The Process Semantics of Time and Space as Anticipation

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Outline

• Relationships in Category Theory
  – Pivotal Role of Adjointness
  – Intension-Extension
  – Static/Dynamic
  – All a Question of Typing

• Natural Composition of Systems
  – Godement
  – Satisfy Complex Requirements

• Time-Space and other Examples
  – Relevance to Anticipation
Purpose

• To attempt to show that the natural relationships declared categorically can satisfy those needed in the real world and provide a basis for anticipation.
Relationships Dominate Category Theory

• Categories
  – Cartesian closed – products
  – Locally cartesian closed – pullbacks/comma/slice
  – Intra-category

• Functors
  – Map from one category to another
  – Inter-category

• Natural Transformations
  – Map from one functor to another
  – Inter-functor
Adjointness

- Perhaps most important relationship is that of adjointness
- Discovered by Kan
- Elaborated by Lawvere
- Provides a more general type of relationship then equivalence
  - Suited to real-world where relationships are not always so simple
Inter-relationship between two Categories L, R through Functors F, G

If adjointness holds, we write \( F \dashv G \)
Features of Adjointness $F \dashv G$

- Free functor ($F$) provides openness
- Underlying functor ($G$) enforces rules
- Natural so one (unique) solution
- Special case
  - $GF(L)$ is the same as $L$ AND
  - $FG(R)$ is the same as $R$
  - Equivalence relation
- Adjointness in general is a relationship less strict than equivalence
  - $1_L \leq GF$ if and only if $FG \leq 1_R$
Example of Adjointness

- If conditions hold, then we can write the adjunction $F \dashv G$
- The adjunction is represented by a 4-tuple: $\langle F, G, \eta, \varepsilon \rangle$
- $\eta$ and $\varepsilon$ are unit and counit respectively
  - $\eta : L \to GFL; \varepsilon : FGR \to R$
  - Measure displacement in mapping on one cycle

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Uses of Adjointness

• Representing intension-extension
• Intension-extension is critical for representing information systems
  – Goes back to Port Royal Logic
• Intension is definition of a system
  – The permanent part that does not change
• Extension is time-varying part of a system
  – The time-varying part that is in constant flux
Example 1

• Banking System
• Intension is definition of structures, rules and procedures that specify how the bank operates; can be a preorder with cycles
• Extension is the data values for the banking operation at a particular time; a partial order
Consider one relationship

- Customer (C) : account (A) in context of holds (H)
- For Intension (I) - Extension (E) define 2 categories and 3 functors:
Definitions

• $C \times_H A$ is the relationship $C \times X \times A$ in the context of $H$
• $C$ is customer, $A$ is account, $H$ is holds
• $C + A$ is all possible values for pairs of $C,A$; by convention written this way but also includes other pools of possible data values (data soup, data types)
Functors pair 1

- $\Sigma$ selects values that exist for $C, A$ in $E$
  - Free functor as performing choice
- $\Delta$ takes values that exist for relationship between $C, A$ in $E$ and checks conformity with definition in $I$ for $C \times_H A$
  - Underlying functor as checking a rule
- If $\Sigma$ and $\Delta$ are adjoint, we write:
  $$\Sigma \dashv \Delta$$
Functors pair 2

- $\Delta$ takes values that exist for relationship between $C$, $A$ in $E$ and checks conformity with definition in $I$ for $C X_{H} A$
  - Because many contexts may exist it is now a free functor as selecting a role (viewpoint is now free)
  - Could have other contexts $H'$, $H''$ ...
  - Note no use of number (Gödel !)
- $\Pi$ selects values that exist for $C X_{H} A$ in $E$
  - Underlying functor as checking that values selected in $E$ match the type definition
- If both pairs of adjoints hold, we write:
  $\Sigma \vdash \Delta \vdash \Pi$
Simultaneity

• In all categorical constructions
  – There is no sequence in the composition
  – The whole structure is evaluated simultaneously (snapped)

• The I-E relationship is an arrow
  \[ \Sigma \dashv \Delta \dashv \Pi \]
  and not any of the categories or functors on their own
Can Build up Relations

