \frac{1}{\frac{1}{C_2} + \frac{1}{C_{fs}}}.$$

The resulting equivalent circuit's components are similar to those of the T network in Section 3.5.3.1. Thus all results obtained in Sections 3.5.3.1 and 3.5.3.2 are applicable if we put $G_1 = \frac{1}{R_1}$, $G_\ell = \frac{1}{R_\ell}$, and replace $C_2$ with $C_{2tot}$ in all formulae derived\(^3\), in particular in (3.31). Then the remaining capacitance $C_2$ of the T network is given by

$$C_2 = \frac{1}{1/C_{2tot} - 1/C_{fs}} = \frac{1}{\frac{\omega}{G_\ell} \sqrt{\frac{G_\ell - G_m}{G_m} - \frac{1}{C_{fs}}}}. \quad (4.7)$$

However, $C_2$ is a variable capacitance, thus the highest value of $C_{2tot}$ is $C_{fs}$, which differs from the case of a resistive load, where the maximum value tended (theoretically) to infinity (but a “real world” variable capacitor's maximum value would be finite).

To illustrate the impact of the preceding restriction to the maximum $C_{2tot}$, we compute the derivative of $C_{2tot}$ with respect to $G_m$ from (3.31),

$$\frac{dC_{2tot}}{dG_m} = \frac{C_2^2}{2\omega (G_\ell - G_m) \sqrt{G_m (G_\ell - G_m)}},$$

which is $> 0$ for the (utmost) valid range of $G_m$ indicated in (3.33). Thus $G_m$ has to be lower than or equal to

$$G_mC_{fs} = \frac{G_\ell}{1 + \left(\frac{G_\ell}{\omega C_{fs}}\right)^2} \left(= \frac{\text{Re}\{Z_\ell\}}{|Z_\ell|} = \text{Re} \left\{ \frac{1}{Z_\ell} \right\} \right), \quad (4.8)$$

ensuring $C_{2tot} \leq C_{fs}$. For AC excitation $\omega > 0$, hence $G_mC_{fs} < G_\ell$ holds (which is easy to understand because $C_{2tot} \to \infty$ if $G_m = R_\ell$, but since $C_{fs} \in (0, \infty)$ and $0 < C_{2tot} \leq C_{fs}$, $C_{2tot} \in (0, \infty)$ holds).

Differing from a purely resistive load, the (utmost) limits of the parameter $G_m$ are given by $0 \leq G_m \leq G_mC_{fs}$ if $G_\ell \leq G_1$ or by $0 \leq G_m \leq \text{Min}\{G_mC_{fs}, G_{mlim}\}$ if $G_\ell > G_1$.

Similar to Section 3.5.3.2, $g_{r, inna}$ increases (or might stay constant) with increasing $G_m$. Thus the highest valid $G_m$ yields the maximum $g_{r, inna}$, hence $G_m = G_mC_{fs}$ if $G_\ell \leq G_1$ or $G_m = \text{Min}\{G_mC_{fs}, G_{mlim}\}$ if $G_\ell > G_1$, respectively. In both cases, the resulting matching network is a L network, because $C_{2tot} = C_{fs}$ and $C_2 \to \infty$ for $G_m = G_mC_{fs}$, or $C_1 \to \infty$ for $G_m = G_{mlim}$. In particular, for $G_m = G_mC_{fs}$ the resulting circuit is a T network, but the type of the external matching network is a L network\(^4\).

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\(^3\)We should note that there is a slight inconsistency in the definition of $G_\ell$ because $G_\ell \neq \text{Re} \left\{ \frac{1}{R_\ell + \frac{1}{\omega C_{fs}}} \right\}$ as we would expect it to be according to the definitions in (2.13) and (2.7).

\(^4\)However, due to finite capacitances, a “real world” matching network would always be a
4.3.2 Complex load exhibiting an inductive imaginary part

4.3.2.1 π network

![Equivalent Circuit Diagram]

Figure 4.3: Resulting π network’s equivalent circuit for matching of a (complex) load admittance consisting of a conductance $G_t$ in parallel to an inductance $L_{tp}$ to a (real) source conductance $G_i$

If the load impedance $Z_t$, hence the load admittance $\frac{1}{Z_t}$, exhibits an inductive imaginary part, thus $\text{Im}\left\{\frac{1}{Z_t}\right\} > 0$ and $\text{Im}\left\{\frac{1}{Z_t}\right\} < 0$, we may describe the load admittance at matching frequency by a conductance $G_t$ in parallel to an inductance $L_{tp}$, where

$$G_t = \text{Re}\left\{\frac{1}{Z_t}\right\}, \quad (4.9)$$

$$L_{tp} = -\frac{1}{\omega \text{Im}\left\{\frac{1}{Z_t}\right\}}. \quad (4.10)$$

To achieve matching of the π network’s input admittance to the source conductance, two completely different solutions arise.

1. (Over-)Compensation of $L_{tp}$ by $C_2$, hence $B_{2\text{tot}} \geq 0$ in Figure 4.3 and thus may be described by a capacitance $C_{2\text{tot}}$ at matching frequency.

Similar to Section 4.3.1.1 $C_{2\text{tot}}$ is given by (3.18). For the remaining capacitance $C_2$ of the π network we obtain

$$C_2 = \frac{1}{\omega^2 L_{tp}} + C_{2\text{tot}} = \frac{1}{\omega} \left(\frac{1}{\omega L_{tp}} + \frac{1}{R_t} \sqrt{\frac{R_t - R_m}{R_m}}\right). \quad (4.11)$$

$C_2$ is a variable capacitance and thus $\geq 0$, in particular, $C_2 \geq \frac{1}{\omega^2 L_{tp}} > 0$. If the susceptance of $C_2$ cancels the susceptance of $L_{tp}$, $B_{2\text{tot}}$ and $C_{2\text{tot}}$ may be zero — just equivalent to the case of a purely resistive load with exception of the value of $C_2$, which is shifted by $\frac{1}{\omega^2 L_{tp}}$. All results obtained in Sections 3.5.2.1 and 3.5.2.2 are applicable, but we have to put $R_t = \frac{1}{G_i}$, $R_t = \frac{1}{G_t}$, and replace

T network and could not be set to $G_m = G_{\text{min}}$, although the maximum capacitances might be remarkably high, e.g. for switchable capacitors in a network similar to [33]. But within this section we investigate the networks' principal behaviour, the impact of minimum and maximum values of $C_1$, $L_s$ (from parallel-to-series transformation of $G_p$ and $L_p$), and $C_2$ will be considered in Section 4.5.2.
C_2 with C_{2tot} in all formulae derived\(^5\). However, compared to Section 4.3.1.1, there is no additional limit to R_m, but a minimum value of C_2 given by \( \frac{1}{\omega L_{tp}} \).

Similar to Section 3.5.2.2, \( g_{\pi, \text{inma}} \) increases (or stays constant) with increasing \( R_m \). Thus the highest valid \( R_m \) yields the maximum \( g_{\pi, \text{inma}} \), hence \( R_m = R_t \) if \( R_t \leq R_t - R_e \) or \( R_m = R_{\text{lim}} \) if \( R_t > R_t - R_e \), respectively. For \( R_m = R_t \) the resulting type of the external matching network is a \( \pi \) network, whose capacitance \( C_2 \) cancels \( L_{tp} \) at matching frequency, hence \( \pi \) network and \( L_{tp} \) build up a L network. For \( R_m = R_{\text{lim}} \), \( C_1 = 0 \) holds, and the resulting type of the external matching network is a L network\(^2\).

We already might assume that the case of (over-)compensation does not guarantee the lowest losses, because the equivalent circuit’s lossless load inductance is “wasted” by (over-)compensation and all inductance needed for matching has to be supplied by the lossy inductor of the matching network.

2. Undercompensation of \( L_{tp} \) by \( C_2 \), hence \( B_{2tot} < 0 \) in Figure 4.3 and thus may be described by an inductance \( L_{2tot} \) at matching frequency.

According to (3.18) and the notes regarding the signs in parentheses/square brackets we obtain

\[
L_{2tot} = \frac{1}{L_{tp}} - \omega^2 C_2 = \frac{R_t}{\omega} \sqrt{\frac{R_m}{R_t - R_m}}.
\]

The remaining capacitance \( C_2 \) of the \( \pi \) network is then given by

\[
C_2 = \frac{1}{\omega} \left( \frac{1}{L_{tp}} - \frac{1}{R_t} \sqrt{\frac{R_t - R_m}{R_m}} \right).
\]

As before, we have to put \( R_t = \frac{1}{G_i} \) and \( R_t = \frac{1}{G_i} \) in all equations derived\(^5\).

Additionally, according to (3.19) and the notes regarding the signs in parentheses/square brackets we obtain for the inductance \( L_s \) of the \( \pi \) network

\[
L_s = \sqrt{(R_m + R_s) (R_t - R_m - R_s)} - \sqrt{R_m (R_t - R_m)}.
\]

The preceding changes of the equivalent network’s elements also involve changes of the limits of parameter \( R_m \).

One additional limit occurs because \( C_2 \) is still a capacitance and \( \geq 0 \). Thus \( L_{2tot} \geq L_{tp} \) holds. Since \( L_{2tot} \) increases with increasing \( R_m \) (square root similar

---

\(^5\) We should note that there is a slight inconsistency in the definition of \( R_t \) because \( R_t \neq \text{Re} \left\{ \frac{1}{G_i + \frac{1}{\omega L_{tp}}} \right\} \) as we would expect it to be according to the definitions in (2.7) and (2.13).
to those of $C_{2\text{rot}}$ in Section 4.3.1.2), $R_m$ has to be higher than\(^6\) or equal to

$$R_{mL_\psi} = \frac{R_\ell}{1 + \left( \frac{R_\ell}{\omega L_{\psi}} \right)^2} = \frac{\text{Re}\left\{ \frac{1}{Z_\ell} \right\}}{\left| \frac{1}{Z_\ell} \right|^2} = \text{Re}\{Z_\ell\}. \quad (4.14)$$

For AC excitation $\omega > 0$ and $R_{mL_\psi} < R_\ell$ holds.

Another additional limit, $R_{mL_\psi 0}$, may occur because $L_s$ is still an inductance and $\geq 0$. In the resistive and capacitive cases, the sum of the square roots in $L_s$ could never be lower than zero, but equal to zero if both square roots would be zero. However, the inductive case involves a difference instead of a sum, thus $L_s$ might be lower than zero, if the absolute value of the first square root is lower than the absolute value of the second square root (then we would have to replace the inductance with a capacitance in order to match). In particular, a valid $R_{mL_\psi 0}$ might occur only if $R_\ell > R_1 - R_{s0}$ and if additionally $R_{mL_\psi 0} > R_{mL_\psi}$. Since this result is not needed further, all derivations are performed in Appendix C.1.1.1.

The major difference between resistive/capacitive and inductive cases is the different behaviour with respect to the highest $g_{\pi, \text{inma}}$. Although all estimations performed in Section 3.5.2.2 and Appendix B.2.1.1 apply, we have to use

$$\frac{\partial (\omega L_s)}{\partial R_m} = \frac{R_1 - 2 (R_m + R_s)}{2 \sqrt{(R_m + R_s) (R_1 - R_m - R_s)}} - \frac{R_\ell - 2 R_m}{2 \sqrt{R_m (R_\ell - R_m)}},$$

$$\frac{\partial (\omega L_s)}{\partial R_s} = \frac{R_1 - 2 (R_m + R_s)}{2 \sqrt{(R_m + R_s) (R_1 - R_m - R_s)}}$$

during replacement of $\frac{\partial (\omega L_s)}{\partial R_m}$ and $\frac{\partial (\omega L_s)}{\partial R_s}$ in the final checks for numerator and denominator in the case “at least one $a_n > 0$”, yielding a difference of the same two terms instead of a sum compared to the resistive/capacitive case. Since we proved that both terms are $\geq 0$ if $0 < b_N \leq 2$, their difference might be positive or negative. Thus the method applied gives no indication for the sign of $\frac{dg_{\pi, \text{inma}}}{dR_m}$.

We could try to reconsider the method used, but would not succeed, because calculation of $g_{\pi, \text{inma}}$ for e.g. a constant $Q$ inductor ($R_s = \frac{1}{Q} \omega L_s$) yields a maximum of $g_{\pi, \text{inma}}$ within the valid range of $R_m$ if $R_\ell < R_1$ ($R_{s0} = 0$ for $R_s = \frac{1}{Q} \omega L_s$). In particular, it is a “real” maximum, where $\frac{dg_{\pi, \text{inma}}}{dR_m} = 0$, and which is not located at one of the range’s limits — however, if $Q$ is sufficiently high, it is very close to $R_m = R_\ell$ and its $g_{\pi, \text{inma}}$ does not differ much from $g_{\pi, \text{inma}}(R_\ell)$. Unfortunately such a maximum’s $g_{\pi, \text{inma}}$ and the value of its

\(^6\)But $R_m < R_\ell$, because $R_m = R_\ell$ would yield $B_{2\text{rot}} = 0$, which is part of the case addressing (over-)compensation where $C_{2\text{rot}} = 0$. Conversely, $R_m = R_{mL_\psi}$ yields $C_2 = 0$ (undercompensation). However, each of those limits may not be within the valid range of $R_m$ if it is limited further by additional conditions, e.g. by minimum/maximum network element values.
CHAPTER 4. EXTENSION OF THE METHOD TO COMPLEX LOADS

location \( R_m \) cannot be computed by using inequalities and estimations. Even worse, the constant \( Q \) inductor involves two solutions for \( R_0(R_m) \) if \( R_t > R_i \), whose according \( \frac{dQ}{dR_m} \) exhibit opposite signs and thus no single inequality or estimation would apply.

It's interesting to note that if such a (single) maximum would occur, it would always be higher than the highest \( g_{\pi,max} \) in the case of (over-)compensation, because both cases join at \( R_m = R_t \), provided that \( R_m \) is not limited further when solving for \( R_0(R_m) \).

Summarising the preceding results we may conclude that obviously the maximum \( g_{\pi,max} \) is yielded if the type of the external network is a \( \pi \) network.

4.3.2.2 T network

![T network diagram]

Figure 4.4: Resulting T network’s equivalent circuit for matching of a (complex) load impedance consisting of a resistance \( R_t \) in series to an inductance \( L_{ts} \) to a (real) source resistance \( R_i \)

If the load impedance \( Z_t \) exhibits an inductive imaginary part, thus \( \text{Im}\{Z_t\} > 0 \), we may describe the load impedance at matching frequency by a resistance \( R_t \) in series to an inductance \( L_{ts} \), where

\[
R_t = \text{Re}\{Z_t\}, \quad (4.15) \quad L_{ts} = \frac{1}{\omega} \text{Im}\{Z_t\}. \quad (4.16)
\]

To achieve matching of the T network’s input impedance to the source resistance, two completely different solutions arise.

1. (Over-)Compensation of \( L_{ts} \) by \( C_2 \), hence \( X_{2tot} \leq 0 \) in Figure 4.4 and thus may be described by a capacitance \( C_{2tot} \) at matching frequency.

Similar to Section 4.3.1.2 \( C_{2tot} \) is given by (3.31). For the remaining capacitance \( C_2 \) of the T network we obtain

\[
C_2 = \frac{1}{\omega^2 L_{ts} + \frac{1}{C_{2tot}}} = \frac{1}{\omega \left( \frac{L_{ts}}{\omega} + \frac{1}{C_t} \sqrt{\frac{G_t}{C_m}} \right)}.
\]

(4.17)

\( C_2 \) is a variable capacitance and thus \( \geq 0 \), but additionally \( C_2 \leq \frac{1}{\omega^2 L_{ts}} \) holds. If the reactance of \( C_2 \) cancels the reactance of \( L_{ts} \), \( X_{2tot} \) may be zero and
$C_{2\text{tot}}$ may tend to infinity — just equivalent to the case of a purely resistive load with exception of the value of $\frac{1}{C_2}$, which is shifted by $\omega^2 L_{\text{ts}}$. All results obtained in Sections 3.5.3.1 and 3.5.3.2 are applicable, but we have to put $G_1 = \frac{1}{R_1}, G_\ell = \frac{1}{R_\ell}$, and replace $C_2$ with $C_{2\text{tot}}$ in all formulae derived\(^7\). However, compared to Section 4.3.1.2, there is no additional limit to $G_m$, but a maximum value of $C_2$ given by $\frac{1}{\omega^2 L_{\text{ts}}}$.

Similar to Section 3.5.3.2, $g_{t,\text{in}}$ increases (or might stay constant) with increasing $G_m$. Thus the highest valid $G_m$ yields the maximum $g_{t,\text{in}}$, hence $G_m = G_\ell$ if $G_\ell \leq G$ or $G_m = G_{\text{min}}$ if $G_\ell > G_1$, respectively. For $G_m = G_\ell$ the resulting type of the external matching network is a T network, whose capacitance $C_2$ cancels $L_{\text{ts}}$ at matching frequency, hence T network and $L_{\text{ts}}$ build up a L network. For $G_m = G_{\text{min}}$, $C_1 \to \infty$ holds, and the resulting type of the external matching network is a L network\(^4\).

We already might assume that the case of (over-)compensation does not guarantee the lowest losses, because the equivalent circuit’s lossless load inductance is “wasted” by (over-)compensation and all inductance needed for matching has to be supplied by the lossy inductor of the matching network.

2. Undercompensation of $L_{\text{ts}}$ by $C_2$, hence $X_{2\text{tot}} > 0$ in Figure 4.4 and thus may be described by an inductance $L_{2\text{tot}}$ at matching frequency.

According to (3.31) and the notes regarding the signs in parentheses/square brackets we obtain

$$L_{2\text{tot}} = L_{\text{ts}} - \frac{1}{\omega^2 C_2} = \frac{1}{\omega G_\ell} \sqrt{\frac{G_\ell - G_m}{G_m}}.$$  

The remaining capacitance $C_2$ of the T network is then given by

$$C_2 = \frac{1}{\omega \left(\omega L_{\text{ts}} - \frac{1}{G_\ell} \sqrt{\frac{G_\ell - G_m}{G_m}}\right)}.$$  

As before, we have to put $G_1 = \frac{1}{R_1}$ and $G_\ell = \frac{1}{R_\ell}$ in all equations derived\(^7\).

Additionally, according to (3.32) and the notes regarding the signs in parentheses/square brackets we obtain for the inductance $L_p$ of the T network

$$L_p = \frac{1}{\omega \left(\sqrt{(G_m + G_p)(G_1 - G_m - G_p)} - \sqrt{G_m(G_\ell - G_m)}\right)}.$$  

The preceding changes of the equivalent network’s elements also involve changes of the limits of parameter $G_m$.

One additional limit occurs because $C_2$ is still a capacitance and $\geq 0$. Thus $L_{2\text{tot}} \leq L_{\text{ts}}$ holds. Since $L_{2\text{tot}}$ decreases with increasing $G_m$ (square root similar

\(^7\) We should note that there is a slight inconsistency in the definition of $G_\ell$ because $G_\ell \neq \text{Re}\left\{\frac{1}{R_\ell + j\omega L_m}\right\}$ as we would expect it to be according to the definitions in (2.13) and (2.7).
to those of $C_{2\text{tot}}$ in Section 4.3.1.1), $G_m$ has to be higher than\(^8\) or equal to

$$G_{mL} = \frac{G_\ell}{1 + (\omega L g G_\ell)^2} \left( \frac{\text{Re}\{Z_\ell\}}{|Z_\ell|^2} = \text{Re}\left\{ \frac{1}{Z_\ell} \right\} \right).$$

(4.20)

For AC excitation $\omega > 0$ and $G_{mL} < G_\ell$ holds.

Another additional limit, $G_{mL/\omega L}$, may occur because $L_\omega$ is still an inductance and $\geq 0$, hence $\frac{1}{L_\omega} \geq 0$. In the resistive and capacitive cases, the sum of the square roots in $\frac{1}{L_\omega}$ could never be lower than zero, but equal to zero if both square roots would be zero. However, the inductive case involves a difference instead of a sum, thus $\frac{1}{L_\omega}$ or $L_\omega$ might be lower than zero, if the absolute value of the first square root is lower than the absolute value of the second square root (then we would have to replace the inductance with a capacitance in order to match). In particular, if $R_{s0} \leq R_1$, as it would be for the usual $R_1 = 50 \Omega$, due to (3.33) no valid $G_{mL/\omega L}$ occurs. Since this result is not needed further, all derivations (including that for $R_{s0} > R_1$) are performed in Appendix C.1.1.2.

The major difference between resistive/capacitive and inductive cases is the different behaviour with respect to the highest $g_{t,\text{inma}}$. Although all estimations performed in Section 3.5.3.2 and Appendix B.2.2.1 apply, we have to use

$$\frac{\partial \left( \frac{1}{L_\omega} \right)}{\partial G_m} = \frac{G_1 - 2 (G_m + G_p)}{2 (G_m + G_p) (G_1 - G_m - G_p)} - \frac{G_\ell - 2 G_m}{2 G_m (G_\ell - G_m)},$$

$$\frac{\partial \left( \frac{1}{L_\omega} \right)}{\partial G_p} = \frac{G_1 - 2 (G_m + G_p)}{2 (G_m + G_p) (G_1 - G_m - G_p)}$$

during replacement of $\frac{\partial \left( \frac{1}{L_\omega} \right)}{\partial G_m}$ and $\frac{\partial \left( \frac{1}{L_\omega} \right)}{\partial G_p}$ in the final checks for numerator and denominator in the case "at least one $a_i > 0"$, yielding a difference of the same two terms instead of a sum compared to the resistive/capacitive case. Since we proved that both terms are $\geq 0$ if $0 < b_N \leq 2$ and $Q \geq 1$, their difference might be positive or negative. Thus the method applied gives no indication for the sign of $\frac{dg_{t,\text{inma}}}{dG_m}$.

We could try to reconsider the method used, but would not succeed, because calculation of $g_{t,\text{inma}}$ for e.g. a constant $Q$ inductor ($R_s = \frac{1}{Q} \omega L_\omega$, hence $G_p = \frac{1}{Q \omega L_\omega}$) yields a maximum of $g_{t,\text{inma}}$ within the valid range of $G_m$ if $G_\ell < G_1$. In particular, it is a "real" maximum, where $\frac{dg_{t,\text{inma}}}{dG_m} = 0$, and which is not located at one of the range's limits — however, if $Q$ is sufficiently high, it is very close to $G_m = G_\ell$ and its $g_{t,\text{inma}}$ does not differ much from $g_{t,\text{inma}}(G_\ell)$. Unfortunately such a maximum's $g_{t,\text{inma}}$ and the value of its location $G_m$ cannot be computed.

\(^8\)But $G_m < G_\ell$, because $G_m = G_\ell$ would yield $X_{2\text{tot}} = 0$, which is part of the case addressing (over-)compensation where $C_{2\text{tot}} \to \infty$. Conversely, $G_m = G_{mL\omega}$ yields $C_2 \to \infty$ (undercompensation). However, each of those limits may not be within the valid range of $G_m$ if it is limited further by additional conditions, e.g. by minimum/maximum network element values.
by using inequalities and estimations. Even worse, the constant $Q$ inductor involves two solutions for $G_p(G_m)$ if $G_\ell > G_i$, whose according $\frac{d\phi_{\ell, inna}}{dG_m}$ exhibit opposite signs and thus no single inequality or estimation would apply.

It’s interesting to note that if such a (single) maximum would occur, it would always be higher than the highest $g_{\ell, inna}$ in the case of (over-)compensation, because both cases join at $G_m = G_\ell$, provided that $G_m$ is not limited further when solving for $G_p(G_m)$.

Summarising the preceding results we may conclude that obviously the maximum $g_{\ell, inna}$ is yielded if the type of the external network is a T network.

### 4.4 Exact analytical solutions for a constant $Q$ central inductor

#### 4.4.1 Explicit analytical solutions, their valid parameter ranges, and important characteristics

##### 4.4.1.1 $\pi$ network

To derive an optimum matching strategy, it is important to know more about the solutions’ behaviour. Since any inductor can be approximated by an inductor with piecewise constant losses as a function of its number of turns, it is advantageous to derive the exact solutions for the case of a constant $Q$ central inductor which is described by (3.1) if $R_{|0} = 0$, $N = 1$, $a_1 = \frac{1}{Q}$, and $b_1 = 1$. Furthermore, this approach enables comparison with several works which derived approximate solutions for this case (assuming the network’s elements to have the values of a lossless network).

Although the exact solution is derived quickly, it’s much more complicated to find the range of the parameter $R_m$ where it’s valid within.

The exact solution is computed by solving $R_s = \frac{1}{Q} \omega L_n$ for $L_n$, putting it in Equation (3.19), and solving the resulting equation for $R_s$, yielding two possible solutions for capacitive/resistive/(over-)compensated inductive loads,

$$R_{\text{scap}} = \frac{R_i - 2 \left( R_m - Q \sqrt{R_m (R_\ell - R_m)} \right) \mp \sqrt{u_\pi}}{2 (1 + Q^2)}, \quad (4.21)$$

where

$$u_\pi = 4 (1 + Q^2) \left( R_i - R_\ell \right) R_m \left( R_i - 2 \left( R_m - Q \sqrt{R_m (R_\ell - R_m)} \right) \right)^2,$$

and two possible solutions for undercompensated inductive loads,

$$R_{\text{ind}} = \frac{R_i - 2 \left( R_m + Q \sqrt{R_m (R_\ell - R_m)} \right) \mp \sqrt{u_\pi}}{2 (1 + Q^2)}, \quad (4.22)$$
where
\[ v_\pi = 4(1 + Q^2)(R_i - R_\ell) R_m + \left( R_i - 2 \left( R_m + Q \sqrt{R_m (R_\ell - R_m)} \right) \right)^2. \]

Which solution is valid in which range of the parameter \( R_m \)?

Searching of those ranges involves a certain search for zeroes of different terms and comparison of the limits found, but only within the utmost limits derived in Section 3.5.2.1 (applying \( \tau_{\theta_0} = 0 \) which holds for \( R_\theta = \frac{1}{Q} \omega L_m \), hence \( 0 \leq R_m \leq \min\{R_i, R_\ell\} \). It is simple math, but very lengthy and thus may only be sketched here.

- The first step is to check which parameter range yields real solutions, in particular, in which parameter range is \( u_\pi \geq 0 \) or \( v_\pi \geq 0 \), respectively.

By a careful inspection of \( u_\pi, v_\pi \) it’s quite obvious that \( u_\pi \) and \( v_\pi \) are always \( \geq 0 \) if \( R_\ell \leq R_i \).

If \( R_\ell > R_i \), it is somewhat more complicated.

If \( R_i < R_\ell \leq \frac{1}{4} ((\sqrt{1 + Q^2} - Q)^2 + 4) R_i \), \( u_\pi > 0 \) within the intervals \( 0 \leq R_m \leq R_{\mu_01} \) and \( R_{\mu_02} \leq R_m \leq R_i \), where
\[
R_{\mu_01} = \frac{R_\ell - Q (\sqrt{1 + Q^2} + Q) R_i - \sqrt{(R_\ell - R_i) (R_\ell + (\sqrt{1 + Q^2} + Q)^2 R_i)}}{2(1 + Q^2)},
\]
\[
R_{\mu_02} = \frac{R_\ell - Q (\sqrt{1 + Q^2} + Q) R_i + \sqrt{(R_\ell - R_i) (R_\ell + (\sqrt{1 + Q^2} + Q)^2 R_i)}}{2(1 + Q^2)}.
\]

If \( R_\ell > \frac{1}{4} ((\sqrt{1 + Q^2} - Q)^2 + 4) R_i \), only the first interval applies.

If \( R_i < R_\ell \leq \frac{1}{4} ((\sqrt{1 + Q^2} + Q)^2 + 4) R_i \), \( v_\pi > 0 \) within the intervals \( 0 \leq R_m \leq R_{\mu_01} \) and \( R_{\mu_02} \leq R_m \leq R_i \), where
\[
R_{\mu_01} = \frac{R_\ell - Q (\sqrt{1 + Q^2} - Q) R_i - \sqrt{(R_\ell - R_i) (R_\ell + (\sqrt{1 + Q^2} - Q)^2 R_i)}}{2(1 + Q^2)}, \tag{4.23}
\]
\[
R_{\mu_02} = \frac{R_\ell - Q (\sqrt{1 + Q^2} - Q) R_i + \sqrt{(R_\ell - R_i) (R_\ell + (\sqrt{1 + Q^2} - Q)^2 R_i)}}{2(1 + Q^2)}.
\]

If \( R_\ell > \frac{1}{4} ((\sqrt{1 + Q^2} + Q)^2 + 4) R_i \), only the first interval applies.

- The second step is to check which parameter range yields positive solutions.
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If \( R_\ell < R_i \), \( R_{\text{scap}^-} \) is never positive. If \( R_\ell = R_i \), \( R_{\text{scap}^-} \) is never positive, too, but \( R_{\text{scap}^-} \) is zero if \( 0 \leq R_m \leq \frac{\sqrt{1+Q^2+Q}}{2\sqrt{1+Q^2}} R_i \). If \( R_\ell > R_i \), \( R_{\text{scap}^-} \) is positive within \( 0 \leq R_m \leq R_{\text{muv01}} \) only.

If \( R_\ell \leq R_i \), \( R_{\text{scap}^+} \) is positive for \( 0 \leq R_m \leq R_\ell \). However, if \( R_\ell = R_i \) it should be noted that \( R_{\text{scap}^+} \) is zero if \( \frac{\sqrt{1+Q^2+Q}}{2\sqrt{1+Q^2}} R_i \leq R_m \leq R_i \). If \( R_\ell > R_i \), \( R_{\text{scap}^+} \) is positive within \( 0 \leq R_m \leq R_{\text{muv01}} \) only.

If \( R_\ell < R_i \), \( R_{\text{ind}^-} \) is never positive. If \( R_\ell = R_i \), \( R_{\text{ind}^-} \) is never positive, too, but \( R_{\text{ind}^-} \) is zero for \( 0 \leq R_m \leq R_{\text{muv01}} \). If \( R_\ell > R_i \), \( R_{\text{ind}^-} \) is positive within \( 0 \leq R_m \leq R_{\text{muv01}} \) only.

If \( R_\ell < R_i \), \( R_{\text{ind}^+} \) is positive for \( 0 \leq R_m \leq R_\ell \). If \( R_\ell = R_i \), \( R_{\text{ind}^+} \) is positive for \( 0 \leq R_m \leq R_{\text{muv01}} \) and zero for \( R_{\text{muv01}} \leq R_m \leq R_i \). If \( R_\ell > R_i \), \( R_{\text{ind}^+} \) is positive within \( 0 \leq R_m \leq R_{\text{muv01}} \) only.

- The third step is to check in which parameter range the solutions are valid with respect to the equation which was solved for. This is performed by putting each solution in Equation (3.19) and checking in which parameter range the resulting \( \frac{1}{Q} \omega L_s \) is equal to the solution itself.

In those cases where the solutions are zero throughout an interval of the parameter \( R_m \), which always involve \( R_\ell = R_i \), it is quite obvious from Equation (3.19) that a zero inductance, hence a valid capacitive solution can only be obtained for \( R_m = R_i \), whereas a valid inductive solution exists throughout that interval.

In the remaining cases it is possible to rewrite \( (R_m + R_s)(R_i - R_m - R_s) \) as \((...)^2\). If \((...) \leq 0\), \( \sqrt{(...)^2} = -(...) \), if \((...) \geq 0\), \( \sqrt{(...)^2} = (...) \). However, for each of the four possible solutions, only one sign yields a valid solution. Hence the parameter’s interval(s) in conjunction with the intervals found in the previous steps limit the applicable range of each solution. The results of the check combined with the limits derived previously are given below:

\( R_{\text{scap}^-} \) is never a valid solution.

If \( 0 < R_\ell \leq R_{\text{dilofRmlim}} \), where

\[
R_{\text{dilofRmlim}} = 2Q(\sqrt{1+Q^2} - Q)R_i,
\]

\( R_{\text{scap}^+} \) is a valid solution for \( 0 \leq R_m \leq R_\ell \). If \( R_{\text{dilofRmlim}} < R_\ell \leq R_1 \), \( R_{\text{scap}^+} \) is a valid solution for \( 0 \leq R_m \leq R_{\text{mlim1}} \) and \( R_{\text{mlim2}} \leq R_m \leq R_\ell \), where

\[
R_{\text{mlim1}} = \frac{R_\ell + 2Q^2R_i - \sqrt{R_\ell^2 + 4Q^2R_i(R_\ell - R_i)}}{2(1+Q^2)},
\]

\[
R_{\text{mlim2}} = \frac{R_\ell + 2Q^2R_i + \sqrt{R_\ell^2 + 4Q^2R_i(R_\ell - R_i)}}{2(1+Q^2)}.
\]
If \( R_\ell = R_i \), \( R_{\text{lim}1} < \frac{\sqrt{1+Q^2}+Q}{2\sqrt{1+Q^2}} R_i \), hence \( R_{\text{scap}+} \) is positive throughout the first interval, and the second interval contracts to a single dot at \( R_m = R_{\text{lim}2} = R_i \). If \( R_\ell > R_i \), \( R_{\text{lim}1} < R_{\text{mv01}} \) and \( R_{\text{scap}+} \) is a valid solution for \( 0 \leq R_m \leq R_{\text{lim}1} \) only.

If \( R_\ell \geq R_i \), \( R_{\text{ind} -} \) is a valid solution for \( 0 \leq R_m \leq R_{\text{mv01}} \).

If \( 0 < R_\ell \leq R_i \), \( R_{\text{ind} +} \) is a valid solution for \( 0 \leq R_m \leq R_\ell \). If \( R_\ell > R_i \), \( R_{\text{ind} +} \) is a valid solution for \( 0 \leq R_m \leq R_{\text{mv01}} \).

It should be noted that the intervals with \( R_{\text{ind} \pm} = 0 \) in the case of \( R_\ell = R_i \) simply describe a parallel resonance of \( C_1 \) and \( L_{2\text{tot}} \).

Within their valid parameter ranges, the network element values could be calculated using (3.17), (3.19), and (3.18). However, it is much easier to use the defining equation of a constant \( Q \) inductor,

\[
L_s = \frac{Q}{\omega} R_s, \tag{4.27}
\]

to compute \( L_s \), and to solve (3.19) for \( \sqrt{(R_m + R_s)(R_i - R_m - R_s)} \), to put the solution in (3.17), and to apply (4.27), yielding

\[
C_1 = \frac{1}{\omega R_i} \frac{QR_s \pm \sqrt{R_m(R_\ell - R_m)}}{R_m + R_s} \tag{4.28}
\]

to compute \( C_1 \). (The minus sign in the numerator applies to \( R_{\text{scap}+} \), the plus sign to \( R_{\text{ind} -} \) and \( R_{\text{ind} +} \).) Then, as usual, \( C_2 \) is computed from (3.18).

The power gain of the \( \pi \) network matched to \( R_i \) at its input is easily calculated by (3.24).

Finally, the answers to two important questions regarding the \( \pi \) network are given.

**What is the physical meaning of the (applicable) limits derived above?**

If \( R_\ell \leq R_{\text{lim}0} \), \( C_1 \) of \( R_{\text{scap}+} \) has a minimum, whose value decreases with increasing \( R_\ell \) until it is zero at \( R_\ell = R_{\text{lim}0} \). For higher \( R_\ell \), it decreases further to negative values. Thus one limit to the left (\( R_{\text{lim}1} \)) and one limit to the right (\( R_{\text{lim}2} \)) exists where \( C_1 = 0 \) until the right limit reaches its maximum valid value of \( R_{\text{lim}2} = R_\ell = R_i \) at \( R_\ell = R_i \). If \( R_\ell > R_i \), only the left limit (\( R_{\text{lim}1} \)) falls within the (utmost) valid range of \( R_m \). (Decreasing of \( R_{\text{lim}1} \) and increasing of \( R_{\text{lim}2} \) with increasing \( R_\ell \) will be derived in Section 4.4.2.)

If \( R_\ell \geq R_i \), two inductive solutions exist. At \( R_{\text{mv01}} \), the upper limit of the valid interval of \( R_m \), both inductive solutions join.

Furthermore, if \( R_\ell \leq R_i \), \( R_{\text{scap}+} \) and \( R_{\text{ind} +} \) join at \( R_m = R_\ell \), where \( C_{2\text{tot}} = 0 \) or \( L_{2\text{tot}} \rightarrow \infty \). They would also join at \( R_m = 0 \), but due to minimum/maximum
element values and finite load susceptance it is outside the parameter’s valid range in practice.

Minimum and solution joining may be observed in the example diagrams depicted in Section 4.4.3.1.

**How does the power gain of the matched π network depend on \( R_m \) for the different (valid) solutions?**

Since the case of a constant \( Q \) factor inductor is included in the loss model (3.1) and the conditions of the proof in Section 3.5.2.2 apply, \( \frac{d g_{\pi,\text{inna}}}{d R_m} \geq 0 \) for \( R_{\text{scap}+} \) throughout its valid range of \( R_m \).

For the two inductive solutions, \( \frac{d g_{\pi,\text{inna}}}{d R_m} \) has to be computed, numerator and denominator checked for zeroes, and the derivative’s sign checked to the right and left of those zeroes. This procedure is quite lengthy and thus only its results are given.

If \( R_\ell < R_i \), there exists a maximum of \( \frac{d g_{\pi,\text{inna}}}{d R_m} \) for \( R_{\text{ind}+} \) at

\[
R_{\text{mmax}+} = \frac{R_\ell \left( R_\ell + (1 + 2Q^2) R_i + \sqrt{(R_\ell + R_i)^2 + 4Q^2 R_\ell R_i}\right)}{2 \left( 1 + Q^2 \right) (R_\ell + R_i)},
\]

(4.29)

where the according power gain’s maximum value is given by

\[
g_{\pi,\text{inna, max}+} = \frac{R_\ell \left( R_\ell + (1 + 2Q^2) R_i + \sqrt{(R_\ell + R_i)^2 + 4Q^2 R_\ell R_i}\right)}{R_i \left( R_i + (1 + 2Q^2) R_\ell + \sqrt{(R_\ell + R_i)^2 + 4Q^2 R_\ell R_i}\right)}. \tag{4.30}
\]

![Diagram a) and b) showing power gain](image)

**Figure 4.5:** Power gain \( g_{\pi,\text{inna}} \) of a lossy π network designed as if it would be lossless (thin lines) and \( g_{\pi,\text{inna}} \) of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing \( R_{\text{ind}+} \) for the exactly matched network — solutions at 7.35 MHz, \( R_i = 50 \Omega \), \( Q = 50 \), and a) \( R_\ell = 5 \Omega \), b) \( R_\ell = 30 \Omega \).

The closer \( R_\ell \) to \( R_i \) or the higher \( Q \), the flatter the maximum will be (approximately). Those dependencies are quite obvious when comparing Figure 4.5 and Figure 4.15. Furthermore, the maximum is compared to the highest power gain of the capacitive solution in the last paragraph of Section 4.4.2.

If \( R_\ell = R_i \), \( g_{\pi,\text{inna}} = 1 \) const. for \( R_{\text{ind}+} \) if \( 0 \leq R_m \leq R_{\text{mv}01} \) and for \( R_{\text{ind}+} \) if \( R_{\text{mv}01} \leq R_m \leq R_i \) (pure parallel resonance of \( C_1 \) and \( L_{\text{2tot}} \)). If \( 0 \leq R_m \leq R_{\text{mv}01} \), \( \frac{d g_{\pi,\text{inna}}}{d R_m} > 0 \) for \( R_{\text{ind}+} \).
If \( R_\ell > R_i \), \( \frac{d g_{\pi \text{ inma}}}{d R_m} < 0 \) for \( R_{\text{ind} -} \) and \( \frac{d g_{\pi \text{ inma}}}{d R_m} > 0 \) for \( R_{\text{ind} +} \) throughout its valid range of \( R_m \).

The behaviour of \( g_{\pi \text{ inma}} \) in the different cases may be observed in the example diagrams depicted in Section 4.4.3.1.

### 4.4.1.2 T network

Since the case of a constant \( Q \) central inductor is one of the rare cases mentioned in Footnote 7 (at the end of Section 3.5.3.1), where \( \pi \) and T network are exactly equivalent, the solutions for \( G_p \) of the T network may easily be derived from those for \( R_s \) of the \( \pi \) network by replacing \( R_m, R_i, R_\ell, u_\pi \), and \( v_\pi \) with \( G_m, G_i, G_\ell, u_\ell \), and \( v_\ell \), respectively.

In a similar manner, \( G_{\mu u01}, G_{\mu u02}, G_{\mu v01}, G_{\mu v02}, G_{\ell \text{ im0Gmlim}}, G_{\text{mlim1}}, \) and \( G_{\text{mlim2}} \) are derived. The solution’s valid ranges are similar to those of the \( \pi \) network if the previously derived limits are used instead.

However, the intervals with \( G_{\text{pind} \pm} = 0 \) in the case of \( G_\ell = G_i \) now describe a series resonance of \( C_1 \) and \( L_{2\text{tot}} \).

Furthermore, there are two more network element values which may be interesting. As with the \( \pi \) network, (3.30), (3.32), and (3.31) could be used to calculate the network element values, but the defining equation of a constant \( Q \) inductor,

\[
L_p = \frac{1}{\omega Q} \frac{1}{G_p}, \tag{4.31}
\]

is more convenient to compute \( L_p \). However, the inductor model (3.1) describes a loss resistance \( R_s \) as a function of an inductor \( L_s \) in series. Thus their values are interesting, too. Applying (4.31) to (A.19) and (A.20) yields

\[
R_s = \frac{1}{1 + Q^2} \frac{1}{G_p}, \tag{4.32}
\]

\[
L_s = \frac{Q}{\omega (1 + Q^2)} \frac{1}{G_p}. \tag{4.33}
\]

Solving (3.32) for \( \sqrt{(G_m + G_p)(G_i - G_m - G_p)} \), putting the solution in (3.30), and applying (4.31) gives

\[
C_1 = \frac{G_i}{\omega} \frac{G_m + G_p}{Q G_p + \sqrt{G_m(G_\ell - G_m)}} \tag{4.34}
\]

to compute \( C_1 \). (The minus sign in the denominator applies to \( G_{\text{cap} +} \), the plus sign to \( G_{\text{pind} -} \) and \( G_{\text{pind} +} \).) Then, as usual, \( C_2 \) is computed from (3.31).

\( C_1, L_p, \) and \( C_2 \) may also be derived from the \( \pi \) network’s element values \( C_1, L_s, \) and \( C_2 \) if they are inverted (except \( \omega \)) and \( R_m, R_i, \) and \( R_\ell \) are replaced with \( G_m, G_i, \) and \( G_\ell \), respectively.

The power gain of the T network matched to \( G_i \) at its input is easily calculated by (3.37).
Finally, the answers to two important questions regarding the T network are given.

**What is the physical meaning of the (applicable) limits derived above?**

If \( G_\ell \leq G_{\text{dimoffGlimin}} \), \( C_1 \) of \( G_{\text{pcap+}} \) has a maximum, whose value increases with increasing \( G_\ell \) until it is a pole at \( G_\ell = G_{\text{dimoffGlimin}} \). For higher \( G_\ell \), two poles occur with sign changes (the inverse of the maximum decreases further to negative values). Thus one limit to the left (\( G_{\text{limin1}} \)) and one limit to the right (\( G_{\text{limin2}} \)) exists where \( C_1 \rightarrow \infty \) until the right limit reaches its maximum valid value of \( G_{\text{limin2}} = G_\ell = G_1 \) at \( G_\ell = G_1 \). If \( G_\ell > G_1 \), only the left limit (\( G_{\text{limin1}} \)) falls within the (utmost) valid range of \( G_m \). (Decreasing of \( G_{\text{limin1}} \) and increasing of \( G_{\text{limin2}} \) with increasing \( G_\ell \) will be derived in Section 4.4.2.)

If \( G_\ell \geq G_1 \), two inductive solutions exist. At \( G_{\text{mv01}} \), the upper limit of the valid interval of \( G_m \), both inductive solutions join.

Furthermore, if \( G_\ell \leq G_1 \), \( G_{\text{pcap+}} \) and \( G_{\text{pind+}} \) join at \( G_m = G_\ell \), where \( C_{2\text{tot}} \rightarrow \infty \) or \( L_{2\text{tot}} = 0 \). They would also join at \( G_m = 0 \), but due to minimum/maximum element values and finite load reactance it is outside the parameter’s valid range in practice.

**How does the power gain of the matched T network depend on \( G_m \) for the different (valid) solutions?**

Due to the exact equivalence of \( \pi \) and T networks in the case of a constant \( Q \) factor inductor, the results derived for the \( \pi \) network apply if \( R_{\text{pcap+}}, R_{\text{pind-}}, R_{\text{pind+}}, g_{\text{r,imma}}, R_{\text{mgmaxind+}}, g_{\text{r,imma,maxind+}}, R_m, R_1 \), and \( R_\ell \) are replaced with \( G_{\text{pcap+}}, G_{\text{pind-}}, G_{\text{pind+}}, g_{\text{r,imma}}, G_{\text{mgmaxind+}}, g_{\text{r,imma,maxind+}}, G_m, G_1 \), and \( G_\ell \), respectively.

However, there is a slight difference between \( \pi \) and T network. In the case of \( G_\ell = G_1 \), \( g_{\text{r,imma}} = 1 = \text{const.} \) for \( G_{\text{pind-}} \) if \( 0 \leq G_m \leq G_{\text{mv01}} \) and for \( G_{\text{pind+}} \) if \( G_{\text{mv01}} \leq G_m \leq G_1 \) as with the \( \pi \) network, but a pure series resonance of \( C_1 \) and \( L_{2\text{tot}} \) occurs instead of the parallel resonance with the \( \pi \) network.

### 4.4.2 Dependency of the valid intervals’ limits on \( Q \) and \( R_\ell \) or \( G_\ell \)

To understand the practical importance of the valid intervals’ limits derived above their dependency on \( Q \) and \( R_\ell \) or \( G_\ell \) has to be investigated. This simply involves calculation of their derivatives with respect to \( Q \) and \( R_\ell \) or \( G_\ell \) and checking the sign of those derivatives. Since \( \pi \) and T network are exactly equivalent here, only the \( \pi \) network derivatives are calculated in the following paragraphs, but the results are given for both types of networks.

\[
\frac{\partial R_{\text{limoffGlimin}}}{\partial Q} = \frac{2(\sqrt{1+Q^2} - Q)^2}{\sqrt{1+Q^2}} R_\ell
\]
$R_{\text{limofRlim}}$ or $G_{\text{limofGlim}}$ increases with increasing $Q$ and tends to $R_i$ or $G_i$ for $Q \to \infty$. (They do not depend on $R_\ell$ or $G_\ell$.)

$$\frac{\partial R_{\text{lim1,2}}}{\partial Q} = \pm \frac{Q \left( 2R_i - R_\ell \pm \sqrt{R_\ell^2 + 4Q^2 R_i (R_\ell - R_i)} \right)^2}{2 (1 + Q^2)^2 \sqrt{R_\ell^2 + 4Q^2 R_i (R_\ell - R_i)}}$$

$R_{\text{lim1}}$ or $G_{\text{lim1}}$ increases with increasing $Q$, whereas $R_{\text{lim2}}$ decreases with increasing $Q$. Both tend to $R_i$ or $G_i$ for $Q \to \infty$. (Note that the range of $R_\ell$ or $G_\ell$ where those limits exist decreases with increasing $Q$ since $R_{\text{limofRlim}}$ or $G_{\text{limofGlim}}$ increases. To compare the limits for increasing $Q$, $R_\ell$ or $G_\ell$ has to be chosen high enough to be valid for those $Q$.)

$$\frac{\partial R_{\text{lim1,2}}}{\partial R_\ell} = \mp \frac{R_{\text{lim1,2}}}{\sqrt{R_\ell^2 + 4Q^2 R_i (R_\ell - R_i)}}$$

$R_{\text{lim1}}$ or $G_{\text{lim1}}$ decreases with increasing $R_\ell$ or $G_\ell$, whereas $R_{\text{lim2}}$ or $G_{\text{lim2}}$ increases with increasing $R_\ell$ or $G_\ell$. Hence $R_{\text{lim1,2}}$ or $G_{\text{lim1,2}}$ are valid only till $R_\ell = R_i$ or $G_\ell = G_i$, where the highest possible $R_{\text{lim2}} = R_i$ or $G_{\text{lim2}} = G_i$ is obtained.

Since $\frac{\partial R_{mv01}}{\partial Q}$ itself and the following the sign check is quite lengthy, only the results are given here. $R_{mv01}$ or $G_{mv01}$ decreases with increasing $Q$. It tends to zero for $Q \to \infty$.

$$\frac{\partial R_{mv01}}{\partial R_\ell} = - \frac{R_{mv01}}{\sqrt{(R_\ell - R_i) \left( R_\ell + (\sqrt{1 + Q^2} - Q)^2 R_i \right)}}$$

$R_{mv01}$ or $G_{mv01}$ decreases with increasing $R_\ell$ ($\geq R_i$) or $G_\ell$ ($\geq G_i$). Thus its highest value is given by

$$R_{mv01}(R_\ell = R_i) = \frac{\sqrt{1 + Q^2} + Q}{2 \sqrt{1 + Q^2}} R_i$$

or

$$G_{mv01}(G_\ell = G_i) = \frac{\sqrt{1 + Q^2} + Q}{2 \sqrt{1 + Q^2}} G_i.$$  \hspace{1cm} (4.35)

For those limits to be valid, $L_{q0} \leq \frac{R_i}{\omega(\sqrt{1 + Q^2} + Q)}$ must hold with the $\pi$ network and $L_{bs} \geq \frac{\sqrt{1 + Q^2} + Q}{\omega G_i}$ with the $T$ network, respectively.
To understand whether those limits are observed in practice additionally the according values of the network’s central inductor $L_s$ have to be considered.

