Theoret. Appl. Mech., Vol.34, No.2, pp. 87–109, Belgrade 2007

Bifurcation of the roots of the characteristic polynomial and the destabilization paradox in friction induced oscillations

O.N. Kirillov *

Abstract

Paradoxical effect of small dissipative and gyroscopic forces on the stability of a linear non-conservative system, which manifests itself through the unpredictable at first sight behavior of the critical non-conservative load, is studied. By means of the analysis of bifurcation of multiple roots of the characteristic polynomial of the non-conservative system, the analytical description of this phenomenon is obtained. As mechanical examples two systems possessing friction induced oscillations are considered: a mass sliding over a conveyor belt and a model of a disc brake describing the onset of squeal during the braking of a vehicle.

Keywords : Friction-induced oscillations, circulatory system, destabilization paradox due to small damping, characteristic polynomial, multiple roots, bifurcation, stability domain, Whitney umbrella singularity.

1 Introduction

1. Consider a linear autonomous non-conservative mechanical system

$$\frac{d^2\mathbf{y}}{dt^2} + \mathbf{D}(\mathbf{k})\frac{d\mathbf{y}}{dt} + \mathbf{A}(q)\mathbf{y} = 0, \tag{1}$$

*Moscow State Lomonosov University, Institute of Mechanics, Michurinskii pr. 1, 119192 Moscow, Russia, e-mail: kirillov@imec.msu.ru

where $\mathbf{y}$ is a vector of generalized coordinates and $\mathbf{D}$ and $\mathbf{A}$ are real square matrices of order m, determining dissipative and gyroscopic and nonconservative positional forces, respectively. It is assumed that the matrix $\mathbf{D}$ is a smooth function of the vector of parameters $\mathbf{k} = (k_1, \ldots, k_{n-1})$ and $\mathbf{D}(0) = 0$, while the matrix $\mathbf{A}$ smoothly depends on the load parameter $q \geq 0$.

Finding a solution to equation (1) in the form $\mathbf{y}=\mathbf{u}\exp(\lambda t)$ we get the generalized eigenvalue problem

$$(\lambda^2\mathbf{I} + \lambda\mathbf{D}(\mathbf{k}) + \mathbf{A}(q))\mathbf{u} = 0, \tag{2}$$

where $\mathbf{I}$ is a unit matrix of order m, $\mathbf{u}$ is an eigenvector, and λ is an eigenvalue. Non-conservative system without gyroscopic and dissipative forces ($\mathbf{k} = 0$)

$$\frac{d^2\mathbf{y}}{dt^2} + \mathbf{A}(q)\mathbf{y} = 0 \tag{3}$$

is called *circulatory* [1, 2]. The spectrum of the circulatory system is symmetrical with respect to the real and imaginary parts of the complex plane; if λ is an eigenvalue of the linear operator $\lambda^2\mathbf{I} + \mathbf{A}(q)$, then $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$, where the overbar indicates complex conjugation, are the eigenvalues too. As a consequence, the circulatory system is stable (in the sense of Lyapunov) if and only if all the eigenvalues λ are purely imaginary and semi-simple [3].

Let us assume that at $q = 0$ the circulatory system (3) is stable. With the increase of the load parameter its eigenvalues move along the imaginary axis. When the parameter q reaches some critical value q_0, two simple purely imaginary eigenvalues merge and originate a double eigenvalue $\lambda_0 = i\omega_0$ with the Jordan chain of length 2. The further increase in the load yields in general the splitting of λ_0 into two simple complex eigenvalues, one of which has a positive real part (flutter instability), Fig 1a. Therefore, the interval $0 \leq q < q_0$ belongs to the stability domain of the circulatory system (3).

As it was revealed in various mechanical problems [1]–[9], [17]–[23], a perturbation of the circulatory system by small dissipative and gyroscopic forces ($\mathbf{k} \neq 0$) generally destroys the interaction of eigenvalues. By that reason at some critical load $q = q_{cr}(k_1, \ldots, k_{n-1})$ one of the eigenvalues moves to the right hand side of the complex plane without interaction

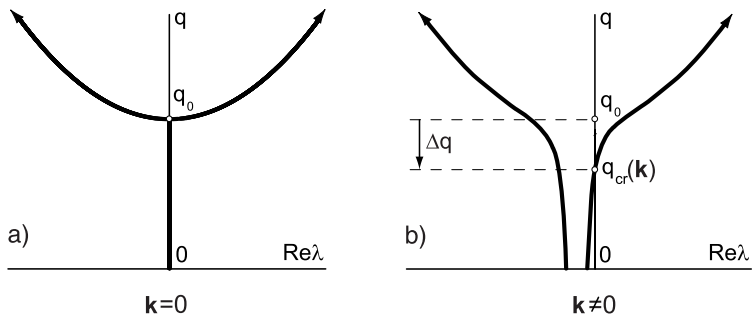

Figure 1: Trajectories of the eigenvalues for undamped and weakly damped non-conservative systems.

with the other eigenvalue, Fig 1b. Moreover, assuming $\mathbf{k} = \epsilon\widetilde{\mathbf{k}}$, where the vector $\widetilde{\mathbf{k}}$ is fixed, and rushing the parameter ϵ to zero one has

$$\widetilde{q}_{cr} \equiv \lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}}) \leq q_0. \tag{4}$$

The inequality similar to (4) was obtained first by Ziegler [1] for the double-linked pendulum loaded by the follower force. He came to an unexpected conclusion, that the critical force of the non-conservative system with infinitely small dissipation is essentially lower, than that of undamped system. This phenomenon has received the name *destabilization paradox* [1]–[9], [22].

Later, it has been noticed on examples of various mechanical systems that the limit of the critical load $\widetilde{q}_{cr}$ depends on a choice of the vector $\widetilde{\mathbf{k}}$. In particular, changing the ratios between the parameters $k_1, \ldots, k_{n-1}$ it is possible to avoid the jump in the critical load and, hence, destabilization (Bolotin's effect [2]). In the works [10, 11, 12, 19, 20, 21, 22] the destabilization paradox was studied for the general finite-dimensional and continuous non-conservative systems. Explicit asymptotic expressions for the critical load q as a function of the vector of parameters $\mathbf{k}$ describing a jump in the critical load were derived. Behavior of eigenvalues of weakly-damped non-conservative systems was analytically described. In the work [10] the structure of the matrix of dissipative and gyroscopic forces stabilizing the circulatory system was determined and the necessary and sufficient stability conditions were found. The results

of the works [10, 11, 12] were obtained by means of the perturbation theory of linear non-self-adjoint operators [13] and used the eigenvectors and associated vectors of the operator of circulatory system.

In the present paper the study of the destabilization paradox is based on the sensitivity analysis of the roots of the characteristic polynomial of the system (1) with coefficients expressed by means of the modified Leverrier algorithm [14]. This allows one avoid computation of the eigenvectors and associated vectors and obtain expressions for the critical load by means of the derivatives of the characteristic polynomial with respect to parameters. In the case of two-dimensional systems the results are expressed directly through the invariants of the matrices $\mathbf{D}$ and $\mathbf{A}$.

