Resilient cluster consensus of multi-agent systems

Yilun Shang

***Abstract*—We investigate the problems of resilient cluster consensus in directed networks under three types of multi-agent dynamics, namely, continuous-time multi-agent systems, discrete-time multi-agent systems, and switched multi-agent systems composed of both continuous-time and discrete-time components. Resilient cluster censoring strategies are proposed to ensure cluster consensus against locally bounded Byzantine nodes in a purely distributed manner, where neither the number/indentity of Byzantine nodes nor the division of clusters is assumed. We do not require complicated algebraic conditions or any balance conditions over inter-cluster structures, distinguishing the current work from previous results on cluster consensus problems besides a fortiori the attack-tolerant feature. Sufficient conditions are established in all the three scenarios based on the graph robustness. Furthermore, we solve the heterogenous cluster robustness problems and resilient scaled cluster consensus problems as extensions. The theoretical results are illustrated through numerical examples including the Santa Fe collaboration network.**

***Index Terms*—Robustness; cluster consensus; continuous-time; discrete-time; switched multi-agent system.**

I. Introduction

Over the past decade, consensus problems of cooperative multi-agent systems have appeared as an emerging research area with broad applications and attracted interests from many fields, such as computer science, control engineering, physics, and social science [1]–[3]. For a networked system, consensus means that the states of all agents in the network converge to a common value based on local information available to each agent. A large amount of consensus algorithms have been reported for varied systems with continuous-time and discrete-time agent dynamics [4], [5].

It is widely known that distributed engineered multi-agent systems are prone to errors and malicious attacks. Therefore, it is imperative to study the resilient consensus of multi-agent systems which can defy the compromise of a group of adversarial nodes within the network. In [6], [7], Zhang et al. and LeBlanc et al. developed an interesting concept of network robustness, referred to as r-robustness, characterizing robust networks on which discrete-time resilient consensus algorithms are designed. The proposed algorithm is purely distributed for normal nodes in the network in the sense that no initial knowledge of the network topology as well as the identities of adversarial nodes is assumed. The results were later extended to a class of hybrid dynamics [8] and second-order multi-agent systems under locally bounded errors [9]. In [10], continuous-time fault-tolerant consensus control for multi-agent systems has been studied in the presence of communication delay. By utilizing mobile detectors, Zhao et al. [11] investigated discrete-time resilient consensus under attacks where the number of Byzantine nodes is not limited by the network connectivity. Recently, Shang [12] considered the resilient consensus of switched multi-agent systems which is composed of a discrete-time subsystem and a continuous-time subsystem activated in turn by a switching rule. It was shown that resilient consensus problem is solvable under arbitrary switching. Other resilient consensus problems, e.g., in terms of memory sampled-data control [13] and robustness against uncertain disturbances [14] have also been studied.

The existing results of most previous work in this field, however, are on resilient consensus with a global agreement. In the real world, agreement is often not unanimous in the sense that agents in the network are partitioned into multiple subgroups (i.e. *clusters*), each of which may reach a common but usually different state asymptotically [15]. For instance, in cooperative team-hunting activities, several groups of predators may surround the prey in different directions for hunting success [16]. Other examples include social learning under environment influenced by different culture backgrounds, robot team coordination, and heterogeneous robotic sorting [17]. As an extension to global consensus, cluster consensus has been intensively studied for both continuous-time systems [15], [18]–[21] and discrete-time systems [22]–[25] to achieve individual consistent states in each cluster, where information exchanges between agents not only within the same cluster but also among different clusters. In [23], for example, cluster consensus is achieved via adaptive inputs when the inter-cluster topologies are balanced meaning that the sum of adjacent weights from each node in one cluster to all nodes in another cluster is kept identical. However, the above works are based on the assumption that all nodes in the network are cooperative, i.e. *normal*, and hence are not capable of coping with resilient consensus problems against attacks. An adversarial node can easily manipulate the network performance as its influence is not appropriately monitored with the existing protocols. One of the main aims in this paper is to, for the first time, propose the *cluster censoring strategy* (c.f. Section II.C) so that the malicious behavior of adversarial nodes is "monitored" and resilient cluster consensus can be guaranteed. Furthermore, to our knowledge, cluster consensus problem has not been solved for switched systems due to the difficulty resulting from coexistence of continuous-time and discrete-time subsystems. In the current work, we put forward new consensus protocols to solve not only cluster consensus but resilient cluster consensus problems in switched multi-agent systems.

The main contributions of this paper are summarized as follows.

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First, two distributed cluster consensus protocols are proposed under which resilient cluster consensus for continuous-time and discrete-time multi-agent systems, respectively, can be reached against locally bounded Byzantine nodes. In the current work, no topological condition is imposed on the inter-connection among clusters, whereas in most previous cluster consensus protocols (see the aforementioned work [15], [18]–[25]) fairly restrictive balance conditions are assumed for the Laplacian matrices describing the inter-cluster topology. Moreover, in the existing cluster consensus problems, each node must be aware of the locations of other nodes (i.e., to which clusters they belong in the network) as the adopted protocol is often stringently dependent on the location of every node in the network. In contrast, our framework works flexibly in the sense that a normal node is not assumed to have knowledge on i) the locations of nodes outside of its own cluster, or ii) the identities and number of Byzantine nodes in the entire network.

Second, based upon the designed continuous-time and discrete-time protocols, resilient cluster consensus is considered over switched multi-agent systems featuring, for example, continuous-time multi-agent systems activated by a computer in a discrete-time manner [12], [26]. Sufficient criteria are provided to ensure cluster consensus against locally bounded Byzantine nodes under arbitrary switching. To the best of our knowledge, this is the first work dealing with (resilient) cluster consensus in switched systems.

Third, we add a further dimension by introducing the resilient scaled cluster consensus problems, where states of normal nodes in each cluster may achieve any prescribed ratios asymptotically instead of a common value [27]. These problems are solved for all continuous-time, discrete-time, and switched multi-agent systems as generalizations. We note that scaled cluster consensus has recently been examined for continuous-time multi-agent systems with first and second order dynamics in [28], where adversarial nodes were not considered.

The rest of the paper is organized as follows. Section II is devoted to preliminaries and formulation of the system models. Main results are provided in Section III. Numerical simulations are given in Section IV to illustrate our theoretical results. Finally, conclusion is drawn in Section V.

II. Preliminaries

Some standard notations will be used throughout the paper. Let $\mathbb{R}$ and $\mathbb{N}$, respectively, be the sets of real numbers and non-negative integers. For a set X, $|X|$ means the number of elements in it, while for a real number x, $|x|$ stands for its absolute value.

A. Graph theory

Consider a directed network containing n agents with associated weighted digraph denoted by $G = (V, E, A)$, where $V = \{1, \cdots, n\}$ represents the set of nodes with $|V| = n$ and $E \subseteq V \times V$ is the set of directed edges. The weighted adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is defined by $a_{ij} > 0$ if $(j, i) \in E$ meaning that node i can receive information directly from node j, and $a_{ij} = 0$ otherwise. Let N and B represent the sets of normal nodes and non-cooperative nodes in G, respectively, with $|N| = n_N$ and $|B| = n_B$. Clearly, we have $V = N \cup B$ and $n = n_N + n_B$. Normal nodes collaborate with their neighbors to reach a consensus value following state update rules specified in (1)-(3) below. Non-cooperative nodes are Byzantine nodes (see Definition 4 below), who aim to manipulate the network performance to thwart consensus. The existence of non-cooperative nodes is not known to the normal nodes as usually in the real world situations. Moreover, G is divided into L clusters G_ℓ ($\ell = 1, 2, \cdots, L$) having node sets $V_1 = \{1, 2, \cdots, n_1\}$, $V_2 = \{n_1+1, n_1+2, \cdots, n_1+n_2\}$, $\cdots$, $V_L = \{n_1+\cdots+n_{L-1}+1, \cdots, n_1+\cdots+n_L\}$, respectively, with $|V_\ell| = n_\ell$ ($1 \le \ell \le L$) and $n = \sum_{\ell=1}^{L} n_\ell$. The edges and weights in each cluster are naturally induced by the overall structure of G. The neighborhood of node $i \in V$ is denoted by $\mathcal{N}_i = \{j \in V : (j, i) \in E\} = \cup_{\ell=1}^{L} \mathcal{N}_{i\ell}$, where $\mathcal{N}_{i\ell} = \{j \in V_\ell : (j, i) \in E\}$ is the set of all neighbors of i in the cluster G_ℓ ($1 \le \ell \le L$). A directed path from node i to j is a sequence of edges $(i, i_1), (i_1, i_2), \cdots, (i_l, j)$ in G with distinct nodes i_k, $k = 1, 2, \cdots, l$. We say that G has a directed spanning tree with root i if for every node j in $V\backslash\{i\}$, there is a directed path from i to j.