In a π network, it is obvious from (4.27) and (4.22) that $L_{\text{indh}}$, the central inductance of the inductive solution employing $R_{\text{indh}}$, is always greater than that of the solution employing $R_{\text{indh}}$, except at $R_m = R_{\text{mv01}}$, where both solutions join. The highest $L_{\text{indh}}$ is $L_{\text{indh}}(R_m = 0) = \frac{Q R_i}{(1+Q^2)\omega}$ (to prove this simply compute $\frac{L_{\text{indh}}(R_m)}{L_{\text{indh}}(0)}$, although $R_m = 0$ is never reached if realistic loads and element values are considered. For useful $Q$ factors of $Q \geq 1$, $L_{\text{indh}}(0)$ decreases with increasing $Q$ and $\omega$.

Similarly, in a T network, the lowest $L_{\text{phd}}$ is $L_{\text{phd}}(G_m = 0) = \frac{Q \omega G_i}{Q \omega G_i}$, which increases with increasing $Q$ for useful $Q$ factors of $Q \geq 1$ and decreases with increasing $\omega$ (to fit the inductor’s series model, the parallel inductance needs to be transformed in a series inductance $L_{\text{phd}}(G_m = 0) = \frac{Q}{\omega G_i}$, but both differ only slightly for sufficiently high $Q$ factors).

Knowing the dependencies given above, which limits are likely to be observed in practice?

If $R_{\ell} \leq R_i$ or $G_{\ell} \leq G_i$, even for a quite low $Q = 5$, $R_{\text{effer}} \approx 0.99 R_i$ or $G_{\text{effer}} \approx 0.99 G_i$, yielding valid $R_{\text{m1,2}}$ for $0.99 R_i \leq R_{\ell} \leq R_i$ or valid $G_{\text{m1,2}}$ for $0.99 G_i \leq G_{\ell} \leq G_i$. Loads that close to the source resistance or admittance would not require matching, except for a compensation of their susceptance or reactance in case of a complex load. For higher $Q$, $R_{\text{effer}}$ or $G_{\text{effer}}$ are even closer to $R_i$ or $G_i$. Thus, for the majority of such loads, $R_{\text{m1,2}}$ or $G_{\text{m1,2}}$ do not occur. However, if $R_{\ell} \geq R_i$ or $G_{\ell} \geq G_i$, in principle $R_{\text{m1}}$ or $G_{\text{m1}}$ are valid, but since $C_i$ is zero (π network) or tends to infinity (T network) at that limit, minimum or maximum element values provide tighter limits to $R_m$ or $G_m$.

Regarding $R_{\text{mv01}}$ or $G_{\text{mv01}}$, it is very unlikely to observe those limits in practice, too. In a π network, $L_{\text{indh}} < 100 \text{nH}$ for $Q > 44.2$ and $0 \leq R_m \leq R_{\text{mv01}}$ at 1.8 MHz — inductances that low may not be realised by an ordinary variable inductor, but only by a variable capacitor in series to a fixed inductor (Figure 3.12 a)), which is disadvantageous with respect to its losses (although the losses may be reduced by using a switchable tapped inductor as proposed in [3]). In a T network, $L_{\text{phd}}$ of the equivalent region may be within the range of variable inductors at high frequencies, e.g. $L_{\text{phd}} > 26.5 \mu \text{H}$ for $Q = 100$ and $0 \leq G_m \leq G_{\text{mv01}}$ at 30 MHz, but moving the lower limit of the valid parameter range below $G_{\text{mv01}}$ would require such high series inductances of the load, e.g. $L_\ell > 53.1 \mu \text{H}$ for $Q = 100$ at 30 MHz, that even a parasitic parallel capacitance of 1 pF would overcompensate those $L_\ell$, yielding for sure an effective series capacitance of that load.
Besides the intervals’ limits, it’s also interesting to compare the inductive maximum obtained for \( R_\ell < R_i \) or \( G_\ell < G_i \) to highest power gain of the capacitive solution at \( R_m = R_\ell \) or \( G_m = G_\ell \), where capacitive and inductive solutions join and the inductive load is fully compensated.

Computing \( \frac{\partial}{\partial R_\ell} \left( \frac{g_{\ell, in, max, ind+}}{g_{\ell, in, ind+} (R_m = R_\ell)} \right) \) or \( \frac{\partial}{\partial R_\ell} \left( \frac{g_{\ell, in, max, ind+}}{g_{\ell, in, ind+} (G_m = G_\ell)} \right) \) yields a maximum ratio of

\[
\left. \frac{g_{\ell, in, max, ind+}}{g_{\ell, in, ind+} (R_m = R_\ell)} \right|_{\max} = \frac{(1+\sqrt{5}+4Q^2+\sqrt{2(3+\sqrt{5}+4(\sqrt{5}-1)Q^2)})}{2(1+Q^2)} \frac{\left(1+(\sqrt{5}-1)Q^2+\sqrt{1+4(\sqrt{5}-2)Q^2}\right)}{2(1+\sqrt{5}+2(\sqrt{5}-1)Q^2+\sqrt{2(3+\sqrt{5}+4(\sqrt{5}-1)Q^2)})}
\]

at \( R_\ell = \frac{\sqrt{5}-1}{2} R_i \approx 0.618 R_i \) or a similar maximum ratio of \( \left. \frac{g_{\ell, in, max, ind+}}{g_{\ell, in, ind+} (G_m = G_\ell)} \right|_{\max} \) at \( G_\ell = \frac{\sqrt{5}-1}{2} G_i \approx 0.618 G_i \approx \frac{1}{1.618} R_i \), respectively.\(^9\)

The maximum ratio itself has a maximum value of approximately 1.059 at \( Q \approx 2.058 \) and quickly approaches 1 for (lower or) higher \( Q \) factors. In particular, the maximum ratio is about 1.026 at \( Q = 10 \), 1.014 at \( Q = 20 \), 1.006 at \( Q = 50 \), and 1.003 at \( Q = 100 \). Thus fully compensating an inductive load does not significantly reduce the power gain compared to the inductive maximum provided the matching network’s variable inductor’s \( Q \) factor is sufficiently high.

### 4.4.3 Example diagrams

#### 4.4.3.1 Solutions of a \( \pi \) network including comparison to the approximate ones

To gain more insight in the solutions’ behaviour, important characteristics of the lossy \( \pi \) network matched exactly at its input are illustrated below in appropriate diagrams. Additionally, the diagrams include the solutions of a lossy \( \pi \) network designed as if it would be lossless, which are quite common in literature — its network elements may be obtained by putting \( R_a = 0 \) in (3.17), (3.19), and (3.18). Then \( R_a \) is given by the definition of a constant \( Q \) factor inductor, \( R_a = \frac{1}{Q} \omega L_a \). Finally, the formula of its power gain is similar to (3.24), but of course with the previously calculated \( R_a \) being put in. It should be noted that there is only one inductive solution if \( R_\ell \leq R_i \), whereas there is no valid inductive solution if \( R_\ell > R_i \).

\(^9\)The derivative of the ratio given is best suited to derive the extremum due to its numerator’s simplicity, whereas e.g. the derivative of the inverse ratio is much more complicated.
The π network was chosen as an example because it exhibits zeroes instead of poles for different network element values, thus the diagrams are easier to understand. The relatively low $Q = 5$ ensures better visibility of the exact solutions’ different “special” characteristics.

At first $C_1$ is considered.

![Diagram](image1)

**Figure 4.6:** $C_1$ of a lossy π network designed as if it would be lossless (thin lines) and of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing $R_{\text{ind}+}$ for the exactly matched network — solutions, a) full range, b) details at 7.35 MHz, $R_l = 50\,\Omega$, $Q = 5$, and $R_\ell = 30\,\Omega$

If $R_\ell \leq R_l$, $C_1$ of the lossy π network designed as if it would be lossless is similar for the capacitive and inductive solution and has a pole at $R_m = 0$. As indicated in Figure 4.6 b) (note the different scale), the lossy π network exactly matched at its input behaves different. $C_1$ of the inductive solution (employing $R_{\text{ind}+}$) has a maximum. If $R_\ell \leq R_{\text{limofRlim}}$, $C_1$ of the capacitive solution has a minimum. At $R_m = 0$, both solutions join without a pole.

![Diagram](image2)

**Figure 4.7:** $C_1$ of a lossy π network designed as if it would be lossless (thin lines) and of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing $R_{\text{ind}+}$ for the exactly matched network — solutions, a) full range, b) details at 7.35 MHz, $R_l = 50\,\Omega$, $Q = 5$, and $R_\ell = R_{\text{limofRlim}} \approx 49.51\,\Omega$

If $R_\ell$ increases, the value of the minimum decreases until it reaches zero at $R_\ell = R_{\text{limofRlim}}$ (refer to Figure 4.7 b) — note the different scale).
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Figure 4.8: $C_1$ of a lossy $\pi$ network designed as if it would be lossless (thin lines) and of a lossy $\pi$ network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing $R_{\text{ind}+}$ for the exactly matched network solutions, a) full range, b) details at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 5$, and $R_\ell = 49.6 \, \Omega$.

If $R_\ell$ increases further, the value of the minimum decreases below zero, yielding two limits of the parameter’s valid range (which guarantee that $C_1 \geq 0$ within), $R_{\text{lim}1}$ to the left and $R_{\text{lim}2}$ to the right of the minimum. (An example is given in Figure 4.8 b) — note the different scale).

Figure 4.9: $C_1$ of a lossy $\pi$ network designed as if it would be lossless (thin lines) and of a lossy $\pi$ network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing $R_{\text{ind}-}$ (long dashes) and $R_{\text{ind}+}$ (short dashes) for the exactly matched network solutions, a) full range, b) details at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 5$, and $R_\ell = R_i = 50 \, \Omega$.

If $R_\ell = R_i$, $R_{\text{lim}2}$ reaches its maximum value, $R_{\text{lim}2} = R_\ell = R_i$, and the second valid interval of the capacitive solution contracts to a single dot (Figure 4.9 b) — note the different scale). Capacitive and inductive (employing $R_{\text{ind}+}$) solution join at $R_m = 0$ and $R_m = R_\ell$ (as they did for lower $R_\ell$).

Additionally, the second inductive solution (employing $R_{\text{ind}-}$) becomes valid. Its $C_1$ tends to infinity if $R_m$ approaches zero. Note that both inductive solutions (employing $R_{\text{ind}+}$, $R_{\text{ind}-}$) join at $R_m = R_{\text{inv}1}$.

If $0 \leq R_m \leq R_{\text{inv}1}$, the inductive solution employing $R_{\text{ind}-}$ and, if $R_{\text{inv}1} \leq R_m \leq R_i$, the inductive solution employing $R_{\text{ind}+}$ is equal to the inductive solution of the lossy $\pi$ network designed as if it would be lossless, resulting from the pure parallel resonance of $C_1$ and $L_{2\text{rot}}$ in this case ($R_a$ and $L_a$ are zero).
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Figure 4.10: $C_1$ of a lossy π network designed as if it would be lossless (thin lines) and of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive — employing $R_{s_{\text{ind}-}}$ (long dashes) and $R_{s_{\text{ind}+}}$ (short dashes) — solutions, a) full range, b) details at 7.35 MHz, $R_l = 50 \, \Omega$, $Q = 5$, and $R_e = 70 \, \Omega$.

If $R_e > R_l$, the capacitive solution’s valid range is limited by $R_{s_{\text{ind}1}}$ only (an example is given in Figure 4.10 a)). Capacitive and inductive (employing $R_{s_{\text{ind}+}}$) solution join at $R_m = 0$.

The second inductive solution (employing $R_{s_{\text{ind}-}}$) remains valid. Its $C_1$ tends to infinity if $R_m$ approaches zero. Note that both inductive solutions (employing $R_{s_{\text{ind}+}}$, $R_{s_{\text{ind}-}}$) join at $R_m = R_{\text{mv01}}$ and that there is no valid inductive solution for the lossy π network designed as if it would be lossless (refer to Figure 4.10 b) — note the different scale).

Next $L_s$ is considered. (Since $R_s = \frac{1}{Q} \Re L_s$, $R_s$ is not shown separately.)

Figure 4.11: $L_s$ of a lossy π network designed as if it would be lossless (thin lines) and of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing $R_{s_{\text{ind}+}}$ for the exactly matched network — solutions at 7.35 MHz, $R_l = 50 \, \Omega$, $Q = 5$, and a) $R_e = 5 \, \Omega$, b) $R_e = 30 \, \Omega$.

If $R_e < R_l$, $L_s$ of the lossy π network designed as if it would be lossless is zero for the capacitive and inductive solution and $L_s$ of the capacitive solution has a maximum. As indicated in Figure 4.11, the lossy π network exactly matched at its input exhibits a nonzero $L_s$ then and $L_s$ of the capacitive/inductive (employing $R_{s_{\text{ind}+}}$) solution has a maximum/minimum. Since $Q > 1$ in the example, $L_s(R_m = 0) < L_s(R_m = R_e)$ for the capacitive and inductive (employing $R_{s_{\text{ind}+}}$) solution if $R_e < \frac{Q^2 + 1}{Q^2 - 1} R_l$. Two examples are given in Figure 4.11.
Since \(\frac{Q^2+1}{Q^2-1}\) increases with increasing \(Q\) and is higher than 0.98 if \(Q > 10\), yielding loads which do not need to be matched (except for a compensation of their susceptance, if present), it should be noted that even for moderate \(Q\) factors \(L_a(R_m = 0) < L_a(R_m = R_\ell)\) is usually observed if \(R_\ell < R_i\).

![Diagram](image)

Figure 4.12: \(L_a\) of a lossy \(\pi\) network designed as if it would be lossless (thin lines) and of a lossy \(\pi\) network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing \(R_{\text{ind}+}\) for the exactly matched network — solutions at 7.35 MHz, \(R_i = 50\, \Omega, Q = 5\), and a) \(R_\ell = \frac{Q^2+1}{Q^2-1} R_i \approx 46.15\, \Omega\) (horizontal dashed line indicates equal \(L_a\)), b) \(R_\ell = R_{\text{lim0fRmlim}} \approx 49.51\, \Omega\)

If \(R_\ell\) increases, \(L_a(R_m = 0)\) and \(L_a(R_m = R_\ell)\) come closer until \(L_a(R_m = 0) = L_a(R_m = R_\ell)\) at \(R_\ell = \frac{Q^2+1}{Q^2-1} R_i\) (\(Q > 1\) in the example) for the capacitive and inductive (employing \(R_{\text{ind}+}\)) solution (refer to Figure 4.12 a)).

If \(R_\ell\) increases further, \(L_a(R_m = 0) > L_a(R_m = R_\ell)\) for the capacitive and inductive (employing \(R_{\text{ind}+}\)) solution (refer to Figure 4.12 b)).

![Diagram](image)

Figure 4.13: \(L_a\) of a lossy \(\pi\) network designed as if it would be lossless (thin lines) and of a lossy \(\pi\) network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing \(R_{\text{ind}+}\) (long dashes) and \(R_{\text{ind}+}\) (short dashes) for the exactly matched network — solutions, a) full range, b) details at 7.35 MHz, \(R_i = 50\, \Omega, Q = 5\), and \(R_\ell = R_i = 50\, \Omega\) (horizontal dashed line indicates equal \(L_a\))

If \(R_\ell = R_i, L_a(R_m = 0) = L_a(R_m = R_{\text{mlim1}})\) for the capacitive solution (Figure 4.13 b) — note the different scale). Since \(R_{\text{mlim2}} = R_\ell = R_i\), the second valid interval of the capacitive solution contracts to a single dot. Capacitive and inductive (employing \(R_{\text{ind}+}\)) solution join at \(R_m = 0\) and \(R_m = R_\ell\) (as they did for lower \(R_\ell\)).

Additionally, the second inductive solution (employing \(R_{\text{ind}+}\)) becomes valid. Note that both inductive solutions (employing \(R_{\text{ind}+}, R_{\text{ind}+}\)) join at \(R_m = R_{\text{mlim2}}\).
If $0 \leq R_m \leq R_{\text{mv01}}$, the inductive solution employing $R_{\text{sind-}}$ and, if $R_{\text{mv01}} \leq R_m \leq R_i$, the inductive solution employing $R_{\text{sind+}}$ is equal to the inductive solution of the lossy π network designed as if it would be lossless, resulting from the pure parallel resonance of $C_1$ and $L_{\text{2rot}}$ in this case ($R_s$ and $L_s$ are zero).

![Graph a)](image1.png)

![Graph b)](image2.png)

Figure 4.14: $L_s$ of a lossy π network designed as if it would be lossless (thin lines) and of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive — employing $R_{\text{sind-}}$ (long dashes) and $R_{\text{sind+}}$ (short dashes) — solutions, a) full range, b) details at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 5$, and $R_\ell = 70 \, \Omega$.

If $R_\ell > R_i$, $L_s(R_m = 0) < L_s(R_m = R_{\text{mv01}})$ for the capacitive solution. Capacitive and inductive (employing $R_{\text{sind+}}$; solution join at $R_m = 0$ (an example is given in Figure 4.14 b) — note the different scale and that the lossy π network designed as if it would be lossless has no valid inductive solution if $R_\ell > R_i$).

The second inductive solution (employing $R_{\text{sind-}}$) remains valid. Its $L_s$ tends to zero if $R_m$ approaches zero. Note that both inductive solutions (employing $R_{\text{sind+}}$, $R_{\text{sind-}}$) join at $R_m = R_{\text{mv01}}$ and that there is no valid inductive solution for the lossy π network designed as if it would be lossless.

Next $g_\pi$ is considered. (Since the value $C_2$ depends on the load susceptibility, it was moved towards the end.)

![Graph a)](image3.png)

![Graph b)](image4.png)

Figure 4.15: Power gain $g_{\pi, \text{ides}}$ of a lossy π network designed as if it would be lossless (thin lines) and $g_{\pi, \text{inna}}$ of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing $R_{\text{sind-}}$ for the exactly matched network — solutions at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 5$, and a) $R_\ell = 5 \, \Omega$, b) $R_\ell = 30 \, \Omega$.
If \( R_\ell < R_i \), the power gain of the capacitive solution increases with increasing \( R_m \) (refer to Figure 4.15).

For the lossy \( \pi \) network designed as if it would be lossless, there exists a maximum of \( g_{\pi, \text{ildes}} \) for the inductive solution at \( R_{\text{mgmaxind}} = \frac{R_\ell R_i}{R_\ell + R_i} \) (the capacitive solution has its maximum \( L_a \) at this \( R_m \)). For the lossy \( \pi \) network exactly matched at its input, \( g_{\pi, \text{inma}, \text{maxinc+}} \), the maximum of \( g_{\pi, \text{inma}} \) in case of the inductive solution employing \( R_{\text{sind+}} \), was given in (4.29).

Since \( g_\pi = \frac{R_\ell}{R_\ell + R_i (1 + R_i^2 B_{\text{2tot}})} \), where \( B_{\text{2tot}} = \omega C_{\text{2tot}} = -\frac{1}{\omega L_{\text{2tot}}} \), holds for any \( \pi \) network, and \( C_{\text{2tot}} \) or \( L_{\text{2tot}} \) are similar at a given \( R_m \) for a lossy \( \pi \) network matched exactly at its input or a lossy \( \pi \) network designed as if it would be lossless, \( g_{\pi, \text{ildes}} > g_{\pi, \text{inma}} \) if the according \( L_a \) is lower (easily seen by comparison of Figure 4.11 and Figure 4.15). Thus, (for low \( Q \) factors) the exactly matched network may be worse, it matches at the expense of power gain!

Considering this result, which may seem surprisingly at a first glance, one should keep in mind that the concept of maximum transferred active power by achieving a complex conjugate match applies only to lossless matching networks! Depending on the losses, lossy matching networks may behave different. However, the match at the network’s input involves other advantages, e.g. reducing of the losses in the line connecting the network to the source.

![Figure 4.16: Power gain \( g_{\pi, \text{ildes}} \) of a lossy \( \pi \) network designed as if it would be lossless (thin lines) and \( g_{\pi, \text{inma}} \) of a lossy \( \pi \) network exactly matched at its input (thick lines), capacitive (solid) and inductive (dashed) — employing \( R_{\text{sind-}} \) (long dashes) and \( R_{\text{sind+}} \) (short dashes) for the exactly matched network — solutions, a) full range, b) details at 7.35 MHz, \( R_i = 50 \Omega \), \( Q = 5 \), and \( R_\ell = R_i = 50 \Omega \)](image)

If \( R_\ell = R_i \), the power gain of the capacitive solution increases with increasing \( R_m \) (refer to Figure 4.16 b) — note the different scale). This also holds for the inductive solution (employing \( R_{\text{sind+}} \)) if \( 0 \leq R_m \leq R_{\text{mv01}} \). Since \( R_{\text{mlin2}} = R_\ell = R_i \), the second valid interval of the capacitive solution contracts to a single dot. Capacitive and inductive (employing \( R_{\text{sind+}} \)) solution join at \( R_m = 0 \) and \( R_m = R_\ell \) (as they did for lower \( R_\ell \)).

Additionally, the second inductive solution (employing \( R_{\text{sind-}} \)) becomes valid. Note that both inductive solutions (employing \( R_{\text{sind+}}, R_{\text{sind-}} \)) join at \( R_m = R_{\text{mv01}} \).
If $0 \leq R_m \leq R_{nv01}$, the inductive solution employing $R_{\text{ind}^-}$ and, if $R_{nv01} \leq R_m \leq R_i$, the inductive solution employing $R_{\text{ind}^+}$ is equal to the inductive solution of the lossy π network designed as if it would be lossless, resulting from the pure parallel resonance of $C_1$ and $L_{2\text{tot}}$ in this case ($R_n$ and $L_n$ are zero and thus the power gain is unity and constant).

![Graph a](image1.png)

![Graph b](image2.png)

Figure 4.17: Power gain $g_{\tau,\text{inna}}$ of a lossy π network designed as if it would be lossless (thin lines) and $g_{\tau,\text{ldes}}$ of a lossy π network exactly matched at its input (thick lines), capacitive (solid) and inductive — employing $R_{\text{ind}^-}$ (long dashes) and $R_{\text{ind}^+}$ (short dashes) — solutions, a) full range, b) details at 7.35 MHz, $R_i = 50 \Omega$, $Q = 5$, and $R_\ell = 70 \Omega$.

If $R_\ell > R_i$, the power gain of the capacitive solution and the inductive solution (employing $R_{\text{ind}^+}$) increases with increasing $R_m$ (refer to Figure 4.17 b) — note the different scale and that the lossy π network designed as if it would be lossless has no valid inductive solution if $R_\ell > R_i$). Capacitive and inductive (employing $R_{\text{ind}^+}$) solution join at $R_m = 0$.

The second inductive solution (employing $R_{\text{ind}^-}$) remains valid. Its power gain decreases from $\frac{B}{R_i}$ at $R_m = 0$ with increasing $R_m$. Note that both inductive solutions (employing $R_{\text{ind}^-}$, $R_{\text{ind}^+}$) join at $R_m = R_{nv01}$ and that there is no valid inductive solution for the lossy π network designed as if it would be lossless.

Furthermore, Figures 4.15 to 4.17 illustrate the huge advantage of $R_m$ parametrisation in case of a constant $Q$ factor inductor — $\frac{dg_{\tau,\text{inna}}}{dR_m}$ of all solutions except the inductive solution employing $R_{\text{ind}^+}$ for $R_\ell < R_i$ exhibit a unique sign over the entire valid parameter range, yielding an optimum network at one of the range’s limits! (The second exception, the pure parallel resonance of $C_1$ and $L_{2\text{tot}}$, where the power gain stays constant, would require a $L_n = 0$ and $R_n = 0$, which is not realisable. Anyhow, it still would fit the concept because the constant gain is achieved at the parameter range’s limits, too.)

Finally $C_2$ is considered. Its value $C_2$ depends on the load susceptance, thus only some useful examples are given below. For the chosen $R_m$ parametrisation, $C_2$ at a given $R_m$ for a lossy π network designed as if it would be lossless and a lossy π network exactly matched at its input is equal. However, the applicable valid parameter ranges differ.
Figure 4.18: $C_2$ of a lossy π network designed as if it would be lossless (thin lines) and of a lossy π network exactly matched at its input (thick lines) at 7.35 MHz, $R_i = 50\,\Omega$, $Q = 5$, $R_\ell = 30\,\Omega$, and a) capacitive (solid) solutions for $C_{\ell p} = 100\,\text{pF}$, b) capacitive (solid, (over-)compensated) and inductive (dashed, undercompensated) — employing $R_{\text{sind+}}$ for the exactly matched network — solutions for $L_{\ell p} = 1\,\mu\text{H}$ (horizontal dashed line indicates $\omega C_2 = \frac{1}{\omega L_{\ell p}}$).

As illustrated in Figure 4.18 a) for an example where $R_{\ell} \leq R_{\text{limoff/mlim}}$, $C_{\ell p}$ limits the maximum possible $R_m$ (the upper limit is below $R_\ell$ in this case). If the load has an inductive imaginary part, (depending on the minimum value of $C_2$,) (over-)compensation (capacitive solution) or undercompensation (inductive solution (employing $R_{\text{sind+}}$ in the example)) may occur, refer to Figure 4.18 b). If valid, capacitive and inductive (employing $R_{\text{sind+}}$) solution join at $R_m = R_\ell$ if $R_\ell \leq R_i$.

Figure 4.19: $C_2$ of a lossy π network designed as if it would be lossless (thin lines) and of a lossy π network exactly matched at its input (thick lines), capacitive solutions at 7.35 MHz, $R_i = 50\,\Omega$, $Q = 5$, $R_\ell = 70\,\Omega$, and a) $C_{\ell p} = 100\,\text{pF}$, b) $C_{\ell p} = 300\,\text{pF}$.

If $R_\ell > R_i$, the upper limit in case of the capacitive solution is given by $R_i$ for a lossy π network designed as if it would be lossless and by $R_{\text{mlim}}$ for a lossy π network exactly matched at its input. Thus a load capacitance may be lower (Figure 4.19 a)) or higher (Figure 4.19 b)) than $C_{\text{tot}}$ at this limit, thus the valid parameter range is not changed or lowered, respectively.
Figure 4.20: $C_2$ of a lossy $\pi$ network designed as if it would be lossless (thin lines) and of a lossy $\pi$ network exactly matched at its input (thick lines), capacitive (solid, (over-)compensated) and inductive — employing $R_{\text{ind}+}$ (long dashes) and $R_{\text{ind}+}$ (short dashes) (both undercompensated) — solutions, a) full range, b) details at 7.35 MHz, $R_i = 50 \Omega$, $Q = 5$, $R_\ell = 70 \Omega$, and $L_{\ell\ell} = 60 \text{nH}$

If $R_\ell > R_i$, quite low $L_{\ell\ell}$ are required for the inductive solutions (employing $R_{\text{ind}+}$, $R_{\text{ind}+}$ with similar $C_2$) to be valid (an example is given in Figure 4.20). Note the different scales and that there is no valid inductive (undercompensated) solution for the lossy $\pi$ network designed as if it would be lossless.

### 4.4.3.2 Power gain’s contour line Smith charts for L, $\pi$, and T networks

In the following diagrams, power gain’s contour line Smith charts for different L, $\pi$, and T networks are shown. Their central inductor has a range of 0.1 $\mu$H - 40 $\mu$H, a constant $Q$ factor of 100, the matching frequency is 7.35 MHz, and the variable capacitor’s range is 5 pF - 500 pF. The contour lines were calculated from the resulting matching networks of 100 x 100 equally spaced load reflection coefficients whose $\text{Re}\{\ell_\ell\}$, $\text{Im}\{\ell_\ell\} \in [-1, 1]$, where only those with $|\ell_\ell| < 1$ were used.

To gain more insight in the networks’ behaviour, a full set of diagrams is given in Appendix C.2.1.1, where the matching frequency is either 1.8 MHz, 7.35 MHz, or 30 MHz and the variable capacitances’ ranges resemble different “typical” variable capacitors (a cheap 5 pF - 250 pF, a better 5 pF - 500 pF, and an expensive 25 pF - 4000 pF vacuum variable).

In case of $\pi$ and T networks, best and worst networks’ power gain is shown.

In case of L networks, it is assumed that the network topology enables implementation of both “complementary” L networks ($C_1$ only or $C_2$ only), e.g. by using appropriate switches.
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Figure 4.21: Simplified equivalent circuit of L networks derived from π networks with parasitic parallel capacitance a) $C_2$ or b) $C_1$ and from T networks with parasitic series capacitance c) $C_2$ or d) $C_1$.

Additionally, the impact of a parasitic capacitance replacing one variable capacitance of a π or T network is considered (refer to Figure 4.21) — the resulting networks yield a more realistic description of "real world" L networks. Based on [3], the value of that parasitic capacitance was chosen to be equal to the minimum/maximum value of the remaining variable capacitance for L networks derived from π networks/T networks.

Since lossy L networks may theoretically exhibit more than one distinct solutions for a given load, it has to be noted that:

- the pure L networks (without parasitic capacitances) and the L networks with a parasitic parallel capacitance derived from π networks yielded only a single valid solution throughout all complete Smith charts for the given network element ranges.

- the L networks with a parasitic series capacitance derived from T networks yielded more than one solution within some region of the Smith charts for the given network element ranges, but the differences in power gain of best and worst solutions were low — for the chosen contour line values both diagrams were merely similar — thus only the best diagrams are shown.

Regarding the latter — a more detailed review of the differences between best and worst L networks' power gain is given in Appendix C.2.1.1.

Figure 4.22: a) Best and b) worst $g_{π, inna}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, 5 pF $\leq C_{1,2} \leq 500$ pF, and 0.1 μH $\leq L_s \leq 40$ μH, undercompensation shaded.
Figure 4.23: $g_{\text{lofr, inna}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded

Figure 4.24: a) Best and b) worst $g_{h, \text{inna}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, undercompensation shaded

Figure 4.25: $g_{\text{loft, inna}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded
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From the preceding diagrams (and those shown in Appendix C.2.1.1), it’s quite obvious that the matching range of $\pi$ and T networks is always greater than that of the derived L networks which seems to be particularly true for L networks derived from T networks at lower frequencies. However, their matching range can be enormously extended if the omitted variable capacitor is replaced with a fixed capacitance similar to the variable capacitor’s minimum/maximum value for L networks derived from $\pi$ networks/T networks as proposed by [3].

Comparison of $\pi$ and T networks yields that in our frequency range of interest, the matching range of $\pi$ networks is lower than that of T networks. Furthermore, $\pi$ networks require high maximum values of the variable capacitances to obtain a “reasonable” matching range at all. But best and worst $\pi$ networks’ power gains are much closer than T networks’ — thus T networks are more vulnerable to careless tuning. As an extreme example, consider the left upper region in Figures 4.24 a) and b) — the best T network’s power gain is between 0.95 and 0.98, the worst T network’s between 0 and 0.1 which is at least 0.35 lower! The $\pi$ networks’ differences are much lower, although increasing if higher maximum capacitance values are used.

Checking the difference between the networks’ best and worst power gains is also useful regarding estimation of the gain’s increase due to a proper matching strategy. Depending on the load, frequency, and maximum capacitance values, values well above 10% are possible. As indicated above, even about 85% may occur in extreme cases. It’s important to note that it’s impossible to distinguish matched networks’ power gains by simply searching for a good match — all network settings which lie within the region defined by the outer contour line(s) (all settings able to match) yield (theoretically) perfect matches!

Regarding best and worst T networks, approximations were given in [65] for the loads at which highest losses of best T networks occur for a given $|r_L|$ — approximately at $G_L = \frac{1 + |r_L|}{1 - |r_L|} G_1$. Footnote 3 of [65] contains further approximations for the lowest losses of best T networks. In [65], no indication was given how the formulae were derived and how the contour Smith charts were calculated. However, using the solutions of Section 4.4.1.2, the values of Table 1 and of Table 2 for $Q = 100$ in [65] (load impedance only tabulated for $Q = 100$) were verified and a Smith chart equivalent to Figure 2 of [65] was drawn. Unfortunately the contour line values chosen by [65] are not suited for comparison — e.g. it’s impossible to decide whether undercompensation was included in case of loads exhibiting an inductive imaginary part, although it’s not likely because the lower bound for best T networks’ losses given in Footnote 3 of [65] applies only if that case is not taken into account. For the network element ranges used to derive all Smith charts above (including a set comparable to that used by [65]), it seems that the lowest losses occur at a circle’s section which describes best T networks with $C_1$ and $L_s$ at their maximum values and varying $C_2$. (That locus may be calculated analytically, but the formulae are quite lengthy — since it was not proven that these are indeed the lowest losses, the formulae are not given herein.) Furthermore, using the solutions of Section 4.4.1.2, it could be proven that if $G_L > G_1$ and $G_m$ is limited by the maximum value of $C_2$, the
highest losses for a given \(|Z_\ell|\) occur at \(Z_\ell = G_\ell = \frac{1 + |Z_\ell|}{1 - |Z_\ell|} G_1\). Calculating numerical examples for the remaining cases yielded highest losses of best \(T\) networks close to the approximations given in [65].

4.5 Impact of minimum/maximum elements' values on power gain, optimum networks

4.5.1 \(\pi\) network

In Sections 3.5.2.2 and 4.3.1.1 we observed that the optimum \(\pi\) network of a purely resistive load or a load exhibiting a capacitive imaginary part is an (external) \(L\) network. However, minimum/maximum elements' values were not considered. Anyhow, the \(L\) network was obtained because the optimisation yielded a maximum power gain if the \(\pi\) network’s parameter \(R_m\) was at its maximum value.

Which changes have to be made if minimum/maximum elements’ values have to be taken into account?

The optimisation remains unchanged. Thus the optimum power gain is still achieved for the highest possible \(R_m\). However, unlike before, it may be limited by the minimum/maximum value of one of the \(\pi\) network’s elements instead of the limits derived so far.

How may we ensure that \(R_m\) is at its highest value?

In Section 4.3.1.1 we proved that \(\frac{dC_{tot}}{dR_m} < 0\) (which also holds for purely resistive loads if \(C_{tot}\) is replaced with \(C_2\)). Thus the value of \(C_{tot}\) has to be at its lowest value for highest \(R_m\). Since \(C_{tot}\) is the sum of \(C_2\) and the (given) parallel load capacitance, this condition holds if \(C_2\) is at its lowest value.

Hence the the optimum \(\pi\) network of a purely resistive load or a load exhibiting a capacitive imaginary part is obtained if \(C_2\) is set to the lowest value where matching is still possible.

However, if the parameter’s range is limited rather by the minimum/maximum value of \(L_n\) or \(C_1\) instead of \(C_2\), there may occur gaps in the range of \(C_2\) where no matching is possible (\(L_n\) and \(C_1\) depend on \(R_n\) and may exhibit a minimum/maximum throughout the valid range of \(R_m\)). Thus if we manually try to find the lowest \(C_2\) where matching is still possible we should initially use the lowest value of \(C_2\) and increase it till a match is achieved.

According to Section 4.3.2.1, the situation changes if loads exhibiting an inductive imaginary part are considered. In case 1 ((over-)compensation), \(C_2\) decreases if \(R_m\) increases, whereas in case 2 (undercompensation), \(C_2\) increases if \(R_m\) increases — both cases join at \(R_m = R_\ell\), which is an (internal) \(L\) network if it’s within the parameter’s valid range given by (3.20) (in particular, if at least \(R_\ell < R_1 - R_{50}\) and
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if those range is not further limited by the minimum/maximum value of one of the
π network’s elements.

*How may we distinguish case 1 ((over-)compensation) from case 2 (undercompensation)?*

Comparision of (4.11) and (4.12) yields that except of the intersection \( C_2 = \frac{1}{\omega^2 L_{tp}} \)
(at \( R_m = R_t \) as indicated above), \( C_2 > \frac{1}{\omega^2 L_{tp}} \) in case 1 and \( C_2 < \frac{1}{\omega^2 L_{tp}} \) in case 2.
Furthermore, by considering the derivative of \( C_2 \) with respect to \( R_m \) we obtain
that \( C_2 \) continuously decreases if in case 1 the parameter \( R_m \) is increased up to
the intersection at \( R_m = R_t \) (if within the parameter’s valid range) and decreases
further if \( R_m \) is decreased from \( R_m = R_t \) in case 2. Hence if we decrease \( C_2 \) and
try to match, the transition from case 1 to case 2 is observed for \( C_2 = \frac{1}{\omega^2 L_{tp}} \) if
\( R_t < R_l - R_{s0} \) and \( R_m = R_t \) is not excluded by the minimum/maximum value of
one of the π network’s elements. If \( R_m = R_t \) is not within the parameter’s valid
range, both cases are still distinguishable by comparison of \( C_2 \) to \( \frac{1}{\omega^2 L_{tp}} \).

With respect to the π network’s power gain case 1 behaves like the case of a purely
resistive load or a load exhibiting a capacitive imaginary part. Thus the highest
power gain is obtained if \( C_2 \) is set to the lowest value where matching is still possible.
Conversely, case 2 behaves different. As indicated in Section 4.3.2.1, it may involve
a maximum which is not located at maximum \( R_m \). In conjunction with the consider-
erations regarding the values of \( C_2 \) and its derivatives with respect to \( R_m \) we may
conclude that if such maximum exists, we should not decrease \( C_2 \) beyond the value
where the power gain maximum occurs. Thus the highest power gain may not be
observed for the lowest \( C_2 \) where matching is still possible.

### 4.5.2 T network

In Sections 3.5.3.2 and 4.3.1.2 we observed that the optimum T network of a purely
resistive load or a load exhibiting a capacitive imaginary part is an (external) \( L \)
network. However, minimum/maximum elements’ values were not considered. Anyhow,
the \( L \) network was obtained because the optimisation yielded a maximum power gain
if the T network’s parameter \( G_m \) was at its maximum value.

*Which changes have to be made if minimum/maximum elements’ values have to be
taken into account?*

The optimisation remains unchanged. Thus the optimum power gain is still achieved
for the highest possible \( G_m \). However, unlike before, it may be limited by the
minimum/maximum value of one of the T network’s elements instead of the limits
derived so far.

*How may we ensure that \( G_m \) is at its highest value?*

In Section 4.3.1.2 we proved that \( \frac{dG_{max}}{dG_m} > 0 \) (which also holds for purely resistive
loads if \( C_{2tot} \) is replaced with \( C_2 \)). Thus the value of \( C_{2tot} \) has to be at its highest
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value for highest \( G_m \). Since \( C_{2\text{rot}} \) is the reciprocal of the sum of the reciprocal of \( C_2 \) and the reciprocal of the (given) serial load capacitance, this condition holds if \( C_2 \) is at its highest value.

Hence the the optimum T network of a purely resistive load or a load exhibiting a capacitive imaginary part is obtained if \( C_2 \) is set to the highest value where matching is still possible.

However, if the parameter’s range is limited rather by the minimum/maximum value of \( L_a \) (from parallel-to-series transformation of \( G_p \) and \( L_p \) or \( C_1 \) instead of \( C_2 \), there may occur gaps in the range of \( C_2 \) where no matching is possible (\( L_p \) and \( C_1 \) depend on \( G_p \) and may exhibit a minimum/maximum throughout the valid range of \( G_m \)). Thus if we manually try to find the highest \( C_2 \) where matching is still possible we should initially use the highest value of \( C_2 \) and decrease it till a match is achieved.

According to Section 4.3.2.2, the situation changes if loads exhibiting an inductive imaginary part are considered. In case 1 ((over-)compensation), \( C_2 \) increases if \( G_m \) increases, whereas in case 2 (undercompensation), \( C_2 \) decreases if \( G_m \) increases — both cases join at \( G_m = G_\ell \), which is an (internal) L network if it’s within the parameter’s valid range given by (3.33) (in particular, if at least \( G_\ell < G_1 \)) and if those range is not further limited by the minimum/maximum value of one of the T network’s elements.

How may we distinguish case 1 ((over-)compensation) from case 2 (undercompensation)?

Comparison of (4.17) and (4.18) yields that except of the intersection \( C_2 = \frac{1}{\omega^2 L_\ell} \) (at \( G_m = G_\ell \) as indicated above), \( C_2 < \frac{1}{\omega^2 L_\ell} \) in case 1 and \( C_2 > \frac{1}{\omega^2 L_\ell} \) in case 2. Furthermore, by considering the derivative of \( C_2 \) with respect to \( G_m \) we obtain that \( C_2 \) continuously increases if in case 1 the parameter \( G_m \) is increased up to the intersection at \( G_m = G_\ell \) (if within the parameter’s valid range) and increases further if \( G_m \) is decreased from \( G_m = G_\ell \) in case 2. Hence if we increase \( C_2 \) and try to match, the transition from case 1 to case 2 is observed for \( C_2 = \frac{1}{\omega^2 L_\ell} \) if \( G_\ell < G_1 \) and \( G_m = G_\ell \) is not excluded by the minimum/maximum value of one of the T network’s elements. If \( G_m = G_\ell \) is not within the parameter’s valid range, both cases are still distinguishable by comparison of \( C_2 \) to \( \frac{1}{\omega^2 L_\ell} \).

With respect to the T network’s power gain case 1 behaves like the case of a purely resistive load or a load exhibiting a capacitive imaginary part. Thus the highest power gain is obtained if \( C_2 \) is set to the highest value where matching is still possible.

Conversely, case 2 behaves different. As indicated in Section 4.3.2.2, it may involve a maximum which is not located at maximum \( G_m \). In conjunction with the considerations regarding the values of \( C_2 \) and its derivatives with respect to \( G_m \) we may conclude that if such maximum exists, we should not increase \( C_2 \) beyond the value where the power gain maximum occurs. Thus the highest power gain may not be observed for the highest \( C_2 \) where matching is still possible.
4.5.3 Conclusion

Deriving a matching strategy for optimum power gain is particularly easy for all cases considered above except for case 2 (undercompensation) if the load exhibits an inductive imaginary part.

For purely resistive loads, load impedances exhibiting a capacitive imaginary part, or if due to the minimum/maximum network elements’ values only (over-)compensation is applicable to load impedances exhibiting an inductive imaginary part, the following matching strategy ensures highest power gain of the network:

- For π networks set $C_2$ to its lowest value and try to match. If no match is achievable, increase the value of $C_2$ until matching is possible.

- For T networks set $C_2$ to its highest value and try to match. If no match is achievable, decrease the value of $C_2$ until matching is possible.

Conversely, if the load exhibits an inductive imaginary part and the minimum/maximum network elements’ values are suitable for undercompensation to apply we need to know the network elements’ values and the frequency to compute their settings for highest power gain by applying the solutions derived in Section 4.4.1.1 or Section 4.4.1.2 to a piecewise constant approximation of the inductor’s measured $Q$ factor and choosing the optimum settings of the best segment.

It should be noted that pure (external) lossy L networks may exhibit some distinct settings for a given load (for example consider Figure 4.8 b)), although usually only one solution remains valid, in particular if network elements’ limits are taken into account. Thus the matching strategy of the type of network the L network was derived from has to be applied, even if both “complementary” L networks ($C_1$ only or $C_2$ only) are realisable (e.g. with a switch) and able to match.

**Is it likely to observe case 2 if the load exhibits an inductive imaginary part?**

Table 4.1: Transition from case 1 ((over-)compensation) to case 2 (undercompensation) at $C_2 = \frac{1}{\omega^2 L_{tp,s}}$ for loads exhibiting an inductive imaginary part.

<table>
<thead>
<tr>
<th>$\omega L_{tp,s}$</th>
<th>1.8 MHz</th>
<th>7.35 MHz</th>
<th>30 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{tp,s}$</td>
<td>$\omega^2 L_{tp,s}$</td>
<td>$L_{tp,s}$</td>
<td>$\omega^2 L_{tp,s}$</td>
</tr>
<tr>
<td>1 Ω</td>
<td>88 nH</td>
<td>88 nF</td>
<td>22 nH</td>
</tr>
<tr>
<td>2 Ω</td>
<td>180 nH</td>
<td>44 nF</td>
<td>43 nH</td>
</tr>
<tr>
<td>5 Ω</td>
<td>440 nH</td>
<td>18 nF</td>
<td>110 nH</td>
</tr>
<tr>
<td>10 Ω</td>
<td>880 nH</td>
<td>8.8 nF</td>
<td>220 nH</td>
</tr>
<tr>
<td>20 Ω</td>
<td>1.8 μH</td>
<td>4.4 nF</td>
<td>430 nH</td>
</tr>
<tr>
<td>50 Ω</td>
<td>4.4 μH</td>
<td>1.8 nF</td>
<td>1.1 μH</td>
</tr>
<tr>
<td>100 Ω</td>
<td>8.8 μH</td>
<td>880 pF</td>
<td>2.2 μH</td>
</tr>
<tr>
<td>200 Ω</td>
<td>18 μH</td>
<td>440 pF</td>
<td>4.3 μH</td>
</tr>
<tr>
<td>500 Ω</td>
<td>44 μH</td>
<td>180 pF</td>
<td>11 μH</td>
</tr>
<tr>
<td>1000 Ω</td>
<td>88 μH</td>
<td>88 pF</td>
<td>22 μH</td>
</tr>
<tr>
<td>2000 Ω</td>
<td>180 μH</td>
<td>44 pF</td>
<td>43 μH</td>
</tr>
</tbody>
</table>
To get a first hint, the values of $C_2 = \frac{1}{\omega^2 L_{tp}}$ at the transition from case 1 ((over-) compensation) to case 2 (undercompensation) were computed in Table 4.1 for different reciprocals $\omega L_{tp}$ of the load's susceptance ($\pi$ network) or reactances $\omega L_{ts}$ of the load (T network) at different frequencies.

Assuming that good vacuum variable capacitors offer an adjustable capacitance range of about 5 pF − 500 pF, expensive ones up to 25 pF − 4 nF (or 100 pF − 5 nF), and parasitic parallel capacitances of about 5 pF may occur, we may conclude that case 2 ($C_2 < \frac{1}{\omega^2 L_{ts}}$) is applicable to a $\pi$ network over a wide range of $\omega L_{tp}$ up to 7.35 MHz provided that minimum/maximum values of $L_s$ or $C_1$ do not involve additional limitations. For higher frequencies, it may be restricted to somewhat lower $\omega L_{tp}$, but its range is still broad, which even holds for a minimum capacitance of 30 pF.

However, the applicable range of case 2 in a T network ($C_2 > \frac{1}{\omega^2 L_{ts}}$) is severely restricted by the maximum capacitance of $C_2$. If it's 5 nF, the restrictions are comparable to those of a 30 pF minimum capacitance in a $\pi$ network, but limit $\omega L_{ts}$ to somewhat higher values at lower frequencies. If the maximum capacitance $C_2$ is 500 pF, or, as in “cheaper” T networks, only 250 pF, the minimum $\omega L_{ts}$ required for case 2 to apply is significantly increased, even at 30 MHz.

Another limit is quite obvious if the variable inductor’s $Q$ factor is assumed to be reasonably high, say $Q \geq 100$. Then, according to Section 4.4.2, a “real world” $\pi$ or T network\(^\text{10}\) is not able to match a $R_t > R_i - R_{s0}$ ($\pi$ network) or $G_t > G_i$ (T network) in our frequency range of interest due to limitations introduced by parasitic elements. (Note: Lossless networks (for which also $R_{s0} = 0$ holds) are not able to match these cases at all).

Summarising the preceding considerations yields that the simple matching strategy proposed above is applicable to the majority of load impedances. However, if the load impedance's imaginary part is inductive, $R_t < R_i - R_{s0}$ in a $\pi$ network or $G_t < G_i$ in a T network, and case 2 (undercompensation) is not excluded by the minimum/maximum values of the network’s elements, explicit knowledge of the network’s elements, the matching frequency, and (e.g.) magnitude and phase of the network’s input reflection coefficient is required to obtain the network’s settings for highest active power transfer. Then it’s also possible to compute the load impedance for any setting and to determine which case applies.

It’s interesting to note that the “simple” part of the matching strategy was already proposed for T networks by [66], where the variable inductor was assumed to have a constant $Q$ factor of $Q = 200$, $Q$-based lossless network design was used, and an iterative calculation algorithm was applied to change the numerical values of $L_p$ and $C_1$ according to the losses introduced by the inductor's finite $Q$ factor. Due to restriction of the T network’s variable capacitances range\(^\text{11}\) to 20 pF − 240 pF,

\(^{10}\) Although “real world” inductor’s $Q$ factor is not constant, it can always be approximated by piecewise constant $Q$ factors. The results of Section 4.4.2 apply to any segment and thus to the whole inductor.

\(^{11}\) Depending on the results, the program used by [66] may further limit the T network’s variable capacitances range to 25 pF − 235 pF or 30 pF − 230 pF.
case 2 was not observed by [66], although it should have been obtainable at higher frequencies (at least at 30 MHz) for $G_\ell < \frac{1}{50\Omega}$ or $\frac{1}{G_\ell} > 50\Omega$.

If we compare the power gain achieved by $\pi$ or T networks using the (complete) optimum matching strategy described above to those of the popular (external) L networks assuming continuously variable network elements (switchable elements will be considered in Section 4.6) and a switch to realise both complementary L networks, we may conclude that according to Sections 4.3.1.1 and 4.3.1.2, an (external) L network is the optimum choice for purely resistive loads or loads exhibiting a capacitative imaginary part, but its matching range is lower compared to the appropriate $\pi$ or T network made of similar network elements (refer to Section 4.4.3.2 and Appendix C.2.1.1) where also a “cure” is given based on [3]). If the load exhibits an inductive imaginary part, it is still the optimum choice if $R_\ell > R_1 - R_\delta 0$ for the L network derived from a $\pi$ network or $G_\ell > G_1$ for the L network derived from a T network, the variable inductor’s $Q$ factor is assumed to be reasonably high, say $Q \geq 100$, and the lossy networks’ “inductive” case 2 solutions are not realisable as indicated above. Then case 1 (overcompensation) applies and the highest power gain is obtained for $R_m = R_m \lim$ or $G_m = G_m \lim$, respectively, which yields an (external) L network. However, if $R_\ell < R_1 - R_\delta 0$ or $G_\ell < G_1$, case 2 (undercompensation) applies and the external L network is obtained for $R_m = R_m L_{\ell p}$ or $G_m = G_m L_{\ell t}$. Unless $R_\ell \ll \omega_0 L_{\ell p}$ or $G_\ell \ll \frac{1}{\omega_0 L_{\ell t}}$, the resulting (external) L network’s power gain may be close to or far away from the optimum. In particular, considering a variable inductor of a reasonably high constant $Q$ factor, say $Q \geq 100$, yields a flat, broad maximum whose power gain is $\leq 0.3\%$ higher (derived in the last paragraph of Section 4.4.2) than the power gain of the internal L network ($C_2$ set to compensate $L_{\ell p, s}$, and $L_{\ell p}$ and $C_1$ tuned to match) where case 1 and case 2 join. Depending on $L_{\ell p, s}$, the external L network’s $R_m L_{\ell p, s}$ may be higher or lower than the optimum $R_m$. If it is higher, its power gain is close to the maximum, if it is lower, its power gain may be much lower than the maximum, although the maximum’s flatness helps to increase the range of $L_{\ell p, s}$ where the external L network does not waste too much active power. The advantages of using an external L network are its low number of network elements and its simple tunability by measuring the magnitude of the network’s input reflection coefficient.

