A UNIFYING PICTURE OF HELICAL AND AZIMUTHAL MAGNETOROTATIONAL INSTABILITY, AND THE UNIVERSAL SIGNIFICANCE OF THE LIU LIMIT

OLEG N. KIRILLOV1, FRANK STEFANI1, AND YASUHIDE FUKUMOTO2

1 Helmholtz-Zentrum Dresden-Rossendorf, P.O. Box 510119, D-01314 Dresden, Germany; o.kirillov@hzdr.de, f.stefani@hzdr.de
2 Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan; yasuhide@imi.kyushu-u.ac.jp

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Abstract

The magnetorotational instability (MRI) plays a key role for cosmic structure formation by triggering turbulence in the rotating flows of accretion disks that would be otherwise hydrodynamically stable. In the limit of small magnetic Prandtl number, the helical and the azimuthal versions of MRI are known to be governed by a quite different scaling behavior than the standard MRI with a vertical applied magnetic field. Using the short-wavelength approximation for an incompressible, resistive, and viscous rotating fluid, we present a unified description of helical and azimuthal MRI, and we identify the universal character of the Liu limit $2(1 - \sqrt{2}) \approx -0.8284$ for the critical Rossby number. From this universal behavior we are also led to the prediction that the instability will be governed by a mode with an azimuthal wavenumber that is proportional to the ratio of axial to azimuthal applied magnetic field, when this ratio becomes large and the Rossby number is close to the Liu limit.

Key words: instabilities – magnetohydrodynamics (MHD) – waves

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1. INTRODUCTION

The magnetorotational instability (MRI) is widely accepted as the main source of turbulence and outward angular momentum transport that is needed for the matter in accretion disks to spiral inward onto the central protostar or black hole (Balbus & Hawley 1991). While the early work on MRI was mainly concerned with ideal magnetohydrodynamics, the last years have seen an increasing interest in the influence of viscosity and electrical resistivity on the MRI (Pessah & Chan 2008). A particular role is thought to be played by the magnetic Prandtl number $(Pm)$, which measures the ratio of viscosity to resistivity. For accretion disks around black holes (BHs), Balbus & Henri (2008) had discussed the transition from large values of $Pm$, in the vicinity of the BH, to small values in the outer part of the disk, with $Pm$ reaching unity for approximately 100 Schwarzschild radii (depending on several parameters, among them the mass of the BH). By invoking a thermal runaway process at the unstable interface between regions with $Pm > 1$ and $Pm < 1$, the authors associated this boundary with the existence of high and low X-ray states (see Remillard & McClintock 2006).

In recent years, a vivid discussion (Lesur & Longaretti 2007; Fromang et al. 2007; Käpylä & Korpi 2011; Oishi & Mac Low 2011) was devoted to the possible decline of the angular momentum transport rate with decreasing $Pm$, and to the intricate roles that are played here by the magnetic Reynolds number, the detailed boundary conditions, and the stratification of the disk.

Besides this relevance to the outer part of accretion disks, to protoplanetary disks (Turner & Sano 2008), and possibly even to planetary cores (Petitdemange et al. 2008), the limit of low and vanishing $Pm$ has acquired some additional interest in connection with the recent liquid metal experiments devoted to the study of MRI (Sisan et al. 2004; Stefani et al. 2006; Nornberg et al. 2010).

While for the standard version of MRI (SMRI), characterized by only a vertical field being applied, the low $Pm$ limit is rather smooth and unspectacular (Pessah & Chan 2008), the addition of an azimuthal field leads to dramatic effects as revealed for the first time by Hollerbach & Rüdiger (2005). The arising helical MRI (HMRI), as we call it now, was shown to work also in the limit of vanishing $Pm$ since it is governed by the Reynolds and Hartmann number, quite in contrast to SMRI, which is governed by the magnetic Reynolds number and the Lundquist number. In this inductionless limit the induced magnetic field can be neglected with respect to the imposed one, whereas the corresponding induced currents have still to be taken into account in the Lorentz force term (see Hollerbach & Rüdiger 2005; Priede 2011).