Definition 1. (reachability) [6], [7] A set $S \subseteq V$ is called r-reachable provided there is a node $i \in S$ such that $|\mathcal{N}_i\backslash S| \ge r$, where $r \in \mathbb{N}$.

Definition 2. (robustness) [6], [7] A digraph G is called r-robust with $r \in \mathbb{N}$ provided for every pair of nonempty and disjoint subsets in V, at least one of them is r-reachable.

Lemma 1. [7] *Suppose that G is r-robust and G' is the graph produced by removing up to s incoming edges of each node in G, where $0 \le s < r$. Then G' is $r-s$-robust. G is 1-robust if and only if it contains a directed spanning tree.*

B. System models

In a network with directed topology, we consider three types of system dynamics, i.e., continuous-time dynamics, discrete-time dynamics, and switched dynamics containing a continuous-time subsystem and a discrete-time subsystem. Let $x_i(t) \in \mathbb{R}$ be the state of node i at time t.

Definition 3. (resilient cluster consensus) The normal nodes in N is said to achieve resilient cluster consensus against non-cooperative nodes in B if, for each $1 \le \ell \le L$, there exists $c_\ell \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = c_\ell$ for all $i \in V_\ell$ and all initial conditions $\{x_i(0)\}_{i\in V_\ell}$.

Remark 1. In the above definition, we do not require $c_{\ell_1} \ne c_{\ell_2}$ for $\ell_1 \ne \ell_2$, which is in line with many of the existing research studies in this field (e.g. [20], [22], [25], [29]). From this perspective, cluster consensus covers global consensus as a special case. On the other hand, inter-cluster separation has also been realized in the literature through leader-following [19], [30] or inter-cluster heterogeneous external input [21], [23], [31].

It is clear that the ordinary cluster consensus problem is recovered if $B = \emptyset$ in Definition 3. For continuous-time agent dynamics, each normal node $i \in N$ adopts the following scheme

$$\dot{x}_i(t) = \varphi_i^C \left(\{x_j^i(t) : j \in \mathcal{N}_i \cup \{i\}\}\right) \tag{1}$$

For discrete-time agent dynamics, each normal node $i \in N$ follows

$$x_i(t+1) = \varphi_i^D\left(\{x_j^i(t) : j \in \mathcal{N}_i \cup \{i\}\}\right) \tag{2}$$

where $x_j^i(t) \in \mathbb{R}$ represents the state value communicate to i from j at time t. For switched multi-agent systems, we take (1) and (2), respectively, as the continuous- and discrete-time subsystems governed by a switching law. In other words, each normal node $i \in N$ follows

$$\begin{cases} \dot{x}_i(t) = \varphi_i^C\left(\{x_j^i(t) : j \in \mathcal{N}_i \cup \{i\}\}\right), \\ \qquad \text{continuous-time subsystem activated at } t; \\ x_i(t) = \varphi_i^D\left(\{x_j^i(t-1) : j \in \mathcal{N}_i \cup \{i\}\}\right), \\ \qquad \text{discrete-time subsystem activated at } t. \end{cases} \tag{3}$$

Naturally, we define $x_j^i(t) = x_j(t)$ for $j \in N$ as a normal node will always communicate the true state in its neighborhood. We also have $x_i^i(t) = x_i(t)$. The system functions φ_i^C and φ_i^D will be designed later so that the normal nodes in the network are able to withstand sabotage of non-cooperative nodes, whose cardinality and identities are unavailable to normal nodes. Non-cooperative nodes in B are potentially capable of exerting arbitrary disruptive behaviors specified as follows.

Definition 4. (Byzantine node) A node $i \in B$ is a Byzantine node if it applies some different system function $\tilde{\varphi}_i^C$ for the continuous-time systems (or $\tilde{\varphi}_i^D$ for the discrete-time systems, or either one for the switched systems, respectively), or it sends disparate values to any of its neighbors at some time $t > 0$.

Byzantine nodes are considered as one of the worst attackers as they often possess a perfect knowledge of the entire system [8], [9], [11]. They are capable of falsifying the information sending to their neighbors either in a point-to-point way or through broadcast communication, and can potentially collude with other Byzantine nodes. It is therefore reasonable to limit the number of these nodes. Given an integer r, we here investigate the r-locally bounded model [7], where $|\mathcal{N}_i \cap B| \le r$ for every normal node $i \in N$. In other words, each normal node has no more than r non-cooperative neighbors.

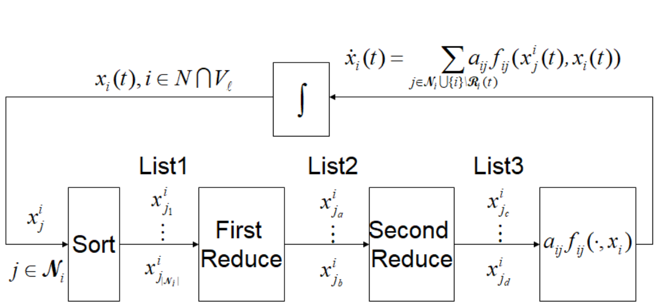

Fig. 1. Data flow for the continuous-time subsystem: List3 $\subseteq$ List2 $\subseteq$ List1; $\mathcal{N}_i := \{j_1, \cdots, j_{|\mathcal{N}_i|}\}$; $x_{j_a}^i$ (if larger than x_i) is the $(r+1)$-th largest value of $\mathcal{N}_{i\ell}$ in List1; $x_{j_b}^i$ (if smaller than x_i) is the $(r+1)$-th smallest value of $\mathcal{N}_{i\ell}$ in List1; $x_{j_c}^i := \max\{\mathcal{C}_i(t) \cup x_i\}$ and $x_{j_d}^i := \min\{\mathcal{C}_i'(t) \cup x_i\}$, where $\mathcal{C}_i(t)$ is contained in List2 and has all values in $\mathcal{N}_{i\ell}$ that are larger than x_i, and similarly $\mathcal{C}_j'(t)$ is contained in List2 and has values in $\mathcal{N}_{i\ell}$ that are smaller than x_i; $\mathcal{N}_i \backslash \mathcal{R}_i(t) := \{j_c, \cdots, j_d\}$.

C. Cluster censoring strategy

In light of the Weighted-Mean-Subsequence-Reduced (W-MSR) scheme [7], [8], [10], we design the following nearest-neighbor based two-round censoring procedure for continuous-time multi-agent system (1) to facilitate resilient cluster consensus (see Fig. 1 for a data flow scheme for the model).

Fix $r \in \mathbb{N}$. First, each normal node $i \in N \cap V_\ell$ $(1 \le \ell \le L)$ receives the values $\{x_j^i(t)\}$ of its neighbors at time t, and ranks the obtained information $\{x_j^i(t)\}_{j \in \mathcal{N}_i}$ in a descending order. Second, the highest values that are higher than $x_i(t)$ in the above sorted list are removed in order (from large to small) until r values from $\mathcal{N}_{i\ell}$ are eliminated; if there are fewer than r such values, all of them are eliminated. The analogous deletion process is conducted for the lower values. Third, denote by $\mathcal{C}_i(t)$ the set of values in $\mathcal{N}_{i\ell}$ that are higher than $x_i(t)$ in the resulting list. We further delete those values in the list that are higher than $\max\{\mathcal{C}_i(t) \cup x_i(t)\}$. The analogous process is applied to the lower values. Let $\mathcal{R}_i(t)$ be the set of nodes that have been eliminated in the above two steps. Fourth, the value of each $i \in N \cap V_\ell$ evolves following $\varphi_i^C(\cdot)$ in (1)

$$\dot{x}_i(t) = \sum_{j \in (\mathcal{N}_i \cup \{i\}) \backslash \mathcal{R}_i(t)} a_{ij} f_{ij}(x_j^i(t), x_i(t)), \tag{4}$$

where we assume that the function $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$ is (iC) locally Lipschitz continuous, (iiC) $f_{ij}(x, y) = 0 \Leftrightarrow x = y$, and that (iiiC) for any nonequal x and y, $(x - y) f_{ij}(x, y) > 0$.

For discrete-time multi-agent system (2), we have a similar two-round censoring process performed at discrete-time steps. The first three steps are exactly the same as above. In the final step, the value of each $i \in N \cap V_\ell$ changes following $\varphi_i^D(\cdot)$ in (2)

$$x_i(t+1) = \sum_{j \in (\mathcal{N}_i \cup \{i\}) \backslash \mathcal{R}_i(t)} w_{ij}(t) x_j^i(t), \tag{5}$$

where we assume that the weight $w_{ij}(t)$ (iD) equals to zero if $j \notin \mathcal{N}_i \cup \{i\}$, (iiD) there is a constant $w \in (0,1)$ such that $w_{ij}(t) \ge w > 0$ for any $j \in (\mathcal{N}_i \cup \{i\}) \backslash \mathcal{R}_i(t)$, and that (iiiD) the equality $\sum_{j \in (\mathcal{N}_i \cup \{i\}) \backslash \mathcal{R}_i(t)} w_{ij}(t) = 1$ holds.