If the variable inductor’s $Q$ factor is sufficiently high, say $Q \geq 100$, a matching strategy may be proposed based on the considerations regarding external L networks which is somewhat simpler than the optimum strategy given above. If the network’s load impedance is purely resistive, exhibits a capacitive imaginary part, or an inductive imaginary part if $R_\ell > R_1 - R_\delta 0$ for the $\pi$ network or $G_\ell > G_1$ for the T network, the “simple” part of the matching strategy described above is used. If the network’s load impedance exhibits an inductive imaginary part if $R_\ell < R_1 - R_\delta 0$ (which is approximately $R_\ell < R_1$ if $R_1 = 50\Omega$ and $Q$ is high) for the $\pi$ network or $G_\ell < G_1$ for the T network, we set the network’s elements as close as possible to the applicable internal L network ($C_2$ is set to compensate as much as possible, but not to overcompensate the inductive part of the load, and $L_{\ell p, s}$ and $C_1$ are tuned to match). Then a magnitude and phase measurement of the network’s in-
put reflection coefficient suffices if we (approximately) determine the network's load impedance by setting all elements of a π network to their minimum value or all elements of a T network to their maximum value and, if required, compensate the load's inductive part using $C_2$. The disadvantages of this matching strategy are that the minimum/maximum values may not be low/high enough at the upper/lower end of our frequency range of interest to determine the load impedance with sufficient accuracy and that in case no match gaps occur in the network's settings there is no indication which limit of the gap would yield the higher power gain. Additionally, "steps" in time varying loads would require a restart from minimum/maximum element values. All disadvantages vanish if we know the network's elements explicitly and measure the matching frequency, but then the amount of simplification compared to the optimum strategy is lowered.

Thus, depending on the affordable effort, the optimum matching strategy or its simplified descendant should be applied to a π or T network to obtain a high power gain. External L networks (which are somehow equal to using only the "simple" part of the optimum matching strategy in π or T networks) do not guarantee high power gains for certain loads, but are a good choice for the majority of loads.

4.6 Applicability of the optimisation method to switchable matching networks

4.6.1 Simplifying assumptions for matching networks using switchable elements

To check if the optimisation method presented above is applicable to matching networks using switchable elements instead of continuously variable elements we need to consider which simplifying assumptions may apply and what the differences are compared to continuously variable networks. (The other remaining case of a fixed inductor in series or in parallel to a continuously variable capacitor is completely described by (3.1) at matching frequency and thus part of the solution derived above.)

Firstly we assume that high-Q capacitors are used in the switchable network, whose losses are small and negligible compared to the losses of the inductor losses, which are usually made of ferrite or iron powder toroidal cores involving additional core losses.

Secondly we assume that the minimum step size of the network's capacitances and inductances is low enough to allow approximation of those capacitances and inductances by continuously variable ones at matching frequency. In particular, this assumption implies the possibility of an "exact" match at the switchable network's input.

However, the losses involved do not change continuously and still exhibit resistance steps. This behaviour is clearly visible if we consider combinations of low inductance
coils, where the connecting wires’ losses may not be small and thus not be negligible—e.g. switching from $80 \, \text{nH} + 160 \, \text{nH} + 320 \, \text{nH} = 560 \, \text{nH}$ (three coils) to $640 \, \text{nH}$ (one coil) will definitely cause a loss resistance step. Since we cannot predict the step’s direction unless we know which cores and wiring are used, switchable networks behave different than continuously variable ones, because the latter’s losses could be described by (3.1), indicating that for an increasing inductance the losses stay constant or increase, but never decrease!

### 4.6.2 \( \pi \) network

If the parameter \( R_m \) is increased by \( \Delta R_m \ll R_m \), thus changed from \( R_m \) to \( R_m + \Delta R_m \), the (switchable) inductance also changes, which is accompanied by a resistance step in \( R_s \).

We have to distinguish two cases:

1. \( R_s(R_m) < R_s(R_m + \Delta R_m) \):
   \[
   \Delta R_m \ll R_m, \quad \text{thus} \quad R_k(R_m + \Delta R_m) \approx \frac{R_k(R_m + \Delta R_m)}{R_m} \quad \text{holds. Hence} \quad \frac{R_k(R_m)}{R_m} < \frac{R_k(R_m + \Delta R_m)}{R_m} \quad \text{or} \quad \frac{1}{1 + \frac{R_k(R_m)}{R_m}} > \frac{1}{1 + \frac{R_k(R_m + \Delta R_m)}{R_m}}.
   \]
   Then \( g_{\pi, \text{inma}}(R_m) > g_{\pi, \text{inma}}(R_m + \Delta R_m) \).

2. \( R_s(R_m) > R_s(R_m + \Delta R_m) \):
   \[
   \Delta R_m \ll R_m, \quad \text{thus} \quad R_k(R_m + \Delta R_m) \approx \frac{R_k(R_m + \Delta R_m)}{R_m} \quad \text{holds. Hence} \quad \frac{R_k(R_m)}{R_m} > \frac{R_k(R_m + \Delta R_m)}{R_m} \quad \text{or} \quad \frac{1}{1 + \frac{R_k(R_m)}{R_m}} < \frac{1}{1 + \frac{R_k(R_m + \Delta R_m)}{R_m}}.
   \]
   Then \( g_{\pi, \text{inma}}(R_m) < g_{\pi, \text{inma}}(R_m + \Delta R_m) \).

Choosing the optimum value of \( R_m \) for continuously variable networks by the method presented above involved proving of \( g_{\pi, \text{inma}}(R_m) \leq g_{\pi, \text{inma}}(R_m + \Delta R_m) \). Since case 2 exhibits similar behaviour of \( g_{\pi, \text{inma}} \), the method would be applicable to it. However, case 1 is critical because \( g_{\pi, \text{inma}} \) steps downward for a certain \( R_m \) and we do not know if \( g_{\pi, \text{inma}} \) will be as high or even higher than its value prior to the step within the remaining range of \( R_m \) unless we know the exact functional description of \( R_s(R_m) \) and the load and source admittance.

Usually both cases are possible—in a continuously variable matching network with a constant \( Q \) inductor and a resistive load the inductance would have a maximum throughout the (utmost) valid range of \( R_m \) indicated in (3.20)—thus it’s quite likely that for increasing \( R_m \) the switchable network’s inductance may pass a certain value twice for increasing and decreasing inductances, and both resistance step directions occur.

Hence there is no common optimisation for \( \pi \) networks using switchable elements. However, if we know the losses involved and load and source admittances, we may optimise the network elements’ values for minimum losses in this particular case.
Only if the resistance steps would be small compared to the total loss resistance for a minimum inductance step between any two possible total switchable inductance values, the continuously variable network's optimisation method would be approximately applicable for the switchable network — but this assumption might not hold for small inductance values where the losses in the connecting wires might not be small and thus not be negligible.

### 4.6.3 T network

If the parameter \( G_m \) is increased by \( \Delta G_m \ll G_m \), thus changed from \( G_m \) to \( G_m + \Delta G_m \), the (switchable) inductance also changes, which is accompanied by a resistance step in \( R_s \), hence an admittance step in \( G_p \).

According to \( (A.16) \) \( G_p = \frac{R_s}{R_s^2 + \omega^2 L_s^2} \) holds. Since we assumed that \( L_s \) might be approximately described by a continuously variable inductance, the inductance steps are small enough to simplify calculations by using \( L_s(G_m) \approx L_s(G_m + \Delta G_m) \). During optimisation, we additionally restricted the inductor's \( Q \) factor to \( Q > 1 \) (which is equal to \( \omega L_s > R_s \)), thus \( \frac{\partial G_p}{\partial R_s} = \frac{(\omega L_s - R_s)(\omega L_s + R_s)}{(R_s^2 + \omega^2 L_s^2)^2} > 0 \) holds. Hence if there is a step in \( R_s \), there is a step of similar direction in \( G_p \).

Again we have to distinguish two cases:

1. \( R_s(G_m) < R_s(G_m + \Delta G_m) \), hence \( G_p(G_m) < G_p(G_m + \Delta G_m) \):
   \[
   \Delta G_m \ll G_m, \text{ thus } \frac{G_p(G_m + \Delta G_m)}{G_m + \Delta G_m} \approx \frac{G_p(G_m + \Delta G_m)}{G_m} \text{ holds. Hence } \frac{G_p(G_m)}{G_m} \ll \frac{1}{G_m} \frac{G_p(G_m + \Delta G_m)}{G_m} \text{ or } \frac{1}{1 + \frac{G_p(G_m + \Delta G_m)}{G_m}} > \frac{1}{1 + \frac{G_p(G_m)}{G_m}}.
   \]
   Then \( g_{t,\text{inma}}(G_m) > g_{t,\text{inma}}(G_m + \Delta G_m) \).

2. \( R_s(G_m) > R_s(G_m + \Delta G_m) \), hence \( G_p(G_m) > G_p(G_m + \Delta G_m) \):
   \[
   \Delta G_m \ll G_m, \text{ thus } \frac{G_p(G_m + \Delta G_m)}{G_m + \Delta G_m} \approx \frac{G_p(G_m + \Delta G_m)}{G_m} \text{ holds. Hence } \frac{G_p(G_m)}{G_m} > \frac{1}{1 + \frac{G_p(G_m + \Delta G_m)}{G_m}} \text{ or } \frac{1}{1 + \frac{G_p(G_m + \Delta G_m)}{G_m}} < \frac{1}{1 + \frac{G_p(G_m)}{G_m}}.
   \]
   Then \( g_{t,\text{inma}}(G_m) < g_{t,\text{inma}}(G_m + \Delta G_m) \).

Choosing the optimum value of \( G_m \) for continuously variable networks by the method presented above involved proving of \( g_{t,\text{inma}}(G_m) \leq g_{t,\text{inma}}(G_m + \Delta G_m) \). Since case 2 exhibits similar behaviour of \( g_{t,\text{inma}} \), the method would be applicable to it. However, case 1 is critical because \( g_{t,\text{inma}} \) steps downward for a certain \( G_m \) and we do not know if \( g_{t,\text{inma}} \) will be as high or even higher than its value prior to the step within the remaining range of \( G_m \) unless we know the exact functional description of \( G_p(G_m) \) and the load and source impedance.

Usually both cases are possible — in a continuously variable matching network with a constant \( Q \) inductor and a resistive load the inductance would have a minimum throughout the (utmost) valid range of \( G_m \) indicated in (3.33) — thus it’s quite likely that for increasing \( G_m \) the switchable network’s inductance may pass a certain value.
twice for decreasing and increasing inductances, and both resistance step directions occur.

Hence there is no common optimisation for $T$ networks using switchable elements. However, if we know the losses involved and load and source admittances, we may optimise the network elements' values for minimum losses in this particular case.

Only if the resistance steps would be small compared to the total loss resistance for a minimum inductance step between any two possible total switchable inductance values, the continuously variable network's optimisation method would be approximately applicable for the switchable network — but this assumption might not hold for small inductance values where the losses in the connecting wires might not be small and thus not be negligible.
Chapter 5

Transmission Lines’ and Total Losses in a Source, Line 1, Matching Network, Line 2, and Load System

5.1 Introduction

The active power transfer of a transmission line (Figure 2.1) or of a system consisting of a matching network enclosed by two transmission lines (Figure 2.3) was already derived in Section 2.6 or 2.7, respectively.

In the following chapter, we optimise the active power transfer of those systems.

Firstly we consider a transmission line, secondly the matching system involving lossless and lossy matching networks to distinguish different loss origins. Optimisation of the matching system includes optimised splitting of the total line length (although the total line length is usually fixed by a given distance between source and load, the position of the matching network may be optimised). Finally an optimum compromise is proposed which is applicable to unknown or changing load impedances.

Throughout the chapter, all transmission lines are considered to have a complex characteristic impedances. Based on the equations derived, we will be able to distinguish parts of the system where we may approximately use real characteristic impedances instead from parts where we always have to use complex ones.

5.2 Definitions/notations

From this chapter on, the following definitions/notations will be used:
Latin letters

\( B_{t, MB} \)  
Load susceptance of a matchbox

\( C_{lp} \)  
(Equivalent) parallel capacitance calculated from the imaginary part of \( \frac{1}{Z_0} \)

\( C_{ls} \)  
(Equivalent) series capacitance calculated from the imaginary part of \( Z_0 \)

\( C_{l_{tot}} \)  
Total source capacitance (including \( C_1 \) of the network and the source's (equivalent) capacitance)

\( C_{lp, MB} \)  
(Equivalent) parallel load capacitance of a matchbox, calculated from \( B_{t, MB} \) (\( \pi \) network)

\( C_{ls, MB} \)  
(Equivalent) series load capacitance of a matchbox, calculated from \( X_{t, MB} \) (\( T \) network)

\( g_{\text{tot}, li, \text{in} ma} \)  
Power gain of the complete matching system whose lossless matchbox is exactly matched at its input

\( g_{\text{tot}, ti, \text{in} ma} \)  
Power gain of the complete matching system whose \( T \) network is exactly matched at its input

\( g_{\text{tot}, pi, \text{in} ma} \)  
Power gain of the complete matching system whose \( \pi \) network is exactly matched at its input

\( g_{L_{opt}} \)  
Power gain of a transmission line with \( X_0 \neq 0 \) whose load is the optimum load impedance \( Z_{opt} \)

\( g_{MB, \text{in} ma} \)  
Power gain of a matchbox exactly matched at its input

\( g_{MB, \text{in} ma}(\ell_{MB}) \)  
Power gain of a matchbox exactly matched at its input as a function of \( \ell_{MB} \)

\( g_{MB, li, \text{in} ma} \)  
Power gain of a lossless matchbox exactly matched at its input

\( g_{MB, li, \text{in} ma}(\ell_{MB}) \)  
Power gain of a lossless matchbox exactly matched at its input as a function of \( \ell_{MB} \)

\( g_{T, L_{opt}} \)  
Transducer power gain of a transmission line with \( X_0 \neq 0 \) whose load is the optimum load impedance \( Z_{opt} \)

\( g_{\text{Tot}, \text{in} ma} \)  
Transducer power gain of the complete matching system whose matchbox is exactly matched at its input

\( G' \)  
Per-unit-length conductance of a transmission line

\( G_{0a} \)  
Inverse of the real part of \( Z_0 \)

\( G_{t, MB} \)  
In case of a \( \pi \) network, load conductance (real part of load admittance) of that network (matchbox), in case of a \( T \) network, \( G_{t, MB} = \frac{1}{R_x MB} \), where \( R_x \) is the load resistance (real part of load impedance of that network (matchbox))

\( I_{2,m, \pm} \)  
Length of transmission line 2 where (depending on sign \( \pm \)) the \( m \)-th maximum or minimum of \( e^{2i\theta_{tot}} g_{\text{tot}, li, \text{in} ma} \) occurs

\( I_{tot} \)  
Total length of lines in a complete matching system (\( I_{tot} = I_1 + I_2 \))

\( L' \)  
Per-unit-length inductance of a transmission line

\( L_{1tot} \)  
Total source inductance (including \( C_1 \) of the network and the source's (equivalent) inductance)

\( L_{lp, MB} \)  
(Equivalent) parallel load inductance of a matchbox, calculated from \( B_{t, MB} \) (\( \pi \) network)

\( L_{ls, MB} \)  
(Equivalent) series load inductance of a matchbox, calculated from \( X_{t, MB} \) (\( T \) network)

\( m \)  
Whole number (integer)

\( P_{act}(z) \)  
Active power transferred to the "load" of a transmission line at location \( z \)

\( P_{act, L_{opt}} \)  
Active power transferred to the input of a transmission line whose \( X_0 \neq 0 \) terminated with \( Z_{opt} \)

\( L_{1T, \text{in} ma} \)  
Reflection coefficient at the input of a \( T \) network exactly matched at its input (referred to \( Z_0 \))

\( L_{1\pi, \text{in} ma} \)  
Reflection coefficient at the input of a \( \pi \) network exactly matched at its input (referred to \( Z_0 \))

\( L_{1MB} \)  
Reflection coefficient at the input of a matchbox (referred to \( Z_0 \))

\( L_{1MB, \text{in} ma} \)  
Reflection coefficient at the input of a matchbox exactly matched at its input (referred to \( Z_0 \))

\( L_{opt} \)  
Resulting valid reflection coefficient of the optimum load impedance of a transmission line with \( X_0 \neq 0 \) (referred to \( Z_0 \))

\( L_{MB} \)  
Reflection coefficient at a matchbox's load (referred to \( Z_0 \))

\( R_{op} \)  
Inverse of the real part of \( \frac{1}{Z_0} \)
$R_{\ell, MB}$  
In case of a T network, load resistance (real part of load impedance) of that network (matchbox), in case of a $\pi$ network, $R_{\ell, MB} = \frac{G_{\ell, MB}}{\omega L_{\ell}}$, where $G_{\ell, MB}$ is the load conductance (real part of load admittance of that network (matchbox)).

$X_{\ell, MB}$  
Load reactance of a matchbox.

$Z_{0\text{MB}}$  
Location along a transmission line, range 0 to length of transmission line.

$Z_{0\text{MB}}$  
Reference impedance of the reflection coefficients at a matchbox's input.

$Z_{0\text{MB}}$  
Reference impedance of the reflection coefficients at a matchbox's output.

$Z_{1}$  
Impedance at the beginning of a transmission line (at its input).

$Z_{\text{lopt}}$  
Optimum source impedance for a transmission line with $X_{0} \neq 0$ whose load is the optimum load impedance $Z_{\text{lopt}}$.

$Z_{\ell MB}$  
Load impedance of a matchbox.

$Z_{\ell \text{lopt}}$  
Resulting optimum load impedance of a transmission line with $X_{0} \neq 0$.

Greek letters

$\varphi_{\ell}$  
Phase of reflection coefficient $\rho_{\ell}$.

5.3 Losses and matching in a source, transmission line, and load system

The power gain $g_{L}$ of a source, transmission line, and load system (depicted in Figure 2.1) is given by (2.23),

\[
\begin{align*}
g_{L} & = e^{-2\alpha l} \frac{R_{0} (1 - |\rho_{\ell}|^{2}) - 2X_{0} \text{Im}\{\rho_{\ell}\}}{R_{0} \left(1 - e^{-4\alpha l} |\rho_{\ell}|^{2}\right) - 2X_{0} e^{-2\alpha l} \text{Im}\{\rho_{\ell} e^{-j2\beta l}\}} \\
g_{L} & = e^{-2\alpha l} \frac{R_{0} (1 - \rho_{\ell}^{*} \rho_{\ell}) + jX_{0} (\rho_{\ell} - \rho_{\ell}^{*})}{R_{0} \left(1 - e^{-4\alpha l} \rho_{\ell}^{*} \rho_{\ell}\right) + jX_{0} e^{-2\alpha l} (\rho_{\ell} e^{-j2\beta l} - \rho_{\ell}^{*} e^{j2\beta l})},
\end{align*}
\]

which will be optimised for a fixed\(^1\) length $l > 0$ by changing $\rho_{\ell}$. If the method described in [26g] is applied (as mentioned at the end of Appendix A.3, calculation of $P_{\text{act}}(z)$ in [26g] does not include all dependencies on $\rho_{\ell}$ and, furthermore, then $P_{\text{act}}(z)$ is optimised in [26g] instead of $g_{L}$, where $P_{\text{act}}(z)$ is the active power transferred to the "load" of a transmission line at location $z$ as defined in (A.8)), we obtain

\[
\left[ \frac{\partial g_{L}}{\partial \rho_{\ell}^{*}} \right] = \frac{\left[jR_{0}X_{0} e^{-2\alpha l} (e^{-2\alpha l} - e^{-j2\beta l}) \rho_{\ell}^{2}ight]}{\left[R_{0} \left(1 - e^{-4\alpha l} \rho_{\ell}^{*} \rho_{\ell}\right) + jX_{0} e^{-2\alpha l} (\rho_{\ell} e^{-j2\beta l} - \rho_{\ell}^{*} e^{j2\beta l})\right]^{2}} - \frac{\left[R_{0} \left(1 - e^{-4\alpha l} \rho_{\ell}^{*} \rho_{\ell}\right) + X_{0} e^{-2\alpha l} (e^{j2\beta l} - e^{-j2\beta l}) \rho_{\ell}^{*}\right]}{\left[R_{0} \left(1 - e^{-4\alpha l} \rho_{\ell}^{*} \rho_{\ell}\right) + jX_{0} e^{-2\alpha l} (\rho_{\ell} e^{-j2\beta l} - \rho_{\ell}^{*} e^{j2\beta l})\right]^{2}},
\]

\(^{1}\) An optimisation with respect to the transmission line's length $l$ is impossible because lossy transmission lines are passive. Thus $\frac{\partial g_{L}}{\partial l} \leq 0$, causing $g_{L}$ to decrease from $g_{L} = 1$ at $l = 0$ for increasing $l$ and a given (constant) $\rho_{\ell}$. $\frac{\partial g_{L}}{\partial l} = 0$ may indicate a (single) maximum if $l = 0$, but only an inflexion point if $l > 0$. 
\[
\frac{\partial g_L}{\partial L_t} = \left[-jR_0X_0e^{-2\alpha l} (e^{-2\alpha l} - e^{i2\beta l})(\tau^*_t)^2 \right. \\
\left. - \left( R_0^2 (1 - e^{-4\alpha l}) + X_0^2 e^{-2\alpha l} (e^{-i2\beta l} - e^{i2\beta l}) \right) \tau^*_t \\
+ jR_0X_0 (1 - e^{-2\alpha l} e^{-i2\beta l}) \right] \left( R_0 (1 - e^{-4\alpha l} L_t \tau^*_t) + jX_0 e^{-2\alpha l} L_t e^{-i2\beta l} \right)^{-1}.
\]

The ratio following the term in squared brackets never equals zero, thus an extremum exists only if the term in squared brackets equals zero. Since the terms in squared brackets of \( \frac{\partial g_L}{\partial C_t} \) and \( \frac{\partial g_L}{\partial \tau_t} \) are just the conjugate complex of each other, both partial derivatives are simultaneously zero, hence the extremum of \( g_L \) requires \( \frac{\partial g_L}{\partial C_t} = 0 = \frac{\partial g_L}{\partial \tau_t} \) to hold and only one of both partial derivatives has to be considered — \( \frac{\partial g_L}{\partial \tau_t} \) is more convenient for optimisation with respect to \( L_t \).

Since derivation of the optimum load and source impedance is quite lengthy, it is performed in Appendix D.2. The results are summarised below.

The solution involves two different cases, in particular:

1. \( X_0 = 0 \)

If \( \alpha > 0 \), the transmission line is either Heaviside’s distortionless transmission line obtained for \( \frac{R'}{\omega C'} = \frac{C'}{\omega C'} > 0 \) and \( Z_0 = Z_{0ll} \) (where \( C' \) is the per-unit-length conductance and \( L' = Z_{0ll}^2 C' \) the per-unit-length inductance of that line, defined in \([26a]\) or the approximation (2.5) is used. Then \( \tau_t = 0 \) uniquely solves the equation, thus \( Z_0 = Z_{0ll} \) yields the maximum \( g_L \). If we terminate the line accordingly, its input resistance \( Z_1 = Z_{0ll} \), hence the source’s impedance has to be \( Z_1 = Z_{0ll} \) to achieve maximum active input power. In particular, optimum source and load impedance is independent of (frequency and) line length.

If \( \alpha = 0 \), the transmission line is lossless, hence \( \frac{R'}{\omega C'} = \frac{C'}{\omega C'} = 0 \). Then \( g_L = 1 \) for any \( Z_0 \). However, according to (2.20) only a choice of \( Z_0 = Z_{0ll} \) combined with \( Z_1 = Z_{0ll} \) yields a source impedance for maximum active power input which is independent of (frequency and) line length — thus no other choice is appropriate.

2. \( X_0 \neq 0 \)

In (D.1), (D.2), and (D.3), the optimum load reflection coefficient is given by

\[
\tau_{t,\text{opt}} = \frac{\sqrt{\left( R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l) \right)^2 - \left| Z_0 \right|^4 - jR_0 X_0 \sinh(2\alpha l) - jX_0^2 \sin(2\beta l)}}{jR_0 X_0 e^{-2\alpha l} e^{-i2\beta l}}, \tag{5.1}
\]

yielding an optimum load impedance of

\[
Z_{t,\text{opt}} = \frac{\sqrt{\left( R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l) \right)^2 - \left| Z_0 \right|^4 - jR_0 X_0 (\cosh(2\alpha l) - \cos(2\beta l))}}{R_0 \sinh(2\alpha l) + X_0 \sin(2\beta l)} \tag{5.2}
\]
and an optimum power gain of

$$g_{L,\text{opt}} = \frac{R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l) - \sqrt{(R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l))^2 - |Z_0|^4}}{|Z_0|^2}. \quad (5.3)$$

If the transmission line is terminated by its optimum load impedance, its input impedance is the complex conjugate of the optimum load impedance.

To achieve maximum active input power, the source impedance has to be equal to the complex conjugate of the line’s input impedance, hence $Z_{\text{opt}} = Z_{L,\text{opt}}$. If the source impedance is chosen accordingly, the active input power is $P_{\text{act, opt}} = P_{\text{act, maxi}}$, thus $g_{T,\text{opt}} = g_{L,\text{opt}}$. Additionally, the line’s output impedance is equal to the complex conjugate of its optimum load impedance. Optimum active power input and transfer involves complex conjugate matching at the line’s input and output.

But, compared to a transmission line whose characteristic impedance is real ($X_0 = 0$), the optimum load in case of a complex characteristic has a severe disadvantage — it depends on frequency and line length — thus, depending on frequency and line length, different load and source impedances would have to be used.

To choose a useful length independent load impedance close to the optimised load, the limits for $l \to 0$ and $l \to \infty$ were considered in Appendix D.2. Those limits give us a hint which length independent choice of source and load impedance is recommendable. $Z_\ell = Z_0 = Z_1$ (which is still frequency dependent), is far from the optimum value if the line’s length is short, but increasingly close to the optimum value\(^2\), if the line’s length increases, in particular, if $\alpha l \geq 1.5$ holds.

Considering a coaxial transmission line whose complex characteristic impedance is given by $Z_0 \approx Z_{\text{lll}} (1 - j \frac{a}{b})$ according to (2.4) where $\frac{a}{b} \ll 1$, the typical choice of $Z_\ell = Z_1 = R_0 \approx Z_{\text{lll}}$ (which is independent of length and frequency!) slightly differs from the preceding one\(^3\).

Nevertheless, since $\alpha l$ should be low to ensure low line losses, it’s not obvious that the choice indicated above is truly recommendable, but we should keep

\(^2\) However, if source and load impedances are chosen accordingly, a complex conjugate match at the line’s input can only be achieved for high values of $l$ because $Z_1 \neq Z_0$ (except for $l \to \infty$), hence $g_{L} = \frac{R_0^2 + X_0^2 e^{-2\alpha l}}{|Z_0|^2} e^{-2\alpha l}$ differs from $g_{T,\text{opt}} = \frac{R_0^2 |Z_0|^2}{R_0^2 + 2R_0X_0 e^{-2\alpha l} \cos(2\beta l) + X_0^2 e^{-4\alpha l}}$ (except for $l \to \infty$).

\(^3\) However, if source and load impedances are equal to $R_0$, thus $Z_\ell = Z_1 = \frac{X_0}{R_0 - X_0}$, a complex conjugate match at the line’s input cannot be achieved for any value of $l$ because $Z_1 \neq R_0$, hence $g_{L} = \frac{R_0^2 + X_0^2 e^{-2\alpha l} (R_0 \cos(2\beta l) - X_0 \sin(2\beta l))}{4R_0^2 + 2X_0^2 e^{-2\alpha l} (R_0 \cos(2\beta l) - X_0 \sin(2\beta l))} e^{-2\alpha l}$ differs from $g_{T,\text{opt}} = \frac{2R_0^2 |Z_0|^2}{16R_0^2 + 4R_0X_0 e^{-2\alpha l} (2R_0 \cos(2\beta l) - X_0 \sin(2\beta l)) + X_0^2 (e^{-2\alpha l} - 2 \cos(2\beta l))} e^{-\alpha l}$.\)
in mind that the goal is not to choose source and load impedances as close as possible to its optimum impedance values, but to achieve a power gain as close as possible to its optimum value.

The latter holds for coaxial transmission line whose complex characteristic impedance is given by \((2.4)\) and \(\frac{a}{\beta} \ll 1\). Then either a choice of source and load impedances equal to \(Z_0^*\) or equal to \(R_0\) ensures a low deviation of power gain and transducer power gain from its optimum values, which would be 1.6% higher at 1.8 MHz and 0.4% higher at 30 MHz if e.g. a RG-213/U coaxial cable would be used.

5.4 Losses and matching in a source, line 1, matching network, line 2, and load system

5.4.1 Power gain of the system

In the preceding section, we investigated the active power transfer in transmission lines whose characteristic impedances were complex. We derived the optimum load impedance for maximum active power transfer and its according optimum source impedance for maximum active input power, which were both equal and depended on line length and frequency.

Compared to a (lossy) transmission line whose characteristic impedance is real or is approximated as real, it’s not obvious that the insertion of a two-port, a matching network which is designed to match the line’s characteristic impedance at its input, is suited to optimise the active power transfer to the load.

However, in our application all transmission lines have a complex characteristic impedance of the kind \(Z_0 \approx Z_{0\|} (1 - j \frac{a}{\beta})\), where \(\frac{a}{\beta} \ll 1\), and the source impedance is real and equal to \(Z_{0\|} = R_0\).

At the end of the preceding section, we discovered that the active power transfer in such transmission lines is close to maximum if source and load impedances are equal to \(R_0\). Hence we suppose that the insertion of a matching network as indicated above is recommendable, decreases line losses, and increases the active power transfer to the load. Another advantage is that the network may be tuned quite easily by measuring its input reflection coefficient.

Besides network element values, the position of the matching network is variable. Thus we have to consider a matching network embedded in two transmission lines of variable lengths connecting to source and load. Both lines have complex characteristic impedances to investigate its impact on active power transfer.

To derive the power gain \(g_{\text{tot,inna}}\) of the complete system depicted in Figure 2.3 consisting of a source (source impedance \(Z_s\)), a transmission line 1 connecting it to the input of a matching network, a transmission line 2 connecting the output of the matching network to the load (load impedance \(Z_L\)). The matching network
is tuned to match the characteristic impedance of line 1 at its input. To denote network specific variables, the index “MB” (matchbox) is used instead of the index “tp” (two-port) which was used in Section 2.7.

To simplify calculations, we assume line 1 and 2 to be similar, hence both complex characteristic impedances, attenuation constants, and phase constants are equal, and we define \( Z_0 = Z_{01} = Z_{02}, \alpha = \alpha_1 = \alpha_2, \) and \( \beta = \beta_1 = \beta_2. \) Then the matching network’s reference impedances should also be similar and equal to the characteristic impedance of the lines, \( Z_0 = Z_{0MB1} = Z_{0MB2}. \)

The lengths of the transmission lines, \( l_1 \) and \( l_2, \) may be chosen arbitrarily, but the total length

\[
l_{\text{tot}} = l_1 + l_2
\]

is given.

We (usually) also need to know \( \tau_{\ell} \) or \( \tau_\ell, \) respectively, which is transformed to \( \tau_{\ell MB} = \tau_{12} = \tau_\ell e^{-2\alpha l_2} e^{-j2\beta l_2} \) at the input of line 2 according to (2.20).

It’s further transformed by the matching network, which is designed (or tuned) to match the characteristic impedance of line 1 at its input, hence

\[
\tau_{1MB} = \tau_{1MB, inna} = 0.
\]

The matched network’s power gain is given by

\[
g_{MB, inna}(\tau_{\ell MB}) = g_{MB, inna}(\tau_\ell e^{-2\alpha l_2} e^{-j2\beta l_2}).
\]

From (2.27) we obtain

\[
g_{\text{tot, inna}} = e^{-2\alpha l_{\text{tot}}} \cdot g_{MB, inna}(\tau_\ell e^{-2\alpha l_2} e^{-j2\beta l_2}) \frac{R_0 (1 - |\tau_\ell|^2) - 2X_0 \text{Im}\{\tau_\ell\}}{R_0 (1 - e^{-4\alpha l_2} |\tau_\ell|^2) - 2X_0 e^{-2\alpha l_2} \text{Im}\{\tau_\ell e^{-j2\beta l_2}\}},
\]

which has to be maximised in the following sections.

However, prior to the optimisation we have to consider the source impedance \( Z_i \) because the active input power will depend on it. The matching network is designed to match the characteristic impedance of line 1 at its input, \( \tau_{1MB} = \tau_{21} = 0. \) Then \( \tau_{11} = 0 \) or \( \tau_{11} = Z_0 \) according to (2.20). If \( \tau_1 \) is chosen arbitrarily, application of (2.26) gives

\[
g_{\text{Tot, inna}} = e^{-2\alpha l_{\text{tot}}} \cdot \frac{R_0 (1 - |\tau_1|^2) - 2X_0 \text{Im}\{\tau_1\}}{|Z_0|^2} \cdot g_{MB, inna}(\tau_\ell e^{-2\alpha l_2} e^{-j2\beta l_2}) \frac{R_0 (1 - |\tau_\ell|^2) - 2X_0 \text{Im}\{\tau_\ell\}}{R_0 (1 - e^{-4\alpha l_2} |\tau_\ell|^2) - 2X_0 e^{-2\alpha l_2} \text{Im}\{\tau_\ell e^{-j2\beta l_2}\}}.
\]

Maximisation of the active input power requires \( Z_i = Z_0^* \), which is equivalent to \( \tau_1 = -j \frac{R_0}{Z_0} \) as indicated in (2.19) and yields an active input power of \( P_{\text{etmax}}. \) Then the source and load related “correction factor” of line 1 in \( g_{\text{Tot, inna}} \) becomes

\[
\frac{R_0 (1 - |\tau_1|^2) - 2X_0 \text{Im}\{\tau_1\}}{|Z_0|^2} = 1.
\]
However, in our application the line’s characteristic impedance is \( Z_0 \approx Z_{0ll} (1 - j \frac{\alpha}{\beta}) \) and the source impedance is \( Z_1 = Z_{0ll} \approx R_0 \).

From (2.9) we obtain \( \tau_1 = \frac{-jX_0}{2R_0 + jX_0} \approx \frac{j\beta}{2(1 - j\frac{\alpha}{\beta})} \approx \frac{\alpha}{2\beta} (j - \frac{\alpha}{\beta}) \). Then the source and load related “correction factor” of line 1 in \( g_{Ttot, inna} \) is given by

\[
\frac{R_0 \left(1 - |\tau_{1MB}|^2\right) - 2X_0 \text{Im} (\tau_{1MB})}{|Z_0|^2} = \frac{4R_0^2}{4R_0^2 + X_0^2} \approx \frac{1}{1 + \frac{\alpha^2}{4\beta^2}} \approx 1 - \frac{\alpha^2}{4\beta^2},
\]

which is close to 1 because \( \frac{\alpha}{\beta} \ll 1 \), e.g. for a RG-213/U coaxial cable it’s 0.999927 at 1.8 MHz and 0.999995 at 30 MHz.

The situation changes if the matching network is designed to match \( Z_{0ll} \), which is approximately equal to \( Z_0 \). Then \( \tau_1 = \tau_{1MB} = \frac{-jX_0}{2R_0 + jX_0} \approx \frac{j\beta}{2(1 - j\frac{\alpha}{\beta})} \approx \frac{\alpha}{2\beta} (j - \frac{\alpha}{\beta}) \) and the matching network’s input is not (exactly) matched to the line’s characteristic impedance.

According to (2.27) and \( g_L \) in Footnote 3, the load related “correction factor” of line 1 in \( g_{Ttot, inna} \) changes from 1 to

\[
\frac{R_0 \left(1 - |\tau_{1MB}|^2\right) - 2X_0 \text{Im} (\tau_{1MB})}{R_0 \left(1 - e^{-2\alpha l_1} |\tau_{1MB}|^2\right) - 2X_0 e^{-2\alpha l_1} \text{Im} (\tau_{1MB} e^{-j2\beta l_1})}
= \frac{2R_0 |Z_0|^2}{2R_0 |Z_0|^2 + X_0^2} \left( R_0 (\sinh(2\alpha l_1) + 2\cos(2\beta l_1)) - X_0 \sin(2\beta l_1) \right)
= \frac{1 + \frac{\alpha^2}{2\beta^2}}{4\beta^2} e^{-2\alpha l_1} \left( \sinh(2\alpha l_1) + 2\cos(2\beta l_1) + \frac{\alpha^2}{2\beta^2} \sin(2\beta l_1) \right)
\approx 1 - \frac{\alpha^2}{4\beta^2} \left( e^{-2\alpha (l_{tot} - l_2)} \left( 4 \cos(2\beta (l_{tot} - l_2)) - e^{-2\alpha (l_{tot} - l_2)} \right) - 3 \right),
\]

where \( l_1 = l_{tot} - l_2 \) was applied. Since \( \frac{\alpha}{\beta} \ll 1 \), the “correction factor” is still close to 1, e.g. for a RG-213/U coaxial cable it’s between 1 and 1.0003 at 1.8 MHz and between 1 and 1.00004 at 30 MHz.

According to (2.26) and \( g_{T1} \) in Footnote 3, the source and load related “correction factor” of line 1 in \( g_{Ttot, inna} \) changes from \( \frac{R_0 \left(1 - |\tau_{1}|^2\right) - 2X_0 \text{Im} (\tau_{1})}{|Z_0|^2} \) to

\[
\frac{R_0 \left(1 - |\tau_{1}|^2\right) - 2X_0 \text{Im} (\tau_{1})}{|Z_0|^2} - \frac{e^{-2\alpha l_1} e^{-j2\beta l_1}}{1 + \frac{\alpha^2}{2\beta^2}} \left( \cos(2\beta (l_{tot} - l_2)) \left( 1 - \frac{\alpha^2}{4\beta^2} \cos(2\beta (l_{tot} - l_2)) \right) - 1 \right),
\]

where \( l_1 = l_{tot} - l_2 \) was applied. Since \( \frac{\alpha}{\beta} \ll 1 \), the “correction factor” is close to 1, e.g. for a RG-213/U coaxial cable it’s between 1 and 1.0006 at 1.8 MHz and between 1 and 1.00002 at 30 MHz.
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Regarding the values of the “correction factors” of line 1, we should keep in mind that both are multiplied by $e^{-2\alpha l_1}$, thus the “complete” factors are always $\leq 1$. Surprisingly the approximate design is slightly better than exact matching and its according source impedance choice, but that just reflects that both cases differ from the optimum derived in the preceding section (since $g_{L,\text{opt}} = g_{T,\text{opt}}$, the “correction factors” of the optimum are equal, e.g. for a RG-213/U coaxial cable they are between 1 and 1.016 at 1.8 MHz and between 1 and 1.0042 at 30 MHz, which is also quite close to 1).

Nevertheless we may conclude that in our frequency range of interest, the “matched” line 1 behaves as an idealised matched line (source impedance $Z_{\text{0ll}}$, characteristic impedance $Z_{\text{0ll}}$ of line 1, matching network’s input matched to $Z_{\text{0ll}}$) would do, in particular if low loss coaxial cables are used. This holds either for termination by an exactly matched network and a source impedance of $Z_0^* = Z_{\text{0ll}} (1 + j \frac{\alpha}{\beta})$ or $Z_{\text{0ll}}$ or for termination by a network matched to $Z_{\text{0ll}}$ and a source impedance of $Z_{\text{0ll}}$, where the latter is usually applied. Hence the active power transfer of the “matched” line 1 does not necessarily involve complex characteristic impedance calculations because the mismatch introduced is small. However, the impact of the small mismatch caused by the source impedance $Z_{\text{0ll}}$ on the matching network’s design will have to be considered later.

Conversely, depending on the load impedance $Z_\ell$, the mismatch of line 2 may be large, thus active power transfer of line 2 requires complex characteristic impedance calculations.

### 5.4.2 Lossless matching network

A lossless matching network is characterised by a power gain equal to unity for any load impedance attached (and any source impedance it is designed to match to), hence

$$g_{\text{MB, ll, inms}} \left( \frac{R_\ell e^{-2\alpha l_2} e^{-j2\beta l_2}}{R_0 (1 - |r_{\ell}|^2) - 2X_0 \operatorname{Im}(r_{\ell})} \right) = 1.$$ 

Then (5.5) becomes

$$g_{\text{tot, ll, inms}} = e^{-2\alpha l_{\text{tot}}} \frac{R_0 (1 - |r_{\ell}|^2) - 2X_0 \operatorname{Im}(r_{\ell})}{R_0 (1 - e^{-4\alpha l_2} |r_{\ell}|^2) - 2X_0 e^{-2\alpha l_2} \operatorname{Im}(r_{\ell} e^{-j2\beta l_2})}.$$  

(5.7)

According to (5.4), $l_{\text{tot}}$ seems to depend on $l_2$. However, in our application the total length $l_{\text{tot}}$ is given, thus $l_1$ is chosen to keep the total length constant and $e^{-2\alpha l_{\text{tot}}}$ does not depend on $l_2$.

To optimise $g_{\text{tot, ll, inms}}$ by varying $l_2$, we have to solve

$$\frac{d(e^{2\alpha l_{\text{tot}}} g_{\text{tot, ll, inms}})}{dl_2} = 0$$

for $l_2$. 


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Application of (5.7) yields

\[
\frac{\partial (e^{2\alpha l_{\text{tot}} g_{\text{tot}, \text{ll, imma}}})}{\partial l_{2}} = -4e^{-2\alpha l_{2}} \frac{R_{0} (1 - |\zeta_{\ell}|^{2}) - 2X_{0} \text{Im}\{\zeta_{\ell}\}}{(R_{0} (1 - e^{-4\alpha l_{2}} |\zeta_{\ell}|^{2}) - 2X_{0} e^{-2\alpha l_{2}} \text{Im}\{\zeta_{\ell} e^{-j2\beta l_{2}}\})^{2}} \cdot [R_{0}\alpha |\zeta_{\ell}|^{2} e^{-2\alpha l_{2}} + X_{0} (\alpha \text{Im}\{\zeta_{\ell} e^{-j2\beta l_{2}}\} + \beta \text{Re}\{\zeta_{\ell} e^{-j2\beta l_{2}}\})].
\]

The derivative equals zero only if the term in squared brackets equals zero, because

\[-4e^{-2\alpha l_{2}} < 0\] and applying (2.9), (2.8), and (2.7) to numerator and denominator of the second term yields

\[
\begin{align*}
R_{0} (1 - |\zeta_{\ell}|^{2}) - 2X_{0} \text{Im}\{\zeta_{\ell}\} & = \frac{4 \text{Re}\{Z_{\ell}\} |Z_{0}|^{2}}{(\text{Re}\{Z_{\ell}\} + R_{0})^{2} + (\text{Im}\{Z_{\ell}\} + X_{0})^{2}} > 0, \\
R_{0} (1 - e^{-4\alpha l_{2}} |\zeta_{\ell}|^{2}) - 2X_{0} e^{-2\alpha l_{2}} \text{Im}\{\zeta_{\ell} e^{-j2\beta l_{2}}\} & = \frac{4 \text{Re}\{Z_{\ell, MB}\} |Z_{0}|^{2}}{(\text{Re}\{Z_{\ell, MB}\} + R_{0})^{2} + (\text{Im}\{Z_{\ell, MB}\} + X_{0})^{2}} > 0,
\end{align*}
\]

which hold since \(R_{0} > 0\) according to [26c], \(\text{Re}\{Z_{\ell}\} > 0\) because the load is passive and it should be possible to transfer active power to the load\(^4\), and \(\text{Re}\{Z_{\ell, MB}\} > 0\) if \(\text{Re}\{Z_{\ell}\} > 0\) since line 2 is lossy and passive (\(Z_{\ell, MB}\) may be computed from \(Z_{\ell, MB} = \zeta_{\ell} e^{-2\alpha l_{2}} e^{-j2\beta l_{2}}\) if (2.10) is applied (\(Z_{\ell, MB}\) was derived in Section 5.4.1)).

Using the definition

\[
\zeta_{\ell} := |\zeta_{\ell}| e^{j\varphi_{\ell}}
\]

we obtain for the term in squared brackets

\[
\begin{align*}
R_{0}\alpha |\zeta_{\ell}|^{2} e^{-2\alpha l_{2}} + X_{0} (\alpha \text{Im}\{\zeta_{\ell} e^{-j2\beta l_{2}}\} + \beta \text{Re}\{\zeta_{\ell} e^{-j2\beta l_{2}}\}) & = R_{0}\alpha |\zeta_{\ell}|^{2} e^{-2\alpha l_{2}} + X_{0} (\alpha |\zeta_{\ell}| \sin(\varphi_{\ell} - 2\beta l_{2}) + \beta |\zeta_{\ell}| \cos(\varphi_{\ell} - 2\beta l_{2})) ,
\end{align*}
\]

which may be rewritten to

\[
\begin{align*}
R_{0}\alpha |\zeta_{\ell}|^{2} e^{-2\alpha l_{2}} + X_{0} (\alpha \text{Im}\{\zeta_{\ell} e^{-j2\beta l_{2}}\} + \beta \text{Re}\{\zeta_{\ell} e^{-j2\beta l_{2}}\}) & = R_{0}\alpha |\zeta_{\ell}|^{2} e^{-2\alpha l_{2}} + X_{0} |\zeta_{\ell}| |\gamma| \cos(\varphi_{\ell} - 2\beta l_{2} - \arctan\left(\frac{\alpha}{\beta}\right)) ,
\end{align*}
\]

if we take into account that \(\beta > 0\) if \(\omega > 0\) according to [26b].

Then we have to distinguish two different cases.

If \(\zeta_{\ell} = 0\), \(e^{2\alpha l_{\text{tot}} g_{\text{tot}, \text{ll, imma}}} = 1 = \text{const.}\) or \(g_{\text{tot}, \text{ll, imma}} = e^{-2\alpha l_{\text{tot}}}\) holds, causing

\[
\frac{\partial (e^{2\alpha l_{\text{tot}} g_{\text{tot}, \text{ll, imma}}})}{\partial l_{2}} = 0\] although no extremum occurs.

\(^4\text{Re}\{Z_{\ell}\} = 0\) would apply if the load is a pure capacitance, a pure inductance, or an ideal short circuit. Although those loads would be entirely passive, no active power could be transferred to such loads.
If \( l_2 \neq 0 \) we have to consider two subcases:

1. \( X_0 = 0 \)

   Then we have to solve
   \[
   0 = R_0 \alpha |r_\ell| \, e^{-2\alpha l_2}.
   \]
   According to \([26c]\) \( R_0 > 0 \). Additionally, \([26b]\) indicates that \( \alpha \geq 0 \).

   If \( \alpha > 0 \), the transmission line is either Heaviside's distortionless transmission line obtained for \( \frac{R'}{\omega L'} = \frac{C'}{\omega C'} > 0 \) and \( Z_0 = Z_{0\parallel} \) (where \( C' \) is the per-unit-length conductance and \( L' = Z_{0\parallel} C' \) the per-unit-length inductance of that line, defined in \([26a]\)) or the approximation (2.5) is used. Since \( r_\ell \neq 0 \) if \( l_2 \in \mathbb{R} \), \( R_0 \alpha |r_\ell| \, e^{-2\alpha l_2} > 0 \) holds. Combined with the preceding considerations regarding the sign of the remaining terms yields \( \frac{d(e^{2\alpha l_2} g_{\text{tot}, \text{ll, inma}})}{dl_2} < 0 \), thus the maximum \( e^{2\alpha l_2} g_{\text{tot}, \text{ll, inma}} \) is obtained if \( l_2 = 0 \).

   If \( \alpha = 0 \), the transmission line is lossless, hence \( \frac{R'}{\omega L'} = \frac{C'}{\omega C'} = 0 \). Then \( e^{2\alpha l_2} g_{\text{tot}, \text{ll, inma}} = 1 = \text{const. or } g_{\text{tot}, \text{ll, inma}} = 1 \) holds, causing \( \frac{d(e^{2\alpha l_2} g_{\text{tot}, \text{ll, inma}})}{dl_2} = 0 \) although no extremum occurs.

2. \( X_0 \neq 0 \)

   We assumed \( l_{\text{tot}} > 0 \) and, according to \([26c]\), \( R_0 > 0 \). Additionally, \([26b]\) indicates that \( \alpha \geq 0 \), but in conjunction with \([26c]\) we may conclude that \( X_0 \neq 0 \) is accompanied by \( \alpha > 0 \), hence we have to consider \( \alpha > 0 \) only.

   Then we have to solve
   \[
   \cos \left( \varphi_{l_2} - 2\beta l_2 - \arctan \left( \frac{\alpha}{\beta} \right) \right) = -\frac{R_0 \alpha |r_\ell| \, e^{-2\alpha l_2}}{X_0 |\gamma|}.
   \]
   Due to periodicity and symmetry of the cosine function and the applicable range of the arc cosine function (which is \([0, \pi]\)) we obtain
   \[
   \pm \left( \varphi_{l_2} - 2\beta l_{2,m,\pm} - \arctan \left( \frac{\alpha}{\beta} \right) + 2\pi m \right) = \arccos \left( -\frac{R_0 \alpha |r_\ell| \, e^{-2\alpha l_{2,m,\pm}}}{X_0 |\gamma|} \right),
   \]
   where \( m \in \mathbb{Z} \).