However, as it was early shown by Liu et al. (2006), the functioning of HMRI is limited to comparably steep rotation profiles with Rossby numbers $Ro < Ro_{Liu} = 2(1 - \sqrt{2}) \approx -0.8284$ (which we henceforth will call the “Liu limit”) and therefore does not extend to the astrophysically important Kepler profiles characterized by $Ro = -0.75$. This essential limitation of the HMRI, together with a variety of further parameter dependencies, was confirmed in the PROMISE experiment by Stefani et al. (2006, 2007, 2009). The intricate, though continuous, transition between SMRI and HMRI, which involves a spectral exceptional point at which the inertial wave branch coalesces with the branch of the slow magnetocoriolis wave, was clarified only recently by Kirillov & Stefani (2010, 2011).

Another surprise in the limit of low $Pm$ was, for the case of a purely or strongly dominant azimuthal magnetic field, the numerical prediction of a non-axisymmetric version of MRI, working apparently in a similar parameter region as HMRI (Hollerbach et al. 2010). Although the occurrence of MRI under the influence of a purely azimuthal magnetic field had been studied much earlier (see Hawley et al. 1995; Ogilvie & Pringle 1996; Papaloizou & Terquem 1997), the crucial effect arose again for the particular combination of low $Pm$ and slightly steeper than Keplerian shear profiles. It has to be noticed that this azimuthal MRI (AMRI), as we call it now, works for azimuthal magnetic fields that are current-free in the considered fluid, quite in contrast to the Tayler instability (Tayler 1973; Seilmayer et al.
Introducing, as a measure of the steepness of the rotation profile, the Rossby number (Ro),

$$\text{Ro} = \frac{R}{2\Omega} \frac{\partial \Omega}{\partial R},$$

we find

$$a = \Omega (1 + \text{Ro}), \quad b = -\Omega R^2 \text{Ro}.$$  \hfill (10)

To study flow and magnetic field perturbations on the background of the magnetized TC flow, we linearize the Navier–Stokes and induction equation in the vicinity of the stationary solution by assuming $u = u_0 + u^\prime$, $p = p_0 + p^\prime$, and $B = B_0 + B^\prime$ leaving only terms of first order with respect to the primed quantities.

Then, by using a short-wavelength approximation (the details of the derivation will be published elsewhere) in the frame of the geometrical optics approach (see, e.g., Landman & Saffman 1987; Dobrokhотов & Shafarevich 1992; Friedlander & Lipton-Lifsicht 2003; Hattori & Fukumoto 2003; Lebowitz & Zweibel 2004; Mizierski & Lyra 2012), we end up with a system of four coupled equations for the perturbations of arbitrary azimuthal dependency that generalize the corresponding equations derived in Kirillov & Stefani (2010). From those four coupled equations, we can deduce the dispersion relation

$$p(\gamma) := \det(\mathcal{H} - \gamma \mathcal{E}) = 0,$$ \hfill (11)

where $\mathcal{E}$ is the unit matrix and the matrix $\mathcal{H}$ has the form

$$\mathcal{H} = \begin{pmatrix} -im\Omega - \omega_\nu & 2\alpha^2\Omega & \frac{m\omega_\nu + \omega_\Delta}{\sqrt{\rho \mu_0}} & \frac{2m\omega_\nu^2}{\sqrt{\rho \mu_0}} \\ -2\Omega(1 + \text{Ro}) & -im\Omega - \omega_\nu & 0 & \frac{i(m\omega_\nu + \omega_\Delta)\sqrt{\rho \mu_0}}{\sqrt{\rho \mu_0}} \\ i(m\omega_\nu + \omega_\Delta)\sqrt{\rho \mu_0} & 2\alpha^2\Omega & -im\Omega - \omega_\nu & 0 \\ 2\Omega \text{Ro} & -im\Omega - \omega_\nu & 0 & -im\Omega - \omega_\nu \end{pmatrix}.$$ \hfill (12)