2 Bifurcation of the roots of the characteristic polynomial

The effect of small dissipative and gyroscopic forces on the stability of the non-conservative system (1), which is on the boundary between the stability and flutter domains, is stabilizing or destabilizing depending on the behavior of the eigenvalues of the circulatory system due to variation of the parameters q and $\mathbf{k}$.

The eigenvalues of the system (1) are the roots of the characteristic polynomial

$$P(\lambda, \mathbf{p}) = \lambda^{2m} + \sum_{s=1}^{2m} a_s \lambda^{2m-s} = \det(\mathbf{I}\lambda^2 + \mathbf{D}(\mathbf{k})\lambda + \mathbf{A}(q)). \tag{5}$$

The coefficients of the polynomial (5) are smooth functions of the real vector of parameters $\mathbf{p} = (k_1, k_2, \ldots, k_{n-1}, q)$ and are expressed through the invariants of the matrices $\mathbf{D}$ and $\mathbf{A}$ by means of the modified Leverrier algorithm [14]

$$\mathbf{C}_0 = \mathbf{I}, \quad a_1 = \mathrm{tr}\mathbf{D}(\mathbf{k}), \quad \mathbf{C}_1 = -\mathbf{D}(\mathbf{k}) + a_1\mathbf{I},$$

$$a_j = \frac{1}{j}\mathrm{tr}(\mathbf{D}(\mathbf{k})\mathbf{C}_{j-1} + 2\mathbf{A}(q)\mathbf{C}_{j-2}), \quad \mathbf{C}_j = -\mathbf{D}(\mathbf{k})\mathbf{C}_{j-1} - \mathbf{A}(q)\mathbf{C}_{j-2} + a_j\mathbf{I},$$

$$j = 2, 3, \ldots, 2m-2,$$

$$\mathbf{D}(\mathbf{k})\mathbf{C}_{2m-2} + \mathbf{A}(q)\mathbf{C}_{2m-3} = a_{2m-1}\mathbf{I}, \quad \mathbf{A}(q)\mathbf{C}_{2m-2} = a_{2m}\mathbf{I}. \tag{6}$$

Let for $\mathbf{p} = \mathbf{p}_0$ the function $P(\lambda, \mathbf{p}_0)$ have a root $\lambda = \lambda_0$. Let us study the behavior of this root with the variation of the vector of parameters $\mathbf{p}$ along a smooth curve

$$\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon\dot{\mathbf{p}} + \frac{\epsilon^2}{2}\ddot{\mathbf{p}} + o(\epsilon^2), \quad \epsilon \geq 0, \tag{7}$$

where the dot indicates differentiation with respect to the parameter ϵ and the derivatives are evaluated for $\epsilon = 0$. Then, the polynomial $P(\lambda, \mathbf{p}(\epsilon))$ is represented in the form

$$P(\lambda, \mathbf{p}(\epsilon)) = \sum_{r=0}^{2m} \frac{(\lambda-\lambda_0)^r}{r!} \left(\frac{\partial^r P}{\partial \lambda^r} + \epsilon \frac{\partial^r P_1}{\partial \lambda^r} + \epsilon^2 \frac{\partial^r P_2}{\partial \lambda^r} + o(\epsilon^2) \right), \tag{8}$$

where

$$\frac{\partial^r P_1}{\partial \lambda^r} = \sum_{s=1}^{n} \frac{\partial^{r+1} P}{\partial \lambda^r \partial p_s} \dot{p}_s, \quad \frac{\partial^r P_2}{\partial \lambda^r} = \frac{1}{2} \sum_{s=1}^{n} \frac{\partial^{r+1} P}{\partial \lambda^r \partial p_s} \ddot{p}_s + \frac{1}{2} \sum_{s,t=1}^{n} \frac{\partial^{r+2} P}{\partial \lambda^r \partial p_s \partial p_t} \dot{p}_s \dot{p}_t, \tag{9}$$

and the partial derivatives are evaluated at $\mathbf{p} = \mathbf{p}_0$, $\lambda = \lambda_0$. For $r = 0$ the formulas (9) give expressions for the quantities P_1 and P_2.

Let $\lambda_0 = i\omega_0$ be a double eigenvalue of the system (1) with the Jordan chain of length 2. Then the splitting of λ_0 due to perturbation (7) is described by the Puiseux series [13]

$$\lambda = \lambda_0 + \lambda_1 \epsilon^{1/2} + \lambda_2 \epsilon + \lambda_3 \epsilon^{3/2} + \lambda_4 \epsilon^2 + \dots. \tag{10}$$

Substituting expansions (8) and (10) into the equation $P(\lambda, \mathbf{p}) = 0$ and collecting the terms with the same powers of ϵ we get the expressions determining the coefficients in the series (10)

$$P(\lambda_0, \mathbf{p}_0) = 0, \tag{11}$$

$$\lambda_1 \frac{\partial P}{\partial \lambda}\bigg|_{\substack{\lambda=\lambda_0 \\ \mathbf{p}=\mathbf{p}_0}} = 0, \tag{12}$$

$$\left(P_1 + \frac{1}{2}\lambda_1^2 \frac{\partial^2 P}{\partial \lambda^2} + \lambda_2 \frac{\partial P}{\partial \lambda} \right)\bigg|_{\substack{\lambda=\lambda_0 \\ \mathbf{p}=\mathbf{p}_0}} = 0, \tag{13}$$

$$\left(\lambda_1\lambda_2\frac{\partial^2 P}{\partial\lambda^2}+\lambda_3\frac{\partial P}{\partial\lambda}+\lambda_1\frac{\partial P_1}{\partial\lambda}+\lambda_1^3\frac{1}{6}\frac{\partial^3 P}{\partial\lambda^3}\right)\Bigg|_{\substack{\lambda=\lambda_0\\ \mathbf{p}=\mathbf{p}_0}}=0. \tag{14}$$

Equations (11), (12) are satisfied since λ_0 is a double root of the characteristic polynomial (5). Taking into account the equation (12) in the expressions (13) and (14) we find the coefficients λ_1 and λ_2

$$\lambda_1^2=-P_1\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1},\quad \lambda_2=\left(\frac{1}{3}\frac{\partial^3 P}{\partial\lambda^3}P_1-\frac{\partial P_1}{\partial\lambda}\frac{\partial^2 P}{\partial\lambda^2}\right)\left(\frac{\partial^2 P}{\partial\lambda^2}\right)^{-2}, \tag{15}$$

where the derivatives are evaluated for $\mathbf{p}=\mathbf{p}_0$, $\lambda=\lambda_0$.

Thus, with the variation of the parameters (7) the double eigenvalue $\lambda_0=i\omega_0$ with the Jordan chain of length 2 splits according to the formula

$$\lambda=i\omega_0\pm i\sqrt{\epsilon P_1\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}}-$$

$$-\frac{\epsilon}{2}\left(\frac{\partial P_1}{\partial\lambda}\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}-\frac{1}{3!}\frac{\partial^3 P}{\partial\lambda^3}\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}P_1\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}\right)+o(\epsilon), \tag{16}$$

if for $\epsilon\neq 0$ the radicand in expression (16) is not zero [9].