   Numerical solution\(^5\) of
   \[
   l_{2,m,\pm} = \frac{\varphi_{l_2} - \arctan \left( \frac{\alpha}{\beta} \right) + \arccos \left( -\frac{R_0 \alpha |r_\ell| \, e^{-2\alpha l_{2,m,\pm}}}{X_0 |\gamma|} \right) + 2\pi m}{2\beta}
   \]
   yields the line length \( l_{2,m,\mp} \) of each extremum, where \( l_{2,m,\pm} \in [0, l_{\text{tot}}] \).

\(^5\) A (positive) real solution requires \( -\frac{R_0 \alpha |r_\ell| \, e^{-2\alpha l_{2,m,\pm}}}{X_0 |\gamma|} \leq 1 \) to hold. Since \( l_2 \leq l_{\text{tot}} \), at least one extremum might exist only if \( l_{\text{tot}} \geq \frac{1}{2\alpha} \ln \left( \frac{R_0 \alpha |r_\ell|}{X_0 |\gamma|} \right) \). Then the initial numerical value is given by \( \text{Max} \left\{ 0, \frac{1}{2\alpha} \ln \left( \frac{R_0 \alpha |r_\ell|}{X_0 |\gamma|} \right) \right\} \).
If the sign of \( R_0 \alpha |z| \frac{2}{\pi} e^{-2\alpha l_2} + X_0 |z| \) changes at \( l_{2,m,\pm} \) from \(-\) to \(+\) or from \(+\) to \(-\), a maximum or minimum exists at \( l_{2,m,\pm} \) (the product of the remaining terms of \( \frac{d(e^{2\alpha l_2 g_{tot, ii, inna}})}{dl_2} \) is always \(<\) 0 as indicated above). Additionally, the sign of this term at \( l_2 = 0 \) is important, because it indicates if the first extremum is a maximum or a minimum, provided an extremum exists within the range \([0, l_{tot}]\). Alternatively we may plot \( e^{2\alpha l_2 g_{tot, ii, inna}} \) as a function of \( l_2 \), which also helps to choose good initial numerical values for the numerical solution.

However, the goal is to derive the \( l_2 \in [0, l_{tot}] \) which yields the highest \( e^{2\alpha l_2 g_{tot, ii, inna}} \). Hence, if one or more maxima exist within the range \([0, l_{tot}]\), those (local) maxima have to be compared to the values at \( l_2 = 0 \) and \( l_2 = l_{tot} \). Then the highest value\(^{6}\) belongs to the optimum length of \( l_2 \). Conversely, if there is no maximum within the range \([0, l_{tot}]\), we just have to compare the values at \( l_2 = 0 \) and \( l_2 = l_{tot} \) to determine the optimum length of \( l_2 \).

If the line’s complex characteristic impedance is given by \( Z_0 \approx Z_{0ii} (1 - j \frac{\alpha}{\beta}) \) according to (2.4), we have to solve

\[
\begin{align*}
 l_{2,m,\pm} &\approx \frac{\varphi_{\pm} - \arctan \left( \frac{\alpha}{\beta} \right) \mp \arccos \left( \frac{\beta |z| e^{-2\alpha l_2, m, \pm}}{|z|} \right) - 2\pi m}{2\beta} \\
 l_{2,m,\pm} &\approx \frac{\varphi_{\pm} - \frac{\alpha}{\beta} \mp \arccos \left( \left( 1 - \frac{\alpha^2}{2\beta^2} \right) |z| e^{-2\alpha l_2, m, \pm} \right) + 2\pi m}{2\beta}
\end{align*}
\]

numerically\(^{7}\) for \( l_{2,m,\pm} \in [0, l_{tot}] \).

Thus the small imaginary part of \( Z_0 \) causes an optimum \( l_{2,m,\pm} > 0 \). Taking into account “real world” coaxial cable data in our frequency range of interest yields maximum values of \( e^{2\alpha l_2 g_{tot, ii, inna}} \) which are just some percent (max. 2% at 1.8 MHz, max. 0.5% at 30 MHz for a RG-213/U) higher\(^{6}\) than 1, the value at \( l_2 = 0 \), but it depends on \( l_{tot} \) if \( l_2 \) may be chosen accordingly. If not, the maximum possible increase of power gain may be significantly lower. It’s interesting to note that increasing of \( l_{tot} \) to enable a choice of \( l_2 = l_{2,m,\pm} \) for maxi-

---

\(^{6}\) Although \( e^{2\alpha l_2 g_{tot, ii, inna}} > 1 \) is possible (\( e^{2\alpha l_2 g_{tot, ii, inna}} = 1 \) at \( l_2 = 0 \)), the factor \( e^{-2\alpha l_2} \) causes the power gain \( g_{tot, ii, inna} \) to be always \( \leq 1 \).

\(^{7}\) As indicated in Footnote \(^{5}\), a (positive) real solution requires \( l_{tot} \geq \frac{1}{\alpha} \ln \left( \frac{\beta |z|}{|z|} \right) \) to hold, because \( \left( \frac{\beta |z| e^{-2\alpha l_2, m, \pm}}{|z|} \right)_{\max} > 1 \) is possible according to Footnote \(^{3}\) of Appendix D.2. Then the initial numerical value is given by Max\( \{0, \frac{1}{\alpha} \ln \left( \frac{\beta |z|}{|z|} \right)\} \).
um $e^{2\lambda_{\text{tot}}, \Pi, \text{inma}} > 1$ is not useful because $g_{\text{tot}, \Pi, \text{inma}}$ decreases if $l_{\text{tot}}$ increases\textsuperscript{8}.

Conversely, if the imaginary part of $Z_0$ would be more significant, hence $\frac{|X_{0}|}{R_0} \ll 1$ (but $\frac{|X_{0}|}{R_0} < 1$ according to [26c]), the maximum value of $e^{2\lambda_{\text{tot}}, \Pi, \text{inma}}$ could be some ten percent higher\textsuperscript{6} than 1. However, as indicated above and in Section 2.4, coaxial cables in our frequency range of interest do not involve such high imaginary parts of $Z_0$.

### 5.4.3 Lossy matching network

#### 5.4.3.1 $\pi$ network

According to Section 5.4.1, the $\pi$ network is used to transform its load admittance (described by its reflection coefficient $\mathcal{L}_{\text{MB}} = \mathcal{L}_\ell e^{-2\alpha L e^{-2\beta L}}$) in the line’s complex characteristic impedance $Z_0$ at its input (hence $\mathcal{L}_{\text{MB, inma}} = \mathcal{L}_{\pi, \text{inma}} = 0$).

To design such a $\pi$ network, real and imaginary part of its load impedance have to be known which might be computed based on the load’s reflection coefficient $\mathcal{L}_{\text{MB}}$ either separately using (2.14) and (2.15) or together using the reciprocal of (2.10).

Performing the former yields

$$R_{\ell, \text{MB}} = \frac{1}{G_{\ell, \text{MB}}} = \frac{|Z_0|^2 |1+\mathcal{L}_\ell e^{-2\alpha L e^{-2\beta L}}|^2}{R_0 (1-e^{-4\alpha L |\mathcal{L}_\ell|^2}) - 2X_0 e^{-2\alpha L} \text{Im}(\mathcal{L}_\ell e^{-2\beta L})^2},$$

$$C_{\ell, \text{MB}} = \begin{cases} \frac{B_{\ell, \text{MB}}}{\omega} = \frac{2R_0 e^{-2\alpha L} \text{Im}(\mathcal{L}_\ell e^{-2\beta L}) + X_0 (1-e^{-4\alpha L |\mathcal{L}_\ell|^2})}{\omega |Z_0|^2 (1+\mathcal{L}_\ell e^{-2\alpha L})^2} & \omega B_{\ell, \text{MB}} = \frac{|Z_0|^2 |1+\mathcal{L}_\ell e^{-2\alpha L e^{-2\beta L}}|^2}{2R_0 e^{-2\alpha L} \text{Im}(\mathcal{L}_\ell e^{-2\beta L}) + X_0 (1-e^{-4\alpha L |\mathcal{L}_\ell|^2})^2} \end{cases}.$$  

Depending on the sign of the load admittance’s imaginary part we then apply the optimisation given in Section 4.3.1.1 (if $C_{\ell, \text{MB}} \geq 0$) or Section 4.3.2.1 (if $L_{\ell, \text{MB}} > 0$) and the appropriate inclusion of minimum/maximum element values in Section 4.5.1. However, due to the line’s complex characteristic impedance, they might have to be extended to complex source admittances.

To check the impact of source admittance transformation by the line on network design, in particular the necessity of extending the optimisation to complex source admittances, we have to consider the achieved susceptances if the source impedance is chosen from the three “useful” cases — the line’s characteristic impedance, its complex conjugate, or its real part — and low loss coaxial cables are used.

However, calculation of those susceptances is quite lengthy and was moved to the appendix to enhance readability. The minimum/maximum (equivalent) capacitances

\textsuperscript{8} Depending on $l_{\text{tot}}$, choosing of optimum lengths involves intervals where either $l_1$ is constant and $l_2$ increases if $l_{\text{tot}}$ increases or $l_2$ is constant and $l_1$ increases if $l_{\text{tot}}$ increases. Since $g_{\text{tot}, \Pi, \text{inma}}$ is continuous at the intervals’ limits, $g_{\text{MB, inma}} = 1 = \text{const.}$, and $g_{\Pi, \text{inma}}$ decreases if $l_{1,2}$ increases (as indicated in Footnote 1), $g_{\text{tot, inma}}$ decreases if $l_{\text{tot}}$ increases.
or inductances obtained from transformation of the three possible source impedances considered herein are derived in Appendix D.3.1.1. The resulting formulae were used to compute the values below.

In our frequency range of interest, coaxial cables involve $\frac{1}{j} \ll 1$. Thus the real part of the source admittance is only slightly changed by line transformation. Regarding its imaginary part, we have to compute its actual values using low loss cable data to determine its impact on optimised network design derived in Sections 4.3.1.1 and 4.3.2.1.

Performing those calculations for low loss coaxial cables (RG-213/U or better) yields:

1. If the source impedance is equal to the line’s characteristic impedance, its (transformed) admittance at the matching network’s input is equal to the reciprocal of the line’s characteristic impedance which exhibits a parallel capacitance of about $30-0.47\,\text{pF}$ at $1.8-30\,\text{MHz}$.

2. If the source impedance is equal to the complex conjugate of the line’s characteristic impedance, its transformed admittance at the matching network’s input exhibits a maximum parallel capacitance of about $91-1.4\,\text{pF}$ or a minimum parallel inductance of about $260-59\,\mu\text{H}$ at $1.8-30\,\text{MHz}$, respectively. Compensation of those inductances would require a parallel capacitance greater or equal than $30-0.48\,\text{pF}$.

3. If the source impedance is equal to the real part of the line’s characteristic impedance, its transformed admittance at the matching network’s input exhibits a maximum parallel capacitance of about $60-0.95\,\text{pF}$ or a minimum parallel inductance\(^9\) of about $7.1-24\,\text{H}$ at $1.8-30\,\text{MHz}$, respectively. Compensation of those inductances would require a parallel capacitance greater or equal than $1.1\,\text{fF}-1.2\,\text{aF}$.

For any source impedance considered above, the maximum parallel capacitance of the transformed source admittance or the parallel capacitance required to compensate the minimum inductance of the transformed source admittance does not exceed (typical) minimum values of variable capacitances and/or (typical) values of parasitic capacitances if the matching frequency is higher than $10\,\text{MHz}$.

But even at lower frequencies, the mismatch introduced by the parallel susceptance is low (in theory, matching is "perfect" in case 1, better than $35\,\text{dB}$ in case 2, and better than $40\,\text{dB}$ in case 3). Additionally, if we take into account that parasitics change the values of the network’s elements and, if a transmission line connects the network’s output to a load, characteristic impedance discontinuities introduced by connectors, cable bending, etc. may change the (calculated) load of the network,

\(^9\) Exact calculation considering the spiral reveals that even for a low loss RG-213/U, the transformed susceptance stays entirely capacitive. At $1.8/10/30\,\text{MHz}$, the maximum parallel capacitance of $59/4.6/0.95\,\text{pF}$ is achieved for a line length of about $27.7/4.97/1.65\,\text{m}$. Further maxima with decreasing maximum parallel capacitances occur at odd multiples of those lengths.
it's always recommendable to tune the designed \(\pi\) network, e.g. by measuring its input reflection coefficient using a directional coupler. However, tuning involves measurement uncertainties which are higher than the mismatch indicated above.

In particular, in our application the matching network is designed or tuned to match \(Z_{0\|} \approx R_0\) at its input, which is further transformed to the line's input. The resulting admittance at the line's input is similar to that derived in case 3 (a line is a reciprocal "network") — in theory the match at the source would be better than 40 dB — of course it's worse in practice.

Thus we may conclude that the \(\pi\) network may be designed to match \(Z_{0\|} \approx R_0\) at its input — the "error" involved is low.

Which power gain is obtained then?

If the \(\pi\) network is designed (or tuned) to match (exactly) the reciprocal of the line's characteristic impedance at its input, \(\mathcal{Z}_{1\text{MB},\text{inma}} = \mathcal{Z}_{1\pi,\text{inma}} = 0\) holds and the \(\pi\) network's power gain is given by \(g_{\text{MB, inma}} = g_{\pi, \text{inma}}\) according to (3.24). Hence (5.5) gives

\[
g_{\text{tot}, \pi, \text{inma}} = e^{-2\alpha L_{\text{tot}}} \frac{1}{1 + \frac{R_0 \left| e^{-\alpha L_{\text{tot}}} \right|}{R_m}} \frac{\frac{R_0 \left(1 - \left| e^{-\alpha L_{\text{tot}}} \right|^2\right) - 2X_0 \text{Im}\left(e^{-\alpha L_{\text{tot}}} \right)}{R_0 \left(1 - e^{-4\alpha L_{\text{tot}}} \left| e^{-\alpha L_{\text{tot}}} \right|^2\right) - 2X_0 e^{-4\alpha L_{\text{tot}}} \text{Im}\left(e^{-2\alpha L_{\text{tot}}} \right)}}{},
\]

(5.9)

where, as indicated in Sections 4.3.1.1 and 4.3.2.1, \(R_m\) is restricted by \(R_{\ell,\text{MB}}\) and \(C_{\ell,\text{MB}}\) or \(L_{\ell,\text{MB}}\), respectively\(^{10}\), or by the minimum/maximum values of the network's elements according to Section 4.5.1.

If the \(\pi\) network is designed to match \(Z_{0\|} \approx R_0\) at its input, (3.24) is still applicable\(^{11}\) (a network's power gain depends on its elements' settings and its load, but not on its source), but \(R_i\) used for design is (slightly) different from the inverse of the real part of the exact admittance to match and the susceptance of the exact admittance to match is neglected. Then, of course, \(\mathcal{Z}_{1\text{MB},\text{inma}} = \mathcal{Z}_{1\pi,\text{inma}} \neq 0\). Thus exact calculation of the total power gain would involve the load related "correction

\(^{10}\) As exact matching involves an extension to complex sources, \(R_m\) may be restricted further and optimisation may be different. For example, matching the line's characteristic impedance (case 1) involves either a total capacitance \(C_{\text{tot}}\) replacing \(C_1\) in all \(\pi\) network design formulae, then, using the definitions of Appendix D.3.1.1, \(C_1 = C_{\text{op}} + C_{\text{tot}} = C_{\text{op}} + \frac{1}{\omega R_{\text{op}}} \sqrt{\frac{R_{\text{op}} - R_m - R_s}{R_m + R_s}}\) (comparable to case 1 in Section 4.3.2.1), or a total inductance \(L_{\text{tot}}\), then \(C_1 = C_{\text{op}} - \frac{1}{\omega L_{\text{tot}}} \sqrt{\frac{R_{\text{op}} - R_m - R_s}{R_m + R_s}}\) and in (3.19) the minus in parentheses applies (comparable to case 2 in Section 4.3.2.1).

The former does not change the range of \(R_m\) or the optimisation (because the minimum capacitance of the variable capacitor is not considered). The latter is not applicable in conjunction with case 2 in Section 4.3.2.1 (transformation of a resistor by three inductors is never capacitive), restricts \(R_m\) further by \(R_m + R_s \geq \frac{R_{\text{op}}}{1 + \frac{1}{\omega L_{\text{tot}}}}\), changes optimisation (comparable to case 2 in Section 4.3.2.1, but in the final checks for numerator and denominator in the case "at least one \(a_n > 0\)" the first term is subtracted from the second term due to different signs in \(\omega L_n\) and its derivatives), requires explicit knowledge of \(R_s\) and may be applicable only in a certain range of \(R_{\ell,\text{MB}}\).

\(^{11}\) It should be noted that the index "inma" is still valid in (3.24) because it indicates that the \(\pi\) network is designed to match a certain impedance (which has not necessarily to be equal to the line's characteristic impedance).
factor$^*$ of line 1, whose value is indicated at the end of Section 5.4.1. However, it's close to 1 due to the small mismatch. Hence the equation for $g_{\text{tot}, \pi, \text{inma}}$ given above is (approximately) applicable and $\pi$ network design is simplified because the susceptance of the exact admittance to match to is not included (the network's input admittance has to resemble a conductance instead of the complex conjugate of an admittance).

Additionally, if we would like to tune the network, e.g. by inserting a directional coupler, the optimisation remains (approximately) unchanged — if the network is tuned, the coupler is (approximately) matched at its input and output. Thus it just introduces an additional, (approximately) constant loss factor.

Unfortunately, there is no common optimisation for the power gain due to several reasons. Firstly, a load exhibiting a capacitive imaginary part may be transformed by line 2 in a load exhibiting an inductive imaginary part and vice versa. Secondly, optimisation involves three concurrent objectives — the optimum load resistance (derived for resistive loads in Section 3.5.2.3) which is equal to $\frac{1}{R_{\text{L,MB}}} - R_{60} \approx Z_{60} - R_{60} \approx Z_{0} - Z_{0}$ in all three cases considered above ($R_{60} \ll Z_{60}$ for useful variable inductors), the restrictions on the valid range of parameter $R_{\text{m}}$ (given in Sections 4.3.1.1 and 4.3.2.1) which depend on the transformed load admittance, hence on $l_2$, and optimum length $l_2$ for minimum line losses (similar to Section 5.4.2, but in conjunction with lossy matching networks). Additionally, both objectives depending on $l_2$ depend on frequency, too.

Thus optimum length $l_2$ and its according optimum $\pi$ network design usually apply only to a given total line length, network characteristics (in particular, explicitly known losses), load admittance, and frequency. Unlike lossless $\pi$ networks, increasing of the total line length may be recommendable for lossy $\pi$ networks if the decrease of the network's losses exceeds the increase of the line's losses. It's interesting to note that the maximum possible loss reduction increases if the inductor's $Q$ factor decreases.

Calculations always require usage of the line's complex characteristic impedance with two exceptions — the matched line's power gain may be computed using the real characteristic impedance approximation and the $\pi$ network may be designed to match the real characteristic impedance.

### 5.4.3.2 T network

According to Section 5.4.1, the T network is used to transform its load impedance (described by its reflection coefficient $Z_{\text{MB}} = Z_{\text{L}} e^{-2a l_2} e^{-j2\beta l_2}$) in the line's complex characteristic impedance $Z_0$ at its input (hence $Z_{\text{L,MB, inma}} = Z_{\text{L, inma}} = 0$).

To design such a T network, real and imaginary part of its load impedance have to be known which might be computed based on the load's reflection coefficient $Z_{\text{MB}}$ either separately using (2.11) and (2.12) or together using (2.10).
Performing the former yields

\[
G_{E, MB} = \frac{1}{R_{E, MB}} = \frac{|1 - e^{-2\alpha_2} e^{-j2\beta_2}|^2}{R_0 (1 - e^{-4\alpha_2} |\xi_{E, MB}|^2) - 2X_0 e^{-2\alpha_2} \text{Im}(\xi_{E, MB} e^{-j2\beta_2})},
\]

\[
C_{E, MB} = \begin{cases} 
-\frac{1}{\omega X_{E, MB}} = -\frac{|1 - e^{-2\alpha_2} e^{-j2\beta_2}|^2}{\omega (2R_0 e^{-2\alpha_2} \text{Im}(\xi_{E, MB} e^{-j2\beta_2}) + X_0 (1 - e^{-4\alpha_2} |\xi_{E, MB}|^2))}, \\
\frac{X_{E, MB}}{\omega} = \frac{2R_0 e^{-2\alpha_2} \text{Im}(\xi_{E, MB} e^{-j2\beta_2}) + X_0 (1 - e^{-4\alpha_2} |\xi_{E, MB}|^2)}{\omega |1 - e^{-2\alpha_2} e^{-j2\beta_2}|^2}
\end{cases}
\]

Depending on the sign of the load impedance’s imaginary part we then apply the optimisation given in Section 4.3.1.2 (if \(C_{E, MB} \geq 0\)) or Section 4.3.2.2 (if \(L_{E, MB} > 0\)) and the appropriate inclusion of minimum/maximum element values in Section 4.5.2. However, due to the line’s complex characteristic impedance, they might have to be extended to complex source admittances.

To check the impact of source impedance transformation by the line on network design, in particular the necessity of extending the optimisation to complex source impedances, we have to consider the achieved reactances if the source impedance is chosen from the three “useful” cases — the line’s characteristic impedance, its complex conjugate, or its real part — and low loss coaxial cables are used.

However, calculation of those reactances is quite lengthy and was moved to the appendix to enhance readability. The minimum/maximum (equivalent) capacitances or inductances obtained from transformation of the three possible source impedances considered herein are derived in Appendix D.3.1.2. The resulting formulae were used to compute the values below.

In our frequency range of interest, coaxial cables involve \(\frac{\eta}{\omega} \ll 1\). Thus the real part of the source impedance is only slightly changed by line transformation. Regarding its imaginary part, we have to compute its actual values using low loss cable data to determine its impact on optimised network design derived in Sections 4.3.1.2 and 4.3.2.2.

Performing those calculations for low loss coaxial cables (RG-213/U or better) yields:

1. If the source impedance is equal to the line’s characteristic impedance, its (transformed) impedance at the matching network’s input is equal to the line’s characteristic impedance which exhibits a series capacitance of about 100 – 24 nF at 1.8 – 30 MHz.

2. If the source impedance is equal to the complex conjugate of the line’s characteristic impedance, its transformed impedance at the matching network’s input exhibits a minimum series capacitance of about 34 – 7.9 nF or a maximum series inductance of about 76 – 1.2 nH at 1.8 – 30 MHz, respectively. Compensation of those inductances would require a series capacitance lower or equal than 100 – 24 nF.
3. If the source impedance is equal to the real part of the line’s characteristic impedance, its transformed impedance at the matching network’s input exhibits a minimum series capacitance of about 52–12 nF or a maximum series inductance of about 2.8 pF – 3.0 FH at 1.8–30 MHz, respectively. Compensation of those inductances would require a series capacitance lower or equal than 2.8–9.5 mF.

For any source impedance considered above, the minimum series capacitance of the transformed source impedance or the series capacitance required to compensate the maximum inductance of the transformed source impedance exceeds (typical) maximum values of variable capacitances by at least one order of magnitude (typical variable capacitors have a maximum capacitance of about 500 pF to 1 nF, although vacuum variables are available up to 5 nF, but those high values are usually required only at the lower end of our frequency range of interest).

Anyhow, the mismatch introduced by the series reactance is low (in theory, matching is “perfect” in case 1, better than 35 dB in case 2, and better than 40 dB in case 3). Additionally, if we take into account that parasitics change the values of the network’s elements and, if a transmission line connects the network’s output to a load, characteristic impedance discontinuities introduced by connectors, cable bending, etc. may change the (calculated) load of the network, it’s always recommendable to tune the designed T network, e.g. by measuring its input reflection coefficient using a directional coupler. However, tuning involves measurement uncertainties which are higher than the mismatch indicated above.

In particular, in our application the matching network is designed or tuned to match \( Z_{0||} \approx R_0 \) at its input, which is further transformed to the line’s input. The resulting impedance at the line’s input is similar to that derived in case 3 (a line is a reciprocal “network”) — in theory the match at the source would be better than 40 dB — of course it’s worse in practice.

Thus we may conclude that the T network may be designed to match \( Z_{0||} \approx R_0 \) at its input — the “error” involved is low.

Which power gain is obtained then?

If the T network is designed (or tuned) to match (exactly) the line’s characteristic impedance at its input, \( Z_{1\text{MB, inna}} = Z_{1t, \text{inna}} = 0 \) holds and the T network’s power gain is given by \( g_{\text{MB, inna}} = g_{t, \text{inna}} \) according to (3.37). Hence (5.5) gives

\[
g_{\text{tot, t, inna}} = e^{-2\alpha_{\text{tot}} \frac{1}{1 + \frac{G_{\ell, M1}}{G_m}}} \cdot \frac{R_0(1-|\ell|^2) - 2\alpha_0 \Im\{\ell\}}{R_0(1-\ell^2 e^{-4\alpha_1}) - 2\alpha_0 e^{-2\alpha_1} \Im\{\ell \ell^* e^{-2\alpha_1}\}},
\]

where, as indicated in Sections 4.3.1.2 and 4.3.2.2, \( G_m \) is restricted by \( G_{\ell, \text{MB}} \) and

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12 Exact calculation considering the spiral reveals that even for a low loss RG-213/U, the transformed reactance stays entirely capacitive. At 1.8/10/30 MHz, the minimum series capacitance of 53/22/12 nF is achieved for a line length of about 27.1/4.92/1.64 m. Further minima with increasing minimum series capacitances occur at odd multiples of those lengths.
where $C_{th,MB}$ or $L_{th,MB}$, respectively, or by the minimum/maximum values of the network’s elements according to Section 4.5.2.

If the T network is designed to match $Z_{00} \approx R_0$ at its input, (3.37) is still applicable (a network’s power gain depends on its elements’ settings and its load, but not on its source), but $G_1$ used for design is (slightly) different from the inverse of the real part of the exact impedance to match and the reactance of the exact impedance to match is neglected. Then, of course, $L_{1MB,imna} = L_{1t,imna} \neq 0$. Thus exact calculation of the total power gain would involve the load related “correction factor” of line 1, whose value is indicated at the end of Section 5.4.1. However, it’s close to 1 due to the small mismatch. Hence the equation for $g_{tot,t,imna}$ given above is (approximately) applicable and T network design is simplified because the reactance of the exact impedance to match is not included (the network’s input impedance has to resemble a resistance instead of the complex conjugate of an impedance).

Additionally, if we would like to tune the network, e.g. by inserting a directional coupler, the optimisation remains (approximately) unchanged — if the network is tuned, the coupler is (approximately) matched at its input and output. Thus it just introduces an additional, (approximately) constant loss factor.

Unfortunately, there is no common optimisation for the power gain due to several reasons. Firstly, a load exhibiting a capacitive imaginary part may be transformed by line 2 in a load exhibiting an inductive imaginary part and vice versa. Secondly, optimisation involves three concurrent objectives — the optimum load conductance (derived for resistive loads in Section 3.5.3.3) which is equal to $\frac{1}{\text{Re} (Z_{1MB})} = \frac{1}{R_0} \approx \frac{1}{Z_{00}}$ in all three cases considered above, the restrictions on the valid range of parameter $G_m$ (given in Sections 4.3.1.2 and 4.3.2.2) which depend on the transformed load impedance, hence on $l_2$, and optimum length $l_2$ for minimum line losses (similar to Section 5.4.2, but in conjunction with lossy matching networks). Additionally, both objectives depending on $l_2$ depend on frequency, too.

As exact matching involves an extension to complex sources, $G_m$ may be restricted further and optimisation may be different. For example, matching the line’s characteristic impedance (case 1) involves either a total capacitance $C_{tot}$ replacing $C_1$ in all T network design formulae, then, using the definitions of Appendix D.3.1.2, $C_1 = \frac{1}{C_0 + C_{tot}} = \frac{1}{C_0 + C_0 - \frac{C_0 - C_m - C_E}{C_0 + C_m + C_E}}$ (comparable to case 1 in Section 4.3.2.2), or a total inductance $L_{tot}$, then $C_1 = \frac{1}{L_0 + L_{tot}} = \frac{1}{L_0 + L_0 \cdot \frac{C_0 - C_m - C_E}{C_0 + C_m + C_E}}$ and in (3.32) the minus in parentheses applies (comparable to case 2 in Section 4.3.2.2).

The former does not change the range of $G_m$ or the optimisation (because the maximum capacitance of the variable capacitor is not considered). The latter is not applicable in conjunction with case 2 in Section 4.3.2.2 (transformation of a resistor by three inductors is never capacitive), restricts $G_m$ further by $G_m + G_p \geq \frac{C_0}{1 + \frac{\omega L_p}{C_0}}$, changes optimisation (comparable to case 2 in Section 4.3.2.2 with similar final checks for numerator and denominator in the case “at least one $a_n > 0$” although signs are different in $\frac{1}{\omega L_p}$ and its derivatives), requires explicit knowledge of $G_p$, and may be applicable only in a certain range of $G_{th,MB}$.

It should be noted that the index “imna” is still valid in (3.37) because it indicates that the T network is designed to match a certain impedance (which has not necessarily to be equal to the line’s characteristic impedance).
Thus optimum length $l_2$ and its according optimum T network design usually apply only to a given total line length, network characteristics (in particular, explicitly known losses), load impedance, and frequency. Unlike lossless T networks, increasing of the total line length may be recommendable for lossy T networks if the decrease of the network’s losses exceeds the increase of the line’s losses. It’s interesting to note that the maximum possible loss reduction increases if the inductor’s $Q$ factor decreases.

Calculations always require usage of the line’s complex characteristic impedance with two exceptions — the matched line’s power gain may be computed using the real characteristic impedance approximation and the T network may be designed to match the real characteristic impedance.

### 5.4.3.3 Optimum compromise

Summarising the preceding sections, we may conclude that optimisation usually applies only to a given total line length, network characteristics, load impedance, and frequency. Calculations always require usage of the line’s complex characteristic impedance with two exceptions — the matched line’s power gain may also be computed using the real characteristic impedance approximation and the network may be designed to match the real characteristic impedance. Although some parts of the optimisation procedure may yield values close to those of a real characteristic impedance, we should keep in mind that the error in power gain calculation may be high, in particular for the mismatched line to the load (an example is given in Figure 2.2), and some issues (as the impact of source impedance transformation) cannot be addressed properly by a real characteristic impedance.

Additionally, optimisation may involve increasing of the total line length if the decrease of the network’s losses exceeds the increase of the line’s losses. It’s interesting to note that the maximum possible loss reduction increases if the inductor’s $Q$ factor decreases.

Due to the optimisation’s dependencies, an optimisation is recommendable only in applications where frequency and load are (approximately) fixed.

Nevertheless, if a matching network is used for varying loads and/or varying frequencies, it’s always a good compromise to put the π or T network as close as possible to the load and set the network’s elements according to one of the matching strategies given in Section 4.5.3.

The former ensures low, but not minimal line losses. Depending on the matching strategy used, the latter ensures maximum or at least high power gain of the network depending on matching strategy and effort involved.

The simplest matching strategy would be to set $C_2$ to the lowest value where matching is possible in a π network or to the highest value where matching is possible in a T network. Alternatively an L network switchable to realise both complementary types could be used. These simple approaches will require measuring the magnitude of the network’s input reflection coefficient only and do not guarantee high power
gains for certain loads, but are a good choice for the majority of loads if the variable inductor's $Q$ factor is sufficiently high (which holds for "well designed" variable air-core inductors).

However, setting $\pi$ or $T$ networks' elements for highest power gain usually requires explicit knowledge of the elements' values, the matching frequency, and measuring magnitude and phase of the network's input reflection coefficient — the appropriate optimum matching strategy is described in Section 4.5.3.
Chapter 6

Experimental Validation of the Network’s Matching Strategy

6.1 Introduction

In the following chapter, the experimental setup designed to test the network’s matching strategy is described. The performed measurements are explained and results are given validating the proposed matching strategy for optimum power gain of the network.

6.2 Experimental setup

Regarding the experimental setup, several different considerations have to be taken into account.

Firstly, all measurements should be repeatable. Thus it’s important to reduce the influence of the (possibly changing) environment and keep all matching elements within a metal housing (although generally shielding would not be required for matching networks). In addition, the motors used to turn the elements need to be capable of setting the elements’ positions with sufficient accuracy. Hence a suitable combination of encoder resolution and gearbox reduction is essential.

Secondly, in order to reduce unwanted losses introduced by parasitics, it’s important to obey some general network design rules. The distance between the inductor’s windings and any metal part (such as walls or the adjacent capacitors) has to be at least five times of the wire’s width and that between an end of the coil and the adjacent metal wall at least half of the coil’s diameter, or, even better, equal to the coil’s diameter. This ensures that the magnetic fields are low enough that the eddy-currents induced in any surrounding metal part does not increase the inductor’s losses significantly. Furthermore, the parasitic capacitances are kept low, too. The latter also holds for the capacitors if they are mounted not too close to the metal walls. Additionally, the cross-section of the “wires” connecting plugs, capacitors,
and the inductor has to be high enough to reduce their parasitic resistance and inductance.

Thirdly, the components themselves and the connectors have to be of good quality. In particular, it's advantageous to use vacuum variable capacitors. They offer a high ratio of maximum to minimum capacitance and introduce low losses. Regarding the variable inductance, a high-power type should be chosen. Furthermore, 7 mm connectors yield the highest measurement accuracy of the vector network analyser obtainable in the frequency range of interest.

Finally, the experimental setup should be convertible to both network types (π and T network) for further measurements.

![Figure 6.1: Matching compartment, T network built with elements of high values](image_url)

Figure 6.1 shows one possible realisation of a T network. It consists of two Jennings Technology CMV1-4000-0005 vacuum variable capacitors (25–4000 pF) and one (IRV) RI-40 variable inductor (0.1 – 40 μH) from Surplus Sales of Nebraska (made of wire of rectangular cross-section). As indicated above, the network is surrounded by a metal housing made of brass.

Note that the capacitors are connected to the plug on the right side, mounted on Teflon holders, and that the wire which makes the ground connection of the variable inductor is not visible from the chosen point of view.

The distance from the inductor's windings to the adjacent variable capacitors and the surrounding walls is about 5 cm (about five times the wire's width of $l_{rad} \approx 0.92$ cm)
and its ends are at least 10 cm apart from the walls (equal to the coil’s mean diameter of \( d_C \approx 10.2 \text{ cm} \)).

The capacitors are mounted about 3 cm apart from the adjacent wall where the according plug is mounted. Since both capacitors (and the inductor) are mounted at an equal height of 10 cm, and the capacitor’s diameter is about 7.6 cm, their distance to the remaining walls is even higher.

![Figure 6.2: Matching compartment, \( \pi \) network built with elements of low values](image)

Figure 6.2 shows one possible realisation of a \( \pi \) network. It consists of two Jennings Technology CSV1-500-0005 vacuum variable capacitors (5–500 pF) and one (IRV) L32017 variable inductor (0.1–24 \( \mu \text{H} \)) from Surplus Sales of Nebraska (made of wire of rectangular cross-section). The latter contained several additional parts, e.g. a potentiometer for position detection, which were removed completely prior to mounting.

Note that the capacitors are connected to the plug on the left side, mounted on one Teflon and one brass holder, and that there is no ground connection of the variable inductor.

Since the diameter of the capacitors and the inductor is lower than those of the T network elements of Figure 6.1, all distances checked above are even higher herein.

In both networks of Figure 6.1 and Figure 6.2, insulating backlash-free Oldham couplings are used to connect the driving axes to the variable elements.
Figure 6.3: Motor, motor control, and power supply compartment

To set the network’s elements to the given values (positions), servo motors in conjunction with position encoders and gearboxes are employed. To prevent interference with the high frequencies of the network, they were mounted in a separate compartment together with the power supply and the start/end point switches (refer to Figure 6.3).

Figure 6.4: Gearbox, motor, position encoder, and motor control

As shown in Figure 6.4, each motor “unit” consists (from left to right) of a Maxon
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planetary gearhead GP42C (91:6 reduction), a Maxon RE36 servo motor (24 V), and an Agilent HEDL5540 position encoder (500 cycles/revolution). The motor is controlled by a Faulhaber MCDC2805 via a RS232 interface. Since the motor control uses edge detection, the resulting resolution is $4 \cdot 500 \text{ counts} \cdot 91/6 \approx 30333 \text{ counts per revolution}$ or $0.012^\circ$ at the output of the gearbox.

However, the gearbox has an average backlash (no load) of $0.4^\circ$ or 33.7 counts. Thus, if a small movement is intended, it has to be performed in two steps. At first, the motor control has to move back beyond the backlash (plus tolerances) and then, in a second step, forward towards the intended position.

Figure 6.5: a) Worm gear driving end point switch actuator and two end point switches, and b) a servopotentiometer

Furthermore, start and end point switches are needed, the former to detect the zero position with sufficient repeatability during power-up, and both after power-up to prevent the motor from moving beyond the element’s physical limits in case of programming errors. Those switches are Baumer Electric MY-COM G75N types offering a repeatability below $1 \mu m$ (since the actuator employs a worm gear with backlash, the total repeatability is worse, but still sufficiently well).

The switching itself is performed using a steel actuator driven by a worm gear (refer to Figure 5.5 a)). Different gears are employed to accommodate for the different total number of turns of the element to be driven. For example, the CMV1-4000-0005 vacuum variable capacitor has about 14 turns requiring a 20:1 worm gear/gear reduction compared to 32 turns of the (IRV) RI-40 variable inductor requiring a 40:1 worm gear/gear reduction. Note the different discrete positions available at the switch holders and the variable mounting of the switches which takes the different gear sizes into account.

Additionally, as shown in Figure 6.5 b), the worm gear drives a servopotentiometer, which will be used in the future to speed up the homing procedure during power-up.
6.3 Measurements performed to test and validate the matching algorithm for optimum power gain of the network

6.3.1 Experimental setup used

In the following measurements, the setup of Figure 6.1 — hence a T network — is used. Although it may seem disadvantageous with respect to the vulnerability of its series variable capacitors to parasitics, it’s mainly the variable inductor and its $Q$ factor we are interested in, and on this behalf it’s advantageous that one end of the variable inductor is connected to ground.

Furthermore, if the frequency used in the measurement is sufficiently low, nearly the whole range of the available inductance is employed. Thus a frequency of 1.8 MHz is used in all measurements, which, in addition, reduces the influence of any parasitics.

6.3.2 Calculation of the network elements (and the coil’s $Q$)

To calculate the values of the network elements (and the coil’s $Q$) at 1.8 MHz, the matching network’s ports were connected to the ports of an Agilent 8753E vector network analyser (VNA) (refer to Figure 6.6). The vector network analyser was set to a source power of -10 dBm, an IF bandwidth of 10 Hz, and a full 12-term adapter removal calibration was performed using an Agilent 85031B 7 mm CalKit and a 48 cm long precision cable with 7 mm connectors (a Micro-Coax UFA210B). The cable serves as an “adapter” resembling the distance of the two 7 mm connectors of the network — thus the bending of the precision cables connecting the network and the vector network analyser is similar during calibration and measurement.

In any measurement, $C_2$ was incremented and the basic matching algorithm tuned the network to a match of at least -66.8 dB (most times even below -75 dB), which is well below the measurement uncertainty of -55 dB. Hence the overall error is dominated by the measurement uncertainty and as low as possible for the given setup.
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If depicted, the error of any calculation given was derived applying sensitivity analysis. In case matched T network formulae are involved, it consists of two parts — the measurement error and the adjustment error (taking into account that a formula of the exactly matched T network was used although the real T network may deviate from the exact match due to measurement uncertainties).

![Graphs showing C1 and C2](image)

**Figure 6.7:** a) $C_1$ and b) $C_2$ of the equivalent T network, calculated from S-parameter measurements (solid line) and the single capacitances measured with a capacitance meter (dashed line).

Calculation of the equivalent T network's elements from S-parameters (applying admittance matrix to S-parameter matrix conversion) yields the capacitances $C_1$ and $C_2$ shown in Figure 6.7. Compared to the values of the single elements measured previously using an AADE L/C meter IIb, they are somewhat higher — this just resembles the parasitic series inductances of the wires connecting those capacitors to the 7 mm connectors and the central inductor.

![Graphs showing L_s and R_s](image)

**Figure 6.8:** a) $L_s$ of the equivalent T network, calculated from S-parameter measurements (solid line), the single inductance measured with an inductance meter (dashed line), and b) $R_s$ calculated from S-parameter measurements and matched T network formulae.

As indicated in Figure 6.8 a), the values of the equivalent inductance (calculated in a manner similar to that of the capacitors) and of the single inductor measured previously using an AADE L/C meter IIb differ in similar manner — this resembles the parasitic parallel capacitances, in particular between the windings of the inductor (the distance between the windings and the brass walls is relatively high).

However, since equivalent and measured values are quite close, the parasitics seem to be quite low for all elements of the network.
Regarding the accuracy of the calculated losses (and of the \( Q \) factor), the useful range of the measurement is limited.

\[
\begin{align*}
\text{a) } & \quad L_s [\mu \text{H}] \\
& \quad 0 \quad 0.00125 \quad 0.0025 \quad 0.00375 \quad 0.005 \\
& \quad 42 \quad 28 \quad 14 \quad 0 \\
\text{b) } & \quad \Delta L_s [\%] \\
& \quad 0 \quad 0.00125 \quad 0.0025 \quad 0.00375 \quad 0.005 \\
& \quad 7.5 \quad 5 \quad 2.5 \quad 0 
\end{align*}
\]

Figure 6.9: a) \( L_s \) and b) its percentage error when calculated from S-parameters

Fortunately the inductance required to match is approximately symmetrical to the middle (\( G_m \approx 0.01 \frac{1}{\Omega} \)) of the full (theoretical) matching range and does not change much towards the middle provided its \( Q \) factor is sufficiently high.

\[
\begin{align*}
\text{a) } & \quad Q \\
& \quad 0 \quad 0.00125 \quad 0.0025 \quad 0.00375 \quad 0.005 \\
& \quad 270 \quad 180 \quad 90 \quad 0 \\
\text{b) } & \quad \Delta Q [\%] \\
& \quad 0 \quad 0.00125 \quad 0.0025 \quad 0.00375 \quad 0.005 \\
& \quad 36 \quad 24 \quad 12 \quad 0 
\end{align*}
\]

Figure 6.10: a) \( Q \) factor of the inductor and b) its percentage error when calculated from S-parameters and matched T network formulae

Thus the lower quarter of the full range contains nearly all possible inductance values (Figure 6.9 a)) and yields a \( Q \) factor whose accuracy is reasonable (Figure 6.10 b)). (If the inductor's \( Q \) factor would be lower, the useful range would be higher.)

However, it should be noted that although the theoretical accuracy is poor, the values of the \( Q \) factor obtained to the right of the middle are similar to those obtained to the left as one would expect from symmetry!

Furthermore, the \( Q \) factor's magnitude is comparable to those given in [67] — although different coils were measured in this work.

From the calculated \( Q \) factors of Figure 6.10 a) and the previously computed \( L_s \) and \( L_p \) (which is not shown), the series resistance \( R_s \) is calculated and depicted in Figure 6.8 b). It behaves as expected.

Finally, as a cross-check, the calculated \( Q \) factors of Figure 6.10 a) were used to compute the values of the elements and \( g_{t,\text{inna}} \) of the T network based on piecewise constant \( Q \) factors between two measured points (equal to the arithmetic mean
of the calculated \(Q\) factors of these points) and the minimum/maximum element values.

![Graphs](image)

Figure 6.11: a) Comparison of \(g_{t,\text{inma}}\) calculated from S-parameters (dashed line) and \(g_{t,\text{inma}}\) calculated from the piecewise constant \(Q\) factors and minimum/maximum element values (solid line) for a load of 50\(\Omega\) and b) its percentage error when calculated from S-parameters.

The resulting \(g_{t,\text{inma}}\) is shown in Figure 6.11 a) and agrees well with the \(g_{t,\text{inma}}\) calculated from S-parameters. In addition, it behaves as expected — the optimum power gain is achieved for the highest possible \(G_m\).

### 6.3.3 Validation of the matching algorithm setting the network to optimum power gain

![Diagram](image)

Figure 6.12: Additional S-parameter measurement setup used to validate the matching algorithm setting the network to optimum power gain.

To validate the matching algorithm setting the network to optimum power gain, two measurements were performed at 1.8 MHz for each increment of \(C_2\).

Firstly, the vector network analyser was set to a source power of 0 dBm, an IF bandwidth of 10 Hz, and a 3-term reflection calibration was performed using an Agilent 85031B 7 mm CalKit. Then the network was terminated with an artificial load as depicted in Figure 6.12 and, as before, the basic matching algorithm was used to tune the network to a match better than -70 dB. After achieving the match, the amplitude of the voltage at the load was measured using a LeCroy WaveRunner 104MXi oscilloscope (10 GS/s, noise filter set to 11 bit resolution — limiting the bandwidth to 80 MHz — and averaging set to 256) with a ZS1000 active probe.
Secondly, the artificial load was replaced with a connection to port 2 of the vector network analyser (similar to Figure 6.6) and all S-parameters were measured using the settings and the according full 12-term adapter removal calibration described in Section 6.3.2.

The second S-parameter measurement enables calculation of the load including parasitics, of the network element values, and of the network’s power gain. Since the amplitude measurement’s accuracy is difficult to estimate and the absolute value of the input power is not known sufficiently well (±1 dB), the first amplitude measurement serves only as a cross-check for the measured power gain.

Two artificial loads were built — 82 Ω in series to 10 μH or 680 pF. However, due to parasitics (and depending on frequency) the actual resistance and reactance is different. Thus the actual values were calculated from the second S-parameter measurement and given in the captions of the figures below.

Figure 6.13: a) Amplitude measured with the oscilloscope and b) comparison of \( g_{t,\text{inma}} \) calculated from that amplitude (dashed line) and \( g_{t,\text{inma}} \) calculated from the piecewise constant \( Q \) factors and minimum/maximum element values (solid line) for a load of 91 Ω in series to 9.7 μH

As shown in Figure 6.13 a), in case of the inductive load there is a broad region where the change of the amplitude at the load is very low — it’s within 6.7 mV or 0.5% of the measured amplitude!

To compare the squared amplitudes with the \( g_{t,\text{inma}} \) calculated from piecewise constant \( Q \) factors of Figure 6.10 a) and minimum/maximum element values (as described above) in Figure 6.13 b), it was assumed that both power gains are equal at the highest \( G_m \) (where the load’s inductance is just compensated).

Note that the values whose \( g_{t,\text{inma}} \) is higher than that of the highest \( G_m \) belong to the inductive solution (undercompensation), whereas the lower ones belong to the capacitive solution (overcompensation).
Figure 6.14: a) Comparison of $g_{	ext{linma}}$ calculated from S-parameters (dashed line) and $g_{	ext{linma}}$ calculated from the piecewise constant $Q$ factors and minimum/maximum element values (solid line) for a load of 91 $\Omega$ in series to 9.7 $\mu$H and b) its percentage measurement error when calculated from S-parameters.

In addition, a similar comparison with $g_{\text{linma}}$ calculated from measured S-parameters is performed in Figure 6.14 a).

Both figures show that the calculation using piecewise constant $Q$ factors of Figure 6.10 a) agrees well with the measured values, hence the matching algorithm setting the network to optimum power gain using the best setting obtained by that calculation will work as expected.

Note that the measured values extend the calculated values at both line's ends — this is related to the minimum/maximum element values used in the calculation, which were derived from Figure 6.7 — however, parasitics of $C_2$ changed and Figure 6.7 does not contain the lowest possible values of $C_1$ and $C_2$.

Figure 6.15: a) Amplitude measured with the oscilloscope and b) comparison of $g_{\text{linma}}$ calculated from that amplitude (dashed line) and $g_{\text{linma}}$ calculated from the piecewise constant $Q$ factors and minimum/maximum element values (solid line) for a load of 84 $\Omega$ in series to 660 $\mu$F.

As shown in Figure 6.15 a), in case of the capacitive load the amplitude at the load continuously decreases.

To compare the squared amplitudes with the $g_{\text{linma}}$ calculated from piecewise constant $Q$ factors of Figure 6.10 a) and minimum/maximum element values (as described above) in Figure 6.15 b), it was assumed that both power gains are equal at the highest $G_m$. 
Figure 6.16: a) Comparison of $g_{t,\text{inma}}$ calculated from S-parameters (dashed line) and $g_{t,\text{inma}}$ calculated from the piecewise constant $Q$ factors and minimum/maximum element values (solid line) for a load of 84 $\Omega$ in series to 660 pF and b) its percentage measurement error when calculated from S-parameters.

In addition, a similar comparison with $g_{t,\text{inma}}$ calculated from measured S-parameters is performed in Figure 6.16 a).

Both figures show that the calculation using piecewise constant $Q$ factors of Figure 6.10 a) agrees well with the measured values. Furthermore, those calculations are not necessary in the capacitive case since, as expected, the highest power gain is achieved for the highest $G_m$ (which was also observed in measurements mentioned in [68]). Hence the matching algorithm setting the network to optimum power gain using the best setting obtained (or the highest value of $C_2$ yielding a match) will work as expected.

Note that once again the measured values extend the calculated values at both line's ends for the same reason than that given above.

### 6.4 Conclusion

In the measurements described above, the proposed matching strategy tuning the network to optimum power gain was validated.

This is not surprising since it’s always possible to replace any $Q$ factor by its piecewise constant approximation with sufficient accuracy provided the intervals are sufficiently small.

In addition, the measurements illustrate the simpler part of the matching strategy. For capacitive loads, the maximum power gain is achieved for the highest $G_m$ obtainable with the network’s elements. For inductive loads — provided the inductive maximum occurs — it’s flat enough due to the relatively high $Q$ factor of the variable inductor that the power gain at the highest $G_m$ (where compensation occurs) is very close to the maximum power gain and thus a good compromise.
Chapter 7

Conclusions and
Recommendations for Further
Work

7.1 Conclusions

Since the optimum matching strategy and matching system optimisation are already summarised in Sections 4.5.3 and 5.4.3.3, respectively, only a brief conclusion is given herein.