In Equation (12) we have used the following definitions for the viscous, resistive, and the two Alfvén frequencies corresponding to the vertical and the azimuthal magnetic field:

$$\omega_\nu = \nu |k|^2$$ \hfill (13)

$$\omega_\eta = \eta |k|^2$$ \hfill (14)

$$\omega_\Delta^2 = \frac{k_z^2 B_\phi^2}{\rho \mu_0 R^2}$$ \hfill (15)

$$\omega_\Lambda^2 = \left(\frac{B_\phi^2}{\rho \mu_0 R^2}\right)^2.$$ \hfill (16)

Note that $|k|^2 = k_r^2 + k_z^2$ and $\alpha = k_z/|k|$, where $k_R$, $m$, and $k_z$ are the radial, azimuthal, and axial wavenumbers of the perturbation. In the absence of the magnetic fields, the dispersion relation determined by the matrix $\mathcal{H}$ reduces to that derived already by Krueger et al. (1966) for the non-axisymmetric perturbations of the hydrodynamical TC flow. Choosing, additionally, $m = 0$, we reproduce the result of Eckhardt & Yao (1995). In the presence of the magnetic fields and $m = 0$, we arrive at the dispersion relation derived by Kirillov & Stefani (2010).

2012), which is a pinch-type instability in a current-carrying conducting medium.

The aim of this paper is to better understand why the scaling behavior of HMRI and AMRI, as well as their restriction to rather steep rotation profiles, is so similar. In order to clarify this point, we restrict our work completely to the short-wavelength approximation (Friedlander & Lipton-Lifsicht 2003; Hattori & Fukumoto 2003), keeping in mind that some of our conclusions will need further confirmation in more realistic simulations.

2. SHORT-WAVELENGTH ANALYSIS OF VICIOUS, RESISTIVE MRI FOR ARBITRARY AZIMUTHAL WAVENUMBERS

We start from the equations of incompressible, viscous, and resistive magnetohydrodynamics, composing the Navier–Stokes equation for the velocity field $u$ and the induction equation for the magnetic field $B$,

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \frac{B}{\mu_0 \rho} \cdot \nabla p + \nu \nabla^2 u$$ \hfill (1)

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \eta \nabla^2 B,$$ \hfill (2)

where $p = p + B^2/2\mu_0$ is the total pressure, $\rho = \text{const}$ the density, $v = \text{const}$ the kinematic viscosity, $\eta = (\mu_0 \sigma)^{-1}$ the magnetic diffusivity, $\sigma$ the conductivity of the fluid, and $\mu_0$ the magnetic permeability of free space. Additionally, the mass continuity equation for incompressible flows and the divergence-free condition for the magnetic induction are used:

$$\nabla \cdot u = 0, \quad \nabla B = 0.$$ \hfill (3)

In the following we consider a rotational flow in the gap between the radii $R_1$ and $R_2 > R_1$, with an imposed magnetic field sustained by currents external to the fluid (here we disregard any version of the Tayler instability and its combination with MRI). Introducing the cylindrical coordinates $(R, \phi, z)$, we consider the stability of a magnetized Taylor–Couette (TC) flow, i.e., a steady-state background flow with the angular velocity profile $\Omega(R)$ in a (generally helical) background magnetic field,

$$u_0(R) = R \Omega(R) e_\phi, \quad p = p_0(R), \quad B_0(R) = B_\phi^0(R) e_\phi + B_z^0 e_z,$$ \hfill (4)

with the azimuthal field component

$$B_\phi^0(R) = \frac{\mu_0 I}{2\pi R},$$ \hfill (5)

supposed to be produced by an axial current $I$, external to the fluid.