Consider a degenerate case when $P_1=0$ and the coefficient λ_1 in the series (10) is zero. Keeping this in mind, substituting the expansions (8) and (10) into the equation $P(\lambda,\mathbf{p})=0$, and collecting the terms with the same powers of ϵ we find

$$\left(P_1+\lambda_2\frac{\partial P}{\partial\lambda}\right)\Bigg|_{\substack{\lambda=\lambda_0\\ \mathbf{p}=\mathbf{p}_0}}=0, \tag{17}$$

$$\lambda_3\frac{\partial P}{\partial\lambda}\Bigg|_{\substack{\lambda=\lambda_0\\ \mathbf{p}=\mathbf{p}_0}}=0, \tag{18}$$

$$\left(P_2+\lambda_2^2\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}+\lambda_2\frac{\partial P_1}{\partial\lambda}+\lambda_4\frac{\partial P}{\partial\lambda}\right)\Bigg|_{\substack{\lambda=\lambda_0\\ \mathbf{p}=\mathbf{p}_0}}=0. \tag{19}$$

By virtue of the condition of the existence of the double eigenvalue and the degeneration condition the equations (17) and (18) are satisfied. The

equation (19) gives the quadratic equation, which serves for determining the coefficient λ_2 in expansion (10)

$$\lambda_2^2 \frac{1}{2} \frac{\partial^2 P}{\partial \lambda^2} + \lambda_2 \frac{\partial P_1}{\partial \lambda} + P_2 = 0, \tag{20}$$

where all the derivatives are evaluated for $\lambda = \lambda_0$, $\mathbf{p} = \mathbf{p}_0$. Thus, in the degenerate case when $P_1 = 0$ the double eigenvalue λ_0 splits according to the formula $\lambda = \lambda_0 + \lambda_2 \epsilon + o(\epsilon)$ with the coefficient λ_2 determined by the equation (20).

Analogously it can be shown that the behavior of simple eigenvalues $\lambda_{0,s}$ due to the variation of parameters (7) is described by the formula [9]

$$\lambda = \lambda_{0,s} + \mu_1 \epsilon + o(\epsilon), \quad \mu_1 = -P_1 \left(\frac{\partial P}{\partial \lambda} \right)^{-1} \Bigg|_{\substack{\lambda=\lambda_{0,s} \\ \mathbf{p}=\mathbf{p}_0}}. \tag{21}$$

Therefore, the asymptotic formulas (10), (16), (20) and (21) with the coefficients expressed through the derivatives of the characteristic polynomial describe the behavior of simple and double eigenvalues due to variation of parameters in regular and degenerate cases.

3 Stability analysis of the non-conservative system

In n-dimensional space of parameters $k_1, \ldots, k_{n-1}, q$ of the system (1) we consider a point $\mathbf{p}_0 = (0, \ldots, 0, q_0)$, assuming that $\pm\lambda_0 = \pm i\omega_0$, $\omega_0 > 0$, are double eigenvalues of the operator $\mathbf{A}(q_0) + \lambda^2 \mathbf{I}$ with the Jordan chain of length 2, while other eigenvalues $\pm\lambda_{0,s} = \pm i\omega_{0,s}$, $\omega_{0,s} > 0$, $s = 1, \ldots, m-2$ are simple and purely imaginary. The non-conservative system corresponding to $\mathbf{k} = 0$, $q = q_0$ is circulatory and the point $\mathbf{p}_0$ belongs to the stability boundary [2, 3].

From the equations (6) it follows that the odd coefficients of the characteristic polynomial $a_{2r-1}(\mathbf{p})$ and the matrices $\mathbf{C}_{2j-1}(\mathbf{p})$ satisfy the expressions

$$a_{2r-1}(\mathbf{p}_0) = 0, \quad \mathbf{C}_{2j-1}(\mathbf{p}_0) = 0, \quad r = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, m-1, \tag{22}$$

where $\mathbf{p}_0 = (0, \ldots, 0, q_0)$. Besides, by means of the induction argument we have

$$\frac{\partial a_{2r}}{\partial k_s}(\mathbf{p}_0) = 0, \ \frac{\partial a_{2r-1}}{\partial q}(\mathbf{p}_0) = 0, \ r = 1, 2, \ldots, m,$$
$$\frac{\partial \mathbf{C}_{2j}}{\partial k_s}(\mathbf{p}_0) = 0, \ \frac{\partial \mathbf{C}_{2j-1}}{\partial q}(\mathbf{p}_0) = 0, \ j = 1, 2, \ldots, m-1. \tag{23}$$

Consider now the formula (21) describing the behavior of the simple eigenvalue $\lambda_{0,s} = i\omega_{0,s}$ with the variation of the parameters. Let us define the scalar $\widetilde{g}_s$ and the vector $\mathbf{g}_s = (g_1^s, g_2^s, \ldots, g_{n-1}^s)$

$$\widetilde{g}_s = -i\frac{\partial P}{\partial q}\left(\frac{\partial P}{\partial \lambda}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_{0,s}\\ \mathbf{p}=\mathbf{p}_0}}, \quad g_j^s = \frac{1}{\omega_{0,s}}\frac{\partial P}{\partial k_j}\left(\frac{\partial P}{\partial \lambda}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_{0,s}\\ \mathbf{p}=\mathbf{p}_0}}. \tag{24}$$

According to the equations (22) and (23) the quantities $\widetilde{g}_s$ and g_j^s are real. With the use of the expressions (7) and (24) equation (21) takes the form

$$\lambda = i\omega_{0,s} - i\widetilde{g}_s(q - q_0) - \omega_{0,s}\langle \mathbf{g}_s, \mathbf{k}\rangle + \ldots, \tag{25}$$

where the angular brackets denote scalar product of real vectors: $\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{j=1}^{n-1} a_j b_j$.

Thus, a small variation of the load parameter q keeps the simple eigenvalues $\lambda_{0,s} = i\omega_{0,s}$ on the imaginary axis, while the variation of the vector of parameters $\mathbf{k}$ corresponding to dissipative and gyroscopic forces moves the eigenvalues $\lambda_{0,s}$ out of the imaginary axis. The eigenvalues are in the left hand side of the complex plane if the vector $\mathbf{k}$ satisfies the inequalities

$$\langle \mathbf{g}_s, \mathbf{k}\rangle > 0, \quad s = 1, 2, \ldots, m-2. \tag{26}$$