For switched multi-agent system (3), we can easily concatenate these two algorithms by invoking the appropriate one according to the the signal of switching law in question. See Fig. 2 for the framework diagram for the nearest-neighbor based two-round censoring strategy.

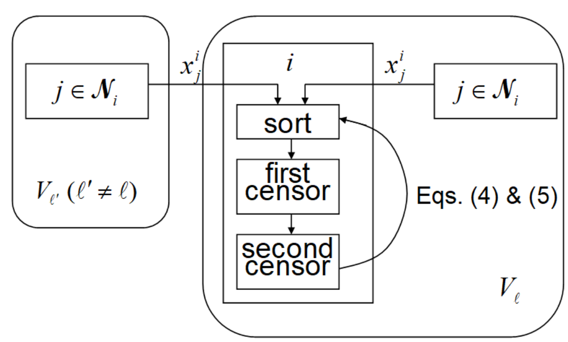

Fig. 2. Framework diagram for the nearest-neighbor based two-round censoring procedure.

Remark 2. For both continuous-time and discrete-time (sub)systems, the above algorithm involves two rounds of censoring. Generally, in the first round of censoring, any neighbor in $\mathcal{N}_i$ having value no less than the r-th largest value of $\mathcal{N}_{i\ell}$ is deleted, and similarly, any neighbor in $\mathcal{N}_i$ having value no more than the r-th smallest value of $\mathcal{N}_{i\ell}$ is deleted; in the second round of censoring, any neighbor in $\mathcal{N}_i$ having value larger than the largest value of $\mathcal{N}_{i\ell}$ in the remaining list is deleted, and similarly, any neighbor in $\mathcal{N}_i$ having value smaller than the smallest value of $\mathcal{N}_{i\ell}$ in the remaining list is deleted. Our strategy can be viewed as a generalization of W-MSR into multiple clusters. In fact, if $L = 1$ (namely, there is only one cluster), then the first round of censoring corresponds to the filter operation in W-MSR and the second round of censoring becomes void. In the multiple clusters scenario, it can be seen that the two-round censorship is necessary to guarantee cluster consensus essentially (c.f. Examples 1 & 2 in Section III). Here, it is also worth clarifying that nodes are deleted at each time step based solely on their values instead of their identities/locations, which are not available for a normal agent. Therefore, both normal and Byzantine nodes can be deleted in the two rounds of censoring in general, and we do not require deleting all Byzantine nodes in our algorithms.

Remark 3. The functions and weights in (4) and (5) have considerable flexibility. For example, we can choose in the continuous-time system $f_{ij}(x, y) = b_{ij}(x - y)$ with $b_{ij} > 0$, which is canonical in the literature on consensus problems (see e.g. [4]). In the discrete-time system, we can take $w_{ij}(t) = (|\mathcal{N}_i|+1-|\mathcal{R}_i(t)|)^{-1}$ to give each neighbor the same weight, or $w_{ij}(t) = a_{ij} \cdot \left(\sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_i(t)} a_{ij}\right)^{-1}$ taking into consideration of the network adjacency matrix.

The strategies proposed above have low complexity and are purely local and distributed as the W-MSR protocol. The designed algorithms enable us to simultaneously cope with the coexistence of normal and Byzantine nodes, the interplay among different clusters, and the switching between discrete- and continuous-time agent dynamics. We refer to the above algorithms as the *cluster censoring strategy* with parameter r in the sequel.

III. RESILIENT CLUSTER CONSENSUS ANALYSIS

In this section, the resilient cluster consensus problem and its generations will be considered under three types of agent dynamics, namely, continuous-time, discrete-time, and switched multi-agent systems.

A. Continuous-time system

To start with, define $\Theta_{M\ell}(t) := \max_{i\in N\cap V_\ell} x_i(t)$ and $\Theta_{m\ell}(t) := \min_{i\in N\cap V_\ell} x_i(t)$ for $t \geq 0$ to be the highest and lowest values of normal nodes in the cluster G_ℓ ($1 \leq \ell \leq L$).

Lemma 2. *Consider the continuous-time multi-agent system (1) under the digraph $G = (V, E)$, in which normal nodes adopt the cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, for any $i \in N \cap V_\ell$ ($1 \leq \ell \leq L$), we have $x_i(t) \in [\Theta_{m\ell}(0), \Theta_{M\ell}(0)]$ for all $t \geq 0$.*

Proof. Fix ℓ and $i \in N \cap V_\ell$. We will show $x_i(t) \leq \Theta_{M\ell}(0)$ for $t \geq 0$. The proof of the lower bound is akin to this.

Suppose the upper bound does not hold. There exists some time $t^* > 0$ such that there exists a node $i_0 \in N \cap V_\ell$, $x_{i_0}(t^*) = \Theta_{M\ell}(0)$ for the first time and $x_i(t) \leq \Theta_{M\ell}(0)$ for all $t \leq t^*$ and all $i \in N\cap V_\ell$. Thus, $\dot{x}_{i_0}(t^*) > 0$. It follows from (4) that

$$\dot{x}_{i_0}(t^*) = \sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0}(t^*)} a_{i_0j} f_{i_0j}(x_j^{i_0}(t^*), x_{i_0}(t^*)).$$

It is not difficult to see whether $j \in (\mathcal{N}_{i_0\ell} \cup \{i_0\})\setminus\mathcal{R}_{i_0}(t^*)$ or $j \in (\mathcal{N}_{i_0\ell'} \cup \{i_0\})\setminus\mathcal{R}_{i_0}(t^*)$ with $\ell' \neq \ell$, we have $x_j^{i_0}(t^*) \leq x_{i_0}(t^*)$ since i_0 is connected to no more than r Byzantine neighbors in G. By the assumptions (iiC) and (iiiC), all components on the right-hand side of the above expression are less than or equal to zero which leads to $\dot{x}_{i_0}(t^*) \leq 0$. We derive a contradiction and the proof is complete. □

Remark 4. The interval $[\Theta_{m\ell}(0), \Theta_{M\ell}(0)]$ is an invariant set for all normal nodes in the cluster G_ℓ. This is referred to as validity or safety condition [7] for a safety sensitive process as the initial interval is often viewed as safe. It is also worth noting that the parameter r in our cluster censoring strategy is known a priori to all normal nodes. The parameter r informs the network robustness condition (Theorem 1 below) and can be extended to cluster-wise heterogeneous values (c.f. Remark 5). However, a more realistic scenario would be to determine the individual value r_i ($i \in V$) through a distributed decision making, for example, using the max consensus process [35].

The two-round censoring strategy plays a key role in the proof Lemma 2. Furthermore, it is intuitively clear from the following examples that the two rounds of removal are necessary to ensure resilient cluster consensus.

Example 1. Suppose $G = G_1 \cup G_2$ and a normal node $i \in N \cap V_1$ has neighborhoods $\mathcal{N}_{i1} = \{i_1, i_2 \cdots, i_r\} \subseteq B$ and $\mathcal{N}_{i2} = \{j\} \subseteq N$. Assume that the values of $x_{i_l}(t)$ ($l = 1, 2 \cdots, r$) and $x_j(t)$ are all larger than $x_i(t)$, $t \geq 0$. If we only cut off r neighbors in $\mathcal{N}_i$ (as in the global consensus case, c.f. [6]–[8]) in the first round of censoring, the node i can easily be controlled by one of its Byzantine neighbor and no consensus will be reached among normal nodes in G_1. Moreover, this cannot be remedied by requiring strong robustness of G_1 since additional (normal) neighbors of i in V_1 may all have lower values than x_i. Example 1 shows that the first round of censoring is essential.

Example 2. Suppose $G = G_1 \cup G_2$, $B = \emptyset$, G_ℓ ($\ell = 1, 2$) are complete digraphs, and a normal node $i \in V_1$ has $\mathcal{N}_{i2} = \{i_1, i_2, \cdots, i_{r+1}\}$. Assume $n_\ell > r+1$ ($\ell = 1, 2$). If the second round of censoring is omitted, then all nodes in G must reach a global consensus by a standard result of consensus problem (see e.g. [4]), which becomes trivial. On the other hand, if there is a Byzantine node in $\mathcal{N}_{i2}$, the states of nodes in G_1 will be ruled by the Byzantine node when the second round of censoring is not in place. Thus, the cluster consensus is not achieved by Definition 3. Obviously, these issues cannot be solved by imposing strong robustness of G_1 since it is already complete. Example 2 highlights the necessity of including the second round of removal.

We will see below in Theorem 1 that our proposed cluster censoring strategy (together with appropriate robust network topologies) is sufficient for resilient cluster consensus.

Assumption 1. Let $\{\tau_p\}_{p\in\mathbb{N}}$ be the sequence of instants when the set $\mathcal{R}_i(t)$ changes for some $i \in N$. Suppose $|\tau_{p+1} - \tau_p| \geq \tau > 0$ for some τ.