The angular velocity profile of the background TC flow is known to have the form

$$\Omega(R) = a + \frac{b}{R^2},$$ \hfill (6)

where $a$ and $b$ are defined as

$$a = \frac{\mu_\Omega - \frac{\hat{\eta}}{\hat{\Omega}}} {1 - \frac{\hat{\eta}}{\hat{\Omega}}}, \quad b = \frac{1 - \frac{\mu_\Omega}{\hat{\Omega}}} {1 - \frac{1}{\hat{\Omega}}} R_1^2 \Omega_1$$ \hfill (7)

with the definitions

$$\hat{\eta} = \frac{R_1}{R_2}, \quad \mu_\Omega = \frac{\Omega_2}{\Omega_1}.$$ \hfill (8)
The dispersion relation (11) generated by the matrix (12) can be rewritten completely in terms of dimensionless numbers, i.e., Rossby number (Ro), magnetic Prandtl number (Pm), ratio of the two Alfvén frequencies (β), Hartmann number (Ha), Reynolds number (Re), and a rescaled azimuthal wavenumber n:

\[ \beta = \frac{\omega_A}{\omega_0}, \quad \text{Re} = \frac{\Omega}{\omega_y}, \quad \text{Ha} = \frac{B_0}{k_\rho \sqrt{\mu_0 \beta \nu}} \]

After rescaling the spectral parameter as

\[ \frac{\omega_0}{\sqrt{\mu_0 \beta \eta}}, \quad \alpha = \frac{\beta}{\eta}, \quad n = \frac{m}{\alpha}, \]

with the following coefficients:

\[ a_0 = 1, \quad a_1 = 2 \left( \sqrt{Pm} + \frac{1}{\sqrt{Pm}} \right), \quad a_2 = 2(\beta^2 Ha^2 - 3Re^2 Pm)n^2 + 4\beta Ha^2 n + 2(1 + (1 + 2\beta^2)Ha^2) + 4Re^2(1 + Ro)Pm + \frac{a_1^2}{4} \]

\[ b_1 = 6nRe(1 + Pm), \quad b_2 = a_1(\beta^2 Ha^2 - 3Re^2 Pm)n^2 + 2a_1 \beta Ha^2 n + a_1(1 + (1 + 2\beta^2)Ha^2) + 8Re^2(1 + Ro)\sqrt{Pm} \]

\[ b_3 = 4n^3 \sqrt{Pm}Re(\beta^2 Ha^2 - Re^2 Pm) + 2nRe(4Pm^2 Re^2(1 + Ro) + (1 + Pm)^2 + 2Pm(1 + Ha^2) - \sqrt{Pm} - (1 - n^2)\beta Ha^2 Re^2 Pm) \]

\[ a_4 = ((\beta^2 Ha^2 - Re^2 Pm)n^2 + 2Ha^2 \beta n + Ha^2 + 2PmRe^2) + 2(2RePmRo + 1)((Ha^2 - Re^2 Pm)n^2 + 2Ha^2 \beta n + Ha^2 - (1 + Pm)^2 Re^2 Pm^2 + 4Re^2(1 + Ro) - (Ha^2 + 2PmRe^2) + Ha^4 + 4 + 4\beta^2 Ha^2) \]

\[ b_4 = 2Re(1 + Pm)((\beta^2 Ha^2 - Re^2 Pm)n^2 + 4ReHa^2 \beta(1 + Pm)n^2 + 2Re(2(1 + Ro)(2RePm - \beta^2 Ha^2(1 - Pm)) + (1 + Ha^2)(1 + Pm)n - 4\beta Ha^2 Re(2 + (1 - Pm)Ro). \]

Note again that this complex algebraic equation of fourth order is valid for perturbations of arbitrary azimuthal wavenumber in magnetized incompressible, viscous, resistive rotating fluids exposed to current-free axial and azimuthal magnetic fields. When n = 0, it reduces to the dispersion relation of HMRI derived by Kirillov & Stefani (2010).

3. INDUCTIONLESS LIMIT

Proceeding quite similarly as in Kirillov & Stefani (2010), we apply the Bilharz criterion (Bilharz 1944) to the complex polynomial (22) and derive the maximum Rossby number, at which flows are prone to MRI, as a function of the remaining dimensionless numbers. In the following, we concentrate on the inductionless limit, i.e., we take the limit \( Pm \to 0 \).