Splitting of the double eigenvalue $\lambda_0 = i\omega_0$ in the regular case is described by the formula (16). Let us define the scalar $\widetilde{f}$ and the vector $\mathbf{f} = (f_1, f_2, \ldots, f_{n-1})$ as

$$\widetilde{f} = \frac{\partial P}{\partial q}\left(\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_0\\ \mathbf{p}=\mathbf{p}_0}}, \quad f_s = \frac{1}{i\omega_0}\frac{\partial P}{\partial k_s}\left(\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_0\\ \mathbf{p}=\mathbf{p}_0}}, \tag{27}$$

and the vector $\mathbf{h} = (h_1, h_2, \ldots, h_{n-1})$ and a scalar $\widetilde{h}$ as

$$h_s = \frac{1}{\omega_0}\left(\frac{1}{2!}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\left(\frac{2\alpha_0}{i\omega_0}\frac{\partial P}{\partial k_s} - \frac{\partial^2 P}{\partial \lambda \partial k_s}\right)\Bigg|_{\substack{\lambda=i\omega_0\\ \mathbf{p}=\mathbf{p}_0}},$$

$$\widetilde{h} = \left(\frac{1}{2!}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1} \left(\frac{\alpha_0}{\omega_0}\frac{\partial P}{\partial q} - i\frac{\partial^2 P}{\partial \lambda \partial q}\right)\Bigg|_{\substack{\lambda=i\omega_0 \\ \mathbf{p}=\mathbf{p}_0}}, \quad \alpha_0 = i\omega_0 \frac{1}{3!}\frac{\partial^3 P}{\partial \lambda^3}\left(\frac{1}{2!}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}, \tag{28}$$

$$s = 1, 2, \ldots, n-1.$$

According to the equations (22) and (23) the quantities given by expressions (27) and (28) are real. Taking into account expressions (27) and (28) we rewrite the equation (16) in the form

$$\lambda = i\omega_0 \pm i\sqrt{\epsilon i\omega_0 \langle \mathbf{f}, \dot{\mathbf{k}}\rangle + \epsilon \widetilde{f}\dot{q}} - \epsilon\frac{1}{2}(\langle \alpha_0 \mathbf{f} - \omega_0 \mathbf{h}, \dot{\mathbf{k}}\rangle + i\widetilde{h}\dot{q}) + o(\epsilon). \tag{29}$$

It follows from the equation (29) that in the generic case the double eigenvalue $\lambda_0 = i\omega_0$ splits into two complex eigenvalues, one of them with positive real part (flutter instability). However, under the conditions $\langle \mathbf{f}, \dot{\mathbf{k}}\rangle = 0$, $\widetilde{f}\dot{q} > 0$ and $\langle \mathbf{h}, \dot{\mathbf{k}}\rangle < 0$ the double eigenvalue splits into two simple eigenvalues with negative real parts (asymptotic stability). Taking into account that in the first approximation $\mathbf{k} = \epsilon\dot{\mathbf{k}}$ and $q = q_0 + \epsilon\dot{q}$ and assuming $\widetilde{f} < 0$, we rewrite the stability conditions as

$$\langle \mathbf{f}, \mathbf{k}\rangle = 0, \quad q < q_0, \quad \langle \mathbf{h}, \mathbf{k}\rangle < 0. \tag{30}$$

In the space of parameters conditions (26) and (30) define a set of directions, which lead from the point $\mathbf{p}_0$ to the asymptotic stability domain, i.e. the tangent cone to the domain at the point $\mathbf{p}_0$. The tangent cone given by expressions (26) and (30) is degenerate, because its dimension is $n-1$, which is less than the dimension of the asymptotic stability domain (n) [9, 15].

To obtain more detailed approximation of the asymptotic stability domain we consider behavior of eigenvalues with the variation of the parameters along the curves (7), assuming that the direction vector of the curves is tangent to the set (30) and orthogonal to the q–axis:

$$\langle \mathbf{f}, \dot{\mathbf{k}}\rangle = 0, \quad \dot{q} = 0, \tag{31}$$

As a consequence, the radicand in formula (29) is zero. In such a degenerate case the splitting of the double eigenvalue λ_0 is described by the formula (10) with $\lambda_1 = 0$ and with the coefficient λ_2, which is determined by the equation (20).

Taking into account expressions (27) and (28), and conditions (31) we rewrite equation (20) in the form

$$\lambda_2^2 - \omega_0\langle \mathbf{h}, \dot{\mathbf{k}}\rangle\lambda_2 + \frac{1}{2}\widetilde{f}\ddot{q} + \omega_0^2\langle \mathbf{G}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle + i\omega_0\left(\frac{1}{2}\langle \mathbf{f}, \ddot{\mathbf{k}}\rangle + \langle \mathbf{H}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle\right) = 0, \quad (32)$$

where the elements of the real matrices $\mathbf{H}$ and $\mathbf{G}$ are determined by the expressions

$$H_{st} = \frac{1}{2\omega_0}\mathrm{Im}\left(\frac{\partial^2 P}{\partial k_s \partial k_t}\right)\left(\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_0\\ \mathbf{p}=\mathbf{p}_0}}, \quad (33)$$

$$G_{st} = \frac{1}{2\omega_0^2}\mathrm{Re}\left(\frac{\partial^2 P}{\partial k_s \partial k_t}\right)\left(\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_0\\ \mathbf{p}=\mathbf{p}_0}}, \quad s,t = 1,2,\ldots,n-1. \quad (34)$$

The conditions for the roots of the complex polynomial (32) to have negative real parts are given by the Bilharz criterion [16]. Taking into account expressions (7) and (31) we write these conditions as

$$q < q_{cr}(\mathbf{k}), \quad \langle \mathbf{h}, \mathbf{k}\rangle < 0, \quad (35)$$

$$q_{cr}(\mathbf{k}) = q_0 + \frac{(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle)^2}{\widetilde{f}\langle \mathbf{h}, \mathbf{k}\rangle^2} - \frac{\omega_0^2}{\widetilde{f}}\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle. \quad (36)$$

The conditions (35) together with the inequalities (26) approximate the asymptotic stability domain of the weakly damped non-conservative system (1) in the vicinity of the point $\mathbf{p}_0 = (0,\ldots,0,q_0)$. If to assume, that at performance of conditions (35) inequalities (26) are automatically satisfied, then equation (36) approximates the boundary of the asymptotic stability domain near the point $\mathbf{p}_0$.

In important particular case, when the non-conservative system (1) has only two degrees of freedom ($m = 2$), the stability conditions (26) corresponding to simple eigenvalues are absent, and stability is determined only by the splitting of the double eigenvalue λ_0. It follows from expressions (5) and (6) that the characteristic polynomial of the system with two degrees of freedom has the form

$$P = \lambda^4 + \lambda^3 \mathrm{tr}\mathbf{D}(\mathbf{k}) + \lambda^2(\mathrm{tr}\mathbf{A}(q) + \det \mathbf{D}(\mathbf{k})) + \lambda(\mathrm{tr}\mathbf{A}(q)\mathrm{tr}\mathbf{D}(\mathbf{k}) - \mathrm{tr}(\mathbf{A}(q)\mathbf{D}(\mathbf{k}))) +$$

$$+\det \mathbf{A}(q). \tag{37}$$

In the absent of the dissipative and gyroscopic forces ($\mathbf{k} = 0$) the solutions to equation (37) are

$$\lambda = \pm\sqrt{-\frac{\mathrm{tr}\mathbf{A}(q)}{2} \pm \frac{1}{2}\sqrt{(\mathrm{tr}\mathbf{A}(q))^2 - 4\det \mathbf{A}(q)}}. \tag{38}$$

For the critical value of the load parameter $q = q_0$ the radicand in equation (38) is zero. This yields a pair of double complex-conjugate eigenvalues

$$\pm\lambda_0 = \pm i\omega_0, \quad \omega_0 = \sqrt{\frac{\mathrm{tr}\mathbf{A}(q_0)}{2}} > 0. \tag{39}$$

Splitting of these eigenvalues at the account of small dissipative and gyroscopic forces ($\mathbf{k} \neq 0$) results in the stability conditions (35).