In this work, a matching strategy was derived that is suitable for automatic and continuous adjustment of L, π, and T matching networks for optimum active power transfer to their load while matching the network’s input impedance to a resistive source applicable in a frequency range of 1.8–30 MHz.

It is split in two parts — one (simple) strategy for purely resistive or capacitive complex loads which always applies — provided the network’s losses are properly described by the general loss model — and a more complicated strategy for inductive complex loads which involves real-time calculations and requires knowledge of the losses of the inductor and all network elements’ values.

Unlike other works existing in the literature, the approach of this work was proven analytically and not limited to constant Q factor central inductors. The model used was validated by comparison to models of several previous works which fit a large amount of measurements including the frequency range of interest.

Moreover, besides using the optimisation based on the model, it’s possible to perform an optimisation calculated from piecewise constant approximation of the inductor’s Q factor and known elements’ values similar to the optimisation derived for inductive complex loads. It never fails provided the network elements’ values and the inductor’s Q factor were measured in advance over the network elements’ ranges in sufficiently small steps.

The network design presented in this work can also be used to estimate the losses in the network’s low loss variable capacitors (see Appendices B.5.1 and B.5.2).
Furthermore, the influence of the transmission lines' lengths (which connect the matching network to source and load) on power gain of the matching system and an optimum split of the total line length was derived, where the transmission lines' characteristic impedance was complex.

However, an optimisation of the whole matching system is recommendable only in case of a fixed load. Varying loads involve a compromise, putting the matching network as close as possible to the load which is similar to the optimum position derived from the real characteristic impedance approximation (losses described solely by the attenuation) of lossy transmission lines.

Finally, an experimental setup was designed to test and validate the optimum matching strategy of the network. The experimental results are close to those predicted theoretically as well as they illustrate the simpler part and the proposed simpler approximations of the matching strategy given in Section 4.5.3.

7.2 Further work

The emphasis of this work was on deriving an optimum matching strategy for π and T networks suitable for any complex load. If the load's imaginary part is inductive, achieving the highest power gain involved a piecewise constant approximation of the known (measured) \( Q \) factor as a function of the inductor's reactance or its number of turns. Then the optimum network settings are computed for each interval, from which the best settings are chosen yielding the optimum network. However, describing the nonlinear \( Q \) factor by a piecewise constant approximation with sufficient accuracy may require a large number of relatively small intervals, thereby significantly increasing calculation time. This may be particularly disadvantageous in real time applications where the network has to match varying loads.

If a piecewise linearisation would be used instead, a lower number of intervals should suffice to approximate the nonlinear \( Q \) factor with similar accuracy, thereby reducing calculation time.

Thus deriving of an exact analytical solution for piecewise linearised losses, in particular for

\[ R_s = R_{s0} + a_1 \omega L_s \quad \text{or} \quad G_p = G_{p0} + a_1 \frac{1}{\omega L_p}, \]

the former for \( \pi \), the latter for \( T \) networks, may be recommendable.

However, solutions and applicable parameter ranges' limits are much more complicated than those of the constant \( Q \) factor approximation. Additional limits are introduced as indicated in Appendices C.1.1.1 and C.1.1.2. Moreover, the network's power gain may stay constant in some parameter region\(^1\) if \( a_1 = 0 \).

---

\(^1\) If, for example, a \( \pi \) network is considered and \( R_s = R_{s0} \), the optimum power gain for resistive (and capacitive complex) loads is given by \( g_{\pi, \text{inms}} = 1 - \frac{R_{s0}}{R_t} = \text{const.} \) if \( R_t > R_t - R_{s0} \) (yielding \( R_{\text{min}} = R_t - R_{s0} = \text{const.} \)).
CHAPTER 7. CONCLUSIONS AND FURTHER WORK

Another (at least theoretically) interesting extension would be the derivation of the exact analytical solutions if all three network elements are lossy and those losses are described by constant $Q$ factors as sketched in Appendices B.3.2 and B.3.3.

Unfortunately, four possible solutions are obtained instead of two, hence it may be very complicated to prove which solution applies and to calculate the according parameter's limits.

Since high power applications — where an optimisation is essential — require usage of low loss variable capacitors, whose losses can be estimated as indicated in Appendices B.5.1 and B.5.2, the question of whether their exact inclusion significantly improves matching network design needs to be investigated.

On the other hand, there are other useful extensions of the network design presented in this work:

The theory may be extended to networks with two (lossy) inductors and one central capacitor (whose losses are negligible). Network parametrisation may be adapted from those given in Appendices B.3.2 and B.3.3. For example, in a $T$ network, the resulting formulæ are similar to those of Appendix B.3.3 except that $L_p$ is replaced with $C_p$, whose loss $G_p = 0$, and $C_{1,2}$ is replaced with $L_{1,2}$, whose losses $R_{s1,2} = \frac{1}{G_{s1,2}}$ are described by individual series loss models given by (3.1). But, since two series loss models are involved, the problem’s difficulty is increased. In addition, the parametrisation cannot be adapted to “remove” the losses from the design formula of $L_2$ or $L_1$ to simplify the extension to complex loads or sources, respectively.

Another possibility is that the method extended to complex loads may be extended further to complex sources as indicated in Footnotes 10 and 13 in Section 5.4.3.1 and 5.4.3.2, respectively, yielding an extension to complex loads and sources. Since minima/maxima of $C_{1\text{tot}}$ may occur (refer to Figures 4.6 to 4.9) and additional inductive solutions (and parameter limits) for $L_{1\text{tot}}$ have to be derived, this task is also quite difficult.

Furthermore, the network design presented in this work may be employed to design loads for optimum network efficiency. For example, if the load’s imaginary part is considered, it’s always recommendable to keep a capacitive imaginary part as small as possible. Conversely, an inductive imaginary part should be kept as high as possible provided the network may be chosen arbitrarily — then either a $T$ network ($\frac{1}{\text{Re}(Z_t)} < G_t$) or a $\pi$ network ($\frac{1}{\text{Re}(Z_\pi)} < R_\pi$) will be the optimum choice.

Moreover, the lossy network design theory derived in this work is not limited to the order three networks considered herein. Higher order networks may be obtained by splitting those networks into lower order ones — then one approach would be to design those networks for optimum power gain depending on the impedances at the interfaces between those (partial) networks whose optimum values would have to be derived. Generally, this would require at least the extension to complex loads and sources mentioned above.
CHAPTER 7. CONCLUSIONS AND FURTHER WORK

At a first glance, matching of a resistive load to a resistive source seems to be obtainable by a combination of a second or third order network designed to match a complex load to a resistive source and a second or third order network designed to match a complex source to a resistive load (both derived within this work) to generate lossy matching networks of up to fifth order — even up to sixth order if the extension to networks with two (lossy) inductors and one central capacitor would be derived. However, since lossy networks achieve a complex conjugate match in one direction only, this approach would not yield the intended result. Although such a combination would still be able to match in one direction, the part of the network with opposite “matching design” direction would be expected to be suboptimal.

In addition, the applicable frequency range may be extended provided the (equivalent) network elements still consist of lumped elements and narrowband matching is required. In particular, the former is true for CMOS integrated RF circuits up to the GHz range. Nowadays, as for wireless applications, integration density is increasing. Thus matching networks are implemented within the integrated circuit. However, as indicated in [69], the inductors employed are quite lossy. Since it’s always possible to use a piecewise constant approximation of the losses if there would be any case where the inductor’s losses would not fit the model, the network design presented in this work is ideally suited to optimise the power gain of those networks.

Finally, the lossy networks’ power gain optimisation strategy derived in this work may serve to implement an improved cost function in a broadband (or higher order) network design optimisation (an example for a filter optimisation employing a cost function combined with other methods is given in [70]), to enhance the parametrisation of such networks, or to find better initial values in a lossy network optimisation like in [71].
Appendix A

RF Basics

A.1 Definitions/notations

From this chapter on, the following definitions/notations will be used:

Latin letters

- \( B_t \) Defined as \(-\omega C_1\) in Figure A.4 and subsequent calculations
- \( B_l \) Defined as \( \frac{1}{\omega L_p} \) in Figure A.6 and subsequent calculations
- \( B_t \) Load susceptance (in parallel to load conductance \( G_t \))
- \( g_{\text{dp}}(\xi_{\text{tp}}, 50\Omega) \) Power gain of a two-port as a function of reflection coefficient \( \xi_{\text{tp}}, 50\Omega \)
- \( I(z) \) Complex amplitude of the current in a transmission line at location \( z \)
- \( I_1 \) Complex amplitude a current of flowing in a network’s input
- \( I_t \) Complex amplitude of a current flowing through the load impedance \( Z_t \)
- \( I \) Complex variable
- \( q \) Complex variable
- \( \xi(z) \) Reflection coefficient at location \( z \) (referred to \( Z_\infty \))
- \( \xi_t \) Reflection coefficient at the input of a T network (referred to \( Z_\infty \))
- \( \xi_{\text{tp}, 50\Omega} \) Reflection coefficient at the input of a two-port referred to 50\(\Omega\)
- \( \xi_{\text{tp}} \) Reflection coefficient at the input of a \( \pi \) network (referred to \( Z_\infty \))
- \( S_{11\text{tp}, 50\Omega} \) S-parameter \( S_{11} \) of a two-port referred to 50\(\Omega\)
- \( S_{12\text{tp}, 50\Omega} \) S-parameter \( S_{12} \) of a two-port referred to 50\(\Omega\)
- \( S_{21\text{tp}, 50\Omega} \) S-parameter \( S_{21} \) of a two-port referred to 50\(\Omega\)
- \( S_{22\text{tp}, 50\Omega} \) S-parameter \( S_{22} \) of a two-port referred to 50\(\Omega\)
- \( V(z) \) Complex amplitude of the voltage along a transmission line at location \( z \)
- \( V_1 \) Complex amplitude of the voltage at the beginning of a transmission line (at its input)
- \( V_{11} \) Complex amplitude of the voltage at a network’s input
- \( V_2 \) Complex amplitude of the voltage at the end of a transmission line (at its load)
- \( V_{12}(z) \) Complex amplitude of the incident voltage at location \( z \) of a transmission line
- \( V_t \) Complex amplitude of the voltage at the load impedance \( Z_t \)
- \( V_s(z) \) Complex amplitude of the reflected voltage at location \( z \) of a transmission line
- \( x \) Real variable
A.2 Active power delivered from a source to a load as a function of available power

If a voltage source of complex amplitude $V_0$ and source impedance $Z_i$ is connected to a load impedance $Z_\ell$, $P_{act\ell}$ is the active power delivered to the load. It will be compared to the available power $P_{act maxi}$, which equals the active power delivered to the load for a conjugate complex match ($Z_\ell = Z_1^*$).

The available power is defined by

$$P_{act maxi} = \frac{|V_0|^2}{8 \text{Re}\{Z_i\}}. \quad (A.1)$$

For the setup shown in Figure A.1, the active power delivered to the load is

$$P_{act\ell} = \frac{1}{2} \text{Re}\{V_\ell I_\ell^*\} = \frac{1}{2} \text{Re}\left\{\frac{Z_\ell V_0}{Z_1 + Z_\ell} \frac{Z_\ell^* V_0^*}{(Z_1^* + Z_\ell^*) Z_1\ell^*}\right\}$$

$$P_{act\ell} = \frac{|V_0|^2}{2} \frac{\text{Re}\{Z_\ell\}}{(\text{Re}\{Z_i\} + \text{Re}\{Z_\ell\})^2 + (\text{Im}\{Z_i\} + \text{Im}\{Z_\ell\})^2}.$$

Applying (A.1) yields

$$P_{act\ell} = P_{act maxi} \frac{4 \text{Re}\{Z_i\} \text{Re}\{Z_\ell\}}{(\text{Re}\{Z_i\} + \text{Re}\{Z_\ell\})^2 + (\text{Im}\{Z_i\} + \text{Im}\{Z_\ell\})^2}.$$

In the next step we will replace real and imaginary parts of source and load impedance with their reflection coefficient representations derived in (2.11) and (2.12).
Doing so gives

\[
P_{\text{act}} = P_{\text{act max}} \frac{4 \frac{R_0 (1 - |r_i|^2) - 2X_0 \text{Im} \{r_i\}}{|1 - r_i|^2} \frac{R_0 (1 - |r_\ell|^2) - 2X_0 \text{Im} \{r_\ell\}}{|1 - r_\ell|^2}}{\left( \frac{R_0 (1 - |r_i|^2) - 2X_0 \text{Im} \{r_i\}}{|1 - r_i|^2} + \frac{R_0 (1 - |r_\ell|^2) - 2X_0 \text{Im} \{r_\ell\}}{|1 - r_\ell|^2} \right)^2 + \left( \frac{2R_0 \text{Im} \{r_i\} + X_0 (1 - |r_i|^2)}{|1 - r_i|^2} + \frac{2R_0 \text{Im} \{r_\ell\} + X_0 (1 - |r_\ell|^2)}{|1 - r_\ell|^2} \right)^2}.\]

Putting

\[
\frac{R_0 (1 - |r_i|^2) - 2X_0 \text{Im} \{r_i\}}{|1 - r_i|^2} = \frac{R_0 (1 - (\text{Re} \{r_i\})^2 - (\text{Im} \{r_i\})^2) - 2X_0 \text{Im} \{r_i\}}{(1 - (\text{Re} \{r_i\})^2 + (\text{Im} \{r_i\})^2)}\]

and

\[
\frac{R_0 (1 - |r_\ell|^2) - 2X_0 \text{Im} \{r_\ell\}}{|1 - r_\ell|^2} = \frac{R_0 (1 - (\text{Re} \{r_\ell\})^2 - (\text{Im} \{r_\ell\})^2) - 2X_0 \text{Im} \{r_\ell\}}{(1 - (\text{Re} \{r_\ell\})^2 + (\text{Im} \{r_\ell\})^2)}\]

in the above equation followed by rearranging yields

\[
P_{\text{act}} = P_{\text{act max}} \frac{[R_0 (1 - (\text{Re} \{r_i\})^2 - (\text{Im} \{r_i\})^2) - 2X_0 \text{Im} \{r_i\}] [R_0 (1 - (\text{Re} \{r_\ell\})^2 - (\text{Im} \{r_\ell\})^2) - 2X_0 \text{Im} \{r_\ell\}]}{(R_0^2 + X_0^2) [2 \text{Im} \{r_i\} \text{Im} \{r_\ell\} + (\text{Im} \{r_i\})^2 (\text{Re} \{r_\ell\})^2 + (1 - \text{Re} \{r_i\} \text{Re} \{r_\ell\})^2 + (\text{Im} \{r_i\})^2 ((\text{Re} \{r_i\})^2 + (\text{Im} \{r_i\})^2)]}.\]

Further rearranging according to

\[
2 \text{Im} \{r_i\} \text{Im} \{r_\ell\} + (\text{Im} \{r_i\})^2 (\text{Re} \{r_\ell\})^2 + (1 - \text{Re} \{r_i\} \text{Re} \{r_\ell\})^2 + (\text{Im} \{r_\ell\})^2 ((\text{Re} \{r_i\})^2 + (\text{Im} \{r_i\})^2)
\]

\[
= 1 - 2 \text{Re} \{r_i\} \text{Re} \{r_\ell\} + 2 \text{Im} \{r_i\} \text{Im} \{r_\ell\} + (\text{Re} \{r_\ell\})^2 + (\text{Im} \{r_\ell\})^2 (\text{Re} \{r_i\})^2 + (\text{Im} \{r_i\})^2
\]

\[
= 1 - r_i r_\ell - r_i^* r_\ell^* + |r_i|^2 |r_\ell|^2
\]

\[
= 1 - 2 \text{Re} \{r_i r_\ell\} + |r_i|^2 |r_\ell|^2
\]

\[
= 1 - 2 \text{Re} \{r_i r_\ell\} + (\text{Re} \{r_i r_\ell\})^2 + (\text{Im} \{r_i r_\ell\})^2
\]

finally gives

\[
P_{\text{act}} = P_{\text{act max}} \frac{[R_0 (1 - |r_i|^2) - 2X_0 \text{Im} \{r_i\}] [R_0 (1 - |r_\ell|^2) - 2X_0 \text{Im} \{r_\ell\}]}{|Z_0|^2 (1 - r_i r_\ell)^2}. \quad (A.2)
\]

For a real characteristic impedance \(X_0 = 0\), \((A.2)\) is equal to \([72a]\).
A.3 Active power in a transmission line and its transducer and power gain

\[ P_{\text{act1}} = P_{\text{act}}(0) \quad P_{\text{act}}(z) = \frac{P_{\text{act}}(l) = P_{\text{act}}}{r(z) = r(l) = r'}. \]

![Diagram](image)

Figure A.2: Definitions and setup for \( P_{\text{act}}(z) \) derivation

Consider a transmission line of length \( l \) in the setup depicted in Figure A.2. Its complex characteristic impedance is \( Z_0 \), its attenuation constant \( \alpha \), and its phase constant \( \beta \). The line is driven by a voltage source of complex amplitude \( V_0 \) and its terminated by a load impedance \( Z' \).

\( V(z) \), the complex amplitude of the voltage along the line at location \( z \), may be computed as the sum of the complex amplitude \( V_i(z) \) of the incident wave’s voltage and the complex amplitude \( V_r(z) \) of the reflected wave’s voltage.

Their ratio, as indicated in [26e], is the reflection coefficient \( r(z) \) at location \( z \),

\[ r(z) := \frac{V_r(z)}{V_i(z)} = \frac{r_2 e^{-2\alpha(l-z)} e^{-j2\beta(l-z)}}{r_2 e^{-2\alpha(l-z)} e^{-j2\beta(l-z)}} = r_2 e^{-2\alpha(l-z)} e^{-j2\beta(l-z)}. \quad (A.3) \]

By using this relationship in conjunction with [26d] (which derives the complex amplitude \( V(z) \) of the voltage or \( I(z) \) of the current along the transmission line at location \( z \)), we obtain

\[ V(z) = V_i(0) e^{-\alpha z} e^{-j\beta z} (1 + r(z)), \quad (A.4) \]

\[ I(z) = \frac{V_i(0) e^{-\alpha z} e^{-j\beta z} (1 - r(z))}{Z_0}. \quad (A.5) \]

The complex amplitude \( V_i \) of the voltage at the line’s input may be computed as a function of the reflection coefficients according to [73], yielding

\[ V_i = \frac{V_0}{2} \left( \frac{(1 - r_i)(1 + r_i)}{1 - r_i r_i} \right), \quad (A.6) \]

where \( r_i = r(0) \).
Due to $V_1 = V(0)$, we obtain by comparison of (A.4) and (A.6) that

$$V_1(0) = \frac{V_0}{2} \frac{1 - \tau_1}{1 - \tau_1 \tau_1}.$$  \hfill (A.7)

[26g] calculates the active power delivered to location $z$ of the line,

$$P_{\text{act}}(z) = \frac{1}{2} \text{Re} \left\{ V(z) \, \Gamma^*(z) \right\}.$$  \hfill (A.8)

Applying (A.4) and (A.5) yields

$$P_{\text{act}}(z) = \frac{1}{2} \text{Re} \left\{ \frac{V_1(0) \, e^{-\alpha z} \, e^{-j\beta z} \, (1 + \tau(z)) \, V_1^*(0) \, e^{-\alpha z} \, e^{j\beta z} \, (1 - \tau^*(z))}{Z_0^*} \right\}$$

$$= \frac{1}{2} |V_1(0)|^2 e^{-2\alpha z} \text{Re} \left\{ \frac{1 + \tau(z) - \tau^*(z) - |\tau(z)|^2}{Z_0^*} \right\}.$$  \hfill (A.9)

This result is partially calculated in [26g]. However, [26g] does not take into account that the complex amplitude of the incident wave's voltage depends on the load impedance or its reflection coefficient $\tau_1$, respectively.

The active power delivered to the load is $P_{\text{act}}(l) = P_{\text{act}}(l)$, hence the transducer power gain $g_{T,L}$ of the line is

$$g_{T,L} := \frac{P_{\text{act}}}{P_{\text{act}}^\text{max}} = e^{-2\alpha l} \frac{[R_0 (1-|\tau_1|^2)-2X_0 \text{Im}\{\tau_1\}][R_0 (1-|\tau(z)|^2)-2X_0 \text{Im}\{\tau(z)\}]}{|Z_0|^2 |1-\tau_1 \tau_1|^2}.$$

In the following sections we will additionally need the power gain $g_L$ of the line or $P_{\text{act}}$, the active power delivered to the load, as a function of $P_{\text{act}}(0)$, the active power delivered to the line's input at $z = 0$.

To derive it, we use (A.2), where $\tau_1$ is replaced with $\tau(0) = \tau_1 = \tau_1 e^{-2\alpha l} e^{-j2\beta l}$ (from (A.3)), in (A.10), yielding

$$g_L := \frac{P_{\text{act}}}{P_{\text{act}}^\text{max}} = e^{-2\alpha l} \frac{R_0 (1-|\tau_1|^2)-2X_0 \text{Im}\{\tau_1\}}{R_0 (1-|\tau_1 e^{-2\alpha l} e^{-j2\beta l}|^2)-2X_0 \text{Im}\{\tau_1 e^{-2\alpha l} e^{-j2\beta l}\}}$$

$$g_L = e^{-2\alpha l} \frac{R_0 (1-|\tau_1|^2)-2X_0 \text{Im}\{\tau_1\}}{R_0 (1-|\tau_1|^2)-2X_0 e^{-2\alpha l} \text{Im}\{\tau_1 e^{-j2\beta l}\}}.$$  \hfill (A.12)

A similar formula is derived in [28] and [29].
A.4 Input reflection coefficient and power gain of L, π, and T networks at a fixed operating frequency

A.4.1 Calculation using network elements’ values

A.4.1.1 π network

Using the definitions of Figure A.3, the input admittance $Y_{1\pi}$ is given by

$$Y_{1\pi} = j\omega C_1 + \frac{1}{R_s + j\omega L_s + \frac{1}{G_t + j\omega C_{2\text{tot}}}} \right.$$ (A.13)

where $G_t = \text{Re} \left\{ \frac{1}{Z_t} \right\}$, $C_{2\text{tot}} = C_2 + \frac{1}{\omega} \text{Im} \left\{ \frac{1}{Z_t} \right\}$.

Applying the preceding admittance and (2.9), we may compute the π network’s input reflection coefficient $\xi_{1\pi}$ referred to $Z_0$, obtaining

$$\xi_{1\pi} = \frac{1 - Y_{1\pi} Z_0}{1 + Y_{1\pi} Z_0}. \right.$$ (A.14)

Since the π network’s power gain $g_1$ is independent of the driving source’s impedance (e.g. according to (A.26)), we may simplify its derivation by choosing the source impedance to cancel all reactive components at the input of the π network.

Figure A.4: Definitions for calculation of the π network’s power gain and equivalent circuit for $X_i = -\omega L_s$, $B_i = -\omega C_1$

---

1There is no need to distinguish $C_{2\text{tot}}$ and $L_{2\text{tot}}$ depending on the sign of $C_2 + \frac{1}{\omega} \text{Im} \left\{ \frac{1}{Z_t} \right\}$. The calculated $Y_{1\pi}$, $\xi_{1\pi}$, or $g_1$ resemble proper values although a formally negative $C_{2\text{tot}}$ might occur.
All reactive components in Figure A.4, namely \( X_1, B_1, \ell, C_{1,2}, \) and \( L_\alpha, \) are lossless. Thus \( \tilde{P}_{\text{act1}} = \tilde{P}_{\text{act1}} = P_{\text{act1}} = \tilde{P}_{\text{act1}} = \tilde{P}_{\text{act1}} \) holds, which equals the incident active power in the equivalent circuit (right half of Figure A.4).

Using the definitions of Figure A.4 yields

\[
P_{\text{act1}} = \frac{1}{2} \text{Re}\{V_1 V_0^*\} = \frac{1}{2} \text{Re}\left\{ |L_0| \left( R_s + \frac{1}{G_\ell + j\omega C_{2\text{tot}}} \right) \right\}
\]

\[
P_{\text{act1}} = \frac{1}{2} |L_0|^2 \left( R_s + \frac{G_\ell}{G_\ell^2 + \omega^2 C_{2\text{tot}}^2} \right).
\]

The active power delivered to the load is given by

\[
P_{\text{act} \ell} = \frac{1}{2} |V_\ell|^2 G_\ell = \frac{1}{2} |L_0|^2 \frac{G_\ell}{G_\ell^2 + \omega^2 C_{2\text{tot}}^2}.
\]

Thus the \( \pi \) network’s power gain is

\[
g_\pi = \frac{P_{\text{act}\ell}}{P_{\text{act1}}} = \frac{G_\ell}{R_s (G_\ell^2 + \omega^2 C_{2\text{tot}}^2) + G_\ell} = \frac{1}{1 + \frac{R_s}{G_\ell^2 + \omega^2 C_{2\text{tot}}^2}} \uparrow. \tag{A.15}
\]

Two \( \pi \) network’s, each for \( C_1 = 0 \) or \( C_2 = 0, \) may be viewed as limits of the \( \pi \) network. Their input impedances and power gains are computed by putting \( C_1 = 0 \) or \( C_2 = 0, \) respectively.

### A.4.1.2 T network

![Figure A.5](image)

**Figure A.5:** Definitions for calculation of the T network’s input reflection coefficient and its equivalent circuit after series-to-parallel transformation of \( R_s \) and \( L_s \)

Calculation of the T network’s input reflection coefficient starts by series-to-parallel-transformation of \( R_s \) and \( L_s \) in \( G_p \) and \( L_p \) (right half of Figure A.5). [34c] computes the parallel components’ values,

\[
\frac{G_\ell}{G_\ell^2 + \omega^2 C_{2\text{tot}}^2} \text{ is the series resistance after parallel-to-series transformation of } G_\ell \text{ and } C_{2\text{tot}} \text{ (calculated using (A.19) after replacing } G_p \text{ with } G_\ell \text{ and } \frac{1}{L_p} \text{ by } \omega C_{2\text{tot}}).}
\]
\[
G_p = \frac{R_s}{R_s^2 + \omega^2 L_s^2}, \quad \text{(A.16)}
\]
\[
L_p = \frac{R_s^2 + \omega^2 L_s^2}{\omega^2 L_s}. \quad \text{(A.17)}
\]

Additionally, we need to examine the transformed network’s element values in case \(R_s, L_s\) tend to zero or infinity. However, \(R_s\) and \(L_s\) are not independent of each other, but obey (3.1). If the limits of \(G_p\) and \(L_p\) are derived using (3.1), we obtain

(i) If \(L_s = 0\) then \(R_s = R_{s0}\):

a) \(R_{s0} = 0\):

- \(0 < b_1 < \frac{1}{2}\): \(G_p \to \infty, L_p \to \infty\) or \(\frac{1}{L_p} = 0\)
  (since \(G_1 - G_m - G_p < 0\), matching is impossible, thus never part of a valid solution)
- \(b_1 = \frac{1}{2}\): \(G_p \to \infty, L_p = \frac{a_3^2}{\omega}\) or \(\frac{1}{L_p} = \frac{\omega}{a_3^2}\)
  (since \(G_1 - G_m - G_p < 0\), matching is impossible, thus never part of a valid solution)
- \(\frac{1}{2} < b_1 < 2\): \(G_p \to \infty, L_p = 0\) or \(\frac{1}{L_p} \to \infty\)
  (since \(G_1 - G_m - G_p < 0\), matching is impossible, thus never part of a valid solution)
- \(b_1 = 2\): \(G_p = a_1, L_p = 0\) or \(\frac{1}{L_p} \to \infty\)
  \((b_1 = b_N = 2 \text{ if } b_N \leq 2)\)
- \(b_1 > 2\): \(G_p = 0, L_p = 0\) or \(\frac{1}{L_p} \to \infty\)
  (impossible if \(b_N \leq 2\))

b) \(R_{s0} > 0\): \(G_p = \frac{1}{R_{s0}}, L_p \to \infty\) or \(\frac{1}{L_p} = 0\).

(ii) If \(L_s \to \infty\) then \(R_s \to \infty\) (except \(R_s = R_{s0}\) if \(R_s = \text{const.}\)), \(G_p = 0\), and \(L_p \to \infty\) or \(\frac{1}{L_p} = 0\).

The inverse parallel-to-series transformation is given by

\[
R_s = \frac{G_p}{G_p^2 + \frac{1}{\omega^2 L_p^2}}, \quad \text{(A.19)}
\]
\[
L_s = \frac{\frac{1}{\omega^2 L_p}}{G_p^2 + \frac{1}{\omega^2 L_p^2}}. \quad \text{(A.20)}
\]

Using the definitions in Figure A.5 yields the input impedance \(Z_{1t}\) of the series-to-
parallel transformed equivalent circuit\(^2\),

\[
Z_{1t} = \frac{1}{j\omega C_1} + \frac{1}{G_p + \frac{1}{j\omega L_p} + \frac{1}{R_e + j\omega C_{2t}}}
\]

where \(R_e = \text{Re}\{Z_e\}, \quad C_{2t} = \frac{1}{C_2^2 - \omega^2 \text{Im}\{Z_e\}}\).

\(^2\)There is no need to distinguish \(C_{2t}\) and \(L_{2t}\) depending on the sign of \(\frac{1}{C_2^2} - \omega \text{Im}\{Z_e\}\). The calculated \(Z_{1t}, \ell_1\), or \(g_1\) resemble proper values although a formally negative \(C_{2t}\) might occur.
Applying the preceding impedance and (2.9), we may compute the T network's input reflection coefficient \( \Gamma_{1t} \) referred to \( Z_0 \), obtaining

\[
\Gamma_{1t} = \frac{Z_{1t} - Z_0}{Z_{1t} + Z_0}.
\]

Since the T network’s power gain \( g_t \) is independent of the driving source’s impedance (e.g. according to (A.26)), we may simplify its derivation by choosing the source impedance to cancel all reactive components at the input of the T network.

![Diagram of T network power gain and equivalent circuit](image)

Figure A.6: Definitions for calculation of the T network’s power gain and equivalent circuit for \( B_1 = \frac{1}{\omega L_p} \), \( X_1 = \frac{1}{\omega C_1} \)

All reactive components in Figure A.6, namely \( B_1, X_{1,t} \), \( C_{1,2} \), and \( L_p \), are lossless.

Thus \( \tilde{P}_{act1} = \tilde{P}_{act1} = P_{act1} = \tilde{P}_{act1} = \tilde{P}_{act1} \) holds, which equals the incident active power in the equivalent circuit (right half of Figure A.6).

Using the definitions of Figure A.6 yields

\[
P_{act1} = \frac{1}{2} \text{Re}\{V_0 I_1^*\} = \frac{1}{2} \text{Re}\left\{|V_0|^2 \left( \frac{G_p + \frac{1}{R_t - \frac{1}{\omega C_{2tot}}} \right) \right\}
\]

\[
P_{act1} = \frac{1}{2} |V_0|^2 \left( G_p + \frac{R_t}{R_t^2 + \frac{1}{\omega^2 C_{2tot}}} \right).
\]

The active power delivered to the load is given by

\[
P_{act\ell} = \frac{1}{2} |I_\ell|^2 R_\ell = \frac{1}{2} |V_0|^2 \frac{R_t}{R_t^2 + \frac{1}{\omega^2 C_{2tot}}}.
\]

Thus the T network’s power gain is

\[
g_t = \frac{P_{act\ell}}{P_{act1}} = \frac{R_t}{G_p \left( R_t^2 + \frac{1}{\omega^2 C_{2tot}} \right) + R_t} = \frac{1}{1 + \frac{G_p}{\frac{R_t}{R_t^2 + \frac{1}{\omega^2 C_{2tot}}}}}.
\]

Two L networks, each for \( C_1 \to \infty \) or \( C_2 \to \infty \), may be viewed as limits of the T network. Their input impedances and power gains are computed by putting \( C_1 \to \infty \) or \( C_2 \to \infty \), respectively.

\[
\frac{R_t}{R_t^2 + \frac{1}{\omega^2 C_{2tot}}} \text{ is the parallel conductance after series-to-parallel transformation of } R_t \text{ and } C_{2tot}\]

(calculated using (A.16) after replacing \( R_\ell \) with \( R_t \) and \( \omega L_0 \) with \( \frac{1}{\omega C_{2tot}} \)).
A.4.2 Calculation using (measured) S-parameters

The following formulae for the calculation of a two port’s input impedance and power gain using (measured) S-parameters applies to any two-port, thus there is no difference in the calculations for the L, π, or T network.

Since common network analysers measure S-parameters referred to a characteristic impedance of 50 Ω for both ports, it will also be the characteristic impedance all reflection coefficients are referred to in the following formulae.

[74a] computes the input reflection coefficient of a two-port tp as a function of its load,

\[
\tau_{1tp,50Ω} = S_{11tp,50Ω} + \frac{S_{21tp,50Ω} S_{12tp,50Ω} \tau_{tp,50Ω}}{1 - S_{22tp,50Ω} \tau_{tp,50Ω}}. \tag{A.24}
\]

The two-port’s power gain may be calculated from [74b] or by applying (A.2) and (A.24) to [72b], yielding

\[
g_{tp}(\tau_{tp,50Ω}) = \frac{|S_{21tp,50Ω}|^2 \left(1 - |\tau_{tp,50Ω}|^2\right)}{|1 - S_{22tp,50Ω} \tau_{tp,50Ω}|^2 \left(1 - |\tau_{1tp,50Ω}|^2\right)}. \tag{A.25}
\]

Using (A.24) and the relationship \(|p|^2 |q|^2 = |pq|^2\) gives

\[
g_{tp}(\tau_{tp,50Ω}) = \frac{|S_{21tp,50Ω}|^2 \left(1 - |\tau_{tp,50Ω}|^2\right)}{|1 - S_{22tp,50Ω} \tau_{tp,50Ω}|^2 - |(1 - S_{22tp,50Ω} \tau_{tp,50Ω}) S_{11tp,50Ω} + S_{21tp,50Ω} S_{12tp,50Ω} \tau_{tp,50Ω}|^2}. \tag{A.26}
\]

The L, π, and T networks considered are reciprocal networks ([75]). Then, according to [26i] the following equality holds,

\[
S_{21tp,50Ω} = S_{12tp,50Ω}. \tag{A.27}
\]

If the networks would additionally be lossless (\(R_a = 0\) or \(G_p = 0\), they would consume no active power, thus

\[
g_{tp}(\tau_{tp,50Ω}) = 1. \tag{A.28}
\]
Appendix B

Lossy L, π, and T Matching Networks

B.1 Definitions/notations

From this chapter on, the following definitions/notations will be used:

Latin letters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{l(p)}$</td>
<td>Lossy capacitance $C_1$ of a π network, index $p$ indicates usage of parallel loss model</td>
</tr>
<tr>
<td>$C_{l(s)}$</td>
<td>Lossy capacitance $C_1$ of a T network, index $s$ indicates usage of series loss model</td>
</tr>
<tr>
<td>$C_{2(p)}$</td>
<td>Lossy capacitance $C_2$ of a π network, index $p$ indicates usage of parallel loss model</td>
</tr>
<tr>
<td>$C_{2(s)}$</td>
<td>Lossy capacitance $C_2$ of a T network, index $s$ indicates usage of series loss model</td>
</tr>
<tr>
<td>$S_{l,imp}$</td>
<td>Power gain of a T network including losses of all elements designed as if its capacitors would be lossless</td>
</tr>
<tr>
<td>$S_{l,des}$</td>
<td>Power gain of a T network designed as if it would be lossless</td>
</tr>
<tr>
<td>$S_{l,out}$</td>
<td>Power gain of a T network which is (exactly) matched at its output</td>
</tr>
<tr>
<td>$S_{π,imp}$</td>
<td>Power gain of a π network including losses of all elements designed as if its capacitors would be lossless</td>
</tr>
<tr>
<td>$S_{π,des}$</td>
<td>Power gain of a π network which is (exactly) matched at its output</td>
</tr>
<tr>
<td>$G_{Irest}$</td>
<td>Resulting total source conductance of the &quot;inner&quot; T network as defined in Appendix B.3.3</td>
</tr>
<tr>
<td>$G_{frest}$</td>
<td>Resulting total load conductance of the &quot;inner&quot; T network as defined in Appendix B.3.3</td>
</tr>
<tr>
<td>$G_{sc}$</td>
<td>Series loss conductance of capacitance $C_1$ of a T network ($G_{sc1} = \frac{1}{R_{sc1}}$)</td>
</tr>
<tr>
<td>$G_{sc2}$</td>
<td>Series loss conductance of capacitance $C_2$ of a T network ($G_{sc2} = \frac{1}{R_{sc2}}$)</td>
</tr>
<tr>
<td>$I_{at}$</td>
<td>Complex amplitude of a current flowing through $R_t$ (and thus through $R_{frest}$) of the T network in Appendix B.3.3</td>
</tr>
<tr>
<td>$I_{as}$</td>
<td>Complex amplitude of a current flowing through $R_s$ of the π network in Appendix B.3.2</td>
</tr>
<tr>
<td>$P_{ac1}$</td>
<td>Active power transferred to the input of a network</td>
</tr>
<tr>
<td>$P_{ac2}$</td>
<td>Active power transferred to a particular part of a network</td>
</tr>
<tr>
<td>$P_{ac3}$</td>
<td>Active power transferred to a particular part of a network</td>
</tr>
<tr>
<td>$Q_1$</td>
<td>$Q$ factor of capacitor $C_1$ of a π or T network</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>$Q$ factor of capacitor $C_2$ of a π or T network</td>
</tr>
<tr>
<td>$Z_{l,des}$</td>
<td>Reflection coefficient at the input of a T network designed as if it would be lossless (referred to $Z_{l,ill}$)</td>
</tr>
<tr>
<td>$Z_{π,des}$</td>
<td>Reflection coefficient at the input of a π network designed as if it would be lossless referred to $Z_{ill}$</td>
</tr>
</tbody>
</table>
B.2 Matching of a resistive load to a resistive source (matching at the network’s input)

B.2.1 π network

B.2.1.1 Optimum value of parameter $R_m$ for maximum power gain

As indicated in the overview of the proof given in Section 3.5.2.2, we begin the actual proof by assuming all $a_n = 0$, thus $R_s = R_{s0}$ throughout the valid range of $R_m$. We then obtain from (3.26)

$\left( \frac{R_s}{R_m} - \frac{dR_s}{dR_m} \right)_{a_n = 0} = \frac{R_{s0}}{R_m}$.

Since $R_{s0}, R_m \geq 0$, numerator and denominator of $\left( \frac{R_s}{R_m} - \frac{dR_s}{dR_m} \right)_{a_n = 0}$ are always $\geq 0$.

If at least one $a_n > 0$, we check numerator and denominator of (3.26) separately:

1. Since $\omega L_s, R_m, R_s, a_n$ are $\geq 0$, and $b_n > 0$, the numerator will always be $\geq 0$ for $R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} \leq 0$.

Conversely, if $R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} > 0$,

$b_N \left( R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} \right) \geq b_n \left( R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} \right)$

holds due to $b_N > b_{N-1} > \ldots > b_1 > 0$. 
From (3.1) we get \( R_s \geq \sum_{n=1}^{N} a_n (\omega L_s)^{b_n} \), thus obtaining

\[
R_s \omega L_s - \sum_{n=1}^{N} a_n b_n (\omega L_s)^{b_n} \left( R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} \right) 
\geq R_s \omega L_s - b_N R_s \left( R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} \right).
\]

After applying (3.27), (3.28), and \( R_s \geq 0 \), we finally have to check

\[
\omega L_s - b_N \left( R_s \frac{\partial (\omega L_s)}{\partial R_s} + R_m \frac{\partial (\omega L_s)}{\partial R_m} \right) 
= \frac{(R_m + R_s)(2 - b_N) (R_t - R_m - R_s) + b_N (R_m + R_s)}{2 \sqrt{(R_m + R_s)(R_t - R_m - R_s)}} 
+ \frac{R_m ((2 - b_N) (R_t - R_m) + b_N R_m)}{2 \sqrt{R_m (R_t - R_m)}},
\]

which is always \( \geq 0 \) throughout the (utmost) valid range of \( R_m \) indicated in (3.20) if \( 0 < b_N \leq 2 \) holds\(^1\).

2. Since \( \omega L_s, R_m, a_n \) are \( \geq 0 \), and \( b_n > 0 \), the denominator will always be \( \geq 0 \) for \( \frac{\partial (\omega L_s)}{\partial R_s} \leq 0 \).

Conversely, if \( \frac{\partial (\omega L_s)}{\partial R_s} > 0 \),

\[ a_n b_N (\omega L_s)^{b_n} \frac{\partial (\omega L_s)}{\partial R_s} \geq a_n b_n (\omega L_s)^{b_n} \frac{\partial (\omega L_s)}{\partial R_s} \]

holds due to \( b_N > b_{N-1} > \ldots > b_1 > 0 \).

From (3.1) we get \( R_s \geq \sum_{n=1}^{N} a_n (\omega L_s)^{b_n} \), thus obtaining

\[
\omega L_s - \sum_{n=1}^{N} a_n b_n (\omega L_s)^{b_n} \frac{\partial (\omega L_s)}{\partial R_s} \geq \omega L_s - b_N R_s \frac{\partial (\omega L_s)}{\partial R_s}.
\]

After applying (3.28), we finally have to check

\[
\omega L_s - b_N R_s \frac{\partial (\omega L_s)}{\partial R_s} 
= \frac{(2 - b_N) (R_m + R_s)(R_t - R_m - R_s) + b_N (R_m (R_t - R_m) + R_s^2)}{2 \sqrt{(R_m + R_s)(R_t - R_m - R_s)}} 
+ \sqrt{R_m (R_t - R_m)},
\]

which is always \( \geq 0 \) throughout the (utmost) valid range of \( R_m \) indicated in (3.20) if \( 0 < b_N \leq 2 \) holds\(^1\).

---

\(^1\) The proof would still be valid if we would put \( b_N = 0 \), but its initial definition in (3.1) was \( b_N > b_{N-1} > \ldots > b_1 > 0 \).
Combining all preceding parts of the proof, we may conclude that (at least) for $0 < b_N \leq 2$ numerator and denominator of $R_m - \frac{dR_s}{dR_m}$ in (3.26) and thus $\frac{dg_{\pi, \text{inna}}}{dR_m}$ itself remains $\geq 0$ throughout the (utmost) valid range of $R_m$ indicated in (3.20).

B.2.1.2 Optimum value of load resistance $R_\ell$ for maximum power gain

In Section 3.5.2.2 the parameter $R_m$ was optimised for maximum $g_{\pi, \text{inna}}$. However, the maximum $g_{\pi, \text{inna}}$ still depends on $R_\ell$ (for the intended application, $R_l$ is fixed and known).

Thus it’s necessary to investigate which $R_\ell$ yields the maximum $g_{\pi, \text{inna}}$ if $R_m$ is chosen for optimum $g_{\pi, \text{inna}}$ according to Section 3.5.2.2.

Which $R_\ell$ is likely to yield the maximum $g_{\pi, \text{inna}}$?

As indicated in Section 3.5.2.1, $L_a = 0$ for $R_\ell = R_l - R_s$, the limit which separates the two possible (utmost) upper limits of $R_m$. Then the lowest $R_s$ occurs ($R_s = R_s$).

Supposing that the highest $g_{\pi, \text{inna}}$ coincides with the lowest $R_s$ requires proving of $\frac{dg_{\pi, \text{inna}}(R_m=R_\ell)}{dR_\ell} \geq 0$ if $R_\ell \leq R_l - R_s$ or $\frac{dg_{\pi, \text{inna}}(R_m=R_{\text{limin}})}{dR_\ell} \leq 0$ if $R_\ell > R_l - R_s$, respectively, thereby obtaining that $g_{\pi, \text{inna}}$ at $R_m = R_\ell = R_l - R_s$ is greater (or equal) than any other $g_{\pi, \text{inma}}$.

According to Section 3.5.2.2, the proof has to be split in two parts:

(i) $R_\ell \leq R_l - R_s$, hence $0 \leq R_m \leq R_\ell$

The highest $g_{\pi, \text{inna}}$ is obtained if $R_m = R_\ell$. $g_{\pi, \text{inna}}$ is given by (3.24),

$$g_{\pi, \text{inna}}(R_m = R_\ell) = \frac{R_\ell}{R_\ell + R_s(R_m = R_\ell)}.$$  

Its derivative with respect to $R_\ell$ is

$$\frac{dg_{\pi, \text{inna}}(R_m=R_\ell)}{dR_\ell} = \frac{R_\ell}{(R_\ell + R_s(R_m=R_\ell))^2} \left( \frac{R_s(R_m=R_\ell)}{R_\ell} - \frac{dR_s(R_m=R_\ell)}{dR_\ell} \right).$$

Since $R_\ell$, $R_s(R_m = R_\ell) \geq 0$, it suffices to check $\frac{R_s(R_m=R_\ell)}{R_\ell} - \frac{dR_s(R_m=R_\ell)}{dR_\ell}$.

Using (3.1), we define an implicit function for $R_s(R_m = R_\ell)$,

$$\tilde{\psi}_\pi := R_s + \left( \sum_{n=1}^{N} a_n (\omega L_s(R_m = R_\ell))^n \right) - R_s(R_m = R_\ell) \equiv 0.$$  

Comparing the implicit function and the factor to check to those of Section 3.5.2.2, it’s quite obvious that the proofs are similar if $R_s$ is replaced with $R_s(R_m = R_\ell)$ and $R_m$ with $R_\ell$. Regarding the derivatives, those with respect to $R_s$ or $R_m$ are replaced with derivatives with respect to $R_s(R_m = R_\ell)$ or $R_\ell$. 


Then \( \frac{R_s(R_m = R_t)}{R_t} - \frac{dR_s(R_m = R_t)}{dR_t} \) is given by (3.26) if the indicated replacements are applied.

To prove that \( \frac{R_s(R_m = R_t)}{R_t} - \frac{dR_s(R_m = R_t)}{dR_t} \geq 0 \) the proof of Section 3.5.2.2 may be adapted accordingly. All estimations are valid if the indicated replacements are made, hence prior to the usage of \( \omega L_s \) and its partial derivatives the terms to check are similar. However, \( \omega L_s \) and its partial derivatives have changed compared to those of Section 3.5.2.2 and are given by

\[
\omega L_s(R_m = R_t) = \sqrt{\left( R_t + R_s(R_m = R_t) \right) \left( R_t - R_t - R_s(R_m = R_t) \right)},
\]

\[
\frac{\partial(\omega L_s(R_m = R_t))}{\partial R_t} = \frac{R_t - 2 \left( R_t + R_s(R_m = R_t) \right)}{2 \sqrt{\left( R_t + R_s(R_m = R_t) \right) \left( R_t - R_t - R_s(R_m = R_t) \right)}},
\]

\[
\frac{\partial(\omega L_s(R_m = R_t))}{\partial R_s(R_m = R_t)} = \frac{R_t - 2 \left( R_t + R_s(R_m = R_t) \right)}{2 \sqrt{\left( R_t + R_s(R_m = R_t) \right) \left( R_t - R_t - R_s(R_m = R_t) \right)}}.
\]

\((L_s(R_m = R_t))\) is obtained by putting \( R_s = R_{s0} \) in (3.19).

Similar to Section 3.5.2.2, we begin the proof by assuming all \( a_n = 0 \), thus \( R_s = R_{s0} \) throughout the valid range of \( R_m \). We then obtain

\[
\left( \frac{R_s(R_m = R_t)}{R_t} - \frac{dR_s(R_m = R_t)}{dR_t} \right)_{a_n = 0 \forall n} = \frac{R_{s0}}{R_t}.
\]

Numerator and denominator of \( \left( \frac{R_s(R_m = R_t)}{R_t} - \frac{dR_s(R_m = R_t)}{dR_t} \right)_{a_n = 0 \forall n} \) are always \( \geq 0 \) because \( R_{s0}, R_t \geq 0 \).

If at least one \( a_n > 0 \), numerator and denominator of \( \left( \frac{R_s(R_m = R_t)}{R_t} - \frac{dR_s(R_m = R_t)}{dR_t} \right) \) are checked separately similar to Section 3.5.2.2, yielding the following terms which have to be proved to be \( \geq 0 \):

1. In case of the numerator \( R_s(R_t) \geq 0, \omega L_s(R_m = R_t), \frac{\partial(\omega L_s(R_m = R_t))}{\partial R_t} \), and \( \frac{\partial(\omega L_s(R_m = R_t))}{\partial R_s(R_m = R_t)} \) are applied, yielding

\[
\omega L_s(R_m = R_t) - b_N \left( R_s(R_m = R_t) \frac{\partial(\omega L_s(R_m = R_t))}{\partial R_s(R_m = R_t)} + R_t \frac{\partial(\omega L_s(R_m = R_t))}{\partial R_t} \right)
\]

\[
= \frac{(R_t + R_s(R_m = R_t)) \left( (2 - b_N) \left( R_t - R_t - R_s(R_m = R_t) \right) + b_N \left( R_t + R_s(R_m = R_t) \right) \right)}{2 \sqrt{(R_t + R_s(R_m = R_t)) \left( R_t - R_t - R_s(R_m = R_t) \right)}},
\]

which is always \( \geq 0 \) if \( 0 < b_N \leq 2 \) holds\(^2\), because (3.20) still holds if \( R_m = R_t \).