• Composition of arrows is natural
  – Godement calculus
• As I-E relation is an arrow
• Can compose one I-E relation with another
• Can build up complex levels of types and definitions with flexible meta levels
Defining Four Levels of Data Typing with Contravariant Functors and Intension-Extension (I-E) Pairs
Composition of I-E pairs

• Higher I-E pair becomes \((\Sigma \vdash \Delta \vdash \Pi)'\)
• Lower I-E pair becomes \((\Sigma \vdash \Delta \vdash \Pi)\)
• Then top-down is
  \[ (\Sigma \vdash \Delta \vdash \Pi) \circ (\Sigma \vdash \Delta \vdash \Pi)' \]
  and bottom-up is
  \[ (\Sigma \vdash \Delta \vdash \Pi)' \circ (\Sigma \vdash \Delta \vdash \Pi) \]
Form of Underlying Categories

• Might represent data structure (static)
  – Pullbacks/ pushouts
  – Comma/ slice categories

• Might represent process (dynamic)
  – Monads/comonads
Pullback

\[
\begin{array}{c}
\text{C} \\
\text{A} \\
\text{C} + \text{A}
\end{array}
\]

\[
\begin{array}{c}
\Pi_c \\
\Sigma \\
\Pi_a \\
\Delta \\
\Pi \\
\end{array}
\]

\[
\begin{array}{c}
\text{C} \\
\text{X_H} \\
\text{A} \\
\text{C} + \text{A}
\end{array}
\]

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Pullback versus I-E

• Saying much the same thing
• Pullback useful as a descriptive diagram
• I-E is more useful algebraically as it:
  – Spells out the exact nature of the relationship
  – Is an arrow which can be composed with other categorical arrows
Monad

Transaction (ACID):
\( T \) is one cycle
\( T^2 \) is two cycles
\( T^3 \) is three cycles

\( T \) is an endofunctor, can be an adjoint:
\( GF \) or \( \Sigma \dashv \Delta \dashv \Pi \), latter as 2 pairs strictly

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Monad

- Monad is an abstract concept:
  \[ \text{Monad} = \langle T, \eta, \mu \rangle \]
  Where \( T \) is the endofunctor -- endofunctor is functor with same source and target (often an adjoint)
  \( \eta \) is the unit of adjunction: change in \( L \) on one cycle
  \( \mu \) is the multiplication: change between \( T^2 \) and \( T \) (on 2\textsuperscript{nd} cycle looking back)
  But can express in more detail:
Adjointness between Monad/Comonad

$$\text{Monad} = \langle T, \eta, \mu \rangle$$

$$T = GF$$

$$\text{Comonad} = \langle S, \epsilon, \delta \rangle$$

$$S = FG$$
Comonad

- Comonad is dual of Monad
- Comonad = $\langle S, \varepsilon, \delta \rangle$
- Where $S$ is the endofunctor (often an adjoint)
- $\varepsilon$ is the counit of adjunction, measuring change in $R$ on one cycle
- $\delta$ is the comultiplication, measuring change between $S$ and $S^2$ (looking forward)
Simultaneity

• Monad/comonad are not handled in a sequential fashion
• Cycles are simultaneous
• Structure satisfying the rules is snapped
Time/Space

I

Σ

Δ

Π

T Xc S

E

T + S

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**Time/Space as I-E**

- $T \& S$ is in the intension
- $T \| S$ is in the extension

- $\Sigma \rightarrow \Delta \rightarrow \Pi$ gives the relationship for a particular context $C$ between:
  - $T \& S$ and $T \| S$

where $T \& S$ is the invariant intension ($I$) and $T \| S$ is the time-varying extension ($E$)

Relativistic: $\eta$, $\varepsilon$ are significant; classical: maybe not so.
Anticipation 1

- So is anticipation the comultiplication
  - $\delta: S \rightarrow S^2$
  - taking the comonad forward one cycle
- This is a tempting conclusion
- But it is not so simple
- Anticipation is not one arrow on its own
- Need to consider the full context
Anticipation 2

• Could better be viewed as looking forward:

\[ \delta: S \rightarrow S^2 (FG \rightarrow FGFG) \]

in the context of the monad/comonad adjointness, in particular of the arrow looking back:

\[ \mu: T^2 \rightarrow T (GFGF \rightarrow GF) \]