In case of $m = 2$ degrees of freedom the quantities $\widetilde{f}$, $\widetilde{h}$ and the components of the vectors $\mathbf{f}$ and $\mathbf{h}$ are expressed directly through the invariants of the matrices $\mathbf{A}$ and $\mathbf{D}$

$$\widetilde{f}=\frac{1}{4\omega_0^2}\mathrm{tr}\left((\mathbf{A}(q_0) - \omega_0^2\mathbf{I})\frac{\partial \mathbf{A}}{\partial q}\right), \quad f_s=\frac{1}{4\omega_0^2}\mathrm{tr}\left((\mathbf{A}(q_0) - \omega_0^2\mathbf{I})\frac{\partial \mathbf{D}}{\partial k_s}\right),$$
$$\widetilde{h}=\frac{1}{4\omega_0^3}\mathrm{tr}\left((\mathbf{A}(q_0) - 3\omega_0^2\mathbf{I})\frac{\partial \mathbf{A}}{\partial q}\right), \quad h_s=\frac{1}{4\omega_0^3}\mathrm{tr}\left((\mathbf{A}(q_0) - 3\omega_0^2\mathbf{I})\frac{\partial \mathbf{D}}{\partial k_s}\right). \tag{40}$$

Components of the real matrices $\mathbf{H}$ and $\mathbf{G}$ are given by the formulas

$$H_{st} = \frac{1}{8\omega_0^2}\mathrm{tr}\left((\mathbf{A}_0 - \omega_0^2\mathbf{I})\frac{\partial^2 \mathbf{D}}{\partial k_s \partial k_t}\right), \quad G_{st}=\frac{1}{8\omega_0^2}\left(\mathrm{tr}\frac{\partial \mathbf{D}}{\partial k_s}\mathrm{tr}\frac{\partial \mathbf{D}}{\partial k_t} - \mathrm{tr}\left(\frac{\partial \mathbf{D}}{\partial k_s}\frac{\partial \mathbf{D}}{\partial k_t}\right)\right),$$
$$s, t = 1, 2, \ldots, n-1. \tag{41}$$

Derivatives in the expressions (40) and (41) are evaluated for $\mathbf{k} = 0$ and $q = q_0$.

Finally we notice that the equation (32) gives also the equations describing the evolution of the eigenvalues on the complex plane

$$(\mathrm{Im}\lambda - \omega_0 - \mathrm{Re}\lambda - a/2)^2 - (\mathrm{Im}\lambda - \omega_0 + \mathrm{Re}\lambda + a/2)^2 = 2d, \tag{42}$$

$$(\mathrm{Re}\lambda + a/2)^4 + \left(c - a^2/4\right)(\mathrm{Re}\lambda + a/2)^2 = d^2/4, \tag{43}$$

$$(\mathrm{Im}\lambda - \omega_0)^4 - \left(c - a^2/4\right)(\mathrm{Im}\lambda - \omega_0)^2 = d^2/4, \tag{44}$$

where the coefficients a, c and d are

$$a = -\omega_0\langle \mathbf{h}, \mathbf{k}\rangle, \quad c = \widetilde{f}(q - q_0) + \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle, \quad d = \omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle). \tag{45}$$

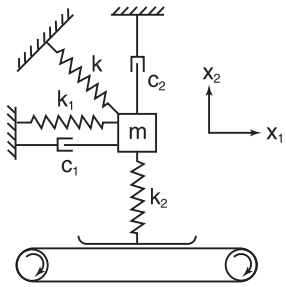

Figure 2: A mass sliding over a conveyor belt.

4 Effect of viscous damping on the stability of a mass sliding over a conveyor belt

As it was mentioned in [17], in a broad variety of engineering systems with sliding contact self-excited friction induced oscillations lead to the generation of strong noise, which in most cases is unacceptable. Among the most well known systems of this class are squealing railway wheels, squeaking door hinges, and automotive braking systems. One of the mechanisms of the noise generation is the mode coupling, i.e. interaction of eigenvalues.

As a simple model of a system possessing friction induced oscillations we consider a single point mass sliding over a conveyor belt (Fig 2), mainly held in position by two linear springs k_1 and k_2 parallel and normal to the belt surface [17, 19]. Parameter k_2 may be regarded as the physical contact stiffness between the objects in relative sliding motion. Moreover, there is another linear spring k oriented at an oblique of 45° relative to the normal direction, see Fig 2. For the friction a Coulomb model is assumed, where the frictional force F_t is proportional to the normal force F_n exerted, $F_t = \mu F_n$, where μ is the kinetic coefficient of friction taken to be constant. Since the normal force is linearly related to the displacement of the mass normal to the contact surface, the equations for perturbations around steady sliding state form the system of equations (1) with two degrees of freedom where the vector $\mathbf{y}$ and the matrices $\mathbf{D}$ and $\mathbf{A}$ are [17, 19]

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2d_1\omega_1 & 0 \\ 0 & 2d_2\omega_2 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \omega_1^2 & -k/2m + \mu(\omega_2^2 - k/2m) \\ -k/2m & \omega_2^2 \end{bmatrix}. \tag{46}$$

The coefficients of linear viscous damping d_1 and d_2 are defined by the expressions

$$d_j = \frac{c_j}{2\omega_j m}, \quad \omega_j^2 = \frac{2k_j + k}{2m}, \quad j = 1, 2. \tag{47}$$

Let us first consider the system without damping ($d_1 = 0,\ d_2 = 0$). Then, solving the equation $(\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A} = 0$ we find the critical value μ_0 of the friction coefficient

$$\mu_0 = \left(\frac{k}{2m} + \frac{2m}{k}\left(\frac{\omega_1^2 - \omega_2^2}{2}\right)^2\right)\left(\omega_2^2 - \frac{k}{2m}\right)^{-1}, \tag{48}$$

and from (21) we get the corresponding critical frequency ω_0 as

$$\omega_0 = \sqrt{(\omega_1^2 + \omega_2^2)/2}. \tag{49}$$

If the system parameters are set to arbitrary but fixed values

$$\omega_1 = 4s^{-1}, \quad \omega_2 = 5s^{-1}, \quad k/(2m) = 5s^{-2}, \tag{50}$$

then μ_0=0.4525 and $\omega_0 = \sqrt{41/2} \cong 4.52769 s^{-1}$. The critical value of the friction coefficient of the damped system is found directly from the Routh-Hurwitz conditions applied to the characteristic polynomial of the system (1), (46)

$$\mu_{cr} = \mu_0 - \frac{m^2(\omega_1^2 - \omega_2^2)^2}{k(2\omega_2^2 m - k)} \frac{(d_1\omega_1 - d_2\omega_2)^2}{(d_1\omega_1 + d_2\omega_2)^2} + \\ + \frac{16m^2 d_1 d_2 \omega_1^2 \omega_2^2}{k(2\omega_2^2 m - k)} \frac{d_1\omega_2 + d_2\omega_1}{d_1\omega_1 + d_2\omega_2}. \tag{51}$$