Theorem 1. *Consider the continuous-time multi-agent system (1) under digraph $G = (V, E, A)$, where each normal node adopts the cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, resilient cluster consensus is achieved if G_ℓ is $2r+1$-robust for $1 \leq \ell \leq L$ and Assumption 1 holds.*

Proof. Let $\Gamma_\ell(t) = \Theta_{M\ell}(t) - \Theta_{m\ell}(t)$ for $\ell = 1, 2, \cdots, L$, and $t \geq 0$. It suffices to show that each $\Gamma_\ell(t)$ tends to zero as t tends to infinity. For this purpose, we fix ℓ. Define the Dini derivative of a function as $D^+\varphi(t) = \limsup_{h\to 0^+}(\varphi(t+h) - \varphi(t))/h$. In view of the cluster censoring strategy, the Dini derivatives of $\Theta_{M\ell}(t)$ and $\Theta_{m\ell}(t)$ along the trajectory of (4) can be calculated as

$$D^+\Theta_{M\ell}(t) = \dot{x}_I(t) = \sum_{j\in(\mathcal{N}_I\cup\{I\})\backslash\mathcal{R}_I(t)} a_{Ij} f_{Ij}(x_j^I(t), x_I(t)) \quad (6)$$

and

$$D^+\Theta_{m\ell}(t) = \dot{x}_J(t) = \sum_{j\in(\mathcal{N}_J\cup\{J\})\backslash\mathcal{R}_J(t)} a_{Jj} f_{Jj}(x_j^J(t), x_J(t)), \quad (7)$$

where $\dot{x}_I(t) = \max_{i\in\mathcal{I}(t)} \dot{x}_i(t)$, $\mathcal{I}(t) = \{i \in N \cap V_\ell : x_i(t) = \Theta_{M\ell}(t)\}$ and $\dot{x}_J(t) = \max_{i\in\mathcal{J}(t)} \dot{x}_i(t)$, $\mathcal{J}(t) = \{i \in N \cap V_\ell : x_i(t) = \Theta_{m\ell}(t)\}$ invoking Dini derivative's theorem [32]. For $j \in (\mathcal{N}_{I\ell} \cup \{I\})\backslash\mathcal{R}_I(t)$, we have $x_I(t) \geq x_j^I(t)$ since node j has at most r Byzantine neighbors in G and r neighbors in $\mathcal{N}_{I\ell}$ will be removed in step 2 of the censoring algorithm. Due to step 3 of the algorithm, $x_I(t) \geq x_j^I(t)$ still holds for $j \in (\mathcal{N}_{I\ell'} \cup \{I\})\backslash\mathcal{R}_I(t)$ with $\ell' \neq \ell$. Since $a_{Ij} > 0$ and $f_{Ij}(x_j^I(t), x_I(t)) \leq 0$, we obtain $D^+\Theta_{M\ell}(t) \leq 0$ by (6). Analogously, $D^+\Theta_{m\ell}(t) \geq 0$ by (7) and hence $D^+\Gamma_\ell(t) = D^+\Theta_{M\ell}(t) - D^+\Theta_{m\ell}(t) \leq 0$.

Suppose that $\lim_{t\to\infty} D^+\Gamma_\ell(t) \neq 0$. There must exist constants $\varepsilon_0 > 0$, $\delta_0 > 0$, and a sequence of instants $\{s_l\}_{l\geq 1}$ with $\lim_{l\to\infty} s_l = \infty$ such that $D^+\Gamma_\ell(s_l) \leq -2\varepsilon_0$ and $|s_{l+1} - s_l| > \delta_0$ for any $l \geq 1$. For any interval I with $I \cap \{\tau_p\}_{p\geq 1} = \emptyset$, $D^+\Gamma_\ell(t)$ is uniformly continuous in I since $D^+\Gamma_\ell(t)$ is continuous and $\dot{x}_i(t)$ is bounded for all normal nodes $i \in N \cap V_\ell$ by (iC). We can choose $\delta_1 > 0$ such that for any t^1 and t^2 in I and $|t^1 - t^2| < \delta_1$, $|D^+\Gamma_\ell(t^1) - D^+\Gamma_\ell(t^2)| < \varepsilon_0$ holds. In addition, it follows from Assumption 1 that δ_1 can be taken small enough such that for every $l \geq 1$, $[s_l - \delta_1, s_l + \delta_1] \subseteq I$ for some I. For any $t \in [s_l - \delta_1, s_l + \delta_1]$, we have

$$\begin{aligned} D^+\Gamma_\ell(t) &= -|D^+\Gamma_\ell(s_l) - (D^+\Gamma_\ell(s_l) - D^+\Gamma_\ell(t))| \\ &\leq -2\varepsilon_0 + \varepsilon_0 = -\varepsilon_0. \end{aligned}$$

Take $0 < \delta < \delta_1$ satisfying $\{[s_l - \delta, s_l + \delta]\}_{l\geq 1}$ are pairwise disjoint intervals. Therefore,

$$\begin{aligned} \int_0^\infty D^+\Gamma_\ell(t)dt &\leq -\lim_{N\to\infty} \sum_{l=1}^{N} \int_{s_l-\delta}^{s_l+\delta} \varepsilon_0 dt \\ &= -2\lim_{N\to\infty} N\varepsilon_0\delta = -\infty. \end{aligned}$$

It is a contradiction against the fact $\Gamma_\ell(t) \geq 0$ for all t. This means the assumption at the outset is not true and we obtain $D^+\Gamma_\ell(t) \to 0$ as $t \to \infty$.

Now, in view of (6) and (7), there exist two constants $c_{M\ell} \geq c_{m\ell}$ such that $\lim_{t\to\infty} \Theta_{M\ell}(t) = \lim_{t\to\infty} x_I(t) = c_{M\ell}$ and $\lim_{t\to\infty} \Theta_{m\ell}(t) = \lim_{t\to\infty} x_J(t) = c_{m\ell}$. Assume that $c_{M\ell} > c_{m\ell}$. Drawing upon Lemma 1 and the assumptions that the clusters G_ℓ $(1 \leq \ell \leq L)$ are $2r+1$-robust, we see that the communication network G_ℓ (under time-varying edge removal of our cluster censoring strategy) is always 1-robust and equivalently has a spanning tree. There is an instant $T > 0$ and $\varepsilon > 0$ such that $x_I(t) > c_{M\ell} - \varepsilon > c_{m\ell} + \varepsilon > x_J(t)$ for $t \geq T$. Noting that $\lim_{t\to\infty} \dot{x}_I(t) = 0$, we obtain $\lim_{t\to\infty} x_j^I(t) - x_I(t) = 0$ for all $j \in (\mathcal{N}_I \cup \{I\})\backslash\mathcal{R}_I(t)$ by (6) and the cluster censoring strategy. Akin to this, we derive $\lim_{t\to\infty} x_j^J(t) - x_J(t) = 0$ for all $j \in (\mathcal{N}_J \cup \{J\})\backslash\mathcal{R}_J(t)$ via (7). Since there is a finite number of nodes in G_ℓ, at some time $T' \geq T$, there must be two directed paths—one connecting the root node q to I and the other connecting q to J such that $x_q(T') > c_{M\ell} - \varepsilon > c_{m\ell} + \varepsilon > x_q(T')$. This is impossible. Hence, $c_{M\ell} = c_{m\ell}$ and the states of nodes in G_ℓ converge to a common limit. This holds for all $\ell = 1, 2, \cdots, L$. The proof is complete. □

Remark 5. It is noteworthy that, compared to the existing cluster consensus problems(see e.g. [15], [18], [20], [22], [23], [25]), neither inter-cluster balance condition nor complicated eigenvalue conditions are required in Theorem 1 owing to the cluster censoring strategy and r-robustness of clusters. Moreover, if for each normal node $i \in N \cap V_\ell$ $(1 \leq \ell \leq L)$ we consider r_ℓ instead of requiring the same r for all normal nodes in G in our cluster censoring strategy, we are able to deal with more general situations where clusters may have heterogeneous robustness. We refer to the resulting algorithm (both for continuous- and discrete-time) as the cluster censoring strategy with parameter $(r_1, r_2, \cdots, r_L)$. Given the sequence $(r_1, r_2, \cdots, r_L)$, we define the $(r_1, r_2, \cdots, r_L)$-locally bounded model, where $|\mathcal{N}_i \cap B| \leq r_\ell$ holds for every node $i \in N \cap V_\ell$ $(1 \leq \ell \leq L)$. Namely, each normal node in the cluster G_ℓ has no more than r_ℓ Byzantine neighbors in the entire network G. The following result can be shown similarly as Theorem 1.