After verifying the facts that the inductionless threshold value of Ro increases monotonically with Re, so that we can take the limit \( Re \to \infty \), and that in this limit the threshold value of Ro increases monotonically with Ha (see Figure 1(a)), so that we can take the limit \( Ha \to \infty \), we obtain the following explicit expression for this maximized (with respect to Ha and Re) critical Rossby number at the threshold of MRI in the inductionless limit:

\[ Ro_{cr}(\beta, n) = \frac{4\beta^4 + (\beta n + 1)^4 - (2\beta^2 + (\beta n + 1)^2)\sqrt{4\beta^4 + (\beta n + 1)^4}}{2\beta^2(\beta n + 1)^2}. \]

The maximum value of the Rossby number, \( Ro_{cr} \), is shown as a red dot in Figure 1(a). When \( n = 0 \), Equation (24) reduces to the threshold of HMRI in the inductionless limit found in Kirillov & Stefani (2011). Given n and \( \beta \), the critical Rossby number calculated with finite values of Re and Ha is always below the majorating value, \( Ro_{cr}(\beta, n) \), determined by Equation (24) (see Figure 1(b)). With the increase of Ha and Re, constrained by the scaling law

\[ Re = 2(1 + \sqrt{2})^3 Ha^3, \]

the threshold shown in Figure 1(b) tends to the majorating surface \( Ro_{cr}(\beta, n) \) shown in Figure 2(a).

The remarkably simple dependence (24) of \( Ro_{cr} \), only on the ratio \( \beta \) of azimuthal to axial field and on the rescaled azimuthal wavenumber n, relies on the appropriate choice of the dimensionless parameters, in particular on “hiding” the wavenumber ratio \( \alpha \) in them. Assuming that we have fixed the sign of \( \beta \), the functional dependence of the maximum critical Rossby number (24) on \( \beta \) and n has a two-saddle line structure as shown in Figure 2(a).

Now, the most important result of this paper is that both saddle lines have the same height everywhere, namely, the Liu limit \( Ro_{lin} = 2(1 - \sqrt{2}) \), independent of the particular combination of \( \beta \) and n. From Equation (24) it can be proved that these saddle lines, with \( Ro_{lin} = 2 - 2\sqrt{2} \), are governed by the two equations

\[ \beta_{max,1}(n) = \frac{1}{\sqrt{2} - n}, \quad Ro_{cr}(\beta_{max,1}) = 2 - 2\sqrt{2} \]

\[ \beta_{max,2}(n) = -\frac{1}{\sqrt{2} + n}, \quad Ro_{cr}(\beta_{max,2}) = 2 - 2\sqrt{2}. \]

In Figure 2(b) the curves \( \beta_{max,1}(n) \) and \( \beta_{max,2}(n) \) are shown in green and red colors, respectively. Note that according to the first of Equations (26), \( n = 0 \) (HMRI) corresponds to \( \beta = 1/\sqrt{2} \), which, being substituted into Equation (25), yields the following scaling law for the optimum combination of Re and Ha in HMRI.
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Figure 1. Inductionless limit $P_m \to 0$: (a) At the given $n = \sqrt{2} - 1/20$ and $\beta = 20$ the threshold of MRI in the $(Re^{-1}, Ha^{-1}, Ro)$ space with the maximum at the singular (red) point $(0, 0, Ro_{\text{Liu}} = 2 - 2\sqrt{2})$. The red line leading to the maximum projects into a curve approximated in the vicinity of the origin by the scaling law (25). (b) The threshold of MRI in the $(\beta, n, Ro)$ space with $Ha = 6$ and $Re$ determined by the scaling law (25) reaches its upper bound at $\beta \to \infty$ (AMRI), which is still below the Liu limit $Ro_{\text{Liu}} = 2 - 2\sqrt{2}$.

(A color version of this figure is available in the online journal.)