Let us now find the approximation of the critical friction using the approach of section 3. For the matrices (46) formulae (40), (41) yield

$$\widetilde{f} = -\frac{k}{4m}\frac{\omega_2^2 - \frac{k}{2m}}{\omega_1^2 + \omega_2^2}, \quad \mathbf{f} = \frac{1}{2}\frac{\omega_1^2 - \omega_2^2}{\omega_1^2 + \omega_2^2}\begin{bmatrix} \omega_1 \\ -\omega_2 \end{bmatrix},$$

$$\mathbf{h} = \frac{-1}{(\omega_1^2 + \omega_2^2)\sqrt{2(\omega_1^2 + \omega_2^2)}}\begin{bmatrix} \omega_1(\omega_1^2 + 3\omega_2^2) \\ \omega_2(\omega_2^2 + 3\omega_1^2) \end{bmatrix}, \quad \mathbf{H} = 0, \quad \mathbf{G} = \frac{\omega_1\omega_2}{\omega_1^2 + \omega_2^2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{52}$$

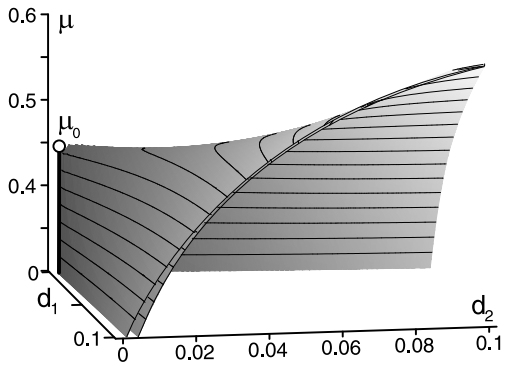

Figure 3: The boundary of the asymptotic stability domain and its approximation.

Using (52) in the expression (36) we find for the values of parameters (50) the critical value of the friction coefficient of the damped system as

$$\mu_{cr}=\frac{181}{400}-\frac{136161}{100}\frac{(4d_1-5d_2)^2}{(364d_1+365d_2)^2}+\frac{82}{5}d_1d_2. \tag{53}$$

Equation (53) defines the boundary of asymptotic stability domain if $\langle \mathbf{h}, \mathbf{k} \rangle < 0$ or equivalently

$$364d_1 + 365d_2 > 0. \tag{54}$$

Approximation of the stability boundary given by equations (53), (54) in comparison with the exact solution (48), (51) is shown in Fig. 3. One can see that the difference between the two surfaces is remarkably small. The critical friction coefficient as a function of the damping parameters d_1 and d_2 has a singularity at the point, corresponding to the system without dissipation at $q = q_0$. This singularity is responsible for the jump in the critical load due to small damping [3, 5–7].

Finally, we find approximate expressions for the trajectories of the eigenvalues of the system with non-zero damping. According to expressions (45) and (52) for the values of the parameters (50) we get

$$a = \frac{364d_1 + 365d_2}{82}, \quad c = -\frac{50}{41}(\mu - \mu_0) + 20d_1d_2, \quad d = -\frac{9\sqrt{82}}{164}(4d_1 - 5d_2). \tag{55}$$

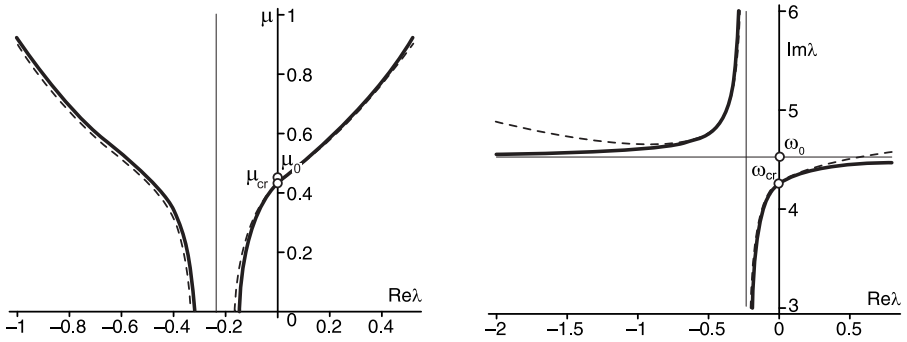

Figure 4: Trajectories of eigenvalues and their approximations for $d_1 = 0.03$ and $d_2 = 0.075$.

With the expressions (55) the equation (42) describes approximately the movement of eigenvalues in the complex plane, and equation (43) gives the real part of the eigenvalues as a function of parameters. For $d_1 = 0.03$ and $d_2 = 0.075$ these trajectories are shown in Fig. 4 by the bold lines. For comparison, the precise numerical solutions of the characteristic equation of the system are presented in Fig. 4 by the dashed lines. We conclude that the approach developed in our paper gives rather good approximation of the critical parameters and eigenvalue trajectories of the non-conservative system with small damping.

5 Two-dimensional model of a disk brake

In [18] a simple two-dimensional model of the squealing disc brake was considered. The rotating disk with the sickness $2h$ is clamped on both sides by springs with an overall stiffness c_3. The rotating speed v_0 of the disc is assumed to be constant, and both springs always remain in contact with the disc. Coulomb friction at the contacts creates a resistant force, which is proportional to the normal pressure with the coefficient μ. The kinetic friction coefficient μ is taken to be constant and does not depend on relative velocity and pressure.

A cross-section of an element of the disc in the plane orthogonal to the disc radius is shown in Fig. 5. It is assumed that the element of mass m with the moment of inertia J can rotate about its center of mass by

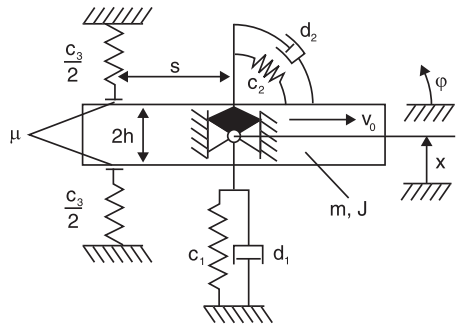

Figure 5: Two-degrees-of-freedom model of a disc brake [18].

angle φ, while the center of mass can be displaced in vertical direction by the quantity x with respect to the equilibrium position. The springs are attached off-centered with a distance s from the center of rotation, Fig. 5. Viscoelastic properties of the disc are modelled by the restoring forces and torques with the stiffness coefficients c_1, c_2 and damping coefficients d_1, d_2, corresponding to the translational and rotational degrees of freedom, Fig. 5.