Theorem 2. *Consider the continuous-time multi-agent system (1) under digraph $G = (V, E, A)$, where each normal node adopts the cluster censoring strategy with parameter $(r_1, r_2, \cdots, r_L)$. Under the $(r_1, r_2, \cdots, r_L)$-locally bounded Byzantine model, resilient cluster consensus is achieved if G_ℓ is $2r_\ell+1$-robust for $1 \leq \ell \leq L$ and Assumption 1 holds.*

Definition 5. (resilient scaled cluster consensus) Given $0 \neq \gamma_i \in \mathbb{R}$ for each node $i \in V$, normal nodes are said to achieve resilient scaled cluster consensus with respect to $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ if for each $1 \leq \ell \leq L$, there exists $c_\ell \in \mathbb{R}$ such that $\lim_{t\to\infty} \gamma_i x_i(t) = c_\ell$ for all $i \in V_\ell$ and all initial conditions $\{x_i(0)\}_{i\in V_\ell}$.

Scaled consensus problems (see e.g. [27], [28]) offers a further dimension of freedom where prescribed ratios rather than a fix common value is achieved. Clearly, the special case of $\gamma_i \equiv 1$ corresponds to the cluster consensus defined in Definition 3. We consider the following scaled cluster censoring strategy with parameter r for the continuous-time multi-agent system (1).

Fix $r \in \mathbb{N}$. First, each normal node $i \in N \cap V_\ell$ $(1 \le \ell \le L)$ receives the values $\{x_j^i(t)\}$ of its neighbors at time t, and ranks the obtained information $\{\gamma_j x_j^i(t)\}_{j\in\mathcal{N}_i}$ in a descending order. Second, the highest values that are higher than $\gamma_i x_i(t)$ in the above ranked list are eliminated in order (from high to low) until r values from $\mathcal{N}_{i\ell}$ are eliminated; If there are fewer than r such values, all of them are deleted. The analogous deletion process is applied to the lower values. Third, denote by $\mathcal{C}_i(t)$ the set of values coming from $\mathcal{N}_{i\ell}$ that is higher than $\gamma_i x_i(t)$ in the resulting list. We further delete those values in the list that are higher than $\max\{\mathcal{C}_i(t) \cup \gamma_i x_i(t)\}$. The analogous process is applied to the lower values. Let $\mathcal{R}_i(t)$ be the set of nodes that have been eliminated in the above two steps. Fourth, the value of each $i \in N \cap V_\ell$ evolves following $\varphi_i^C(\cdot)$ in (1)

$$\dot{x}_i(t) = \mathrm{sgn}(\gamma_i) \sum_{j\in(\mathcal{N}_i\cup\{i\})\backslash\mathcal{R}_i(t)} a_{ij} f_{ij}(\gamma_j x_j^i(t), \gamma_i x_i(t)), \tag{8}$$

where $\mathrm{sgn}(\cdot)$ represents the signum function and the function $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$ possesses the same three conditions as in Section II.B.

The following corollary follows from the same line of argument as in Theorem 1 by re-defining $\Theta_{M\ell}(t) := \max_{i\in N\cap V_\ell} \gamma_i x_i(t)$ and $\Theta_{m\ell}(t) := \min_{i\in N\cap V_\ell} \gamma_i x_i(t)$ for $t \ge 0$.

Corollary 1. *Consider the continuous-time multi-agent system (1) under digraph $G = (V, E, A)$, where each normal node adopts the scaled cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, resilient scaled cluster consensus with respect to $(\gamma_1, \cdots, \gamma_n)$ is achieved if G_ℓ is $2r + 1$-robust for $1 \le \ell \le L$ and Assumption 1 holds.*

A heterogeneous version of the resilient scaled cluster consensus in the spirit of Theorem 2 can also be derived. We omit here due to the space of limitation.

B. Discrete-time system

For discrete-time system (2), we have a slightly stronger result in the same spirit of Lemma 2. Recall that $\Theta_{M\ell}(t) := \max_{i\in N\cap V_\ell} x_i(t)$ and $\Theta_{m\ell}(t) := \min_{i\in N\cap V_\ell} x_i(t)$ for $t \ge 0$ are the highest and lowest values of normal nodes, respectively, in the cluster G_ℓ $(1 \le \ell \le L)$.

Lemma 3. *Consider the discrete-time multi-agent system (2) under the digraph $G = (V, E)$, in which normal nodes adopt the cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, for any $i \in N \cap V_\ell$ $(1 \le \ell \le L)$, we have $x_i(t + 1) \in [\Theta_{m\ell}(t), \Theta_{M\ell}(t)]$ for all $t \ge 0$.*

Proof. Fix ℓ and $i \in N\cap V_\ell$. We will show $x_i(t+1) \le \Theta_{M\ell}(t)$ for $t \ge 0$. The lower bound can be shown likewise. It follows from (5) that $x_i(t + 1)$ is a convex combination of values $\{x_j^i(t)\}_{j\in(\mathcal{N}_i\cup\{i\})\backslash\mathcal{R}_i(t)}$. For $j \in (\mathcal{N}_{i\ell} \cup \{i\})\backslash\mathcal{R}_i(t)$, $x_j^i(t) \le \Theta_{M\ell}(t)$ since r nodes in $\mathcal{N}_{i\ell}$ are deleted in step 2 of the cluster censoring strategy and the network is r-locally bounded. For $j \in (\mathcal{N}_{i\ell'} \cup \{i\})\backslash\mathcal{R}_i(t)$ with $\ell' \ne \ell$, $x_j^i(t) \le \Theta_{M\ell}(t)$ still holds for that the removal in step 3 ensures either $x_j^i(t) \le x_i(t)$ or $x_j^i(t) \le x_k(t)$ for some $k \in N \cap \mathcal{N}_{i\ell}$. Hence, $x_i(t + 1) \le \Theta_{M\ell}(t)$. □

As shown in the examples in Section III.A, the two-round censoring strategy is essential for reaching resilient cluster consensus in discrete-time system. The sufficiency is summarized in the following result.

Theorem 3. *Consider the discrete-time multi-agent system (2) under digraph $G = (V, E, A)$, where each normal node adopts the cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, resilient cluster consensus is achieved if G_ℓ is $2r + 1$-robust for $1 \le \ell \le L$.*

Proof. Fix $\ell \in \{1, 2, \cdots, L\}$. Thanks to Lemma 2, we set $c_{M\ell} := \lim_{t\to\infty} \Theta_{M\ell}(t) \ge c_{m\ell} := \lim_{t\to\infty} \Theta_{m\ell}(t)$ since both limits exist. Similarly as the continuous-time scenario, we aim to show $c_{M\ell} = c_{m\ell}$ by the method of contradiction.

Suppose on the contrary that $c_{M\ell} > c_{m\ell}$. Select $\varepsilon_0 > 0$ satisfying $c_{M\ell} - \varepsilon_0 > c_{m\ell} + \varepsilon_0$. Given $\varepsilon_s > 0$, we introduce $H_{M\ell}(t, \varepsilon_s) = \{i \in N \cap V_\ell : x_i(t) > c_{M\ell} - \varepsilon_s\}$ and $H_{m\ell}(t, \varepsilon_s) = \{i \in N \cap V_\ell : x_i(t) < c_{m\ell} + \varepsilon_s\}$ for $t > 0$ similarly as in [7]. Clearly, $H_{M\ell}(t, \varepsilon_0) \cap H_{m\ell}(t, \varepsilon_0) = \emptyset$. Choose $\varepsilon < \frac{w^{n_\ell}\varepsilon_0}{(1-w^{n_\ell})}$ and $0 < \varepsilon < \varepsilon_0$, where $0 < w < 1$ is given in (iiD). Take $t_\varepsilon > 0$ be the time step satisfying for $t \ge t_\varepsilon$, $\Theta_{M\ell}(t) < c_{M\ell} + \varepsilon$ and $\Theta_{m\ell}(t) > c_{m\ell} - \varepsilon$.