Figure 2. (a) The critical Rossby number maximized with respect to $Ha$ and $Re$, given by Equation (24), in dependence on $\beta$ and $n$. (b) The lines $\beta_{\text{max}, 1}$ (green) and $\beta_{\text{max}, 2}$ (red) at which the function $Ro_{\text{crt}}(\beta, n)$ attains its maximal value $Ro_{\text{Liu}} = 2 - 2\sqrt{2}$.

(A color version of this figure is available in the online journal.)

(Kirillov & Stefani 2010):

$$Re = \frac{2 + \sqrt{2}}{2} Ha^3.$$ (27)

Therefore, even in the case of non-axisymmetric perturbations, the maximum possible value of the Rossby number prone to the MRI caused by the helical magnetic field in the inductionless limit is still $Ro_{\text{Liu}} = 2 - 2\sqrt{2}$, exactly as in the case of HMRI, which is an instability with respect to the axisymmetric perturbations ($n = 0$). The relations (26) between $\beta$ and $n$ that correspond to the Liu limit give a sort of the resonance conditions between the components of the wavevector of the three-dimensional perturbation and the components of the helical magnetic field.

On the basis of Equations (26) connecting $\beta$ and the rescaled azimuthal wavenumber $n$, we can ask now for the structure of the solution in terms of the original, “physical” azimuthal wavenumber $m$. From the definition $\alpha = k_z/|k|$ we see immediately that $\alpha$ can take on only values between $-1$ and $+1$. The solution structure can thus be visualized as in Figure 3. For example, from the first of Equations (26) we see that for large values of $\beta$ (AMRI), the only possible integer solution is the $m = 1$ mode, whose corresponding wavenumber ratio converges to $\alpha = 1/\sqrt{2}$ (see also Figure 2(b)). There is a lower limit of $\beta = 1 + \sqrt{2} \approx 2.41$ for this $m = 1$ mode. Lowering $\beta$ further, we find next the $m = 0$ mode (HMRI) to dominate at the Liu limit, restricted only to $\beta = 1/\sqrt{2}$ (Kirillov...
to illustrate the mode structure also in dependence on mentioned above. critical Rossby number, illustrating the resonance phenomenon \( \frac{\omega_A \phi}{\omega_A} > 1 \) when there are only two (AMRI) that corotate with the smaller values of the positive or negative \( \beta/\alpha \) when \( L_1 \) limit there are only two (AMRI) that corotate with the basic flow that rotates counterclockwise with respect to the horizon- in the sense that the magnetized TC flow is unstable at the transition between standard MRI and HMRI involves a spectral exceptional point at which the branches of the inertial wave and the slow magneto-Coriolis wave coalesce. It is this exceptional point that resolves the apparent paradox that the transition between the two versions is continuous (Hollerbach & Rüdiger 2005) despite the fact that HMRI can be characterized as a weakly destabilized inertial wave (Liu et al. 2006), while standard MRI is known to be a destabilized slow magneto-Coriolis wave. We believe that, with varying \( P_m \), quite similar transitions will also occur for AMRI (and the higher \( m \) modes discussed in this paper), although we have to leave the detailed investigation for further work.

As a side remark, the determination of such limits is even more complicated by the necessity to distinguish, for traveling waves as in HMRI, between convective and absolute (or global) instabilities, which has been thoroughly discussed by Priede & Gerbeth (2009) and was shown to be experimentally important by Stefani et al. (2009).

From the strictly astrophysical point of view, our support for the Liu limit may appear disappointing, since it would exclude any relevance of the inductionless versions of MRI to accretion disks with Keplerian rotation. A subtle question in this respect is, however, connected with the saturation mechanism of the MRI that could, possibly, lead to modified shear profiles. With the main focus on low \( P_m \) flows, Umurhan (2010) considered the possibility that the saturation of MRI could lead to modified flow structures within parts of steeper shear, sandwiched with parts of shallower shear. Moreover, for protoplanetary disks (characterized by low \( P_m \)) the simulations of Kato et al. (2009) had also shown the appearance of layers with increased shear.
By virtue of a possible sudden onset within such segments of steepening shear, the low $P_m$ and inductionless MRI versions could thus play a certain role in real astrophysical settings.

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