Clamping of a rotating flexible disc can cause its transverse vibrations yielding an unpleasant sound (squeal). Small vibrations of the system around the equilibrium position are described by the linear differential equation (1), where $\mathbf{y} = [x, \varphi]^T$, and the matrices $\mathbf{D}$ and $\mathbf{A}$ have the form [18, 20].

$$\mathbf{D}=\frac{1}{mJ}\begin{bmatrix} Jd_1 & 0 \\ 0 & md_2 \end{bmatrix}, \quad \mathbf{A}=\frac{1}{mJ}\begin{bmatrix} J(c_1+c_3) & -Jc_3s \\ m(\mu h-s)c_3 & m(s-\mu h)c_3s+mc_2 \end{bmatrix}. \tag{56}$$

As in the previous example we find first the critical value of the friction coefficient μ_0 of the undamped system ($d_1 = 0, d_2 = 0$)

$$\mu_0 = \frac{s}{h}+\frac{mc_2 + J(c_3 - c_1) - 2\sqrt{c_3J(mc_2 - Jc_1)}}{hsmc_3}. \tag{57}$$

When the parameter μ approaches its critical value (57), two eigenfrequencies merge and originate a double eigenfrequency ω_0:

$$\omega_0^2 = \frac{c_1}{m}+\frac{\sqrt{c_3J(mc_2 - Jc_1)}}{Jm}. \tag{58}$$

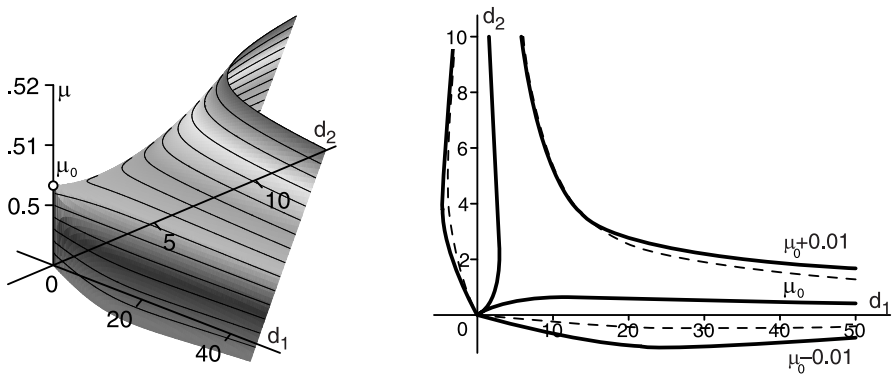

Figure 6: Critical value of the friction coefficient μ as a function of the parameters d_1 and d_2 and its level curves for the values of parameters (59).

With the further increase in the friction coefficient the double eigenvalue $i\omega_0$ splits into two simple, one of them having positive real part (squeal).

For the values of the parameters given in [18, 20]

$$m{=}50kg,\ J{=}10kgm^2,\ c_1{=}10Nm^{-1},\ c_2{=}10Nm,\ c_3{=}60Nm^{-1},\ s{=}1m,\ h{=}2m. \tag{59}$$

the critical friction coefficient and the critical frequency are

$$\mu_0 = \frac{2}{3} - \frac{1}{15}\sqrt{6} \simeq 0.50337, \tag{60}$$

$$\omega_0 = \frac{1}{5}\sqrt{5 + 10\sqrt{6}} \simeq 1.08618s^{-1}. \tag{61}$$

The critical value of the friction coefficient μ_{cr} for the damped system $(d_1 \neq 0, d_2 \neq 0)$ is found with the use of the Routh-Hurwitz criterion applied to the characteristic polynomial of the system (1), (56)

$$\mu_{cr} = \frac{s}{h} + \frac{mc_2 - Jc_1}{mhsc_3} + \frac{Jd_1^2 + md_1d_2}{2m^2hsc_3} + \frac{J^2d_1^2 + m^2d_2^2}{2m^2hsd_1d_2} -$$

$$- \frac{(Jd_1 + md_2)\sqrt{(c_3(Jd_1 - md_2) + d_2d_1^2)^2 - 4md_1d_2(c_1c_3J - mc_2c_3 + d_1d_2c_1)}}{2d_1d_2m^2hsc_3}. \tag{62}$$

The surface (62) and its level curves are shown in Fig. 6. For the values of parameters given by equations (59) and $d_1 = 1Nsm^{-1}$, $d_2 = 1Nms$ equation (62) yields

$$\mu_{cr} = \frac{24803}{30000} - \frac{\sqrt{10553201}}{10000} \simeq 0.50191 < \mu_0 \simeq 0.50337. \tag{63}$$

The inequality (63) reflects the destabilization of the non-conservative system by small dissipative forces because the critical value of the friction coefficient falls abruptly. Substituting expressions (59) and (63) into the characteristic equation of the system (1), (56), we find the critical frequency corresponding to the critical friction coefficient (63)

$$\omega_{cr} \simeq 1.15304 s^{-1}. \tag{64}$$

Comparing (61) and (64) one concludes that the critical frequency of the disc vibrations also jumps due to the influence of small dissipative forces.

To find the approximations of the critical load and frequency we should first calculate the quantity $\widetilde{f}$, the matrix $\mathbf{G}$

$$\widetilde{f} = \frac{c_3 sh(J(\omega_0^2 m - c_1) - 2\sqrt{c_3 J(mc_2 - Jc_1)})}{4mJ^2\omega_0^2}, \quad \mathbf{G} = \frac{1}{8\omega_0^2 mJ}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{65}$$

and the vectors $\mathbf{f}$ and $\mathbf{h}$

$$\mathbf{f} = \left(\frac{c_1 + c_3 - \omega_0^2 m}{4\omega_0^2 m^2}, \quad \frac{J(c_1 - c_3 - \omega_0^2 m) + 2\sqrt{c_3 J(mc_2 - Jc_1)}}{4\omega_0^2 mJ^2} \right), \tag{66}$$

$$\mathbf{h} = \left(\frac{c_1 + c_3 - 3\omega_0^2 m}{4\omega_0^3 m^2}, \quad \frac{J(c_1 - c_3 - 3\omega_0^2 m) + 2\sqrt{c_3 J(mc_2 - Jc_1)}}{4\omega_0^3 mJ^2} \right). \tag{67}$$

Using expressions (66) and (67) in the equation (36) and taking into account (59) we find the critical friction coefficient in the form

$$\mu_{cr}(d_1, d_2) = \mu_0 - \frac{1153\sqrt{6} - 2448}{3000} \frac{(d_1 - 5d_2)^2}{(5d_1 + (13 + 7\sqrt{6})d_2)^2} + \frac{12 + \sqrt{6}}{72000} d_1 d_2. \tag{68}$$

The surface of the critical friction coefficient (62) and its approximation (68) are shown in Fig. 6

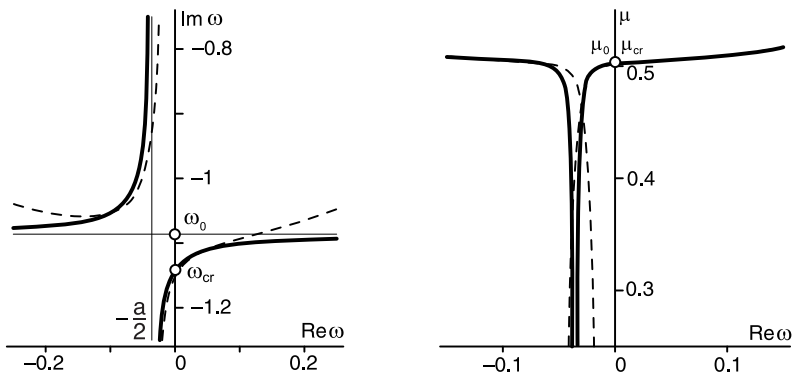

Figure 7: Trajectories of the eigenvalues and their approximations for $d_1 = 1Nsm^{-1}$, $d_2 = 1Nms$.