2. In case of the denominator \( \omega L_s(R_m = R_t) \) and \( \frac{\partial(\omega L_s(R_m = R_t))}{\partial R_s(R_m = R_t)} \) are applied, yielding

\[
\omega L_s(R_m = R_t) - b_N R_s(R_m = R_t) \frac{\partial(\omega L_s(R_m = R_t))}{\partial R_s(R_m = R_t)}
\]

\[
= \frac{(2 - b_N) \left( R_t + R_s(R_m = R_t) \right) \left( R_t - R_t - R_s(R_m = R_t) \right) + b_N \left( R_t + R_s(R_m = R_t) \right) \left( R_t - R_t + R_s(R_m = R_t) \right)}{2 \sqrt{(R_t + R_s(R_m = R_t)) \left( R_t - R_t - R_s(R_m = R_t) \right)}},
\]

\(^2\) The proof would still be valid if we would put \( b_N = 0 \), but its initial definition in (3.1) was \( b_N > b_{N-1} > \ldots > b_1 > 0 \).
which is always \( \geq 0 \) if \( 0 < b_N \leq 2 \) holds\(^2\), because (3.20) still holds if \( R_m = R_\ell \).

Combining all preceding parts of the proof, we may conclude that (at least) for \( 0 < b_N \leq 2 \) numerator and denominator of \( \frac{R_s(R_m = R_\ell)}{R_\ell} - \frac{dR_s(R_m = R_\ell)}{dR_\ell} \) and thus \( \frac{R_s(R_m = R_\ell)}{R_\ell} - \frac{dR_s(R_m = R_\ell)}{dR_\ell} \) itself remains \( \geq 0 \) if \( R_\ell \leq R_i - R_o0 \).

(ii) \( R_\ell > R_i - R_o0 \), hence \( 0 \leq R_m \leq R_{\text{milim}} \)

The highest \( g_{\pi, \text{inna}} \) is obtained if \( R_m = R_{\text{milim}} = R_i - R_o(R_m = R_{\text{milim}}) \). \( g_{\pi, \text{inna}} \) is given by (3.24),

\[
g_{\pi, \text{inna}}(R_m = R_{\text{milim}}) = \frac{R_{\text{milim}}}{R_i}.
\]

Its derivative with respect to \( R_\ell \) is

\[
\frac{dg_{\pi, \text{inna}}(R_m = R_{\text{milim}})}{dR_\ell} = \frac{1}{R_i} \frac{dR_{\text{milim}}}{dR_\ell}.
\]

Since \( R_i \geq 0 \), it suffices to check \( \frac{dR_{\text{milim}}}{dR_\ell} \).

As \( R_s(R_m = R_{\text{milim}}) \) is a function of \( L_s(R_m = R_{\text{milim}}) \), which itself is a function of \( R_\ell \) and implicitly\(^3\) of \( R_s \) according to (3.19), we define the implicit function \( \psi_{\pi} \equiv 0 \) from (3.21), (3.1), and (3.19),

\[
\psi_{\pi} := R_o0 + \left( \sum_{n=1}^{N} a_n \left( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \right)^{b_n} \right) - R_i + R_{\text{milim}} \equiv 0.
\]

Then we compute the implicit function's partial derivatives with respect to \( R_\ell \) and \( R_{\text{milim}} \),

\[
\frac{\partial \psi_{\pi}}{\partial R_\ell} = \sum_{n=1}^{N} a_n b_n \left( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \right)^{b_n-1} R_{\text{milim}} = \frac{\sum_{n=1}^{N} a_n b_n \left( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \right)^{b_n-1} R_{\text{milim}}}{2 \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})}}
\]

\[
\frac{\partial \psi_{\pi}}{\partial R_{\text{milim}}} = \sum_{n=1}^{N} a_n b_n \left( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \right)^{b_n-1}(R_\ell - 2 R_{\text{milim}}) = \frac{\sum_{n=1}^{N} a_n b_n \left( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \right)^{b_n} (R_\ell - 2 R_{\text{milim}})}{2 \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})}} + 1.
\]

Applying the implicit function theorem [64] and multiplying numerator and denominator of the resulting fraction by \( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \) gives

\[
\frac{dR_{\text{milim}}}{dR_\ell} = -\frac{\sum_{n=1}^{N} a_n b_n \left( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \right)^{b_n} R_{\text{milim}}}{2 R_{\text{milim}}(R_\ell - R_{\text{milim}}) + \sum_{n=1}^{N} a_n b_n \left( \sqrt{R_{\text{milim}}(R_\ell - R_{\text{milim}})} \right)^{b_n} (R_\ell - 2 R_{\text{milim}})}.
\]

As indicated above, supposed the maximum \( g_{\pi, \text{inna}} \) occurs at \( R_\ell = R_i - R_o0 \), we have to prove that \( \frac{dR_{\text{milim}}}{dR_\ell} \leq 0 \) if \( R_\ell > R_i - R_o0 \).

---

\(^2\)In particular, \( R_s(R_m = R_{\text{milim}}) \) and \( L_s(R_m = R_{\text{milim}}) \) are both functions of \( R_\ell \) only. However, we usually may not be able to solve for \( R_s(R_m = R_{\text{milim}}) \) explicitly after applying (3.19) to (3.1).
For ease of understanding, an overview of the following proof is given below. To prove that numerator and denominator are \( \leq 0 \) for certain \( b_N \), we firstly assume all \( a_n = 0 \) or \( R_s = R_{s0} \) and calculate \( \frac{dR_{\text{lim}}}{dR_t} \), which will equal zero for any \( R_t > R_i - R_{s0} \).

Secondly, we assume that at least one \( a_n > 0 \), hence \( R_s > R_{s0} \) (according to (3.1)). Then the numerator of \( \frac{dR_{\text{lim}}}{dR_t} \) is always \( \geq 0 \) and all terms of the denominator are \( \geq 0 \) except \( R_t - 2R_{\text{lim}} \). Thus, if \( R_t \geq 2R_{\text{lim}} \), the denominator is always \( \geq 0 \). If \( R_t < 2R_{\text{lim}} \), we have to use certain inequalities to prove that (at least) for \( b_N \leq 2 \) the numerator will be \( \geq 0 \). However, since \( b_N \) was restricted to \( b_N \leq 2 \) in Section 3.5.2.2, it suffices to check the minimum estimated value of the denominator at \( b_N = 2 \). The positive (or zero) numerator and denominator yield a \( \frac{dR_{\text{lim}}}{dR_t} \leq 0 \).

We then know that for all possible values of \( R_{s0}, a_n \), and \( b_n \) according to (3.1),

\[
\frac{d\varphi_{\text{inner}}(R_{s0} = R_{\text{lim}})}{dR_t} \leq 0 \text{ if } R_t > R_i - R_{s0} \text{ and (at least) } b_N \leq 2.
\]

As indicated above, we begin the actual proof by assuming all \( a_n = 0 \), thus \( R_s = R_{s0} \). We then obtain

\[
\left. \frac{dR_{\text{lim}}}{dR_t} \right|_{a_n=0 \forall n} = 0.
\]

Then \( R_{\text{lim}} = R_i - R_{s0} \) for any \( R_t > R_i - R_{s0} \) independently of \( R_t \).

If at least one \( a_n > 0 \), the numerator of \( \frac{dR_{\text{lim}}}{dR_t} \) is always \( \geq 0 \) and all terms of the denominator are \( \geq 0 \) except \( R_t - 2R_{\text{lim}} \).

Thus, if \( R_t \geq 2R_{\text{lim}} \), the denominator is always \( \geq 0 \), too. Conversely, if \( R_t < 2R_{\text{lim}} \), we obtain from (3.21), (3.1) and (3.19) that

\[
R_s(R_{\text{lim}}) - R_{s0} = R_i - R_{\text{lim}} - R_{s0} = \sum_{n=1}^{N} a_n \left( \sqrt{R_{\text{lim}}(R_t - R_{\text{lim}})} \right)^{b_n}.
\]

Hence,

\[
2R_{\text{lim}}(R_t - R_{\text{lim}}) + \sum_{n=1}^{N} a_n b_n \left( \sqrt{R_{\text{lim}}(R_t - R_{\text{lim}})} \right)^{b_n} (R_t - 2R_{\text{lim}}) \geq 2R_{\text{lim}}(R_t - R_{\text{lim}}) + b_N(R_t - R_{\text{lim}} - R_{s0})(R_t - 2R_{\text{lim}})
\]

holds due to \( b_N > b_{N-1} > \ldots > b_1 > 0 \).

Since \( b_N \) was restricted to \( b_N \leq 2 \) in Section 3.5.2.2, it suffices to check if the minimum estimated value is \( \geq 0 \). Its minimum occurs at \( b_N = 2 \), yielding

\[
2R_{\text{lim}}(R_t - R_{\text{lim}}) + 2(R_i - R_{\text{lim}} - R_{s0})(R_t - 2R_{\text{lim}}) = 2((R_i - R_{s0})(R_t - (R_i - R_{s0})) + (R_i - R_{\text{lim}} - R_{s0})^2),
\]

which is always \( \geq 0 \) if \( R_t > R_i - R_{s0} \).

Combining all preceding parts of the proof, we may conclude that numerator and denominator of \( \frac{dR_{\text{lim}}}{dR_t} \) are \( \geq 0 \) and thus \( \frac{dR_{\text{lim}}}{dR_t} \) and \( \frac{d\varphi_{\text{inner}}(R_{s0} = R_{\text{lim}})}{dR_t} \) remain \( \leq 0 \) if \( R_t > R_i - R_{s0} \) and (at least) \( b_N \leq 2 \).
Summarising both proofs yields a maximum $g_{\pi, \text{inma}}$ at $R_t = R_i - R_{\text{in}}$ if $0 < b_N \leq 2$ and if $R_{\text{in}}$ was chosen according to Section 3.5.2.2. If $R_t$ is lower or higher, $g_{\pi, \text{inma}}$ decreases except for $R_i = R_{\text{in}}$, where it stays constant at its maximum value if $R_t$ increases further.

### B.2.2 T network

#### B.2.2.1 Optimum value of parameter $G_m$ for maximum power gain

As indicated in the overview of the proof given in Section 3.5.3.2, we begin the actual proof by assuming all $a_n = 0$, thus $R_i = R_{\text{in}}$ throughout the valid range of $G_m$. We then obtain from (3.39)

$$
\left( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \right)_{a_n=0} = \frac{G_p \left( \frac{1}{\omega L_p^2} - G_p^2 - 2 \frac{1}{\omega L_p} \left( G_p \frac{\partial (1/L_p)}{\partial G_p} + G_m \frac{\partial (1/L_p)}{\partial G_m} \right) \right)}{G_m \left( \frac{1}{\omega L_p^2} - G_p^2 - 2 \frac{1}{\omega L_p} G_p \frac{\partial (1/L_p)}{\partial G_p} \right)}.
$$

Applying (3.32), (3.40), and (3.41) yields

$$
\left( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \right)_{a_n=0} = \frac{G_p (G_m + G_p)}{\sqrt{G_m (G_t - G_m)}} \cdot \frac{2 \sqrt{G_m (G_t - G_m)} \sqrt{(G_m + G_p)(G_i - G_m - G_p) + G_m (G_i - G_m - G_p + G_t - G_m) + G_p (G_t - G_m)}}{G_m (G_t - G_m + G_i - G_m) \sqrt{(G_m + G_p)(G_i - G_m - G_p) + G_m (G_i - G_m - G_p + G_t - G_m) + G_p (G_t - G_m)}}
$$

Since $G_i$, $G_t$, $G_m$, $G_p \geq 0$, and (3.33) applies, numerator and denominator of

$$
\left( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \right)_{a_n=0} \text{ are always } \geq 0.
$$

If at least one $a_n > 0$, we begin by deriving some inequalities which we will use to check numerator and denominator of (3.39) separately.

By thoroughly reviewing numerator and denominator, we identify two common terms. For the second one we can prove that

$$
2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega^2 L_p^2} - G_p^2 \right) \sum_{n=1}^{N} a_n b_n \lambda^{b_n-1} \geq 0
$$

holds if $0 < b_N \leq 2$.

The proof has to be split in two cases.

If $G_p \geq \frac{1}{\omega L_p}$ and $G_p$, $\omega L_p$, $a_n \geq 0$, and $b_n > 0$, the above estimate is valid for any $b_N$. 

However, if \( G_p < \frac{1}{\omega L_p} \), several estimates are needed. Since \( b_N > b_{N-1} > \ldots > b_1 > 0 \) and \( R_s \geq \sum_{n=1}^{N} a_n (\omega L_s)^{b_n} \) (from (3.1)), we obtain after applying (A.16) and (A.17)

\[
2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega^2 L_p^2} - G_p^2 \right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1} \geq 2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega^2 L_p^2} - G_p^2 \right) b_N \frac{\sum_{n=1}^{N} a_n \Lambda^{b_n}}{\Lambda} \geq 2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega^2 L_p^2} - G_p^2 \right) b_N \frac{\sum_{n=1}^{N} a_n \Lambda^{b_n}}{\Lambda} \geq G_p \left( (2 - b_N) \frac{1}{\omega^2 L_p^2} + b_N G_p^2 \right) \left( \frac{2}{\omega L_p} \right),
\]

which is always \( \geq 0 \) if \( 0 < b_N \leq 2 \) holds\(^4\).

**From now on** we limit the inductor’s \( Q \) factor to \( Q \geq 1 \), which just indicates that \( \omega L_s \geq R_s \) or \( \frac{1}{\omega L_p} \geq G_p \) (from (A.17) and (A.16)) and is likely to assume for a useful variable inductor.

Combined with the preceding estimate, we finally obtain

\[
2 G_p \frac{1}{\omega L_p} \geq 2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega^2 L_p^2} - G_p^2 \right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1} \geq 0,
\]

which we will use in the following proof.

After deriving all necessary estimates, we check numerator and denominator of (3.39) separately:

1. Since \( \omega L_p, G_m, G_p, a_n \) are \( \geq 0 \), \( b_n > 0 \), and the preceding estimates apply, the numerator will always be \( \geq 0 \) for \( G_p \frac{\partial (\frac{1}{\omega L_p})}{\partial G_p} + G_m \frac{\partial (\frac{1}{\omega L_p})}{\partial G_m} \leq 0 \).

Conversely, if \( G_p \frac{\partial (\frac{1}{\omega L_p})}{\partial G_p} + G_m \frac{\partial (\frac{1}{\omega L_p})}{\partial G_m} > 0 \), the preceding estimates apply and

\[
G_p \left( \frac{1}{\omega^2 L_p^2} - G_p^2 + 2 G_p \frac{1}{\omega L_p} \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1} \right) - \left( 2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega^2 L_p^2} - G_p^2 \right) \sum_{n=1}^{N} a_n b_n \Lambda^{b_n-1} \right) \\
\cdot \left( G_p \frac{\partial (\frac{1}{\omega L_p})}{\partial G_p} + G_m \frac{\partial (\frac{1}{\omega L_p})}{\partial G_m} \right) \geq G_p \left( \frac{1}{\omega^2 L_p^2} - G_p^2 \right) - 2 G_p \frac{1}{\omega L_p} \left( G_p \frac{\partial (\frac{1}{\omega L_p})}{\partial G_p} + G_m \frac{\partial (\frac{1}{\omega L_p})}{\partial G_m} \right)
\]

\(^4\)The proof would still be valid if we would put \( b_N = 0 \), but its initial definition in (3.1) was \( b_N > b_{N-1} > \ldots > b_1 > 0 \).
holds.

After applying (3.32), (3.40), (3.41), and $G_m \geq 0$, we finally have to check

$$
\frac{1}{\omega L_p} - G_p^2 - 2 \frac{1}{\omega L_p} \left( G_p \frac{\partial (1/L_p)}{\partial G_p} + G_m \frac{\partial (1/L_p)}{\partial G_m} \right)
$$

$$= G_m \left( G_m + G_p \right) \left( 2 + \frac{G_m (G_l - G_m - G_p + G_{l - G_m}) + G_p (G_l - G_m)}{\sqrt{G_m (G_l - G_m)} \sqrt{(G_m + G_p) (G_l - G_m - G_p)}} \right),
$$

which is always $\geq 0$ throughout the (utmost) valid range of $G_m$ indicated in (3.33).

2. Since $\omega L_p$, $G_m$, $G_p$, $a_n$ are $\geq 0$, $b_n > 0$, and the preceding estimates apply, the denominator will always be $\geq 0$ for $\frac{\partial (1/L_p)}{\partial G_p} \leq 0$.

Conversely, if $\frac{\partial (1/L_p)}{\partial G_p} > 0$, the preceding estimates apply and

$$
G_m \left( \frac{1}{\omega L_p} - G_p^2 + 2 G_p \frac{1}{\omega L_p} \left( \sum_{n=1}^{N} a_n b_n A^{b_n-1} \right) 
- \left( 2 G_p \frac{1}{\omega L_p} - \left( \frac{1}{\omega L_p} - G_p^2 \right) \sum_{n=1}^{N} a_n b_n A^{b_n-1} \right) \frac{\partial (1/L_p)}{\partial G_p} \right)
\geq G_m \left( \frac{1}{\omega^2 L_p} - G_p^2 - 2 \frac{1}{\omega L_p} G_p \frac{\partial (1/L_p)}{\partial G_p} \right)
$$

holds.

After applying (3.32), (3.41), and $G_m \geq 0$, we finally have to check

$$
\frac{1}{\omega^2 L_p} - G_p^2 - 2 \frac{1}{\omega L_p} G_p \frac{\partial (1/L_p)}{\partial G_p}
$$

$$= G_m \left( G_l - G_m + G_{l - G_m} \right) + \frac{(G_l G_p + 2 G_m (G_l - G_m - G_p)) \sqrt{G_m (G_l - G_m)}}{\sqrt{(G_m + G_p) (G_l - G_m - G_p)}},
$$

which is always $\geq 0$ throughout the (utmost) valid range of $G_m$ indicated in (3.33).

Combining all preceding parts of the proof, we may conclude that (at least) for $0 < b_N \leq 2$ and $Q \geq 1$ (hence $\omega L_s \geq R_s$ or $\frac{1}{\omega L_p} \geq G_p$), numerator and denominator of $\frac{G_m}{G_m} - \frac{dG_m}{dG_m}$ in (3.39) and thus $\frac{dG_m}{dG_m}$ itself remains $\geq 0$ throughout the (utmost) valid range of $G_m$ indicated in (3.33).
B.2.2.2 Optimum value of load conductance $G_\ell$ for maximum power gain

In Section 3.5.3.2 the parameter $G_m$ was optimised for maximum $g_{t,\text{inna}}$. However, the maximum $g_{t,\text{inna}}$ still depends on $G_\ell$ (for the intended application, $G_1$ is fixed and known).

Thus it’s necessary to investigate which $G_\ell$ yields the maximum $g_{t,\text{inna}}$ if $G_m$ is chosen for optimum $g_{t,\text{inna}}$ according to Section 3.5.3.2.

Which $G_\ell$ is likely to yield the maximum $g_{t,\text{inna}}$?

As indicated in Section 3.5.3.1, $L_{ap} \to \infty$ for $G_\ell = G_1$, the limit which separates the two possible (utmost) upper limits of $G_m$. Then, according to case (ii) in (A.18), the lowest $G_p$ occurs ($G_p = 0$).

Supposing that the highest $g_{t,\text{inna}}$ coincides with the lowest $G_p$ requires proving of
\[
\frac{dg_{t,\text{inna}}(G_m=G_\ell)}{dG_\ell} \geq 0 \text{ if } G_\ell \leq G_1 \text{ or } \frac{dg_{t,\text{inna}}(G_m=G_m_{\text{im}})}{dG_\ell} \leq 0 \text{ if } G_\ell > G_1,
\]
respectively, thereby obtaining that $g_{t,\text{inna}}$ at $G_m = G_\ell = G_1$ is greater (or equal) than any other $g_{t,\text{inna}}$.

According to Section 3.5.3.2, the proof has to be split in two parts:

(i) $G_\ell \leq G_1$, hence $0 \leq G_m \leq G_\ell$

The highest $g_{t,\text{inna}}$ is obtained if $G_m = G_\ell$. $g_{t,\text{inna}}$ is given by (3.37),
\[
g_{t,\text{inna}}(G_m = G_\ell) = \frac{G_\ell}{G_\ell + G_p(G_m = G_\ell)}.
\]

Its derivative with respect to $G_\ell$ is
\[
\frac{dg_{t,\text{inna}}(G_m=G_\ell)}{dG_\ell} = \frac{G_\ell}{(G_\ell + G_p(G_m = G_\ell))^2} \left( \frac{G_p(G_m = G_\ell)}{G_\ell} - \frac{dG_p(G_m = G_\ell)}{dG_\ell} \right).
\]

Since $G_\ell$, $G_p(G_m = G_\ell) \geq 0$, it suffices to check $\frac{G_p(G_m = G_\ell)}{G_\ell} - \frac{dG_p(G_m = G_\ell)}{dG_\ell}$.

Using (3.1) and (A.19), we define an implicit function for $R_{\ell}(G_m = G_\ell)$,
\[
\tilde{\psi}_\ell := R_{\ell,0} - \left( \sum_{n=1}^{N} a_n \left( \frac{\omega L_p(G_m = G_\ell)}{G_\ell^2(G_m = G_\ell) + \omega^2 L_p^2(G_m = G_\ell)} \right)^b_n \right) - \frac{G_p(G_m = G_\ell)}{G_p^2(G_m = G_\ell) + \omega^2 L_p^2(G_m = G_\ell)} \equiv 0.
\]
Comparing the implicit function and the factor to check to those of Section 3.5.3.2, it’s quite obvious that the proofs are similar if $G_p$ is replaced with $G_p(G_m = G_t)$ and $G_m$ with $G_t$. Regarding the derivatives, those with respect to $G_p$ or $G_m$ are replaced with derivatives with respect to $G_p(G_m = G_t)$ or $G_t$.

Then $\frac{G_p(G_m = G_t)}{G_t} - \frac{dG_p(G_m = G_t)}{dG_t}$ is given by (3.39) if the indicated replacements are applied.

To prove that $\frac{G_p(G_m = G_t)}{G_t} - \frac{dG_p(G_m = G_t)}{dG_t} \geq 0$ the proof of Section 3.5.3.2 may be adapted accordingly. All estimations are valid if the indicated replacements are made, hence prior to the usage of $\frac{1}{\omega L_p}$ and its partial derivatives the terms to check are similar. However, $\frac{1}{\omega L_p}$ and its partial derivatives have changed compared to those of Section 3.5.3.2 and are given by

$$\frac{1}{\omega L_p(G_m = G_t)} = \sqrt{(G_t + G_p(G_m = G_t))(G_t - G_p(G_m = G_t))},$$

$$\frac{\partial}{\partial G_t} \left( \frac{1}{\omega L_p(G_m = G_t)} \right) = \frac{G_t - 2(G_t + G_p(G_m = G_t))}{2 \sqrt{(G_t + G_p(G_m = G_t))(G_t - G_p(G_m = G_t))}},$$

$$\frac{\partial}{\partial G_p(G_m = G_t)} \left( \frac{1}{\omega L_p(G_m = G_t)} \right) = \frac{G_t - 2(G_t + G_p(G_m = G_t))}{2 \sqrt{(G_t + G_p(G_m = G_t))(G_t - G_p(G_m = G_t))}}.$$

$(L_p(G_m = G_t))$ is obtained by putting $G_m = G_t$ in (3.32)).

Similar to Section 3.5.3.2, we begin the proof by assuming all $a_n = 0$, thus $R_s = R_{s0}$ throughout the valid range of $G_m$. We then obtain

$$\left( \frac{G_p(G_m = G_t)}{G_t} - \frac{dG_p(G_m = G_t)}{dG_t} \right) \bigg|_{a_n = 0} = \frac{G_p(G_m = G_t)}{G_t} - \frac{1}{\omega L_p(G_m = G_t)} \left( G_p(G_m = G_t) - G_p(G_m = G_t) \frac{\partial}{\partial G_p(G_m = G_t)} \frac{1}{\omega L_p(G_m = G_t)} \right).$$

Applying $\frac{1}{\omega L_p(G_m = G_t)}$ and its partial derivatives gives

$$\left( \frac{G_p(G_m = G_t)}{G_t} - \frac{dG_p(G_m = G_t)}{dG_t} \right) \bigg|_{a_n = 0} = \frac{G_p(G_m = G_t)}{G_t} \left( G_t + 2G_p(G_m = G_t) \right),$$

which is always $\geq 0$ if $G_t \leq G_1$.

If at least one $a_n > 0$, numerator and denominator of $\frac{G_p(G_m = G_t)}{G_t} - \frac{dG_p(G_m = G_t)}{dG_t}$ are checked separately similar to Section 3.5.3.2, yielding the following terms which have to be proved to be $\geq 0$ if $\omega L_p(G_m = G_t) \geq R_s(G_m = G_t)$ or $\omega L_p(G_m = G_t) \geq G_p(G_m = G_t)$ and $0 < b_N \leq 2$ holds:
1. In case of the numerator $G_p(G_m = G_\ell) \geq 0$, 
\[
\frac{1}{\omega L_p(G_m = G_\ell)} \frac{\delta(L_p(G_m = G_\ell))}{\delta G_\ell},
\]
and \(\frac{\delta(L_p(G_m = G_\ell))}{\delta G_\ell}\) are applied, yielding 
\[
\omega^2 L_p^2(G_m = G_\ell) - G_p^2(G_m = G_\ell) - 2 \frac{1}{\omega L_p(G_m = G_\ell)} G_p(G_m = G_\ell) \frac{\delta(L_p(G_m = G_\ell))}{\delta G_\ell} + G_\ell \frac{\delta(L_p(G_m = G_\ell))}{\delta G_\ell}
\]
\[
= G_\ell (G_\ell + 2 G_p(G_m = G_\ell)),
\]
which is always $\geq 0$.

2. In case of the denominator $G_\ell \geq 0$, 
\[
\frac{1}{\omega L_p(G_m = G_\ell)}\]
and \(\frac{\delta(L_p(G_m = G_\ell))}{\delta G_\ell}\) are applied, yielding 
\[
\omega^2 L_p^2(G_m = G_\ell) - G_p^2(G_m = G_\ell) - 2 \frac{1}{\omega L_p(G_m = G_\ell)} G_p(G_m = G_\ell) \frac{\delta(L_p(G_m = G_\ell))}{\delta G_\ell}
\]
\[
= G_\ell (G_1 - G_\ell),
\]
which is always $\geq 0$ if $G_\ell \leq G_1$.

Combining all preceding parts of the proof, we may conclude that (at least) for $0 < b_N \leq 2$ and $Q \geq 1$ (hence $\omega L_s \geq R_s$ or \(\frac{1}{L_p} \geq G_p\)), numerator and denominator of 
\[
\frac{G_p(G_m = G_\ell)}{G_\ell} - \frac{dG_p(G_m = G_\ell)}{dG_\ell}
\]
and thus \(\frac{dG_{t, \text{inma}}(G_m = G_\ell)}{dG_\ell}\) itself remains $\geq 0$ if $G_\ell \leq G_1$.

(ii) $G_\ell > G_1$, hence $0 \leq G_m \leq G_{\text{mlim}}$

The highest $g_{t, \text{inma}}$ is obtained if $G_m = G_{\text{mlim}} = G_1 - G_p(G_m = G_{\text{mlim}})$. $g_{t, \text{inma}}$ is given by (3.37),
\[
g_{t, \text{inma}}(G_m = G_{\text{mlim}}) = \frac{G_{\text{mlim}}}{G_1}
\]
Its derivative with respect to $G_\ell$ is
\[
\frac{d g_{t, \text{inma}}(G_m = G_{\text{mlim}})}{dG_\ell} = \frac{1}{G_1} \frac{d G_{\text{mlim}}}{dG_\ell}.
\]
Since $G_1 \geq 0$, it suffices to check $\frac{d G_{\text{mlim}}}{dG_\ell}$.
As $R_s(G_m = G_{\text{mlim}})$ is a function of $L_s(G_m = G_{\text{mlim}})$, and both are functions of $G_p(G_m = G_{\text{mlim}})$ and $L_p(G_m = G_{\text{mlim}})$ according to (A.19), (A.20), and $G_\ell$ according to (3.32), where $G_p(G_m = G_{\text{mlim}})$ itself is a function of $G_\ell$ and implicitly\(^5\) of $G_p(G_m = G_{\text{mlim}})$, we define the implicit function $\tilde{\psi}_t \equiv 0$ from (3.1), (A.19), (A.20), (3.32), and (3.34),

$$
\tilde{\psi}_t := R_{s0} + \left( \sum_{n=1}^{N} a_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n} \right) - \frac{G_1 - G_{\text{mlim}}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \equiv 0.
$$

Then we compute the implicit function's partial derivatives with respect to $G_\ell$ and $G_{\text{mlim}}$,

$$
\frac{\partial \tilde{\psi}_t}{\partial G_\ell} = \left( \sum_{n=1}^{N} a_n b_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1} \right) \frac{G_{\text{mlim}} ((G_1 - G_{\text{mlim}})^2 - G_{\text{mlim}} (G_\ell - G_{\text{mlim}}))}{2 \sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}}) ((G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}}))^2}}
+ \frac{G_{\text{mlim}} (G_1 - G_{\text{mlim}})}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})^2},
$$

$$
\frac{\partial \tilde{\psi}_t}{\partial G_{\text{mlim}}} = \left( \sum_{n=1}^{N} a_n b_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1} \right) \frac{G_\ell (G_\ell - G_{\text{mlim}}) + 2 G_{\text{mlim}} G_1 (G_\ell - G_1)}{2 \sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}}) ((G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}}))^2}}
+ \frac{G_1 (G_\ell - G_1)}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})^2}.
$$

Applying the implicit function theorem [64] gives

$$
\frac{dG_{\text{mlim}}}{dG_\ell} = \frac{G_{\text{mlim}} \left( \frac{2 (G_1 - G_{\text{mlim}}) \sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} + (G_1 - G_{\text{mlim}})^2 - G_{\text{mlim}} (G_\ell - G_{\text{mlim}}) \right)}{2 \sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}} \sum_{n=1}^{N} a_n b_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1}.
$$

\(^5\)In particular, $R_s(G_m = G_{\text{mlim}})$, $L_s(G_m = G_{\text{mlim}})$, $G_p(G_m = G_{\text{mlim}})$, and $L_p(G_m = G_{\text{mlim}})$ are all functions of $G_\ell$ only. However, we usually may not be able to solve for $G_p(G_m = G_{\text{mlim}})$ explicitly after applying (A.19), (A.20), and (3.32) to (3.1).
As indicated above, supposed the maximum $g_{t, \text{inna}}$ occurs at $G_t = G_i$, we have to prove that $\frac{dG_{\text{mlim}}}{dG_t} \leq 0$ if $G_t > G_i$.

For ease of understanding, an overview of the following proof is given below.

To prove that numerator and denominator are $\leq 0$ for certain $b_n$, we abbreviate $\tilde{S}_t = \sum_{n=1}^{N} a_nb_n \left( \frac{\sqrt{G_{\text{mlim}} (G_t-G_{\text{mlim}})}}{(G_i-G_{\text{mlim}})^2+G_{\text{mlim}} (G_t-G_{\text{mlim}})} \right)^{b_n-1}$ and obtain

$$\frac{dG_{\text{mlim}}}{dG_t} = -\frac{G_{\text{mlim}} \left( 2(G_i-G_{\text{mlim}}) \sqrt{G_{\text{mlim}} (G_t-G_{\text{mlim}})} + \left( (G_i-G_{\text{mlim}})^2 - G_{\text{mlim}} (G_t-G_{\text{mlim}}) \right) \tilde{S}_t \right)}{2 G_i (G_t-G_i) \sqrt{G_{\text{mlim}} (G_t-G_{\text{mlim}})} + \left( G_t (G_i^2 - G_t G_{\text{mlim}}) + 2G_{\text{mlim}} (G_t-G_i) \right) \tilde{S}_t}.$$  

We firstly assume all $a_n = 0$ or $R_s = R_{s0}$, hence $\tilde{S}_t = 0$ and calculate $\frac{dG_{\text{mlim}}}{dG_t}$, which will be $\leq 0$ for any $G_t > G_i$.

Secondly, we assume that at least one $a_n > 0$, hence $R_s > R_{s0}$ (according to (3.1)) or $\tilde{S}_t > 0$. We then prove that $b_N \frac{G_i-G_{\text{mlim}}}{\sqrt{G_{\text{mlim}} (G_t-G_{\text{mlim}})}} \geq \tilde{S}_t$ and check numerator and denominator separately:

1. If $(G_i - G_{\text{mlim}})^2 - G_{\text{mlim}} (G_t - G_{\text{mlim}}) \geq 0$, the numerator is always $\geq 0$. Conversely, if $(G_i-G_{\text{mlim}})^2 - G_{\text{mlim}} (G_t-G_{\text{mlim}}) < 0$, we have to use the preceding estimate. Since $b_N$ was restricted to $b_N \leq 2$ in Section 3.5.3.2, it suffices to check the minimum estimated value of the numerator at $b_N = 2$, which will be $\geq 0$.

2. If $G_t \left( G_i^2 - G_t G_{\text{mlim}} \right) + 2G_{\text{mlim}} G_i (G_t - G_i) \geq 0$, hence $G_{\text{mlim}} \leq \frac{G_t G_i^2}{(G_i-G_t)^2+G_i^2}$, the denominator is always $\geq 0$. Conversely, if $G_{\text{mlim}} > \frac{G_t G_i^2}{(G_i-G_t)^2+G_i^2}$, we have to use the preceding estimate. Since $b_N$ was restricted to $b_N \leq 2$ in Section 3.5.3.2, it suffices to check the minimum estimated value of the denominator at $b_N = 2$. To prove that this value is $\geq 0$, we consider $0 \leq G_{\text{mlim}} \leq G_i < G_t$ yielding $G_t \geq 2G_i$ if $G_{\text{mlim}} > \frac{G_t G_i^2}{(G_i-G_t)^2+G_i^2}$.

We then know that for all possible values of $R_{s0}$, $a_n$, and $b_n$ according to (3.1), $\frac{dG_{\text{mlim}}}{dG_t}$ and thus $\frac{d g_{t, \text{inna}} (G_m=G_{\text{mlim}})}{dG_t}$ itself is always $\leq 0$ if $G_t > G_i$ and (at least) $b_N \leq 2$.

As indicated above, we begin the actual proof by assuming all $a_n = 0$, thus $R_s = R_{s0}$ for any valid $G_m$. We then obtain

$$\left. \frac{dG_{\text{mlim}}}{dG_t} \right|_{a_n=0} = -\frac{G_{\text{mlim}} \left( 2(G_i-G_{\text{mlim}}) \sqrt{G_{\text{mlim}} (G_t-G_{\text{mlim}})} \right)}{2 G_i (G_t-G_i) \sqrt{G_{\text{mlim}} (G_t-G_{\text{mlim}})}}.$$  

Numerator and denominator of this fraction are always $\geq 0$ if $G_t > G_i$, hence $\frac{dG_{\text{mlim}}}{dG_t}$ is always $\leq 0$.

If at least one $a_n > 0$, we begin by deriving an inequality which we will use to check numerator and denominator separately.
Since \( b_N > b_{N-1} > \ldots > b_1 > 0 \),
\[
\sum_{n=1}^{N} a_n b_N \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1} \geq \sum_{n=1}^{N} a_n b_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1}
\]
holds.

Furthermore, from the definition of the implicit function we obtain
\[
\frac{G_1 - G_{\text{mlim}}}{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}} \geq \sum_{n=1}^{N} a_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n} \quad \text{or}
\]
\[
\frac{G_1 - G_{\text{mlim}}}{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}} \geq \sum_{n=1}^{N} a_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1}.
\]

Combining both estimates yields
\[
b_N \frac{G_1 - G_{\text{mlim}}}{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}} \geq \sum_{n=1}^{N} a_n b_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1}.
\]

After deriving the required estimate, we check numerator and denominator of \( \frac{dG_{\text{mlim}}}{dG_\ell} \) separately:

1. If \( (G_1 - G_{\text{mlim}})^2 - G_{\text{mlim}} (G_\ell - G_{\text{mlim}}) \geq 0 \), the numerator of \( \frac{dG_{\text{mlim}}}{dG_\ell} \) is always \( \geq 0 \).

Conversely, if \( (G_1 - G_{\text{mlim}})^2 - G_{\text{mlim}} (G_\ell - G_{\text{mlim}}) < 0 \), the preceding estimate applies and
\[
G_{\text{mlim}} \left( 2(G_1 - G_{\text{mlim}}) \sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right.
\]
\[
+ \left( (G_1 - G_{\text{mlim}})^2 - G_{\text{mlim}} (G_\ell - G_{\text{mlim}}) \right)
\]
\[
\cdot \sum_{n=1}^{N} a_n b_n \left( \frac{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}}{(G_1 - G_{\text{mlim}})^2 + G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)^{b_n-1} \bigg) \geq G_{\text{mlim}} \left( \frac{2(G_1 - G_{\text{mlim}}) G_{\text{mlim}} (G_\ell - G_{\text{mlim}})}{\sqrt{G_{\text{mlim}} (G_\ell - G_{\text{mlim}})} \right)
\]
\[
+ \left( (G_1 - G_{\text{mlim}})^2 - G_{\text{mlim}} (G_\ell - G_{\text{mlim}}) \right) b_N (G_1 - G_{\text{mlim}}) \bigg). \]
As \( 0 \leq G_{\text{mim}} \leq G_i < G_\ell \), \( G_{\text{mim}}, G_i - G_{\text{mim}}, \) and the denominator of the fractions are \( \geq 0 \). Since \( b_N \) was restricted to \( b_N \leq 2 \) in Section 3.5.3.2, it suffices to check if the minimum estimated value is \( \geq 0 \). Its minimum occurs at \( b_N = 2 \), yielding
\[
2 G_{\text{mim}} (G_\ell - G_{\text{mim}}) + \left((G_i - G_{\text{mim}})^2 - G_{\text{mim}} (G_\ell - G_{\text{mim}})\right)2
= 2 (G_i - G_{\text{mim}})^2,
\]
which is always \( \geq 0 \) according to (3.33).
Hence the numerator of \( \frac{dG_{\text{mim}}}{dG_\ell} \) is always \( \geq 0 \) if \( G_\ell > G_i \) and (at least) \( b_N \leq 2 \).

2. The denominator of \( \frac{dG_{\text{mim}}}{dG_\ell} \) is always \( \geq 0 \) if
\[
G_\ell (G_i^2 - G_\ell G_{\text{mim}}) + 2 G_{\text{mim}} G_i (G_\ell - G_i)
= -((G_\ell - G_i)^2 + G_i^2) \left(G_{\text{mim}} - \frac{G_\ell G_i^2}{(G_\ell - G_i)^2 + G_i^2}\right)
\]
is \( \geq 0 \), which is equivalent to \( G_{\text{mim}} \leq \frac{G_\ell G_i^2}{(G_\ell - G_i)^2 + G_i^2} \).

Conversely, if \( G_\ell (G_i^2 - G_\ell G_{\text{mim}}) + 2 G_{\text{mim}} G_i (G_\ell - G_i) < 0 \) or \( G_{\text{mim}} > \frac{G_\ell G_i^2}{(G_\ell - G_i)^2 + G_i^2} \), the preceding estimate applies and
\[
\frac{2 G_1 (G_\ell - G_i) \sqrt{G_{\text{mim}} (G_\ell - G_{\text{mim}})}}{G_{\text{mim}} (G_\ell - G_{\text{mim}}) + 2 G_{\text{mim}} G_i (G_\ell - G_i)}
\cdot \sum_{n=1}^{N} a_n b_n \left(\frac{\sqrt{G_{\text{mim}}}}{(G_\ell - G_{\text{mim}})^2 + G_{\text{mim}} (G_\ell - G_{\text{mim}})}\right)^{b_n - 1}
\geq \frac{2 G_1 (G_\ell - G_i) G_{\text{mim}} (G_\ell - G_{\text{mim}})}{\sqrt{G_{\text{mim}} (G_\ell - G_{\text{mim}})}}
+ \left(G_\ell (G_i^2 - G_\ell G_{\text{mim}}) + 2 G_{\text{mim}} G_i (G_\ell - G_i)\right) b_N (G_i - G_{\text{mim}}) \left(\frac{G_{\text{mim}}}{G_{\text{mim}} (G_\ell - G_{\text{mim}})}\right).
\]
The denominator of the fractions is \( \geq 0 \). Since \( b_N \) was restricted to \( b_N \leq 2 \) in Section 3.5.3.2, it suffices to check if the minimum estimated value is \( \geq 0 \). Its minimum occurs at \( b_N = 2 \), yielding
\[
2 G_1 (G_\ell - G_i) G_{\text{mim}} (G_\ell - G_{\text{mim}})
+ (G_\ell (G_i^2 - G_\ell G_{\text{mim}}) + 2 G_{\text{mim}} G_i (G_\ell - G_i)) 2 (G_i - G_{\text{mim}})
= 2 G_i^2 (G_i^2 + (G_i - G_{\text{mim}})^2) + 2 (G_i^3 + G_{\text{mim}}^2 (G_\ell - G_i)) (G_\ell - 2 G_i),
\]
which is always \( \geq 0 \) if \( G_\ell > 2 G_i \). The latter can be proved if \( G_{\text{mim}} > \frac{G_\ell G_i^2}{(G_\ell - G_i)^2 + G_i^2} \) and \( 0 \leq G_{\text{mim}} \leq G_i < G_\ell \) are considered, hence
\[
0 < G_\ell - G_{\text{mim}} < G_\ell - \frac{G_\ell G_i^2}{(G_\ell - G_i)^2 + G_i^2} = \frac{G_\ell (G_\ell - G_i)^2}{(G_\ell - G_i)^2 + G_i^2},
\]
\[
0 \leq G_i - G_{\text{mim}} < G_i - \frac{G_i G_i^2}{(G_\ell - G_i)^2 + G_i^2} = \frac{G_i (G_\ell - G_i) (G_\ell - 2 G_i)}{(G_\ell - G_i)^2 + G_i^2}.
\]
Since \( \frac{G_t G_i^2}{(G_t-G_i)^2+G_i^2} \) is always \( \leq G_t \), but is \( \leq G_i \) only if \( G_t \geq 2G_i \), a valid \( G_{\text{lim}} > \frac{G_t G_i^2}{(G_t-G_i)^2+G_i^2} \) may exist\(^6\) only if \( G_t \geq 2G_i \).

Hence the denominator of \( \frac{dG_{\text{lim}}}{dG_t} \) is always \( \geq 0 \) if \( G_t > G_i \) and (at least) \( b_N \leq 2 \).

Combining all preceding parts of the proof, we may conclude that numerator and denominator of \( \frac{dG_{\text{lim}}}{dG_t} \) are \( \geq 0 \) and thus \( \frac{dG_{\text{lim}}}{dG_t} \) and \( \frac{d\theta_{\text{inma}}(G_m=G_{\text{lim}})}{dG_t} \) remain \( \leq 0 \) if \( G_t > G_i \), and (at least) \( b_N \leq 2 \).

Summarising both proofs yields a maximum \( g_{t,\text{inma}} \) at \( G_t = G_i \) if the inductor’s \( Q \) factor is limited to \( Q \geq 1 \) (hence \( \omega L_a \geq R_s \) or \( \frac{1}{\omega T_p} \geq G_p \) ), \( 0 < b_N \leq 2 \) and if \( G_m \) was chosen according to Section 3.5.3.2. If \( G_t \) is lower or higher, \( g_{t,\text{inma}} \) decreases.

### B.3 Matching of a resistive load to a resistive source (matching at the network’s input) including the losses of all network elements

#### B.3.1 Introduction

As indicated in Section 3.5.1, the power gain of \( \pi \) or \( T \) networks may be derived in a simpler manner than that used in Sections 3.5.2.1 and 3.5.3.1. This simpler approach is e.g. applied by [51b] and based on taking into account that active power may be transferred to resistive circuit elements only. Due to its simplicity, it’s ideally suited to include the losses of all network elements.

To illustrate the method, it’s applied to \( \pi \) and \( T \) networks which are designed to match a resistive load to a resistive source at the network’s input.

#### B.3.2 \( \pi \) network

![\pi network's equivalent circuit including losses of all network elements](image)

Figure B.1: \( \pi \) network’s equivalent circuit including losses of all network elements

Consider the \( \pi \) network of Figure B.1 which takes into account the losses of all network elements. Comparison to the network designed in Section 3.5.2.1 yields

\(^6\)It should be noted that depending on the given values of \( R_s, G_t, \) and \( G_i, G_{\text{lim}} \leq \frac{G_t G_i^2}{(G_t-G_i)^2+G_i^2} \) may also occur if \( G_t \geq 2G_i \).
that it is still applicable for the “inner” $\pi$ network consisting of $C_1$, $R_s$, $L_s$, and $C_2$, which transforms the load $R_{\text{tress}} = \frac{1}{\frac{1}{R_{\text{pC1}}} + \frac{1}{R_t}}$ to $R_{\text{tress}} = \frac{1}{\frac{1}{R_t} - \frac{1}{R_{\text{pC1}}}}$ at the network’s input.

The resulting input resistance $R_{\text{tress}}$ the “inner” $\pi$ network is matched to is chosen to obtain the required input resistance $R_i$ of the total network whose input resistance at matching frequency is equal to $R_{\text{tress}}$ in parallel to $R_{\text{pC1}}$.

Figure B.2: Definitions and setup for power gain derivation of the $\pi$ network including losses of all network elements

Thus the $\pi$ network transforms its load $R_t$ to $R_i$ at the network’s input if its elements are set according to the design formulae derived in Section 3.5.2.1 where $R_i$ is replaced with $R_{\text{tress}}$ and $R_t$ with $R_{\text{tress}}$. Then, at matching frequency, the $\pi$ network’s resulting equivalent circuit is given by Figure B.2. Similar to Section 3.5.2.1, $L_s$ is split in $L_{s1,2}$ to illustrate reactive cancellation in each of the two L networks building the $\pi$ network.

If the $\pi$ network’s input is matched to $R_i$, the voltage at its input is equal to $\frac{V_0}{2}$. Hence the active power delivered to its input is given by

$$P_{\text{act1}} = \frac{1}{2} \frac{|V_0|^2}{R_i} = \frac{1}{8} \frac{V_0^2}{R_i} \quad (2.22) \quad P_{\text{actmax}}.$$

$P_{\text{act2}}$, the active power transferred to the “inner” $\pi$ network, is equal to $P_{\text{act1}}$ diminished by the active power $R_{\text{pC1}}$ consumes, thus

$$P_{\text{act2}} = P_{\text{act1}} - \frac{1}{2} \frac{|V_{\text{act}}|^2}{R_{\text{pC1}}} = P_{\text{actmax}} \left(1 - \frac{R_i}{R_{\text{pC1}}}\right).$$

Since $C_1$ and $L_{s1}$ are purely reactive, they do not consume any active power, and $P_{\text{act2}} = P_{\text{act2}}$.

Figure B.3: Equivalent circuit to derive $P_{\text{act3}}$ and $P_{\text{actf}}$ according to [51b]
For our chosen parametrisation, $R_m$ is the input resistance of the partial network to the right of $R_s$. Then, as indicated in Figure B.3 a), the same current $I_{sr}$ flows through $R_s$ and $R_m$. Thus we obtain $\tilde{P}_{act2} = \frac{1}{2} (R_m + R_s) |I_{sr}|^2$ and $P_{act3} = \frac{1}{2} R_m |I_{sr}|^2$, or

$$P_{act3} = \frac{R_m}{R_m + R_s} \tilde{P}_{act2} = P_{actmaxi} \frac{1}{1 + \frac{R_s}{R_m}} \left(1 - \frac{R_l}{R_{pC1}}\right).$$

Again $L_{s2}$ and $C_2$ are purely reactive, do not consume any active power, and $\tilde{P}_{act3} = P_{act3}$.

Then, as shown in Figure B.3 b), the same voltage $V_{pr}$ is applied to $R_{pC2}$ and $R_\ell$. Thus $\tilde{P}_{act3} = \frac{1}{2} |V_{pr}|^2 \left(\frac{1}{R_{pC2}} + \frac{1}{R_\ell}\right)$ and $P_{act} = \frac{1}{2} |V_{pr}|^2 \frac{1}{R_\ell}$, or

$$P_{act} = \frac{\tilde{P}_{act3}}{1 + \frac{R_\ell}{R_{pC2}}} = P_{actmaxi} \frac{1}{1 + \frac{R_s}{R_m}} \frac{1 - \frac{R_l}{R_{pC1}}}{1 + \frac{R_\ell}{R_{pC2}}}.$$

Summarising the preceding considerations yields the power gain of the $\pi$ network including losses of all network elements,

$$g_{\pi, inna} = \frac{1}{1 + \frac{R_s}{R_m}} \frac{1 - \frac{R_l}{R_{pC1}}}{1 + \frac{R_\ell}{R_{pC2}}}.$$  \hspace{1cm} (B.1)

Despite its apparent simplicity, the formula is quite complicated because usually $R_{pC1,2}$ and $R_s$ are functions of the parameter $R_m$.

Even for the “simple” case of constant $Q$ factors of all network elements, hence $R_{pC1} = \frac{Q_1}{\omega C_{1(p)}}$, $R_s = \frac{1}{Q} \omega L_s$, and $R_{pC2} = \frac{Q_2}{\omega C_{2(p)}}$, explicit derivation of those functions is somewhat difficult.

One approach is to put $\frac{Q_2}{\omega C_{2}}$ and the design formula of $C_2$ (given by (3.18) if $R_\ell$ is replaced with $R_{frest}$) in $R_{frest} = \frac{1}{R_{pC2} + R_\ell}$ and solve the resulting equation for $R_{frest}$ — the solution has to be real, positive, and $< R_\ell$.

Then we perform similar calculations for $R_{pC1}$ and $R_{frest}$, put the solutions in $R_s = \frac{1}{Q} \omega L_s$, where $L_s$ is given by (3.19) if $R_l$ is replaced with $R_{frest}$ and $R_\ell$ with $R_{frest}$, and solve for $R_s$. However, more than one solution is obtained. Thus we additionally need to derive which solution is real and applies in which range of $R_m$.