Signify by $G_\ell^N = (N\cap V_\ell, E_\ell^N)$ the subgraph of G_ℓ induced by the normal nodes in N, where E_ℓ^N is composed of directed edges between any normal nodes in V_ℓ. We know that G_ℓ^N is $r+1$-robust in r-locally bounded model as G_ℓ is $2r+1$-robust by assumption. Considering $H_{M\ell}(t, \varepsilon_0)$ and $H_{m\ell}(t, \varepsilon_0)$, we know that there must exist a node in $H_{M\ell}(t_\varepsilon, \varepsilon_0)$ or in $H_{m\ell}(t_\varepsilon, \varepsilon_0)$ that has more than or equal to $r + 1$ normal neighbors not in its own set (but still within G_ℓ). Firstly, assume $i \in H_{M\ell}(t_\varepsilon, \varepsilon_0)$ has more than or equal to $r + 1$ normal neighbors in $G_\ell\backslash H_{M\ell}(t_\varepsilon, \varepsilon_0)$. By definition, the states of these neighbors can no exceed $c_{M\ell} - \varepsilon_0$. Since one of these states will be adopted by i according to the cluster censoring strategy, we derive

$$\begin{aligned} x_i(t_\varepsilon + 1) &\le (1 - w)\Theta_{M\ell}(t_\varepsilon) + w(c_{M\ell} - \varepsilon_0) \\ &\le c_{M\ell} - w\varepsilon_0 + (1 - w)\varepsilon, \end{aligned} \tag{9}$$

noticing that $\Theta_{M\ell}(t_\varepsilon) \le c_{M\ell} + \varepsilon$, normal node's values are expressed in terms of convex combinations with coefficients bounded by w, and that maximum value adopted by i at t_ε is less than or equal to $\Theta_{M\ell}(t_\varepsilon)$ by applying our cluster censoring strategy with parameter r. Noting that any normal node in $G_\ell\backslash H_{M\ell}(t_\varepsilon, \varepsilon_0)$ adopts its own value which cannot exceeding $c_{M\ell} - \varepsilon_0$, we see that the above estimate also holds for such nodes. In an analogous manner, $i \in H_{m\ell}(t_\varepsilon, \varepsilon_0)$ has more than or equal to $r + 1$ normal neighbors in $G_\ell\backslash H_{m\ell}(t_\varepsilon, \varepsilon_0)$, we have $x_i(t_\varepsilon + 1) \ge c_{m\ell} + w\varepsilon_0 - (1 - w)\varepsilon$, which also holds for normal nodes in $G_\ell\backslash H_{m\ell}(t_\varepsilon, \varepsilon_0)$.

We take $\varepsilon_1 = w\varepsilon_0 - (1 - w)\varepsilon \in (\varepsilon, \varepsilon_0)$. Recall that $H_{M\ell}(t_\varepsilon + 1, \varepsilon_1) \cap H_{m\ell}(t_\varepsilon + 1, \varepsilon_1) = \emptyset$ and we have either $|H_{M\ell}(t_\varepsilon + 1, \varepsilon_1)| < |H_{M\ell}(t_\varepsilon, \varepsilon_0)|$ or $|H_{m\ell}(t_\varepsilon + 1, \varepsilon_1)| < |H_{m\ell}(t_\varepsilon, \varepsilon_0)|$ is true. For $s \ge 1$, recursively defining $\varepsilon_s = w\varepsilon_{s-1} - (1 - w)\varepsilon$, we obtain $\varepsilon_s < \varepsilon_{s-1}$. The above discussion still holds valid at every time step $t_\varepsilon + s$ provided $H_{M\ell}(t_\varepsilon + s, \varepsilon_s)$ and $H_{m\ell}(t_\varepsilon + s, \varepsilon_s)$ are non-empty. As G_ℓ^N has at most n_ℓ normal nodes, there is some $T \le n_\ell$ such that either $H_{M\ell}(t_\varepsilon + T, \varepsilon_T)$ or $H_{m\ell}(t_\varepsilon + T, \varepsilon_T)$ becomes empty.

According to the definition of ε, we have $\varepsilon_T = w\varepsilon_{T-1} - (1-w)\varepsilon = w^T\varepsilon_0 - (1-w^T)\varepsilon \geq w^{n_\ell}\varepsilon_0 - (1-w^{n_\ell})\varepsilon > 0$. Consequently, normal nodes in G_ℓ at $t_\varepsilon + T$ possess values no more than $c_{M\ell} - \varepsilon_T < c_{M\ell}$ or possess values no less than $c_{m\ell} + \varepsilon_T > c_{m\ell}$. However, recalling that $c_{M\ell} := \lim_{t\to\infty}\Theta_{M\ell}(t)$ and $c_{m\ell} := \lim_{t\to\infty}\Theta_{m\ell}(t)$ by definition, we derive a contradiction. The proof is then complete. □

Akin to Theorem 2, we likewise have the following extension regarding heterogenous robustness of clusters.

Theorem 4. *Consider the discrete-time multi-agent system (2) under digraph $G = (V, E, A)$, where each normal node adopts the cluster censoring strategy with parameter $(r_1, r_2, \cdots, r_L)$. Under the $(r_1, r_2, \cdots, r_L)$-locally bounded Byzantine model, resilient cluster consensus is achieved if G_ℓ is $2r_\ell + 1$-robust for $1 \leq \ell \leq L$.*

Next, in terms of resilient scaled cluster consensus (i.e., Def. 5), we propose a two-round censoring process with parameter r for discrete-time multi-agent system (2) as follows. The first three steps are exactly the same as described in Section III.A for continuous-time system (1). In the final step, each $i \in N \cap V_\ell$ updates its value applying the following $\varphi_i^D(\cdot)$ in (2)

$$x_i(t+1) = \mathrm{sgn}(\gamma_i) \sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_i(t)} w_{ij}(t)\gamma_j x_j^i(t), \quad (10)$$

where the weights $w_{ij}(t)$ satisfies the same conditions (iD) and (iiD) in Section II.B, and retrofitted (iiiD') $\sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_i(t)} |\gamma_i| w_{ij}(t) = 1$.

By re-defining $\Theta_{M\ell}(t) := \max_{i\in N\cap V_\ell}\gamma_i x_i(t)$ and $\Theta_{m\ell}(t) := \min_{i\in N\cap V_\ell}\gamma_i x_i(t)$ for $t \geq 0$, we can similarly prove the following corollary.

Corollary 2. *Consider the discrete-time multi-agent system (2) under digraph $G = (V, E, A)$, where each normal node adopts the scaled cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, resilient scaled cluster consensus with respect to $(\gamma_1, \cdots, \gamma_n)$ is achieved if G_ℓ is $2r + 1$-robust for $1 \leq \ell \leq L$.*

A heterogeneous version of the resilient scaled cluster consensus akin to Theorem 4 can also be derived.

C. Switched system

For switched multi-agent system (3), we will see that the validity conditions for continuous-time and discrete-time systems delineated respectively in Lemma 2 and Lemma 3 remain valid essentially.

Lemma 4. *Consider the switched multi-agent system (3) under the digraph $G = (V, E)$, in which normal nodes adopt the cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, for any $i \in N \cap V_\ell$ ($1 \leq \ell \leq L$), we have $x_i(t) \in [\Theta_{m\ell}(0), \Theta_{M\ell}(0)]$ when (1) is activated at t, and $x_i(t+1) \in [\Theta_{m\ell}(t), \Theta_{M\ell}(t)]$ when (2) is activated on $[t, t+1]$.*

Proof. The same line of reasoning as in Lemma 2 and Lemma 3 can be applied in general. When the continuous-time subsystem is at work at time t, we obtain $x_i(t) \in [\Theta_{m\ell}(0), \Theta_{M\ell}(0)]$ by noting that $\Theta_{m\ell}(0)$ is increasing and $\Theta_{M\ell}(0)$ is decreasing whenever (2) is in place during $[0, t)$. When the discrete-time subsystem is at work during $[t, t+1]$, we obtain $x_i(t+1) \in [\Theta_{m\ell}(t), \Theta_{M\ell}(t)]$ exactly as in Lemma 3. □

Cluster consensus problems for switched multi-agent systems are known to be notoriously difficult due to the complicated and disparate algebraic conditions raised for continuous-time cluster consensus problems (e.g. [15], [18]) and discrete-time cluster consensus problems (e.g. [22], [23]). Interestingly, in the framework of resilient cluster consensus, we are able to mix both subsystems to derive succinct sufficient conditions that guarantee the cluster consensus under arbitrary switching. This is achieved by dividing the state evolution process into two classes, each of which is dominated by either continuous- or discrete-time dynamics.

Theorem 5. *Consider the switched multi-agent system (3) under digraph $G = (V, E, A)$, where each normal node adopts the cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, resilient cluster consensus is achieved under arbitrary switching if G_ℓ is $2r + 1$-robust for $1 \leq \ell \leq L$ and Assumption 1 holds.*

Proof. To mimic a general switching rule, consider a time sequence $0 \leq t_1 \leq \bar{t}_1 \leq t_2 \leq \bar{t}_2 \leq \cdots \leq t_k \leq \bar{t}_k \leq \cdots$, where the continuous-time subsystem (1) is at work during $(t_k, \bar{t}_k]$ and the discrete-time subsystem (2) is at work during $(\bar{t}_{k-1}, t_k]$. We will consider the following two cases: (I) there exists $k_1 \in \mathbb{N}$ and $\Delta > 0$ such that $\bar{t}_k - t_k \geq \Delta$ for all $k \geq k_1$; and (II) $\lim_{k\to\infty}\bar{t}_k - t_k = 0$.