From the expression (68) it follows that the critical friction coefficient does not decrease when $d_1 = 5d_2$. For the damping coefficients $d_1 = 1Nsm^{-1}$, $d_2 = 1Nms$ equation (68) gives

$$\mu_{cr} = \frac{119777}{6000} - \frac{63559}{8000}\sqrt{6} \simeq 0.50194. \tag{69}$$

The corresponding critical frequency obtained from the asymptotic formula (42) is

$$\omega_{cr} = \frac{14\sqrt{6} - 29}{25}\sqrt{5 + 10\sqrt{6}} \simeq 1.14980s^{-1}. \tag{70}$$

Comparison of expressions (63), (64) and (69), (70) shows that the approximations of the critical values of parameters obtained from the study of the bifurcation of the multiple roots of the characteristic polynomial are in a good agreement with the exact solutions.

Using the expressions (65)–(67) in the equations (42)–(44) one can find an approximation of the trajectories of the eigenvalues on the complex plane. For the values of the parameters (59) the coefficients (45) take the form

$$a = \frac{38 - 7\sqrt{6}}{2300}d_1 + \frac{43 + 6\sqrt{6}}{920}d_2, \quad c = \frac{6(\sqrt{6} - 12)}{23}(\mu - \mu_0) + \frac{d_1 d_2}{2000},$$

$$d = \left(-\frac{33 + 3\sqrt{6}}{2300} d_2 + \frac{-15 + 7\sqrt{6}}{11500} d_1 \right) \sqrt{5 + 10\sqrt{6}}. \tag{71}$$

The trajectories of the eigenvalues calculated numerically in [18] for $d_1 = 1Nsm^{-1}$, $d_2 = 1Nms$ and their approximations (42), (43), (71) are shown in Fig. 7 by the dashed and solid lines, respectively.

Acknowledgements

The work has been supported by the Alexander von Humboldt Stiftung and by the grant CRDF–BRHE Y1-MP-06-19.

References

[1] Ziegler H.: Die Stabilitätskriterien der Elastomechanik. Ing.-Arch. **20**, 49–56 (1952).

[2] Bolotin V.V. Non-conservative Problems of the Theory of Elastic Stability. Pergamon Press, Oxford. 1963.

[3] Seyranian A.P., Kirillov O.N.: Bifurcation diagrams and stability boundaries of circulatory systems. Theor. and Appl. Mechs. **26**, 135–168 (2001).

[4] Bolotin V.V., Zhinzher N.I.: Effects of damping on stability of elastic systems subjected to nonconservative forces. Int. J. Solids Structures. **5**(9), 965–989 (1969).

[5] Seyranian A.P.: Destabilization paradox in stability problems of non-conservative systems. Advances in Mechanics. **13**(2), 89–124 (1990).

[6] Bloch A.M., Krishnaprasad P.S., Marsden J.E., Ratiu T.S.: Dissipation induced instabilities. Annales de l'Institut Henri Poincaré. **11**(1), 37–90 (1994).

[7] Zhinzher N.I.: Effect of dissipative forces with incomplete dissipation on the stability of elastic sysytems. Izv. Ross. Acad. Nauk. MTT. **19**(1), 149–155 (1994).

[8] Seyranian A.P.: On stabilization of non-conservative systems by dissipative forces and uncertainty of critical load. Doklady Akademii Nauk. **348**(3), 323–326 (1996).

[9] Seyranian A.P., Mailybaev A.A. Multiparameter stability theory with mechanical applications. World Scientific. Singapore. 2003.

[10] Kirillov O.N., Seyranian A.P.: Stabilization and destabilization of a circulatory system by small velocity-dependent forces. Journal of Sound and Vibration. **283**(3–5), 781–800 (2005).

[11] Kirillov O.N.: A theory of the destabilization paradox in non-conservative systems. Acta Mechanica. **174**(3–4), 145–166 (2005).

[12] Kirillov O.N., Seyranian A.P.: Dissipation induced instabilities of continuous non-conservative systems. Doklady Mathematics. **71**(3), 470–475 (2005).

[13] Vishik M.I., Lyusternik L.A.: Solution of Some Perturbation Problems in the Case of Matrices and Selfadjoint or Non-selfadjoint Equations. Russian Math. Surveys. **15**(3), 1–73 (1960).

[14] Barnett S.: Leverrier's algorithm: a new proof and extensions. SIAM J. Matrix Anal. Appl. **10**(4), 551–556 (1989).

[15] Arnold V.I. Geometrical Methods in the Theory of Ordinary Differential Equations. Springer, New York and Berlin. 1983.

[16] Bilharz H.: Bemerkung zu einem Satz von Hurwitz. Z. Angew. Math. Mech. **24**, 77–82 (1944).

[17] Hoffmann N., Gaul L.: Effects of damping on mode-coupling instability in friction induced oscillations: imperfect merging of modes and viscous instability. Z. Angew. Math. Mech. **83**. 524–534 (2003).

[18] Popp K., Rudolph M., Kröger M., Lindner M.: Mechanisms to generate and to avoid friction induced vibrations. VDI-Berichte **1736**, VDI-Verlag Düsseldorf, 1–15 (2002).

[19] Kirillov O.N.: Sensitivity analysis of the roots of the characteristic polynomial and stability of non-conservative systems. Proceedings of the International Conference "Physics and Control". St.-Petersburg. Russia. August 24-26, 206–211 (2005).

[20] Kirillov O.N.: On the stability of non-conservative systems with small dissipation. Contemporary Mathematics and its Applications. **36**, 107–117 (2005). (in Russian)

[21] Kirillov O.N.: Gyroscopic stabilization of non-conservative systems. Physics Letters A. **359**(3), 204–210 (2006).

[22] Kirillov O.N.: Destabilization paradox due to breaking the Hamiltonian and reversible symmetry. International Journal of Non-Linear Mechanics. **42**(1), 71–87 (2007).

[23] Sinou J.-J., Jezequel L.: Mode coupling instability in friction-induced vibrations and its dependency on system parameters including damping. European Journal of Mechanics A/Solids. **26**, 106–122 (2007).

Submitted on November 2006.

Bifurkacija korena karakterističnog polinoma i destabilizacioni paradoks u vibracijama izazvanim trenjem

Proučava se paradoksalni efekt malih disipativnih i žiroskopskih sila na stabilnost linearnog nekonzervativnog sistema. Ovaj se manifestuje, na prvi pogled, nepredvidivim ponašanjem kritične nekonzervativne sile. Analitički opis ove pojave se dobija analizom bifurkacije višestrukih korenova karakterističnog polinoma nekonzervativnog sistema. Dva sistema koji poseduju vibracije izazvane trenjem se posmatraju kao mehanički primeri i to: masa koja klizi preko konvejerske trake kao i model disk kočnice koji opisuje početak cviljenja tokom kočenja vozila.

doi:10.2298/TAM0702087K Math.Subj.Class.: 34C23, 34K18, 35B32, 70F40