Since the derivations sketched are quite lengthy, time consuming, and not needed within this work, they are not included herein.
B.3.3 T network

Consider the T network of Figure B.4 which takes into account the losses of all network elements. Comparison to the network designed in Section 3.5.3.1 yields that it is still applicable for the “inner” T network consisting of \( C_1 \), \( G_p \), \( L_p \), and \( C_2 \), which transforms the load \( G_{\text{rest}} = \frac{1}{R_{\text{rest}}} = \frac{1}{R_{ac2} + R_t} = \frac{1}{G_{ac2} + G_t} \) to \( G_{\text{rest}} = \frac{1}{R_{\text{rest}}} = \frac{1}{R_t - R_{ac1}} = \frac{1}{R_t - \frac{1}{G_{ac1}}} \) at the network’s input (in the formulae, the definitions \( G_1 = \frac{1}{R_i} \), \( G_t = \frac{1}{R_t} \), and \( G_{ac1,2} = \frac{1}{R_{ac1,2}} \) were used).

The resulting input conductance \( G_{\text{rest}} \) the “inner” T network is matched to is chosen to obtain the required input conductance \( G_1 = \frac{1}{R_i} \) of the total network whose input conductance at matching frequency is equal to \( G_{\text{rest}} \) in series to \( G_{ac1} \).

Thus the T network transforms its load \( G_t = \frac{1}{R_t} \) to \( G_1 = \frac{1}{R_1} \) at the network’s input if its elements are set according to the design formulae derived in Section 3.5.3.1 where \( G_1 \) is replaced with \( G_{\text{rest}} \) and \( G_t \) with \( G_{\text{rest}} \). Then, at matching frequency, the T network’s resulting equivalent circuit is given by Figure B.5. Similar to Section 3.5.3.1, \( L_p \) is split in \( L_{p1,2} \) to illustrate susceptive cancellation in each of the two L networks building the T network.

If the T network’s input is matched to \( R_t = \frac{1}{G_t} \), the voltage at its input is equal to \( \frac{V_o}{2} \). Hence the active power delivered to its input is given by

\[
P_{\text{act1}} = \frac{1}{2} \left| \frac{V_o}{2} \right|^2 = \frac{1}{8} \frac{V_o^2}{R_t} \overset{(2.22)}{=} P_{\text{actmaxi}}
\]

and the current flowing into the “inner” T network is equal to \( \frac{V_o}{2R_t} \).
$P_{\text{act}2}$, the active power transferred to the “inner” T network, is equal to $P_{\text{act}1}$ diminished by the active power $R_{\text{s}C_1}$ consumes, thus

$$P_{\text{act}2} = P_{\text{act}1} - \frac{1}{2} R_{\text{s}C_1} \left[ \frac{|V_{\text{pt}}|^2}{4R_i^2} \right] = P_{\text{act maxi}} \left( 1 - \frac{R_{\text{s}C_1}}{R_i} \right).$$

Since $C_1$ and $L_{\text{st}}$ are purely reactive, they do not consume any active power, and $P_{\text{act}2} = P_{\text{act}3}$.

![Equivalent Circuit](image)

Figure B.6: Equivalent circuit to derive $P_{\text{act}3}$ and $P_{\text{act}t}$ according to [51b]

For our chosen parametrisation, $G_m$ is the input conductance of the partial network to the right of $G_p$. Then, as indicated in Figure B.6(a), the same voltage $V_{\text{pt}}$ is applied to $G_p$ and $G_m$. Thus we obtain $\tilde{P}_{\text{act}2} = \frac{1}{2} |V_{\text{pt}}|^2 (G_m + G_p)$ and $P_{\text{act}3} = \frac{1}{2} |V_{\text{pt}}|^2 G_m$, or

$$P_{\text{act}3} = \frac{G_m}{G_m + G_p} \tilde{P}_{\text{act}2} = P_{\text{act maxi}} \frac{1}{1 + \frac{G_p}{G_m}} \left( 1 - \frac{R_{\text{s}C_1}}{R_i} \right).$$

Again $L_{\text{pt}}$ and $C_2$ are purely reactive, do not consume any active power, and $\tilde{P}_{\text{act}3} = P_{\text{act}3}$.

Then, as shown in Figure B.6(b), the same current $I_{\text{st}}$ flows through $R_{\text{s}C_2}$ and $R_t$. Thus $\tilde{P}_{\text{act}3} = \frac{1}{2} (R_{\text{s}C_2} + R_t) |I_{\text{st}}|^2$ and $P_{\text{act}t} = \frac{1}{2} R_t |I_{\text{st}}|^2$, or

$$P_{\text{act}t} = \frac{\tilde{P}_{\text{act}3}}{1 + \frac{R_{\text{s}C_2}}{R_t}} = P_{\text{act maxi}} \frac{1}{1 + \frac{G_{\text{p}}}{G_m}} \frac{1 - \frac{R_{\text{s}C_1}}{R_i}}{1 + \frac{R_{\text{s}C_2}}{R_t}}.$$

Summarising the preceding considerations yields the power gain of the T network including losses of all network elements,

$$g_{\text{t, inma}} = \frac{1}{1 + \frac{G_p}{G_m}} \frac{1 - \frac{C_1}{G_{\text{s}C_1}}}{1 + \frac{G_{\text{p}}}{G_m}} \frac{1 - \frac{C_1}{G_{\text{s}C_2}}}{1 + \frac{G_{\text{p}}}{G_m}}. \quad \text{(B.2)}$$

Despite its apparent simplicity, the formula is quite complicated because usually $R_{\text{s}C_{1,2}}$ and $G_p$ are functions of the parameter $G_m$. 


Even for the “simple” case of constant $Q$ factors of all network elements, hence $G_{sc1} = Q_1 \omega C_{1(s)}$, $G_p = \frac{1}{Q_\omega L_p}$, and $G_{sc2} = Q_2 \omega C_{2(s)}$, explicit derivation of those functions is somewhat difficult.

One approach is to put $Q_2 \omega C_2$ and the design formula of $C_2$ (given by (3.31) if $G_\ell$ is replaced with $G_{\ell\text{rest}}$) in $G_{\ell\text{rest}} = \frac{1}{G_{sc2} + G_\ell}$ and solve the resulting equation for $G_{\ell\text{rest}}$ — the solution has to be real, positive, and $< G_\ell$.

Then we perform similar calculations for $G_{sc1}$ and $G_{\ell\text{rest}}$, put the solutions in $G_p = \frac{1}{Q_\omega L_p}$, where $L_p$ is given by (3.32) if $G_i$ is replaced with $G_{\ell\text{rest}}$ and $G_\ell$ with $G_{\ell\text{rest}}$, and solve for $G_p$. However, more than one solution is obtained. Thus we additionally need to derive which solution is real and applies in which range of $G_m$.

Since the derivations sketched are quite lengthy, time consuming, and not needed within this work, they are not included herein.

B.4 Matching of a resistive source to a resistive load (matching at the network’s output)

B.4.1 $\pi$ network

To derive the elements’ values for matching of a (real) source conductance $G_1 = \frac{1}{R_1}$ to a (real) load conductance $G_\ell = \frac{1}{R_\ell}$ if a $\pi$ network with a lossy “central” inductor is used, calculations are based on the (transformed) $\pi$ network’s equivalent circuit in Figure 3.14.

As in Section 3.5.2.1, an approach similar to [13] (derivation for a lossless $\pi$ network) is used, where reactive cancellation is applied to each of the two L networks — $C_{m1}$ cancels $L_{s1}$ and $C_{m2}$ cancels $L_{s2}$.

Additionally, the remaining resistances have to be equal. To match the source to the load, we may choose $R_{m1} = R_m$. At matching frequency, $Z_{11} = R_m$ holds and thus $Z_{21} = Z_{11} + R_s = R_m + R_s$. To achieve a (conjugate complex) match of the source to the load, $Z_{22}$ must be equal to $R_{n2}$, hence $R_{m2} = R_m + R_s$. Conversely, $Z_{2\ell} = R_{m2} = R_m + R_s$ and $Z_{1\ell} = Z_{2\ell} + R_s = R_m + 2R_s$ at matching frequency, which is not equal to $R_{m1} = R_m$ as it would be for a conjugate complex match of the load to the source.

Different choices of $R_{m1,2}$ are also possible, but those shown above are ideally suited for the extension to complex sources.

Using the preceding choice for $R_{m1,2}$, we would obtain $C_{1,2}$ (and $C_{m1,2}$) from parallel-to-series transformation equations for $R_{m1,2}$. Applying reactive cancellation would then yield $L_{s1,2}$ (and $L_s$). The calculations are not shown here because we alternatively may use the formulae from [13] if they are rewritten in our resistance notation and if we use $R_{m1} = R_m$ for the left L network and $R_{m2} = R_m + R_s$ for the right L network ([13] derives the lossless case, where $R_{m1} = R_{m2} = R_m$).
Since $R_{m1,2}$ was obtained from parallel-to-series transformation of passive components (source resistance $R_i > 0$ and load resistance $R_\ell > 0$), $R_{m1,2} \geq 0$ holds. Then the calculations described above yield the capacitances $C_{1,2}$ and the inductance $L_s$ of the original $\pi$ network in Figure 3.13 a), where

$$C_1 = \left(\frac{+}{-}\right) \frac{1}{\omega R_i} \sqrt{\frac{R_i - R_m}{R_m}} \quad (3.3) \quad C_2 = \left(\frac{+}{-}\right) \frac{1}{\omega R_\ell} \sqrt{\frac{R_\ell - R_m - R_s}{R_m + R_s}}, \quad (B.4)$$

$$L_s = \frac{\left(\frac{+}{-}\right) \sqrt{R_m (R_i - R_m)}}{\omega} \left(\frac{+}{-}\right) \sqrt{(R_m + R_s) (R_\ell - R_m - R_s)}. \quad (B.5)$$

(The negative signs in parentheses/square brackets would apply if we would use (lossless) variable inductors $L_{1,2} = -\frac{1}{\omega^2 C_{1,2}}$. Then a negative $L_s$ might occur, which would be just equal to a variable capacitor $C_s = -\frac{1}{\omega^2 L_s}$. As indicated by using parentheses/square brackets, reactive cancellation causes a negative sign of the first square root in $L_s$, if $L_1$ is used instead of $C_1$, or a negative sign of the second square root in $L_s$, if $L_2$ is used instead of $C_2$, respectively. We will need these changes during the extension to inductive sources.)

Comparing the formulae to those of Section 3.5.2.1 yields that they are similar except that $R_i$, $R_\ell$, and the indices of the capacitances are exchanged. Thus the conditions to $R_m$ are changed accordingly. Hence, for valid $R_m$ the designability conditions are

$$0 \leq R_m \leq R_i \quad \text{and} \quad 0 \leq R_m + R_s \leq R_\ell. \quad (B.6)$$

As $R_s \geq R_{s0}$ (from (3.1)), we immediately obtain that $R_{s0}$ is limited by $R_{s0} \leq R_\ell$. If $R_{s0}$ would be higher, the $\pi$ network would not be able to match any source.

Similar to Section 3.5.2.1, there is an upper limit of $R_m$ which is given by $R_i$ if $R_i \leq R_\ell - R_{s0}$ or by $R_{m\text{lim}}$ if $P_i > R_\ell - R_{s0}$, respectively. $R_{m\text{lim}}$ is the highest parameter $R_m$ which solves

$$R_{m\text{lim}} = R_\ell - R_s(R_{m\text{lim}}) \quad (B.7)$$

for $0 \leq R_{m\text{lim}} < R_i$ and involves valid network elements. The change in the upper limit occurs at $R_i = R_\ell - R_{s0}$, where, as before, a match would not need any additional components ($L_s = 0$ and $C_{1,2} = 0$ would match). It’s interesting to note that (for a resistive load) there is no other solution where $L_s = 0$.\(^1\)

Depending on the particular $R_s$ used (given as a function of $\omega L_s$ or $R_m$), additional restrictions to $R_m$ might apply, limiting $R_m$ even further or involving intervals within the limits derived above where matching is impossible. However, (B.6) holds throughout the valid range of $R_m$.

\(^1\)Theoretically $L_s = 0$ might additionally occur for $R_m = 0$ if $R_s(R_m = 0) = 0$, which would be possible for $R_{s0} = 0$. However, since $L_s = 0$, the source resistor would not be transformed. Hence matching would be impossible unless $R_i = R_\ell - R_{s0} = R_\ell = R_m = 0$ — this solution is already described by (a limit of) $R_{m\text{lim}}$.\(^1\)
Additionally, the upper limits in (B.6) separate π and L networks — $C_1 = 0$ if $R_m = R_i$ or $C_2 = 0$ if $R_m = R_{m\text{lim}}$, respectively.

After designing the π network’s output impedance to match the load resistance $R_L$, we compute the active power $P_{\text{actf}}$ delivered to its load resistance $R_L$. Since the capacitive and inductive reactances in Figure 3.14 cancel at matching frequency, we obtain

$$P_{\text{actf}} = \frac{1}{2} \frac{R_m^2}{(R_{m1} + R_s + R_{m2})^2} \frac{|V_0|^2}{4 \left( R_m + R_s \right)^2} \frac{(3.16)}{1 + \omega^2 R_L C_1^2} \frac{|V_0|^2}{8 R_i R_m + R_s} \frac{R_m}{R_m + R_s} \frac{(2.22)}{P_{\text{actmax}}} \frac{1}{1 + \frac{R_m}{R_m}}. \quad (B.8)$$

For a given $R_m$, the network elements’ values $C_{1,2}$ and $L_s$ differ from those matching the load to the source in Section 3.5.2.1. Since $R_s \neq 0$ prevents matching of the load to the source if the network is designed for matching the source to the load, the active power $P_{\text{acts}}$ delivered to $R_s$ is also not equal to that derived in Section 3.5.2.1. However, the same current flows through $R_s$ and $R_{m2} = R_m + R_s$, hence it’s given by

$$P_{\text{acts}} = P_{\text{actf}} \frac{R_m}{R_m + R_s} \frac{(B.8)}{P_{\text{actmax}}} \frac{R_m}{R_s} \left( 1 + \frac{R_m}{R_m} \right)^2. \quad (B.9)$$

For optimisation, we will need the power gain $g_{\pi,\text{outma}}$ of the matched π network. The active power delivered to the network’s input is $P_{\text{act1}} = P_{\text{acts}} + P_{\text{actf}}$ (refer to Figure A.4). Thus we obtain

$$g_{\pi,\text{outma}} = \frac{P_{\text{actf}}}{P_{\text{act1}}} = \frac{P_{\text{actf}}}{P_{\text{acts}} + P_{\text{actf}}} \frac{(B.8), (B.9)}{R_m + R_s}{R_m + 2 R_s}{1 + \frac{R_m}{R_m}}. \quad (B.10)$$

(Alternatively we could derive $g_{\pi,\text{outma}}$ from (A.15) if we put $C_{2\text{tot}} = C_2$. However, we gain more insight in the circuit’s behaviour if it is computed as shown above.)

Comparing the preceding results to those of Section 3.5.2.1 yields that in the formulae of $L_s$, $R_{m\text{lim}}$ and the limits of $R_m$, $R_i$, and $R_L$ are exchanged, whereas for $C_{1,2}$ additionally the indices have to be exchanged.

Since

$$\frac{dg_{\pi,\text{outma}}}{dR_m} \frac{(B.10)}{R_m}{R_m + 2 R_s}{\frac{R_m}{dR_m}}{\frac{dR_s}{dR_m}}, \quad (B.11)$$

deriving of the optimum $R_m$ requires simply checking of the sign of $\frac{R_s}{R_m} - \frac{dR_s}{dR_m}$ similar to Section 3.5.2.2.

Conversely, deriving the optimum source resistance $R_i$ if $R_m$ is chosen for maximum $g_{\pi,\text{outma}}$ involves

$$g_{\pi,\text{outma}}(R_m = R_i) = \frac{R_i + R_s(R_m = R_i)}{R_i + 2 R_s(R_m = R_i)}.$$
and
\[
\frac{dg_{\pi,\text{out}}(R_m = R_t)}{dR_i} = \frac{R_i}{(R_i + 2 R_R(R_m = R_t))^2} \left( \frac{R_R(R_m = R_t)}{R_i} - \frac{d(R_R(R_m = R_t))}{dR_i} \right)
\]
if \( R_i \) is the upper limit of \( R_m \) or
\[
g_{\pi,\text{out}}(R_m = R_{\text{milim}}) = \frac{R_t}{2 R_t - R_{\text{milim}}}
\]
and
\[
\frac{dg_{\pi,\text{out}}(R_m = R_{\text{milim}})}{dR_i} = \frac{R_t}{(2 R_t - R_{\text{milim}})^2} \frac{dR_{\text{milim}}}{dR_i}
\]
if \( R_{\text{milim}} \) is the upper limit of \( R_m \).

Thus, similar to Section 3.5.2.3 and Appendix B.2.1.2, the optimisation with respect to \( R_i \) requires simply checking of the sign of \( \frac{R_R(R_m = R_t)}{R_i} - \frac{d(R_R(R_m = R_t))}{dR_i} \) or \( \frac{dR_{\text{milim}}}{dR_i} \). Hence the proofs of Sections 3.5.2.2, 3.5.2.3 and Appendices B.2.1.1, B.2.1.2 apply if \( R_i \) and \( R_t \) are exchanged.

Furthermore, the extension to complex sources can be performed in a similar manner as the extension to complex loads, in particular because in addition to \( R_i, R_t \), the indices of \( C_{1,2} \) are exchanged compared to the case of matching the load to the source.

### B.4.2 T network

To derive the elements' values for matching of a (real) source resistance \( R_i = \frac{1}{G_i} \) to a (real) load resistance \( R_t = \frac{1}{G_t} \), if a T network with a lossy "central" inductor is used, calculations are based on the (transformed) T network's equivalent circuit in Figure 3.15.

As in Section 3.5.3.1, an approach similar to [13] (derivation for a lossless T network) is used, where susceptive cancellation is applied to each of the two L networks — \( C_{m1} \) cancels \( L_{p1} \) and \( C_{m2} \) cancels \( L_{p2} \).

Additionally, the remaining conductances have to be equal. To match the source to the load, we may choose \( G_{m1} = G_m \). At matching frequency, \( Y_{11} = G_m \) holds and thus \( Y_{21} = Y_{11} + G_p = G_m + G_p \). To achieve a (conjugate complex) match of the source to the load, \( Y_{21} \) must be equal to \( G_{m2} \), hence \( G_{m2} = G_m + G_p \). Conversely, \( Y_{22} = G_{m2} = G_m + G_p \) and \( Y_{12} = Y_{22} + G_p = G_m + 2G_p \) at matching frequency, which is not equal to \( G_{m1} = G_m \) as it would be for a conjugate complex match of the load to the source.

Different choices of \( G_{m1,2} \) are also possible, but those shown above are ideally suited for the extension to complex sources.

Using the preceding choice for \( G_{m1,2} \), we would obtain \( C_{1,2} \) (and \( C_{m1,2} \)) from series-to-parallel transformation equations for \( G_{m1,2} \). Applying susceptive cancellation would then yield \( L_{p1,2} \) (and \( L_p \)). The calculations are not shown here because we
alternatively may use the formulae from [13] if they are rewritten in our conductance notation and if we use \( G_{m1} = G_m \) for the left L network and \( G_{m2} = G_m + G_p \) for the right L network ([13] derives the lossless case in resistive notation, where \( G_{m1} = G_{m2} = G_m \)).

Since \( G_{m1,2} \) was obtained from series-to-parallel transformation of passive components (source conductance \( G_i > 0 \) and load conductance \( G_\ell > 0 \)), \( G_{m1,2} \geq 0 \) holds. Then the calculations described above yield the capacitances \( C_{1,2} \) and the inductance \( L_p \) of the original T network in Figure 3.13 b), where

\[
C_i = \left( \frac{G_i}{\omega} \right)^{\frac{1}{2}} \sqrt{\frac{G_m}{G_i - G_m}}, \quad (B.12) \quad C_2 = \left( \frac{G_\ell}{\omega} \right)^{\frac{1}{2}} \sqrt{\frac{G_m + G_p}{G_\ell - G_m - G_p}}, \quad (B.13)
\]

\[
L_p = \frac{1}{\omega \left( \sqrt{i} \sqrt{G_m (G_i - G_m)} \sqrt{i} \sqrt{(G_m + G_p) (G_\ell - G_m - G_p)} \right)} \quad (B.14)
\]

(The negative signs in parentheses/square brackets would apply if we would use (lossless) variable inductors \( L_{1,2} = \frac{1}{\omega^2 G_{1,2}} \). Then a negative \( L_p \) might occur, which would be just equal to a variable capacitor: \( C_p = -\frac{1}{\omega^2 L_p} \). As indicated by using parentheses/square brackets, susceptive cancellation causes a negative sign of the first square root in \( L_p \), if \( L_1 \) is used instead of \( C_1 \), or a negative sign of the second square root in \( L_p \), if \( L_2 \) is used instead of \( C_2 \), respectively. We would need these changes during the extension to inductive sources.)

Comparing the formulae to those of Section 3.5.3.1 yields that they are similar except that \( G_i \), \( G_\ell \), and the indices of the capacitances are exchanged. Thus the conditions to \( G_m \) are changed accordingly. Hence, for valid \( G_m \), the designability conditions are

\[
0 \leq G_m \leq G_i \quad \text{and} \quad 0 \leq G_m + G_p \leq G_\ell. \quad (B.15)
\]

Similar to Section 3.5.3.1, there is an upper limit of \( G_m \) which is given by \( G_i \) if \( G_i \leq G_\ell \) or by \( G_{m\text{lim}} \) if \( G_i > G_\ell \), respectively. \( G_{m\text{lim}} \) is the highest parameter \( G_m \) which solves

\[
G_{m\text{lim}} = G_\ell - G_p (G_{m\text{lim}}) \quad (B.16)
\]

for \( 0 \leq G_{m\text{lim}} < G_i \) and involves valid network elements. The change in the upper limit occurs at \( G_i = G_\ell \), where, as before, a match would not need any additional components (\( L_a \to \infty \), hence \( L_p \to \infty \), and \( C_{1,2} \to \infty \) would match). It's interesting to note that (for a resistive load) there is no other solution where \( L_{n,p} \to \infty \) or \( \frac{1}{L_p} = 0^\dagger \). 

\(^\dagger\)Theoretically \( \frac{1}{L_p} = 0 \) might additionally occur for \( G_m = 0 \) if \( G_p (G_m = 0) = 0 \). However, since \( L_{n,p} \to \infty \left( \frac{1}{L_p} = 0 \right) \) and \( G_p = 0 \) is case (ii) in (A.18)), the load conductor would not be transformed. Hence matching would be impossible unless \( G_i = G_\ell = G_m = 0 \) — this solution is
Depending on the particular $R_s$ used, hence the particular $G_p$ used (given as a function of $\omega L_p$ or $G_m$), additional restrictions to $G_m$ might apply, limiting $G_m$ even further or involving intervals within the limits derived above where matching is impossible. However, (3.33) holds throughout the valid range of $G_m$.

Additionally, the upper limits in (B.15) separate T and L networks --- $C_1 \to \infty$ if $G_m = G_1$ or $C_2 \to \infty$ if $G_m = G_{\text{lim}}$, respectively.

After designing the T network’s output impedance to match the load conductance $G_L$, we compute the active power $P_{\text{act}}$ delivered to its load conductance $G_L$. Since the capacitive and inductive susceptances in Figure 3.15 cancel at matching frequency, we obtain

\[
P_{\text{act}} = \frac{1}{2} G_m \frac{L_0}{(G_{m1} + G_p + G_{m2})}^2 \quad \text{(3.29)} \quad = \frac{1}{2} \left( G_m + G_p \right) \frac{|V_0|^2}{4(G_m + G_p)^2}
\]

For a given $G_m$, the network elements’ values $C_{1,2}$ and $L_p$ differ from those matching the load to the source in Section 3.5.3.1. Since $G_p \neq 0$, thus $R_s \neq 0$, prevents matching of the load to the source if the network is designed for matching the source to the load, the active power $P_{\text{act}} = P_{\text{acts}}$ delivered to $G_p$ or $R_s$ is also not equal to that derived in Section 3.5.3.1. However, the same voltage is applied to $G_p$ and $G_{m2} = G_m + G_s$, hence it’s given by

\[
P_{\text{act}} = P_{\text{act}} \frac{G_p}{G_m + G_p} \quad \text{(B.17)} \quad = P_{\text{actmaxi}} \frac{G_m}{1 + \frac{G_m}{G_p}}.
\]

For optimisation, we will need the power gain $g_{t,\text{outma}}$ of the matched T network. The active power delivered to the network’s input is $P_{\text{act}} = P_{\text{actp}} + P_{\text{actt}}$ (refer to Figure A.6). Thus we obtain

\[
g_{t,\text{outma}} = \frac{P_{\text{actt}}}{P_{\text{act1}}} = \frac{P_{\text{actt}}}{P_{\text{actp}} + P_{\text{actt}}} \quad \text{(B.17)} \quad \frac{G_m + G_p}{G_m + 2G_p} = \frac{1 + \frac{G_m}{G_p}}{1 + 2 \frac{G_m}{G_p}}.
\]

(Alternatively we could derive $g_{t,\text{outma}}$ from (A.23) if we put $C_{\text{act}} = C_2$. However, we gain more insight in the circuit’s behaviour if it is computed as shown above.)

Comparing the preceding results to those of Section 3.5.3.1 yields that in the formulae of $L_p$, $G_{\text{lim}}$ and the limits of $G_m$, $G_1$, and $G_L$ are exchanged, whereas for $C_{1,2}$ additionally the indices have to be exchanged.

---

already described by (a limit of) $G_{\text{lim}}$.

Unique to the T network, case (i) b) in (A.18) describes a second point where $L_{s0} = 0$ or $L_p \to \infty$, but $L_s = 0$, if $G_L \geq \frac{1}{R_{s0}}$ for $R_{s0} > 0$ and $G_m = G_{\text{lim}} = G_L + \frac{1}{R_{s0}} = G_1$, hence $G_p = \frac{1}{R_{s0}}$ and $C_{1,2} \to \infty$. Compared to $L_s \to \infty$, which was lossless, $L_s = 0$ involves losses in $R_{s0}$. 
Since
\[
\frac{dg_{t, \text{outma}}}{dG_m} \overset{(B.19)}{=} \frac{G_m}{(G_m + 2G_p)^2} \left( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \right),
\]
 deriving of the optimum \( R_m \) requires simply checking of the sign of \( \frac{G_p}{G_m} - \frac{dG_p}{dG_m} \) similar to Section 3.5.3.2.

Conversely, deriving the optimum source conductance \( G_1 \) if \( G_m \) is chosen for maximum \( g_{t, \text{outma}} \) involves
\[
g_{t, \text{outma}}(G_m = G_1) = \frac{G_1 + G_p(G_m = G_1)}{G_1 + 2G_p(G_m = G_1)}
\]
and
\[
\frac{dg_{t, \text{outma}}(G_m = G_1)}{dG_i} = \frac{G_i}{(G_i + 2G_p(G_m = G_1))^2} \left( \frac{G_p(G_m = G_1)}{G_i} - \frac{d(G_p(G_m = G_1))}{dG_i} \right)
\]
if \( G_i \) is the upper limit of \( G_m \) or
\[
g_{t, \text{outma}}(G_m = G_{\text{mlim}}) = \frac{G_1}{2G_1 - G_{\text{mlim}}}
\]
and
\[
\frac{dg_{t, \text{outma}}(G_m = G_{\text{mlim}})}{dG_i} = \frac{G_1}{(2G_1 - G_{\text{mlim}})^2} \frac{dG_{\text{mlim}}}{dG_i}
\]
if \( G_{\text{mlim}} \) is the upper limit of \( G_m \).

Thus, similar to Section 3.5.3.3 and Appendix B.2.2.2, the optimisation with respect to \( G_1 \) requires simply checking of the sign of \( \frac{G_p(G_m = G_1)}{G_i} - \frac{d(G_p(G_m = G_1))}{dG_i} \) or \( \frac{dG_{\text{mlim}}}{dG_i} \). Hence the proofs of Sections 3.5.3.2, 3.5.3.3 and Appendices B.2.2.1, B.2.2.2 apply if \( G_i \) and \( G_1 \) are exchanged.

Furthermore, the extension to complex sources can be performed in a similar manner as the extension to complex loads, in particular because in addition to \( G_1 \), \( G_{\ell} \), the indices of \( C_{1,2} \) are exchanged compared to the case of matching the load to the source.

### B.5 Approximate matching of a resistive load to a resistive source (approximate matching at the network’s input), network with losses in all elements designed as if capacitors were lossless

#### B.5.1 \( \pi \) network

If the \( \pi \) network of Figure B.1 (which takes into account the losses of all network elements) is designed as if the capacitors were lossless (network elements given by
(3.17), (3.18), and (3.19)), its input is not exactly matched, but its power gain $g_{\pi, \text{ilcapdes}}$ may be calculated exactly by applying the method used in Appendix B.3.2 (and [51b]), yielding
\[
\left(1 + \frac{R_m \left(1 + \frac{R_L}{R_{pC2}}\right)}{1 + \left(2 + \frac{R_L}{R_{pC2}}\right) R_m} \right)^2 + \left(\frac{R_m (R_m + R_0) (R_1 - R_2 - R_0)}{1 + \left(2 + \frac{R_L}{R_{pC2}}\right) R_m} \right)
\cdot \left(1 + \frac{R_L}{R_{pC2}}\right) = \frac{1}{g_{\pi, \text{ilcapdes}}}.
\]  
\tag{B.21}
\]
(The negative sign in square brackets would apply for an extension to inductive loads if the case of undercompensation would be considered.)

Of course it's also possible to derive the $\pi$ network's nonzero input reflection coefficient exactly. However, the resulting formula is quite lengthy and thus not given herein.

### B.5.2 T network

If the T network of Figure B.4 (which takes into account the losses of all network elements) is designed as if the capacitors were lossless (network elements given by (3.30), (3.31), and (3.32)), its input is not exactly matched, but its power gain $g_{t, \text{ilcapdes}}$ may be calculated exactly by applying the method used in Appendix B.3.3 (and [51b]), yielding
\[
\left(1 + \frac{G_m \left(1 + \frac{G_L}{G_{mC2}}\right)}{1 + \left(2 + \frac{G_L}{G_{mC2}}\right) G_m} \right)^2 + \left(\frac{G_m (G_m + G_0) (G_1 - G_2 - G_0)}{1 + \left(2 + \frac{G_L}{G_{mC2}}\right) G_m} \right)
\cdot \left(1 + \frac{G_L}{G_{mC2}}\right) = \frac{1}{g_{t, \text{ilcapdes}}}.
\]  
\tag{B.22}
\]
(The negative sign in square brackets would apply for an extension to inductive loads if the case of undercompensation would be considered.)

Of course it's also possible to derive the T network's nonzero input reflection coefficient exactly. However, the resulting formula is quite lengthy and thus not given herein.
Appendix C

Extension of the Method to Complex Loads

C.1 Matching of a complex load to a resistive source (matching at the network’s input)

C.1.1 Complex load exhibiting an inductive imaginary part

C.1.1.1 π network

For complex loads exhibiting an inductive imaginary part, the formula of the inductance $L_s$ differs from that of complex loads exhibiting a capacitive imaginary part. In particular, the second square root is subtracted instead of added, introducing additional zeros of $L_s$ which will be derived in the following section.

From (3.1), the definition of $R_s$, we obtain

$$R_s(L_s = 0) = R_{s0}. \quad (C.1)$$

Putting this result in (4.13) yields

$$0 = \frac{\sqrt{(R_m + R_{s0}) (R_l - R_m - R_{s0})} - \sqrt{R_m (R_l - R_m)}}{\omega}.$$

In our frequency range of interest, the angular frequency $\omega$ stays finite. Thus $L_s$ can only be zero if both square roots are equal. In particular, both square roots may be zero (similar to the capacitive case) or may have the same positive value.
Solving the former for $R_m$ yields $R_{\text{mlim}}$ as defined in (3.21) if $R_\ell = R_i - R_{s0}$. If $R_\ell \neq R_i - R_{s0}$, both square roots cannot be zero simultaneously\(^1\).\(^2\).

Solving the latter in case of $R_{s0} = 0$ requires $R_i = R_\ell$ to hold and yields all $0 \leq R_m \leq R_i = R_\ell$ ($C_1$ and $L_{2\text{rot}}$ in parallel resonance), introducing no additional limits compared to (3.20).

Conversely, solving the latter in case of $R_{s0} > 0$ requires
\[
(R_{mL_0} + R_{s0}) (R_i - R_{mL_0} - R_{s0}) = R_{mL_0} (R_\ell - R_{mL_0})
\]
to be solved for $R_m$, yielding
\[
R_{mL_0} = R_{s0} \frac{R_i - R_{s0}}{R_\ell - R_i + 2R_{s0}}.
\] (C.2)

If $R_\ell = R_i - 2R_{s0}$, there is no $R_{mL_0}$ which solves the equation unless $R_{s0} = R_i$. But, since then $R_\ell = -R_{s0}$, it would have to be $\leq 0$ which would be out of the load’s valid range.

Additionally, $R_{mL_0}$ has to be $> 0$ and $< \text{Min}\{R_\ell, R_{\text{mlim}}\}$ to introduce a new limit within the (utmost) valid range of $R_m$.

The numerator of $R_{mL_0}$ is $\geq 0$ because $0 \leq R_{s0} \leq R_i$ as derived in Section 3.5.2.1.

If $R_\ell - R_i + 2R_{s0} < 0$, hence $0 \leq R_\ell < R_i - 2R_{s0}$ if $0 < R_{s0} < \frac{R_i}{2}$, the denominator of $R_{mL_0}$ is $< 0$ and thus $R_{mL_0} < 0$, introducing no additional limit of $R_m$.

If $R_\ell - R_i + 2R_{s0} > 0$, hence $R_\ell > R_i - 2R_{s0} \geq 0$ if $0 < R_{s0} \leq \frac{R_i}{2}$ or any $R_\ell \geq 0$ if $\frac{R_i}{2} \leq R_{s0} < R_i$, the denominator of $R_{mL_0}$ is $> 0$ and thus $R_{mL_0} > 0$. But we still have to check whether $R_{mL_0}$ stays within the (utmost) limits of $R_m$ given by (3.20).

Since $R_s(R_{mL_0}) = R_{s0}$, (3.20) holds only if the following inequalities hold\(^3\)
\[
R_\ell - R_{mL_0} = (R_\ell + R_{s0}) \frac{R_\ell - R_i + R_{s0}}{R_\ell - R_i + 2R_{s0}} > 0,
\]
\[
R_i - R_{mL_0} - R_{s0} = (R_i - R_{s0}) \frac{R_i - R_\ell + R_{s0}}{R_i - R_\ell + 2R_{s0}} > 0.
\]

Obviously both inequalities hold (under the preceding assumptions) only if $R_\ell > R_i - R_{s0}$. Since $0 \leq R_{s0} \leq R_i$ implies that $R_i - R_{s0} \geq R_i - 2R_{s0}$ and $R_i - R_{s0} \geq 0$, a valid $R_{mL_0}$ is introduced for all $0 < R_{s0} < R_i$ and real (purely resistive) loads, thus $0 < R_{mL_0} < R_{\text{mlim}}$ (refer to Section 3.5.2.1). In case of a complex load exhibiting an inductive imaginary part a valid $R_{mL_0}$ would additionally require $R_{mL_0} > R_{\text{mLdp}}$ to hold, which has to be checked separately for the given value of $L_{\text{dp}}$.

Finally it should be noted that $R_{mL_0} = R_{\text{mlim}} = R_i - R_{s0}$ if $R_\ell = R_i - R_{s0}$.

\(^1\)At a first glance, there seems to be another solution of $L_a = 0$ at $R_m = 0$ if $R_{s0} = 0$. However, it’s usually no valid solution because then $L_a = 0$ and $R_a = 0$, hence the matching network would just consist of $C_1 \to \infty$ in parallel to $L_{2\text{rot}} = 0$. Each of those elements would act as a short, but they may compensate each other (in a parallel resonance) if the limit of $R_m \to 0$ is considered and $R_i = R_\ell$. But, as $R_i = R_\ell$ would require no matching network at all and the indicated elements’ values cannot be realised, the possible solution at $R_m = 0$ is not considered further.

\(^2\)For real loads or complex loads exhibiting a capacitive imaginary part, $R_m = 0$ is no valid solution for $L_a = 0$ if $R_{s0} = 0$ because the matching network would then consist of a $C_1 \to \infty$ in parallel to a $C_2 \to \infty$ or $C_{2\text{rot}} \to \infty$ (instead of $L_{2\text{rot}} = 0$). Such a matching network would be an infinite capacitance in parallel to the load resistance $R_\ell$, which would not match $R_\ell$ to $R_i$.

\(^3\)Since a new limit is checked for, there are “>” in the inequalities instead of the “≥” in (3.20).
APPENDIX C. EXTENSION OF THE METHOD TO COMPLEX LOADS

C.1.1.2 T network

For complex loads exhibiting an inductive imaginary part, the formula of the inductance \( L_p \) differs from that of complex loads exhibiting a capacitive imaginary part. In particular, the second square root is subtracted instead of added, introducing additional zeros of \( \frac{1}{L_p} \) which will be derived in the following section.

From (3.1), the definition of \( R_s \), and the considerations regarding certain series-to-parallel transformed network elements’ values in (A.18) we obtain

\[
G_p \left( \frac{1}{L_p} = 0 \right) = \begin{cases} 
0 & \text{if } L_s \to \infty \\
\frac{1}{R_m} & \text{if } L_s = 0 \text{ and } R_{s0} > 0
\end{cases}
\]  \hspace{1cm} (C.3)

Putting this result in the reciprocal of (4.19) yields

\[
0 = \begin{cases} 
\omega \left( \sqrt{G_m (G_t - G_m) - \sqrt{G_m (G_t - G_m)}} \right) & \text{if } L_s \to \infty \\
\omega \left( \sqrt{G_m + \frac{1}{R_{s0}}} \left( G_t - G_m - \frac{1}{R_{s0}} \right) - \sqrt{G_m (G_t - G_m)} \right) & \text{if } L_s = 0 \text{ and } R_{s0} > 0
\end{cases}
\]

In our frequency range of interest, the angular frequency \( \omega \) stays finite. Thus \( \frac{1}{L_p} \) can only be zero if both square roots are equal. In particular, both square roots may be zero (similar to the capacitive case) or may have the same positive value.

Solving the former for \( G_m \) yields \( G_{mim} \) as defined in (3.34) if \( G_t = G_i \). If \( G_t \neq G_i \), both square roots cannot be zero simultaneously\(^4\). However, if \( L_s = 0 \) there exists an additional solution as indicated in Section 3.5.3.1, Footnote \(^4\).

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\(^4\)At a first glance, there seems to be another solution of \( \frac{1}{L_p} = 0 \) at \( G_m = 0 \). However, it’s usually no valid solution because then \( \frac{1}{L_p} = 0 \) and:

- Either \( L_s \to \infty \) and thus \( G_p = 0 \), hence the matching network would just consist of \( C_1 = 0 \) in series to \( L_{2tot} \to \infty \). Each of those elements would act as an open circuit, but they may compensate each other (in a series resonance) if the limit of \( G_m \to 0 \) is considered and \( G_1 = G_t \). But \( G_1 = G_t \) would require no matching network at all and the indicated elements’ values cannot be realised.

- Or \( L_s = 0, R_{s0} > 0 \), and thus \( G_p = \frac{1}{R_{s0}} \), hence the matching network would consist of finite and nonzero \( C_1 \) and \( G_p \), but \( L_{2tot} \to \infty \), which would not match \( G_t \) to \( G_i \).

Thus the possible solution at \( G_m = 0 \) is not considered further.

\(^5\)For real loads or complex loads exhibiting a capacitive imaginary part, \( G_m = 0 \) is no valid solution for \( \frac{1}{L_p} = 0 \) because then:

- Either \( L_s \to \infty \) and thus \( G_p = 0 \), hence the matching network would consist of a \( C_1 = 0 \) in series to a \( C_2 = 0 \) or \( C_{2tot} = 0 \) (instead of \( L_{2tot} \to \infty \)). Such a matching network would be a zero capacitance in series to the load admittance \( G_t \), which would not match \( G_t \) to \( G_i \).

- Or \( L_s = 0, R_{s0} > 0 \), and thus \( G_p = \frac{1}{R_{s0}} \), hence the matching network would consist of finite and nonzero \( C_1 \) and \( G_p \), but \( C_2 = 0 \) or \( C_{2tot} = 0 \) (instead of \( L_{2tot} \to \infty \)), which would not match \( G_t \) to \( G_i \).
Solving the latter in case of \( L_x \to \infty \) requires \( G_i = G_{\ell} \) to hold and yields all \( 0 \leq G_m \leq G_i = G_{\ell} \) (\( C_1 \) and \( L_{2\text{tot}} \) in series resonance), introducing no additional limits compared to (3.33).

Conversely, solving the latter in case of \( L_x = 0 \) requires
\[
\sqrt{\left(G_m + \frac{1}{R_{\text{so}}}\right) \left(G_i - G_m - \frac{1}{R_{\text{so}}}\right)} = \sqrt{G_m \left(G_{\ell} - G_m\right)}
\]
to be solved for \( G_m \) where \( R_{\text{so}} > 0 \), yielding
\[
G_{m1/L_{p0}} = \frac{1}{R_{\text{so}}} \frac{G_i - \frac{1}{R_{\text{so}}}}{G_{\ell} - G_i + 2 \frac{1}{R_{\text{so}}}}.
\]
(C.4)

If \( G_{\ell} = G_i - 2 \frac{1}{R_{\text{so}}} \), there is no \( G_{m1/L_{p0}} \) which solves the equation unless \( \frac{1}{R_{\text{so}}} = G_i \) or \( R_{\text{so}} = R_i \). But, since then \( G_{\ell} = \frac{1}{R_{\text{so}}} \), it would have to be \( \leq 0 \) which would be out of the load’s valid range.

Additionally, \( G_{m1/L_{p0}} \) has to be > 0 and \( < \text{Min}\{G_{\ell}, G_{\text{mlim}}\} \) to introduce a new limit within the (utmost) valid range of \( G_m \).

Compared to the \( \pi \) network, the \( T \) network introduces no limit for \( R_{\text{so}} \) (refer to Section 3.5.3.1). Thus \( 0 < \frac{1}{R_{\text{so}}} < G_i \) and \( G_i \geq \frac{1}{R_{\text{so}}} \) have to be considered — then the numerator of \( G_{m1/L_{p0}} \) is positive or negative, respectively.

1. \( 0 < \frac{1}{R_{\text{so}}} < G_i \) or \( R_{\text{so}} > R_i \), thus the numerator of \( G_{m1/L_{p0}} \) is > 0

   If \( G_{\ell} - G_i + 2 \frac{1}{R_{\text{so}}} < 0 \), hence \( 0 \leq G_{\ell} < G_i - 2 \frac{1}{R_{\text{so}}} \) if \( 0 < \frac{1}{R_{\text{so}}} < \frac{G_i}{2} \), the denominator of \( G_{m1/L_{p0}} \) is < 0 and thus \( G_{m1/L_{p0}} < 0 \), introducing no additional limit of \( G_m \).

   If \( G_{\ell} - G_i + 2 \frac{1}{R_{\text{so}}} > 0 \), hence \( G_{\ell} > G_i - 2 \frac{1}{R_{\text{so}}} \geq 0 \) if \( 0 < \frac{1}{R_{\text{so}}} \leq \frac{G_i}{2} \) or any \( G_{\ell} \geq 0 \) if \( \frac{G_i}{2} \leq \frac{1}{R_{\text{so}}} < G_i \), the denominator of \( G_{m1/L_{p0}} \) is > 0 and thus \( G_{m1/L_{p0}} > 0 \).

   But we still have to check whether \( G_{m1/L_{p0}} \) stays within the (utmost) limits of \( G_m \) given by (3.33). Since \( G_p(G_{m1/L_{p0}}) = \frac{1}{R_{\text{so}}} \), (3.33) holds only if the following inequalities hold

\[
G_{\ell} - G_{m1/L_{p0}} = \left(G_{\ell} + \frac{1}{R_{\text{so}}} \right) \frac{G_{\ell} - G_i + \frac{1}{R_{\text{so}}}}{G_{\ell} - G_i + 2 \frac{1}{R_{\text{so}}}} > 0,
\]
\[
G_i - G_{m1/L_{p0}} - \frac{1}{R_{\text{so}}} = \left( G_i - \frac{1}{R_{\text{so}}} \right) \frac{G_{\ell} - G_i + \frac{1}{R_{\text{so}}}}{G_{\ell} - G_i + 2 \frac{1}{R_{\text{so}}}} > 0.
\]

Obviously both inequalities hold (under the preceding assumptions) only if \( G_{\ell} > G_i - \frac{1}{R_{\text{so}}} \). Since \( 0 < \frac{1}{R_{\text{so}}} < G_i \) implies that \( G_i - \frac{1}{R_{\text{so}}} > G_i - 2 \frac{1}{R_{\text{so}}} \) and \( G_i - \frac{1}{R_{\text{so}}} > 0 \), a valid \( G_{m1/L_{p0}} \) is introduced for all \( 0 < \frac{1}{R_{\text{so}}} < G_i \) or \( R_{\text{so}} > R_i \) and purely resistive loads, thus \( 0 < G_{m1/L_{p0}} < G_{\text{mlim}} \) (refer to Section 3.5.3.1).

In case of a complex load exhibiting an inductive imaginary part \( G_{m1/L_{p0}} \) would additionally require \( C_{m1/L_{p0}} > C_{mL_{es}} \) to hold, which has to be checked separately for the given value of \( L_{es} \).

\[ \text{Since a new limit is checked for, there are } > \text{ in the inequalities instead of the } \geq \text{ in (3.33).} \]
2. \( \frac{1}{R_{w0}} \geq G_i \) or \( 0 \leq R_{w0} \leq R_i \), thus the numerator of \( G_{m1/L_p0} \) is \( \leq 0 \). In this case there exists no valid \( G_{m1/L_p0} \) fulfilling (3.33), because the same differences as in case 1. occur, where both fractions are similar, but under the assumptions of case 2. the factor \( G_i + \frac{1}{R_{w0}} \) is always \( > 0 \), whereas \( G_i - \frac{1}{R_{w0}} \) is always \( \leq 0 \). Hence, independently of the fraction’s sign, one of both inequalities in (3.33) cannot apply. Therefore no new limit of \( G_m \) is introduced.

Since variable air-core inductors consist of a low number of windings and the ferrite or iron powder cores of switchable inductors are designed for low losses, the assumption \( R_{w0} \leq R_i \) usually applies if \( R_i = 50 \Omega \).

Finally it should be noted that besides the case of \( L_s \rightarrow \infty \) and \( G_t = G_i \) there may be another solution yielding \( G_{m1/L_p0} = G_{m\text{lim}} = G_i = \frac{1}{R_{w0}} \) if \( G_i \geq \frac{1}{R_{w0}} \), \( G_t = G_i - \frac{1}{R_{w0}} \) and \( L_s = 0 \).

### C.2 Exact analytical solutions for a constant \( Q \) central inductor

#### C.2.1 Example diagrams

##### C.2.1.1 Power gain’s contour line Smith charts for \( L \), \( \pi \), and \( T \) networks

In the following diagrams, power gain’s contour line Smith charts for different \( L \), \( \pi \), and \( T \) networks are shown. Their central inductor has a range of 0.1 \( \mu \text{H} \) – 40 \( \mu \text{H} \), a constant \( Q \) factor of 100, and the matching frequency is either 1.8 MHz, 7.35 MHz, or 30 MHz. The variable capacitances’ ranges resemble different “typical” variable capacitors (a cheap 5 pF – 250 pF, a better 5 pF – 500 pF, and an expensive 25 pF – 4000 pF vacuum variable). The contour lines were calculated from the resulting matching networks of 100×100 equally spaced load reflection coefficients whose \( \text{Re}\{Z_\ell\}, \text{Im}\{Z_\ell\} \in [-1, 1] \), where only those with \( |Z_\ell| < 1 \) were used.

In case of \( \pi \) and \( T \) networks, best and worst networks’ power gain is shown.

In case of \( L \) networks, it is assumed that the network topology enables implementation of both “complementary” \( L \) networks (\( C_1 \) only or \( C_2 \) only), e.g. by using appropriate switches.

Additionally, the impact of a parasitic capacitance replacing one variable capacitance of a \( \pi \) or \( T \) network is considered (refer to Figure 4.21) — the resulting networks yield a more realistic description of “real world” \( L \) networks. Based on [3], the value of that parasitic capacitance was chosen to be equal to the minimum/maximum value of the remaining variable capacitance for \( L \) networks derived from \( \pi \) networks/T networks.
Since lossy L networks may theoretically exhibit more than one distinct solutions for a given load, it has to be noted that

- the pure L networks (without parasitic capacitances) and the L networks with a parasitic parallel capacitance derived from π networks yielded only a single valid solution throughout all complete Smith charts for the given network element ranges

- the L networks with a parasitic series capacitance derived from T networks yielded more than one solution within some region of the Smith charts for the given network element ranges, but the differences in power gain of best and worst solutions were low — for the chosen contour line values both diagrams were merely similar — thus only the best diagrams are shown.

To illustrate the latter — best and worst L networks' power gain differs by $< 0.026$ at $1.8 \text{ MHz}$, by $< 0.054$ at $7.35 \text{ MHz}$, and by $< 0.0084$ at $30 \text{ MHz}$ for $5 \text{ pF} \leq C_{1,2} \leq 250 \text{ pF}$ with $250 \text{ pF}$ parasitic series capacitance, by $< 0.04$ at $1.8 \text{ MHz}$, by $< 0.017$ at $7.35 \text{ MHz}$, and by $< 0.0042$ at $30 \text{ MHz}$ for $5 \text{ pF} \leq C_{1,2} \leq 500 \text{ pF}$ with $500 \text{ pF}$ parasitic series capacitance, or by $< 0.0063$ at $1.8 \text{ MHz}$, by $< 0.0017$ at $7.35 \text{ MHz}$, and by $< 0.0005$ at $30 \text{ MHz}$ for $25 \text{ pF} \leq C_{1,2} \leq 4000 \text{ pF}$ with $4000 \text{ pF}$ parasitic series capacitance.