Fix $\ell \in \{1, 2, \cdots, L\}$. In the case (I), fixing $t \geq t_{k_1}$ and following the proof of Theorem 1, we obtain $D^+\Gamma_\ell(t) = D^+\Theta_{M\ell}(t) - D^+\Theta_{m\ell}(t) \leq 0$ when $t \in (t_k, \bar{t}_k]$ for $k \in \mathbb{N}$. On the other hand, when $t \in (\bar{t}_{k-1}, t_k]$ for $k \in \mathbb{N}$, we have $D^+\Gamma_\ell(t) = D^+\Theta_{M\ell}(t) - D^+\Theta_{m\ell}(t) \leq 0$ in view of Lemma 4. Suppose that $\lim_{t\to\infty} D^+\Gamma_\ell(t) \neq 0$. As in Theorem 1, there exist constants $\varepsilon_0 > 0$, $\delta_0 > 0$, and a sequence of instants $\{s_l\}_{l\geq1}$ where the continuous-time subsystem (1) is at work satisfying $\lim_{l\to\infty} s_l = \infty$, $D^+\Gamma_\ell(s_l) \leq -2\varepsilon_0$ and $|s_{l+1} - s_l| > \delta_0$ for any $l \geq 1$. For any interval $I \subseteq (t_k, \bar{t}_k]$ for $k \geq k_1$ with $I \cap \{\tau_p\}_{p\geq1} = \emptyset$, we can produce the same contradiction as in Theorem 1. Thus, $D^+\Gamma_\ell(t) \to 0$ as $t \to \infty$. Following the same proof of Theorem 1, we see that the states of nodes in G_ℓ converge to a common limit. This concludes the case (I).

Next, we consider the case (II), where the discrete-time system (2) would govern the system evolution. Assume $c_{M\ell} > c_{m\ell}$. Recall that $x_i(t)$ is continuous for $i \in N \cap V_\ell$ when (1) is at work. Following the argument of Theorem 3, we can choose $\varepsilon < \frac{3w^{n_\ell}\varepsilon_0}{4(1-w^{n_\ell})}$ and $0 < \varepsilon < \varepsilon_0$, where $0 < w < 1$ is given in (iiD). By our assumption and continuity, there exists $k_0 \in \mathbb{N}$ satisfying $x_i(t+1) \in [\Theta_{m\ell}(t) - \varepsilon/3, \Theta_{M\ell}(t) + \varepsilon/3]$ for all $t \geq \bar{t}_{k_0}$ and $i \in N \cap V_\ell$, irrespective of the subsystems at work during $[t, t+1]$. Take $t_\varepsilon > \bar{t}_{k_0}$ be the time step satisfying for $t \geq t_\varepsilon$, $\Theta_{M\ell}(t) < c_{M\ell} + \varepsilon$ and $\Theta_{m\ell}(t) > c_{m\ell} - \varepsilon$.

Arguing analogously as in Theorem 3, we now replace the upper bound (9) by

$$\begin{aligned} x_i(t_\varepsilon + 1) &\leq (1-w)[\Theta_{M\ell}(t_\varepsilon) + \varepsilon/3] + w(c_{M\ell} - \varepsilon_0) \\ &\leq c_{M\ell} - w\varepsilon_0 + 4(1-w)\varepsilon/3. \end{aligned}$$

On the other hand, the lower bound becomes $x_i(t_\varepsilon + 1) \geq c_{m\ell} + w\varepsilon_0 - 4(1-w)\varepsilon/3$, which also holds for normal nodes in $G_\ell \backslash H_{m\ell}(t_\varepsilon, \varepsilon_0)$. For $s \geq 1$, by recursively defining $\varepsilon_s = w\varepsilon_{s-1} - 4(1-w)\varepsilon/3$, we are able to follow the proof of Theorem 3 and lead to a contradiction against the definitions of $c_{M\ell}$ and $c_{m\ell}$. This proves $c_{M\ell} = c_{m\ell}$ and concludes the theorem in the case of (II). □

The heterogenous version of resilient cluster consensus over switched multi-agent systems reads as follows.

Theorem 6. *Consider the switched multi-agent system (3) under digraph $G = (V, E, A)$, where each normal node adopts the cluster censoring strategy with parameter $(r_1, r_2, \cdots, r_L)$. Under the $(r_1, r_2, \cdots, r_L)$-locally bounded Byzantine model, resilient cluster consensus is achieved under arbitrary switching if G_ℓ is $2r_\ell + 1$-robust for $1 \leq \ell \leq L$ and Assumption 1 holds.*

By adopting the continuous-time and discrete-time scaled cluster censoring strategies for continuous-time and discrete-time subsystems in (3), respectively, we readily obtain the switched scaled cluster censoring strategy. The following corollary can be proved similarly following Theorem 5.

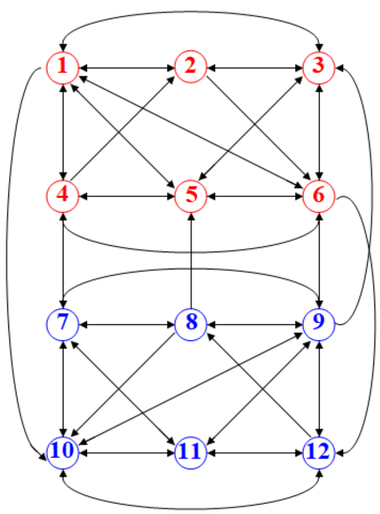

Fig. 3. A digraph G with two clusters G_1 over $V_1 = \{1, 2, \cdots, 6\}$ and G_2 over $V_2 = \{7, 8, \cdots, 12\}$ for Example 1. Both G_1 and G_2 are 3-robust.

Corollary 3. *Consider the switched multi-agent system (3) under digraph $G = (V, E, A)$, where each normal node adopts the switched scaled cluster censoring strategy with parameter r. Under the r-locally bounded Byzantine model, resilient scaled cluster consensus with respect to $(\gamma_1, \cdots, \gamma_n)$ is achieved if G_ℓ is $2r + 1$-robust for $1 \leq \ell \leq L$ and Assumption 1 holds.*

A heterogeneous version of the resilient switched scaled cluster consensus similar to Theorem 6 can also be derived.

Remark 6. In all three classes of systems considered above, nodes potentially have their own dynamics are viewed as Byzantine and hence are potentially to be overcome by the proposed censoring strategies. However, it is worth mentioning that in some practical industrial applications, formation tracking control is of importance; see e.g. [19], [33], [34]. To accommodate leaders in the network, further mechanism to retain them should be adopted in addition to W-MSR. A possible solution can be the introduction of trusted nodes, which do not go through the filtering.

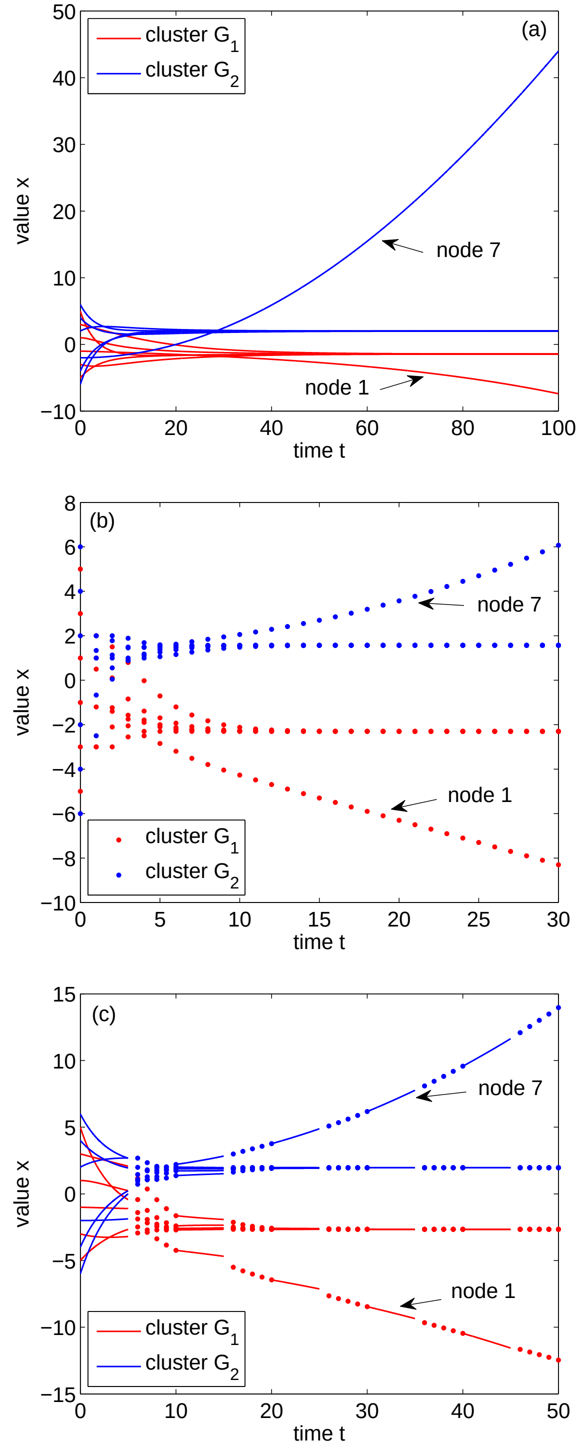

Fig. 4. Resilient cluster consensus over digraph G of Example 1 in the presence of Byzantine nodes $1 \in G_1$ and $7 \in G_2$ for (a) continuous-time multi-agent system (1); (b) discrete-time multi-agent system (2); and (c) switched multi-agent system (3).