Figure C.1: a) Best and b) worst $g_{\text{lof,ima}}$ at $1.8 \text{ MHz}$, $R_0 = 50 \Omega$, $Q = 100$, $5 \text{ pF} \leq C_{1,2} \leq 500 \text{ pF}$, and $0.1 \mu\text{H} \leq L_0 \leq 40 \mu\text{H}$, with $500 \text{ pF}$ parasitics, undercompensation shaded in gray

Anyhow, a small difference can be seen in one of the examples given above. As depicted in Figure C.1, the $> 0.98$ contour line is visible for the best networks in Figure C.1 a) (above the right edge of the $> 0.95$ contour line), whereas there is no $> 0.98$ contour line at all for the worst networks in Figure C.1 b).
Figure C.2: a) Best and b) worst $g_{\pi,\text{inma}}$ at 1.8 MHz, $R_i = 50 \, \Omega$, $Q = 100$, 5 pF $\leq C_{1,2} \leq 250$ pF, and $0.1 \, \mu H \leq L_a \leq 40 \, \mu H$, undercompensation shaded

Figure C.3: a) Best and b) worst $g_{\pi,\text{inma}}$ at 1.8 MHz, $R_i = 50 \, \Omega$, $Q = 100$, 5 pF $\leq C_{1,2} \leq 500$ pF, and $0.1 \, \mu H \leq L_a \leq 40 \, \mu H$, undercompensation shaded

Figure C.4: a) Best and b) worst $g_{\pi,\text{inma}}$ at 1.8 MHz, $R_i = 50 \, \Omega$, $Q = 100$, 25 pF $\leq C_{1,2} \leq 4000$ pF, and $0.1 \, \mu H \leq L_a \leq 40 \, \mu H$, undercomp. shaded
Figure C.5: $g_{\text{lof}, \text{inma}}$ at 1.8 MHz, $R_i = 50 \Omega$, $Q = 100$, $5 \text{ pF} \leq C_{1,2} \leq 250 \text{ pF}$, and $0.1 \mu H \leq L_a \leq 40 \mu H$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.6: $g_{\text{lof}, \text{inma}}$ at 1.8 MHz, $R_i = 50 \Omega$, $Q = 100$, $5 \text{ pF} \leq C_{1,2} \leq 500 \text{ pF}$, and $0.1 \mu H \leq L_a \leq 40 \mu H$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.7: $g_{\text{lof}, \text{inma}}$ at 1.8 MHz, $R_i = 50 \Omega$, $Q = 100$, $25 \text{ pF} \leq C_{1,2} \leq 4000 \text{ pF}$, $0.1 \mu H \leq L_a \leq 40 \mu H$, a) without, b) with 25 pF parasitics, uc. shaded
APPENDIX C. EXTENSION OF THE METHOD TO COMPLEX LOADS

Figure C.8: a) Best and b) worst \( g_{t, 	ext{inna}} \) at 1.8 MHz, \( R_i = 50 \, \Omega \), \( Q = 100 \), 5 pF \( \leq C_{1,2} \leq 250 \) pF, and 0.1 \( \mu H \) \( \leq L_a \leq 40 \) \( \mu H \), undercompensation shaded

Figure C.9: a) Best and b) worst \( g_{t, 	ext{inna}} \) at 1.8 MHz, \( R_i = 50 \, \Omega \), \( Q = 100 \), 5 pF \( \leq C_{1,2} \leq 500 \) pF, and 0.1 \( \mu H \) \( \leq L_a \leq 40 \) \( \mu H \), undercompensation shaded

Figure C.10: a) Best and b) worst \( g_{t, 	ext{inna}} \) at 1.8 MHz, \( R_i = 50 \, \Omega \), \( Q = 100 \), 25 pF \( \leq C_{1,2} \leq 4000 \) pF, and 0.1 \( \mu H \) \( \leq L_a \leq 40 \) \( \mu H \), undercomp. shaded
Figure C.11: $g_{d, \text{inma}}$ at 1.8 MHz, $R_1 = 50 \Omega$, $Q = 100$, $5 \text{ pF} \leq C_{1,2} \leq 250 \text{ pF}$, and $0.1 \mu H \leq L_\alpha \leq 40 \mu H$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.12: $g_{d, \text{inma}}$ at 1.8 MHz, $R_1 = 50 \Omega$, $Q = 100$, $5 \text{ pF} \leq C_{1,2} \leq 500 \text{ pF}$, and $0.1 \mu H \leq L_\alpha \leq 40 \mu H$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.13: $g_{d, \text{inma}}$ at 1.8 MHz, $R_1 = 50 \Omega$, $Q = 100$, $25 \text{ pF} \leq C_{1,2} \leq 4000 \text{ pF}$, $0.1 \mu H \leq L_\alpha \leq 40 \mu H$, a) without, b) with 25 pF parasitics, uc. shaded
Figure C.14: a) Best and b) worst $g_{\pi,\text{inna}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 250 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_n \leq 40 \, \mu\text{H}$, undercompensation shaded

Figure C.15: a) Best and b) worst $g_{\pi,\text{inna}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_n \leq 40 \, \mu\text{H}$, undercompensation shaded

Figure C.16: a) Best and b) worst $g_{\pi,\text{inna}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $25 \, \text{pF} \leq C_{1,2} \leq 4000 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_n \leq 40 \, \mu\text{H}$, undercomp. shaded
Figure C.17: $g_{ofr,inma}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 250 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.18: $g_{ofr,inma}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.19: $g_{ofr,inma}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $25 \, \text{pF} \leq C_{1,2} \leq 4000 \, \text{pF}$, $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 25 pF parasitics, uc. shaded
Figure C.20: a) Best and b) worst $g_{t,\text{inma}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 250 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_s \leq 40 \, \mu\text{H}$, undercompensation shaded

Figure C.21: a) Best and b) worst $g_{t,\text{inma}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_s \leq 40 \, \mu\text{H}$, undercompensation shaded

Figure C.22: a) Best and b) worst $g_{t,\text{inma}}$ at 7.35 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $25 \, \text{pF} \leq C_{1,2} \leq 4000 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_s \leq 40 \, \mu\text{H}$, undercomp. shaded
Figure C.23: $g_{\text{out, inna}}$ at 7.35 MHz, $R_t = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 250 \, \text{pF}$, and $0.1 \, \mu H \leq L_u \leq 40 \, \mu H$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.24: $g_{\text{out, inna}}$ at 7.35 MHz, $R_t = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu H \leq L_u \leq 40 \, \mu H$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.25: $g_{\text{out, inna}}$ at 7.35 MHz, $R_t = 50 \, \Omega$, $Q = 100$, $25 \, \text{pF} \leq C_{1,2} \leq 4000 \, \text{pF}$, $0.1 \, \mu H \leq L_u \leq 40 \, \mu H$, a) without, b) with 25 pF parasitics, uc. shaded
Figure C.26: a) Best and b) worst $g_{\chi,inma}$ at 30 MHz, $R_i = 50 \Omega$, $Q = 100$, $5 \text{ pF} \leq C_{1,2} \leq 250 \text{ pF}$, and $0.1 \mu\text{H} \leq L_a \leq 40 \mu\text{H}$, undercompensation shaded

Figure C.27: a) Best and b) worst $g_{\chi,inma}$ at 30 MHz, $R_i = 50 \Omega$, $Q = 100$, $5 \text{ pF} \leq C_{1,2} \leq 500 \text{ pF}$, and $0.1 \mu\text{H} \leq L_a \leq 40 \mu\text{H}$, undercompensation shaded

Figure C.28: a) Best and b) worst $g_{\chi,inma}$ at 30 MHz, $R_i = 50 \Omega$, $Q = 100$, $25 \text{ pF} \leq C_{1,2} \leq 4000 \text{ pF}$, and $0.1 \mu\text{H} \leq L_a \leq 40 \mu\text{H}$, undercomp. shaded
Figure C.29: $g_{\text{lof, inna}}$ at 30 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 250 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.30: $g_{\text{lof, inna}}$ at 30 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.31: $g_{\text{lof, inna}}$ at 30 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $25 \, \text{pF} \leq C_{1,2} \leq 4000 \, \text{pF}$, $0.1 \, \mu\text{H} \leq L_a \leq 40 \, \mu\text{H}$, a) without, b) with 25 pF parasitics, uc. shaded
Figure C.32: a) Best and b) worst $g_{t, \text{inma}}$ at 30 MHz, $R_1 = 50 \, \Omega$, $Q = 100$, 5 pF $\leq C_{1,2} \leq 250$ pF, and 0.1 $\mu$H $\leq L_\alpha \leq 40$ $\mu$H, undercompensation shaded

Figure C.33: a) Best and b) worst $g_{t, \text{inma}}$ at 30 MHz, $R_1 = 50 \, \Omega$, $Q = 100$, 5 pF $\leq C_{1,2} \leq 500$ pF, and 0.1 $\mu$H $\leq L_\alpha \leq 40$ $\mu$H, undercompensation shaded

Figure C.34: a) Best and b) worst $g_{t, \text{inma}}$ at 30 MHz, $R_1 = 50 \, \Omega$, $Q = 100$, 25 pF $\leq C_{1,2} \leq 4000$ pF, and 0.1 $\mu$H $\leq L_\alpha \leq 40$ $\mu$H, undercomp. shaded
Figure C.35: $g_{\text{off,inma}}$ at 30 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 250 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_n \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.36: $g_{\text{off,inma}}$ at 30 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $5 \, \text{pF} \leq C_{1,2} \leq 500 \, \text{pF}$, and $0.1 \, \mu\text{H} \leq L_n \leq 40 \, \mu\text{H}$, a) without, b) with 5 pF parasitics, uc. shaded

Figure C.37: $g_{\text{off,inma}}$ at 30 MHz, $R_i = 50 \, \Omega$, $Q = 100$, $25 \, \text{pF} \leq C_{1,2} \leq 4000 \, \text{pF}$, $0.1 \, \mu\text{H} \leq L_n \leq 40 \, \mu\text{H}$, a) without, b) with 25 pF parasitics, uc. shaded
Appendix D

Transmission Lines’ and Total Losses in a Source, Line 1, Matching Network, Line 2, and Load System

D.1 Definitions/notations

From this chapter on, the following definitions/notations will be used:

Latin letters

\( a \)  
Quadratic coefficient of \( e_{\text{opt}} \) in Appendix D.2

\( b \)  
Linear coefficient of \( e_{\text{opt}} \) in Appendix D.2

\( c \)  
Constant coefficient of \( e_{\text{opt}} \) in Appendix D.2

\( C_{\text{IIMBmax}} \)  
Minimum (equivalent) parallel capacitance calculated from the limiting circle of the transformation spiral of \( Y_{\text{IMB}} \) (\( \pi \) network)

\( C_{\text{IIMBmin}} \)  
Minimum (equivalent) parallel capacitance calculated from the limiting circle of the transformation spiral of \( Y_{\text{IMB}} \) (\( \pi \) network)

\( C_{\text{IIMBmax}} \)  
Minimum (equivalent) series capacitance calculated from the limiting circle of the transformation spiral of \( Z_{\text{IMB}} \) (\( T \) network)

\( C_{\text{IImin}} \)  
Minimum (equivalent) series capacitance calculated from the limiting circle of the transformation spiral of \( Z_{\text{IMB}} \) (\( T \) network)

\( L_{\text{IIMBmax}} \)  
Minimum (equivalent) parallel inductance calculated from the limiting circle of the transformation spiral of \( Y_{\text{IMB}} \) (\( \pi \) network)

\( L_{\text{IIMBmin}} \)  
Minimum (equivalent) parallel inductance calculated from the limiting circle of the transformation spiral of \( Y_{\text{IMB}} \) (\( \pi \) network)

\( L_{\text{IIMBmax}} \)  
Minimum (equivalent) series inductance calculated from the limiting circle of the transformation spiral of \( Z_{\text{IMB}} \) (\( T \) network)

\( L_{\text{IImin}} \)  
Minimum (equivalent) series inductance calculated from the limiting circle of the transformation spiral of \( Z_{\text{IMB}} \) (\( T \) network)

\( \Sigma_{\text{opt}} \)  
Input reflection coefficient of a transmission line with \( X_0 \neq 0 \) whose load is the optimum load impedance \( Z_{\text{opt}} \) (referred to \( Z_0 \))

\( \Sigma_{\text{opt}} \)  
Optimum source reflection coefficient for a transmission line with \( X_0 \neq 0 \) whose load is the optimum load impedance \( Z_{\text{opt}} \) (referred to \( Z_0 \))

\( \Sigma_{\text{opt}} \pm \)  
Reflection coefficient of the two solutions for optimum load impedance of a transmission line with \( X_0 \neq 0 \) distinguished by the sign in front of the square root (referred to \( Z_0 \))

\( Y_{\text{IIMB}} \)  
Source admittance of a matchbox (\( \pi \) network)
\textbf{APPENDIX D. LINES’ AND TOTAL LOSSES IN A MATCHING SYSTEM} 191

\begin{align*}
Z_{\text{opt}} & \quad \text{Input impedance of a transmission line with } X_0 \neq 0 \text{ whose load is the optimum load impedance } Z_{\text{opt}} \\
Z_{\text{MB}} & \quad \text{Source impedance of a matchbox (T network)} \\
Z_{\text{opt}} \pm & \quad \text{Two solutions for the optimum load impedance of a transmission line with } X_0 \neq 0 \text{ distinguished by the by the sign in front of the square root}
\end{align*}

\textbf{Greek letters}

\begin{align*}
\delta_C & \quad \text{arsinh} \left( \frac{C'}{C''} \right) \\
\delta_R & \quad \text{arsinh} \left( \frac{R'}{R''} \right)
\end{align*}

\textbf{D.2 Losses and matching in a source, transmission line, and load system}

To optimise \( g_L \) as described in Section 5.3, we have to solve

\[
0 = \left( R_0^2 \right) \left( 1 - e^{-2al} \right) \left( e^{-2al} \right) \left( e^{-2al} \right) L_t
\]

or, slightly rearranged,

\[
0 = \left( R_0 \right) \left( 1 - e^{-2al} \right) \left( e^{-2al} \right) L_t
\]

involving two different cases:

1. \( X_0 = 0 \)

Then we have to solve

\[
0 = 2 e^{-2al} R_0^2 \sinh(2al) L_t.
\]

We assumed \( l > 0 \) and, according to [26c], \( R_0 > 0 \). Additionally, [26b] indicates that \( \alpha \geq 0 \).

If \( \alpha > 0 \), the transmission line is either Heaviside’s distortionless transmission line obtained for \( \frac{R'}{\omega L'} = \frac{C'}{\omega C'} > 0 \) and \( Z_0 = Z_{00} \) (where \( C' \) is the per-unit-length conductance and \( L' = Z_{00}^2 C' \) the per-unit-length inductance of that line, defined in [26a]) or the approximation (2.5) is used. Then \( \tau = 0 \) uniquely solves the equation, thus \( Z_\ell = Z_{00} \) yields the maximum \( g_L \). If we terminate the line accordingly, its input resistance \( Z_1 = Z_{00} \), hence the source’s impedance has to be \( Z_1 = Z_{00} \) to achieve maximum active input power. In particular, optimum source and load impedance is independent of (frequency and) line length.

If \( \alpha = 0 \), the transmission line is lossless, hence \( \frac{R'}{\omega L'} = \frac{C'}{\omega C'} = 0 \). Then \( g_L = 1 \) for any \( Z_\ell \). However, according to (2.20) only a choice of \( Z_\ell = Z_0 = Z_{00} \)
combined with \( Z_1 = Z_{01} \) yields a source impedance for maximum active power input which is independent of (frequency and) line length — thus no other choice is appropriate.

2. \( X_0 \neq 0 \)

We assumed \( l > 0 \) and, according to [26c], \( R_0 > 0 \). Additionally, [26b] indicates that \( \alpha \geq 0 \), but in conjunction with [26c] we may conclude that \( X_0 \neq 0 \) is accompanied by \( \alpha > 0 \), hence we have to consider \( \alpha > 0 \) only.

Then all coefficients of \( r_1^2, r_1, \) and \( r_0^2 = 1 \) in the equation to solve are \( \neq 0 \) except for \( l \to \infty \) (another exception would be \( l = 0 \), but we assumed \( l > 0 \)).

Solving for \( l_{\text{opt}} \) if \( l \in (0, \infty) \) yields two possible solutions,

\[
\begin{align*}
  l_{\text{opt}} & = \frac{2 e^{-2\alpha l} (R_0^2 \sin(2\alpha l) + j X_0^2 \sin(2\beta l))}{2j R_0 X_0 e^{-2\alpha l} e^{-j2\beta l} (1 - e^{-2\alpha l} e^{j2\beta l})} \\
  & \pm \frac{\sqrt{4 e^{-4\alpha l} (R_0^2 \sin(2\alpha l) + j X_0^2 \sin(2\beta l))^2 + 4 R_0^2 X_0^2 e^{-2\alpha l} e^{-j2\beta l} (1 - e^{-2\alpha l} e^{j2\beta l})^2}}{2j R_0 X_0 e^{-2\alpha l} e^{-j2\beta l} (1 - e^{-2\alpha l} e^{j2\beta l})^2}.
\end{align*}
\]

Prior to deciding which solution is valid we have to check the argument of the square root. It may be rewritten to

\[
\begin{align*}
  4 e^{-4\alpha l} (R_0^2 \sin(2\alpha l) + j X_0^2 \sin(2\beta l))^2 & + 4 R_0^2 X_0^2 e^{-2\alpha l} e^{-j2\beta l} (1 - e^{-2\alpha l} e^{j2\beta l})^2 \\
  = & \ 4 e^{-4\alpha l} \left( R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l) - 1 \right) \\
  & \cdot \left( R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l) - 1 \right) \\
  = & \ 2 e^{-4\alpha l} \left( R_0^2 \cosh(4\alpha l) - 1 + 4 R_0^2 X_0^2 \cosh(2\alpha l) \cos(2\beta l) - 1 \right) \\
  & + X_0^4 \cos(4\beta l) - 1 \right) \\
  = & \ 4 e^{-4\alpha l} \left( (R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l))^2 - |Z_0|^4 \right).
\end{align*}
\]

To determine its sign, we check the first rearrangement.

The factor \( 4 e^{-4\alpha l} \) is always \( > 0 \). The first term in parentheses is also always \( > 0 \) because \( R_0^2 \cosh(2\alpha l) + 1 \in [2 R_0^2, \infty) \), \( X_0^2 \cos(2\beta l) + 1 \in [0, 2 X_0^2] \), and \( R_0 > 0 \) holds (the latter according to [26c]).

The second term in parentheses equals zero if \( l = 0 \). To derive its values' range if \( l > 0 \) we compute its derivative with respect to \( l \),

\[
\frac{d}{dl} \left( R_0^2 \cosh(2\alpha l) - 1 + X_0^2 \cos(2\beta l) - 1 \right) = 2 (\alpha R_0^2 \sinh(2\alpha l) - \beta X_0^2 \sin(2\beta l)).
\]
The derivative equals zero if \( l = 0 \). Since we assumed \( \alpha > 0 \) and \( \beta > 0 \) (implied by \( \omega > 0 \))\(^1\), \( \sinh(2\alpha l) > 2\alpha l \) and \( \sin(2\beta l) < 2\beta l \) holds if \( l > 0 \), thus
\[
\frac{d(R_0^2(\cosh(2\alpha l) - 1) + X_0^2(\cos(2\beta l) - 1))}{dl} > 4l(\alpha^2 R_0^2 - \beta^2 X_0^2) \geq 0.
\]
The latter inequality applies because we obtain from \([26b]\) and \([26c]\) that
\[
\alpha^2 R_0^2 - \beta^2 X_0^2 = \left(\frac{\omega L'}{\cosh(\delta C)}\right)^2 \left(\sinh^2\left(\frac{\delta_R + \delta_G}{2}\right) - \cosh^2\left(\frac{\delta_R - \delta_G}{2}\right)\right).
\]
where \( \delta_R = \text{arsinh}\left(\frac{R'}{\omega L'}\right) \), \( \delta_G = \text{arsinh}\left(\frac{G'}{\omega C'}\right) \), \( L' \), \( C' > 0 \), and \( R', G' \geq 0 \), \( \alpha^2 \delta_R, \delta_G \geq 0 \), respectively. Hence \( \delta_R + \delta_G \geq \max\{\delta_R, \delta_G\} \) holds, thus \( \sinh\left(\frac{\delta_R + \delta_G}{2}\right) \geq \sinh\left(\frac{\delta_R - \delta_G}{2}\right) \) and \( \cosh\left(\frac{\delta_R + \delta_G}{2}\right) \geq \cosh\left(\frac{\delta_R - \delta_G}{2}\right) \) \( \alpha^2 R_0^2 - \beta^2 X_0^2 \geq 0 \).

Summarising the preceding derivations, the second term in parentheses of the first rearrangement equals zero if \( l = 0 \) and increases with increasing \( l \), hence it’s always \( > 0 \) if \( l > 0 \).

Thus the argument of the square root in \( L_{\text{opt}} \) stays \( > 0 \) if \( l > 0 \) — then the square root is always real.

Finally we have to check which of the possible solutions \( L_{\text{opt}+} \) is a valid solution. Since we consider passive loads, \( \text{Re}\{Z_{\text{opt}}\} \geq 0 \) holds. Additionally, \( |L_{\text{opt}+}| \) is restricted according to \([26f]\) or, even more accurately\(^2\), to
\[
|L_{\text{opt}+}| \leq \frac{|Z_0|}{R_0} |X_0| < 1 + \sqrt{2}.
\]

---

\(^1\) Combining \([26b]\) and \([26c]\) yields \( \alpha > 0 \), \( \beta > 0 \), \( R_0 > 0 \) and \( R_0 > |X_0| \) or \( R_0^2 > X_0^2 \), respectively. Additionally, as indicated above, \( \alpha^2 R_0^2 - \beta^2 X_0^2 \geq 0 \) holds, which is equivalent to \( \alpha R_0 - |X_0| \geq 0 \).

Unlike \( R_0 \) and \( X_0 \), there is no common relation between \( \alpha \) and \( \beta \), which may be proved easily if we rewrite \( \alpha \), \( \beta \) (and \( R_0, X_0 \)) using trigonometric functions (cos, sin, and arctan) instead of the hyperbolic functions (cosh, sinh, and arsinh) used by \([26b]\). By comparison of the two different descriptions we obtain \( \text{arsinh}\left(\frac{1}{x}\right) = \ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right) \) for all \( x \geq 0 \), which may alternatively be applied to derive the following relations between \( \alpha \) and \( \beta \) from \([26b]\).

Then we may conclude that \( \alpha < \beta \) if \( \frac{R'}{\omega L'} < \frac{1}{\omega C'} \) or \( \frac{G'}{\omega C'} < \frac{1}{\omega L'} \), \( \alpha = \beta \) if \( \frac{R'}{\omega L'} = \frac{1}{\omega C'} \) or \( \frac{G'}{\omega C'} = \frac{1}{\omega L'} \), and \( \alpha > \beta \) if \( \frac{R'}{\omega L'} > \frac{1}{\omega C'} \) or \( \frac{G'}{\omega C'} > \frac{1}{\omega L'} \) (\( R', C' \geq 0 \), \( L', C', \omega > 0 \)).

In particular, \( \alpha < \beta \) if \( R' > 0 \) and \( G' = 0 \), \( C' > 0 \) and \( R' = 0 \), or \( \frac{R'}{\omega L'} < \frac{G'}{\omega C'} \) or \( \frac{G'}{\omega C'} < \frac{1}{\omega L'} \). The latter holds for coaxial transmission lines in our frequency range of interest, where \( \frac{R'}{\omega L'} \gg \frac{G'}{\omega C'} \) and \( \frac{R'}{\omega L'} \ll \frac{G'}{\omega C'} \) could be true.

\(^2\) Due to assuming \( \alpha > 0 \), at least one of \( R' \) and \( \delta_R \) or \( \delta_G \), respectively, has to be \( > 0 \).

\(^3\) \([26f]\) derives \( |R_{\text{max}}| \approx 1 + \sqrt{2} \) by considering the maximum phase angle of the line’s characteristic impedance. However, a more thorough investigation of passive impedances’ reflection coefficients yields their locus in the complex reflection coefficient plane — they all lie within a circle of radius \( |Z_0|/R_0 \) and centre \( -jX_0/R_0 \). The limiting circle describes the locus of purely imaginary impedances’ reflection coefficients. According to \((2.19)\) its centre is the reflection coefficient of the conjugate complex of the line’s characteristic impedance. From \([26c]\) we obtain \( R_0 > 0 \) and \( R_0 > |X_0| \), hence the minimum magnitude of the reflection coefficient equals zero (matching), the maximum magnitude equals \( |Z_0|/R_0 \) \( < 1 + \sqrt{2} \). It’s interesting to note that these relations hold for any “lines” involving transmission lines’ differential equations.
The solution involving subtraction of the square root, \( \mathcal{L}_{\text{opt}^-} \), violates both conditions. If we take into account that the square root in \( \mathcal{L}_{\text{opt}^-} \) is always real for \( l > 0 \), we firstly obtain from (2.11)

\[
\text{Re}\{Z_{\text{opt}^-}\} = -\frac{\sqrt{R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l)}}{R_0 \sinh(2\alpha l) + X_0 \sin(2\beta l)} - |Z_0|^4,
\]

which is always \(< 0\) if \( l > 0 \) since \( R_0 \sinh(2\alpha l) + X_0 \sin(2\beta l) \geq R_0 \sinh(2\alpha l) - |X_0 \sin(2\beta l)| \geq 2l (\alpha R_0 - \beta |X_0|) \geq 0 \) if \( l > 0 \) (the latter inequality holds according to Footnote 1).\(^1\) Secondly \( |\mathcal{L}_{\text{opt}^-}| \to \infty \) for \( l \to \infty \), since the denominator of both ratios in \( \mathcal{L}_{\text{opt}^-} \) tends to zero, and both numerators to \(-\infty\).

Conversely, the solution involving summing of the square root, \( \mathcal{L}_{\text{opt}^+} \), yields

\[
\text{Re}\{Z_{\text{opt}^+}\} = \frac{\sqrt{R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l)} |Z_0|^4}{R_0 \sinh(2\alpha l) + X_0 \sin(2\beta l)},
\]

which is always \(> 0\) if \( l > 0 \). Additionally, \( |\mathcal{L}_{\text{opt}^+}| \) tends to the proper limit for \( l \to \infty \) because then \( \alpha \mathcal{L}_{\text{opt}^+}^2 + b \mathcal{L}_{\text{opt}^+} + c = 0 \), where \( a \to 0 \), \( b \to R_0^2 \), and \( c \to jR_0X_0 \). Since \( b \) is real and \( b > 0 \), application of l’Hospitale’s rule gives

\[
\lim_{a \to 0} \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \lim_{a \to 0} \frac{-4c}{4\sqrt{b^2 - 4ac}} = -\frac{c}{b}, \text{ thus } \mathcal{L}_{\text{opt}^+} \text{ is a valid solution}^4.
\]

Summarising the preceding considerations, we may conclude that

\[
\mathcal{L}_{\text{opt}} = \frac{\sqrt{(R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l))^2 - |Z_0|^4} - R_0 \sinh(2\alpha l) - jX_0 \sin(2\beta l)}}{jR_0X_0 e^{-j2\beta l} (1 - e^{-2\alpha l} e^{j2\beta l})} \quad (D.1)
\]

is the unique valid solution if \( l \in (0, \infty) \).

Application of (2.11) and (2.12) yields the optimum load impedance,

\[
Z_{\text{opt}} = \frac{\sqrt{(R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l))^2 - |Z_0|^4} - jR_0X_0 (\cosh(2\alpha l) - \cos(2\beta l))}}{R_0 \sinh(2\alpha l) + X_0 \sin(2\beta l)} \quad (D.2)
\]

The transmission line’s input impedance is given by (2.20), hence \( Z_{\text{opt}} = \mathcal{L}_{\text{opt}} e^{-2\alpha l} e^{-j2\beta l} \). If the line is terminated by the optimum load impedance indicated above, its input reflection coefficient is \( \mathcal{R}_{\text{opt}} = \frac{R_0 r_{\text{opt}} - jX_0}{R_0 - jX_0 z_{\text{opt}}} \). Then, according to (2.18), its input impedance is \( Z_{\text{opt}} = Z_{\text{opt}} \) — if the transmission line is terminated by its optimum load impedance, its input impedance is the complex conjugate of the optimum load impedance!

From (2.23) we finally compute the optimum power gain,

\[
\mathcal{G}_{\text{opt}} = \frac{R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l) - \sqrt{(R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l))^2 - |Z_0|^4}}{|Z_0|^2} \quad (D.3)
\]

However, to achieve maximum active input power, the source impedance has to be equal to the complex conjugate of the line’s input impedance, hence

\(^4\) Application of l’Hospitale’s rule requires \(-b + \sqrt{b^2 - 4ac} \to 0 \) for \( a \to 0 \), hence \( b = |b| \) or \( b \geq 0 \) if \( b \) is real. Conversely, if \( b \leq 0 \), the proper limit would be achieved by the solution involving subtraction of the square root.
\[ Z_{\text{opt}} = Z_{\text{opt}}^* = Z_{\ell_{\text{opt}}} \] or \[ \tau_{\opt} = \tau_{\text{opt}} = \tau_{\text{opt}}. \] If the source impedance is chosen accordingly, the active input power is \( P_{\text{act, opt}} = P_{\text{act, max}} \), thus \( g_{\tau, \text{opt}} = g_{\text{opt}}. \) Additionally, the line’s output impedance is equal to the complex conjugate of its optimum load impedance because \( \tau_{\text{opt}} e^{-2\alpha l} e^{-i2\beta l} \) in conjunction with \( \tau_{\text{opt}} = \tau_{\text{opt}} \) describes an impedance of \( Z_{\text{opt}}^* \). Optimum active power input and transfer involves complex conjugate matching at the line’s in- and output!

Comparison of the preceding optimum\(^5\) derived for a complex characteristic impedance\(^6\) where \( X_0 \neq 0 \) reveals a severe disadvantage compared to that of a real characteristic impedance where \( X_0 = 0 \) — the complex characteristic impedance’s optimum load depends on frequency and line length — thus, depending on frequency and line length, different load and source impedances would have to be used.

To choose a useful length independent load impedance close to the optimised load, we consider its limits for \( l \to 0 \) and \( l \to \infty \).

If \( l = 0 \), the equation to be solved for the optimum \( \tau_{\ell} \) changes to \( 0 \tau_{\ell}^2 + 0 \tau_{\ell} + 0 = 0 \), hence any \( \tau_{\ell} \) or \( Z_{\ell} \) would involve optimum active power transfer — which holds since there is no line at all. To ensure maximum active input power, simply a source impedance of \( Z_1 = Z_{\ell} \) is required.

Conversely, \( \tau_{\text{opt}} \) or \( Z_{\text{opt}} \) tend to particular limits for \( l \to 0 \), because they take into account the tiny, but nonzero line length, for which an optimum still exists.

If \( \frac{X_0}{R_0} \neq -\frac{\alpha}{\beta} \) (more precisely, if \( \frac{X_0}{R_0} > -\frac{\alpha}{\beta} \) according to Footnote 1\), we obtain

\[
\begin{align*}
\lim_{l \to 0} \tau_{\text{opt}} &= \sqrt{\frac{\left| Z_0 \right|^2 \left( \alpha^2 R_0^2 - \beta^2 X_0^2 \right)}{R_0 X_0 \left( \beta + j\alpha \right)}} = \lim_{l \to 0} \tau_{\text{opt}}, \\
\lim_{l \to 0} Z_{\text{opt}} &= \sqrt{\frac{\left| Z_0 \right|^2}{\beta R_0 + j\beta X_0}} = \lim_{l \to 0} Z_{\text{opt}}, \\
\lim_{l \to 0} g_{\text{opt}} &= 1 = \lim_{l \to 0} g_{\tau, \text{opt}}, \\
\end{align*}
\]

if \( \frac{X_0}{R_0} = -\frac{\alpha}{\beta} \) (which requires \( \frac{\alpha}{\beta} < 1 \) because \( \frac{|X_0|}{R_0} < 1 \) according to Footnote 1),

\[
\begin{align*}
\lim_{l \to 0} \tau_{\text{opt}} &= 1 = \lim_{l \to 0} \tau_{\text{opt}}, \\
\Re\{Z_{\text{opt}}\}, \quad \Re\{Z_{\text{opt}}\} &\to \infty \quad \text{for} \quad l \to 0,
\end{align*}
\]

---

\(^5\)Considering Footnote 1 of Section 5.3, we would suppose that \( g_{\text{opt}} \) decreases if \( l \) increases.

To prove it, we compute \( \frac{dg_{\text{opt}}}{dl} = -2 g_{\text{opt}} \sqrt{\frac{\alpha R_0^2 \sinh(2\alpha l) - \beta X_0^2 \sin(2\beta l)}{\sqrt{(R_0^2 \cosh(2\alpha l) + X_0^2 \cos(2\beta l))^2 - |Z_0|^4}}}. \) Since \( g_{\text{opt}} > 0 \) (line and load are entirely passive and \( g_{\text{opt}} \to 0 \) only if \( l \to \infty \)) and \( \alpha R_0^2 \sinh(2\alpha l) - \beta X_0^2 \sin(2\beta l) > 0 \) if \( l > 0 \) (as indicated above), \( \frac{dg_{\text{opt}}}{dl} < 0 \) if \( l > 0 \) (where \( \frac{dg_{\text{opt}}}{dl} \to 0 \) for \( l \to \infty \)). However, \( \frac{dg_{\text{opt}}}{dl} \bigg|_{l=0} = -2 \sqrt{\frac{\alpha R_0^2 - \beta X_0^2}{|Z_0|^4}} \) may equal zero if \( \alpha R_0^2 - \beta X_0^2 = 0 \), which (approximately) holds for coaxial transmission lines in our frequency range of interest.

\(^6\)It’s interesting to note that the optimum reflection coefficient’s limit for \( X_0 \to 0 \) is equal to the optimum reflection coefficient derived for \( X_0 = 0 \) and \( \alpha > 0 \).
\[ \lim_{l \to 0} g_{\text{Lopt}} = 1 = \lim_{l \to 0} g_{T, \text{Lopt}}. \]

In particular, the latter applies if (2.4), hence \( Z_0 \approx Z_{\text{oll}} (1 - j \frac{\alpha}{\beta}) \), (approximately) describes the line’s complex characteristic impedance. Then the limit of the optimum load impedance tends to an open circuit\(^7\).

For \( l \to \infty \) the equation to be solved for the optimum \( \tau_{\ell} \) changes to \( 0 \tau_{\ell}^2 + R_0^2 \tau_{\ell} + j R_0 X_0 = 0 \). Since we already proved that \( \tau_{\text{Lopt}} \) tends to the proper solution, we obtain by solving for \( \tau_{\ell} \),

\[
\lim_{l \to \infty} \tau_{\text{Lopt}} = - \frac{j X_0}{R_0} = \lim_{l \to \infty} \tau_{\text{opt}},
\]

which describes a

\[
\lim_{l \to \infty} Z_{\text{Lopt}} = Z_0 = \lim_{l \to \infty} Z_{\text{opt}}
\]

according to (2.19).

Since we assumed \( \alpha > 0 \), the optimum power gain becomes

\[
\lim_{l \to \infty} g_{\text{Lopt}} = 0 = \lim_{l \to \infty} g_{T, \text{Lopt}}.
\]

### D.3 Losses and matching in a source, line 1, matching network, line 2, and load system

#### D.3.1 Lossy matching network

##### D.3.1.1 \( \pi \) network

To check the impact of source admittance transformation by the line on network design, in particular the necessity of extending the optimisation to complex source admittances, we have to consider the achieved susceptances if low loss coaxial cables are used.

If the source impedance equals the line’s complex characteristic impedance, hence \( Z_0 \approx Z_{\text{oll}} (1 - j \frac{\alpha}{\beta}) \) according to (2.4), its (transformed) source admittance \( \hat{Y}_{\text{IMB}} \) at matching frequency \( \omega \) is equal to

\[
\hat{Y}_{\text{IMB}} = \frac{1}{Z_0} \approx \frac{1}{Z_{\text{oll}} \left(1 + \frac{\alpha^2}{\beta^2}\right)} + j \omega \frac{\alpha}{\beta} Z_{\text{oll}} \left(1 + \frac{\alpha^2}{\beta^2}\right) \approx \frac{1}{Z_{\text{oll}}} + j \omega \frac{\alpha}{\beta} \frac{1}{Z_{\text{oll}}} =: \frac{1}{R_{\text{op}}} + j \omega C_{\text{op}},
\]

which describes a conductance \( \frac{1}{R_{\text{op}}} = \frac{1}{Z_{\text{oll}}} \) in parallel to a capacitance \( C_{\text{op}} = \frac{\alpha}{\omega Z_{\text{oll}}} \).

\(^7\)Although no active power may be transferred to an (ideal) open circuit, we have to keep in mind that optimisation required \( l > 0 \), hence at least tiny line lengths. Additionally \( Z_0 \approx Z_{\text{oll}} (1 - j \frac{\alpha}{\beta}) \) approximately describes the coaxial transmission line’s complex characteristic impedance. Thus the optimum source and load impedances are close to, but not equal to an open circuit.
Conversely, if the source is designed for maximum active power input, its impedance equals the complex conjugate of the line’s characteristic impedance, hence its admittance exhibits the same conductance, but in parallel to an inductance of \( \frac{Z_{\text{mb}}(1 + \frac{1}{\omega^2})}{\omega \frac{\alpha}{\beta}} \approx \frac{Z_{\text{mb}}}{\omega \frac{\alpha}{\beta}} \). However, it’s further transformed by line 1 to \( \tilde{Y}_{\text{MB}} \) at the matching network’s input. The minimum/maximum real and imaginary part of the transformed source admittance may be derived by considering its limiting circle\(^8\) of the transformation spiral in the complex reflection coefficient plane. We obtain

\[
\begin{align*}
\text{Min} \left\{ \frac{1}{\text{Re} \left( \tilde{Y}_{\text{MB}} \right)} \right\} &= Z_{\text{mb}} \frac{1 - \left( \frac{\alpha}{\beta} \right)^2}{1 + \frac{2}{\sqrt{1 + (\frac{\beta}{\alpha})^2}}} \approx \frac{Z_{\text{mb}}}{1 + \frac{2}{\sqrt{1 + (\frac{\beta}{\alpha})^2}}}, \\
\tilde{C}_{\text{MBpmin}} &= \frac{\alpha}{\beta}, \\
\tilde{C}_{\text{MBpmax}} &= \frac{\frac{\alpha}{\beta} \frac{2}{\sqrt{1 + (\frac{\beta}{\alpha})^2}} + 1}{\omega Z_{\text{mb}}} \approx \frac{3 \frac{\alpha}{\beta}}{\omega Z_{\text{mb}}}, \\
\tilde{L}_{\text{MBpmin}} &= \frac{Z_{\text{mb}} \left( 1 - \left( \frac{\alpha}{\beta} \right)^2 \right)}{\omega \frac{\alpha}{\beta} \left( \frac{2}{\sqrt{1 + (\frac{\beta}{\alpha})^2}} - 1 \right)} \approx \frac{Z_{\text{mb}}}{\omega \frac{\alpha}{\beta}}, \\
\tilde{L}_{\text{MBpmax}} &\to \infty.
\end{align*}
\]

However, in our application coaxial cables with \( \alpha << \beta \) are used, thus a generator of source impedance \( Z_{\text{mb}} \approx R_0 \) is chosen which is further transformed by line 1 to \( \tilde{Y}_{\text{MB}} \) at the matching network’s input. The minimum/maximum real and imaginary part of the transformed source admittance may be derived in a similar manner by considering its limiting circle\(^9\) of the transformation spiral in the complex reflection

\(^8\)The limiting circle, the outer limit of the transformation spiral, is obtained if all \( e^{-2a_{11}} \) coefficients in the (transformed) reflection coefficient or its according admittance are replaced with 1. Doing so yields an outer limit of the transformed source admittance given by a circle of radius \( \frac{1}{R_0} \frac{1}{1 - \left( \frac{\alpha}{R_0} \right)^2} \frac{1}{\sqrt{1 + (\frac{\beta}{R_0})^2}} \) and centre \( \frac{1}{R_0} \frac{1}{1 - \left( \frac{\alpha}{R_0} \right)^2} \left( 1 - \frac{j \frac{\beta}{R_0} }{R_0} \right) \). (If \( |X_{11}| > \frac{1}{\sqrt{3}} \approx 0.58 \) the circle also encloses admittances exhibiting a negative real part — which indicates that in such a case involving high attenuation coefficients the transformation spiral stays far away from its limiting circle — source admittance, line, and hence transformed source admittance are entirely passive.)

\(^9\)The limiting circle, the outer limit of the transformation spiral, is obtained if all \( e^{-2a_{11}} \) coefficients in the (transformed) reflection coefficient or its according admittance are replaced with 1. Doing so yields an outer limit of the transformed source admittance given by a circle of radius \( \frac{1}{R_0} \frac{1}{\sqrt{2 \cos \left( \frac{1}{3} \arccos \left( -\frac{5}{32} \right) \right) - 1}} \approx 0.95 \) the circle also encloses admittances exhibiting a negative real part — which indicates that in such a case involving high attenuation coefficients the transformation spiral stays far away from its limiting circle — source admittance, line, and hence transformed source admittance are entirely passive.)
coefficient plane. We obtain

\[
\text{Min} \left\{ \frac{1}{\text{Re}\{\Sigma_{\text{IMB}}\}} \right\} = Z_{\text{ol}} \frac{1 + \left( \frac{\alpha}{\beta} \right)^2}{1 + \frac{1}{2} \left( \frac{\alpha}{\beta} \right)^2 \pm \frac{\alpha}{\beta} \sqrt{1 + \frac{1}{4} \left( \frac{\alpha}{\beta} \right)^2} \sqrt{1 + \left( \frac{\alpha}{\beta} \right)^2}} \approx Z_{\text{ol}} \frac{1 \pm \frac{\alpha}{\beta}}{1 + \frac{\alpha}{\beta}}.
\]

\[C_{\text{IMB},\text{min}} = 0,
\]

\[C_{\text{IMB},\text{max}} = \frac{1}{\omega Z_{\text{ol}}} \frac{\alpha}{\beta} \sqrt{1 + \frac{1}{4} \left( \frac{\alpha}{\beta} \right)^2} \sqrt{1 + \left( \frac{\alpha}{\beta} \right)^2} + 1 + \frac{1}{2} \left( \frac{\alpha}{\beta} \right)^2 \approx \frac{2 \alpha}{\omega Z_{\text{ol}}},
\]

\[L_{\text{IMB},\text{min}} = \frac{Z_{\text{ol}} \left( 1 + \left( \frac{\alpha}{\beta} \right)^2 \right)}{\omega \frac{\alpha}{\beta} \left( \sqrt{1 + \frac{1}{4} \left( \frac{\alpha}{\beta} \right)^2} \sqrt{1 + \left( \frac{\alpha}{\beta} \right)^2} - 1 - \frac{1}{2} \left( \frac{\alpha}{\beta} \right)^2 \right)} \approx \frac{8 Z_{\text{ol}}}{\omega \left( \frac{\alpha}{\beta} \right)^3},
\]

\[L_{\text{IMB},\text{max}} \rightarrow \infty.
\]

**D.3.1.2 T network**

To check the impact of source impedance transformation by the line on network design, in particular the necessity of extending the optimisation to complex source impedances, we have to consider the achieved reactances if low loss coaxial cables are used.

If the source impedance equals the line’s complex characteristic impedance, hence \(Z_0 \approx Z_{\text{ol}} (1 - j \frac{\alpha}{\beta})\) according to (2.4), its (transformed) source impedance \(\tilde{Z}_{\text{IMB}}\) at matching frequency \(\omega\) is equal to

\[
\tilde{Z}_{\text{IMB}} = Z_0 \approx Z_{\text{ol}} - j Z_{\text{ol}} \frac{\alpha}{\beta} = Z_{\text{ol}} + \frac{1}{j \omega Z_{\text{ol}}} \frac{\alpha}{\beta} = \frac{1}{G_0} + \frac{1}{j \omega C_0},
\]

which describes a resistance \(\frac{1}{G_0} = Z_{\text{ol}}\) in series to a capacitance \(C_0 = \frac{1}{\omega Z_{\text{ol}} \frac{\alpha}{\beta}}\).

Conversely, if the source is designed for maximum active power input, its impedance equals the complex conjugate of the line’s characteristic impedance, hence it exhibits the same resistance, but in series to an inductance \(\frac{1}{\omega Z_{\text{ol}} \frac{\alpha}{\beta}}\). However, it’s further transformed by line 1 to \(\tilde{Z}_{\text{IMB}}\) at the matching network’s input. The minimum/maximum real and imaginary part of the transformed source impedance may be derived by considering its limiting circle\(^{10}\) of the transformation spiral in the

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\(^{10}\) The limiting circle, the outer limit of the transformation spiral, is obtained if all \(e^{-2\alpha \beta}\) coefficients in the (transformed) reflection coefficient or its according impedance are replaced with 1. Doing so yields an outer limit of the transformed source impedance given by a circle of radius \(R_0 = \frac{x_0}{2 \frac{x_0}{R_0} - 1} \sqrt{1 + \left( \frac{x_0}{R_0} \right)^2}\) and centre \(R_0 = \frac{1 + (\frac{x_0}{R_0})^2}{1 - (\frac{x_0}{R_0})^2}\). (If \(1 \frac{x_0}{R_0} > 1 \sqrt{3} \approx 0.58\) the circle also encloses impedances exhibiting a negative real part — which indicates that in such a case involving high attenuation coefficients the transformation spiral stays far away from its limiting circle — source impedance, line, and hence transformed source impedance are entirely passive.)
complex reflection coefficient plane. We obtain

\[
\begin{align*}
\operatorname{Max} \left\{ \operatorname{Re} \{ \tilde{Z}_{\text{IMB}} \} \right\} &= Z_{\text{oll}} \left( 1 + \frac{a^2}{\beta} \right)^2 \left( 1 \pm \frac{2 \alpha}{\beta} \sqrt{1 + \left( \frac{a}{\beta} \right)^2} \right) \approx Z_{\text{oll}} \left( 1 \pm \frac{2 \alpha}{\beta} \right), \\
\tilde{C}_{\text{MBmax}} &= \frac{1 - \left( \frac{a}{\beta} \right)^2}{1 - \left( \frac{a}{\beta} \right)^2} \left( 1 + \frac{2 \alpha}{\beta} \sqrt{1 + \left( \frac{a}{\beta} \right)^2} + 1 \right) \approx \frac{1}{3 \omega \frac{a}{\beta} Z_{\text{oll}}}, \\
\tilde{L}_{\text{MBmax}} &= \frac{1}{\omega \frac{a}{\beta} Z_{\text{oll}}} \left( 1 + \frac{2 \alpha}{\beta} \sqrt{1 + \left( \frac{a}{\beta} \right)^2} - 1 \right) \approx \frac{1}{\omega \frac{a}{\beta} Z_{\text{oll}}}. 
\end{align*}
\]

However, in our application coaxial cables with \( \alpha \ll \beta \) are used, thus a generator of source impedance \( Z_{\text{oll}} \approx R_0 \) is chosen which is further transformed by line 1 to \( Z_{\text{IMB}} \) at the matching network's input. The minimum/maximum real and imaginary part of the transformed source impedance may be derived in a similar manner by considering its limiting circle\(^{11}\) of the transformation spiral in the complex reflection coefficient plane. We obtain

\[
\begin{align*}
\operatorname{Max} \left\{ \operatorname{Re} \{ \tilde{Z}_{\text{IMB}} \} \right\} &= Z_{\text{oll}} \left( 1 + \frac{1}{4} \left( \frac{a}{\beta} \right)^2 \sqrt{1 + \frac{1}{4} \left( \frac{a}{\beta} \right)^2} \sqrt{1 + \frac{3}{4} \left( \frac{a}{\beta} \right)^2} \right) \approx Z_{\text{oll}} \left( 1 \pm \frac{\alpha}{\beta} \right), \\
\tilde{C}_{\text{MBmin}} &= \frac{1}{\omega \frac{a}{\beta} Z_{\text{oll}}} \left( \sqrt{1 + \frac{3}{4} \left( \frac{a}{\beta} \right)^2} \sqrt{1 + \frac{1}{4} \left( \frac{a}{\beta} \right)^2 + 1} \right) \approx \frac{1}{2 \omega \frac{a}{\beta} Z_{\text{oll}}}, \\
\tilde{C}_{\text{MBmax}} &= \infty, \\
\tilde{L}_{\text{MBmin}} &= 0, \\
\tilde{L}_{\text{MBmax}} &= \frac{1}{\omega \frac{a}{\beta} Z_{\text{oll}}} \left( \sqrt{1 + \frac{3}{4} \left( \frac{a}{\beta} \right)^2} \sqrt{1 + \frac{1}{4} \left( \frac{a}{\beta} \right)^2 - 1} \right) \approx \frac{3}{8 \omega Z_{\text{oll}}}.
\end{align*}
\]

\(^{11}\)The limiting circle, the outer limit of the transformation spiral, is obtained if all \( e^{-2 \omega t} \) coefficients in the (transformed) reflection coefficient or its according impedance are replaced with 1. Doing so yields an outer limit of the transformed source impedance given by a circle of radius \( R_0 \left( \frac{X_0}{\omega} \right)^2 \sqrt{1 + 4 \left( \frac{R_0}{\omega X_0} \right)^2} \) and centre \( R_0 \left( 1 + \frac{1}{2} \left( \frac{X_0}{X_0} \right)^2 \right) \left( 1 + \frac{j X_0}{\omega X_0} \right) \). (If \( \frac{X_0}{R_0} > 0.25 \), \( \sqrt{2} \cos \left( \frac{1}{3} \arccos \left( -\frac{2}{3} \right) \right) - 1 \approx 0.95 \) the circle also encloses impedances exhibiting a negative real part — which indicates that in such a case involving high attenuation coefficients the transformation spiral stays far away from its limiting circle — source impedance, line, and hence transformed source impedance are entirely passive.)
Appendix E

Publications of This Work


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