IV. Numerical simulations

Example 1. Consider a digraph $G = (V, E, A)$ with $V = \{1, 2, \cdots, 12\}$ and the adjacency matrix A being a binary matrix; see Fig. 3. G_1 and G_2 are two interconnected clusters, which are both 3-robust. Let the Byzantine node set

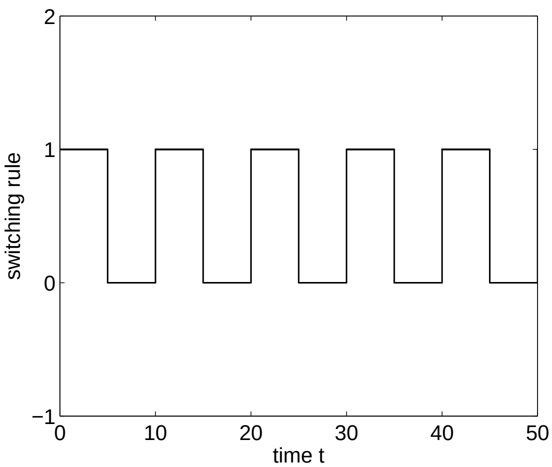

Fig. 5. The switching rule for the switched multi-agent system (3).

$B = \{1, 7\}$ and normal node set $N = V \backslash B$. The initial state configuration is taken as $x_1(0) = -1$, $x_2(0) = 1$, $x_3(0) = 3$, $x_4(0) = -3$, $x_5(0) = 5$, $x_6(0) = -5$, $x_7(0) = -2$, $x_8(0) = 2$, $x_9(0) = 4$, $x_{10}(0) = -4$, $x_{11}(0) = 6$, $x_{12}(0) = -6$.

For continuous-time multi-agent system (1), we take $f_{ij}(x, y) = (x-y)/10$. The Byzantine nodes 1 and 7 have their own dynamics $\dot{x}_1(t) = x_1(t)/50$ and $\dot{x}_7(t) = \sin(t/100)$. Here, node 1 follows a linear time-invariant system while node 7 a non-linear non-autonomous system. By taking $r = 1$, Theorem 1 indicates that resilient cluster consensus can be achieved by using the proposed cluster censoring strategy with parameter 1. The result shown in Fig. 4(a) agrees well with our theoretical prediction.

For discrete-time multi-agent system (2), the weights are taken as $w_{ij}(t) = a_{ij} \cdot \big(\sum_{j \in (\mathcal{N}_i \cup \{i\}) \backslash \mathcal{R}_i(t)} a_{ij}\big)^{-1}$. The Byzantine nodes 1 and 7 follows their own dynamics $x_1(t+1) = -t/5 + (x_2(t) + x_4(t))/2$ and $x_7(t+1) = t^2/200 + (x_8(t) + x_9(t) + x_{10}(t))/3$. By invoking the cluster censoring strategy with parameter 1, cluster consensus is observed in Fig. 4(b). This is in line with the observation in Theorem 3.

For switched-time multi-agent system (3) with the switching rule shown in Fig. 5, we observe from Fig. 4(c) that the resilient cluster consensus has been reached in the presence of Byzantine nodes 1 and 7 obeying the dynamics (both continuous- and discrete-time, respectively) described above. This again agrees with the theoretical result of Theorem 5.

It is worth noting that both Byzantine nodes 1 and 7 in digraph G are able to influence nodes outside their own cluster, and normal node 4 is affected by both Byzantine nodes. This implies a highly intricate network topology, where resilient global consensus cannot be guaranteed for any one of the three multi-agent systems by using previous fault-tolerant algorithms (e.g. [12]).

Example 2. In this example, we consider a digraph $G = (V, E, A)$ with $V = \{1, 2, \cdots, 18\}$ and the adjacency matrix A being a binary matrix; see Fig. 6. Note that G only contains directed edges. It is direct to check that G_1 and G_2 are two 3-robust clusters. Let the Byzantine node set $B = \{1, 18\}$ and normal node set $N = V \backslash B$. The initial state configuration is taken as $x_1(0) = -1$, $x_2(0) = -3$, $x_3(0) = 4.5$, $x_4(0) = 1$, $x_5(0) = -5$, $x_6(0) = 5$, $x_7(0) = 3.5$, $x_8(0) = 6$, $x_9(0) = -6$, $x_{10}(0) = -2$, $x_{11}(0) = 3$, $x_{12}(0) = 2$, $x_{13}(0) = -7$, $x_{14}(0) = 7$, $x_{15}(0) = -4$, $x_{16}(0) = -6.5$, $x_{17}(0) = 0$, $x_{18}(0) = -2.5$.

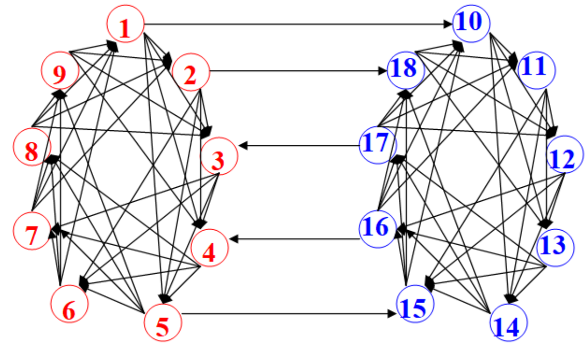

Fig. 6. A digraph G with two clusters G_1 over $V_1 = \{1, 2, \cdots, 9\}$ and G_2 over $V_2 = \{10, 11, \cdots, 18\}$ for Example 2. Both G_1 and G_2 are 3-robust.

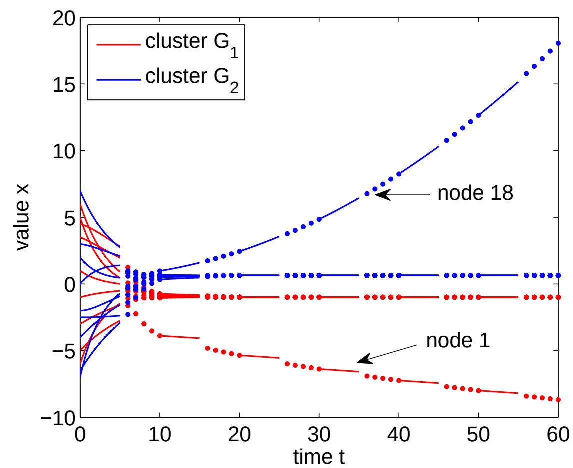

Fig. 7. Resilient cluster consensus over digraph G of Example 2 in the presence of Byzantine nodes $1 \in G_1$ and $18 \in G_2$ for switched multi-agent system (3).

We choose $f_{ij}(x, y) = (x-y)/10$ in the continuous-time subsystem (1) and choose weights $w_{ij}(t) = a_{ij} \cdot \big(\sum_{j \in (\mathcal{N}_i \cup \{i\}) \backslash \mathcal{R}_i(t)} a_{ij}\big)^{-1}$ in the discrete-time subsystem (2). The same switching law as described in Fig. 5 is adopted here. The Byzantine node 1 in G_1 has its own dynamics given by $\dot{x}_1(t) = (x_6(t) + x_7(t))/50$ and $x_1(t+1) = -\sqrt{t} + (x_7(t) + x_8(t))/2$ for continuous-time and discrete-time subsystems, respectively. The Byzantine node 18 in G_2 follows $\dot{x}_{18}(t) = \sin(t/100)$ and $x_{18}(t+1) = t^2/200 + (x_2(t) + x_{16}(t) + x_{17}(t))/3$ for continuous-time and discrete-time subsystems, respectively. We observe the resilient cluster consensus in Fig. 7 by using our cluster censoring strategy with parameter $r = 1$ tolerating the manipulation of nodes 1 and 18 as one would expect.

Example 3. In this example, we consider a researcher collaboration network in the interdisciplinary institute in Santa Fe [36], [37]. Fig. 8 shows a subnetwork consisting 9 collaborators working in two different fields: those work in agent-based

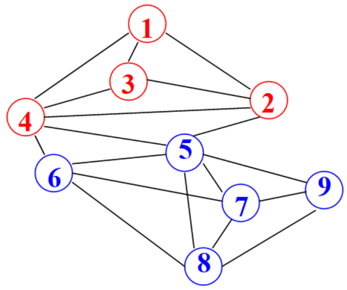

Fig. 8. Scientist collaboration network G with two clusters G_1 over $V_1 = \{1, \cdots, 4\}$ and G_2 over $V_2 = \{5, \cdots, 9\}$ for Example 3. G_1 is 1-robust, G_2 is 3-robust, and $G\backslash\{6\}$ is 2-robust.

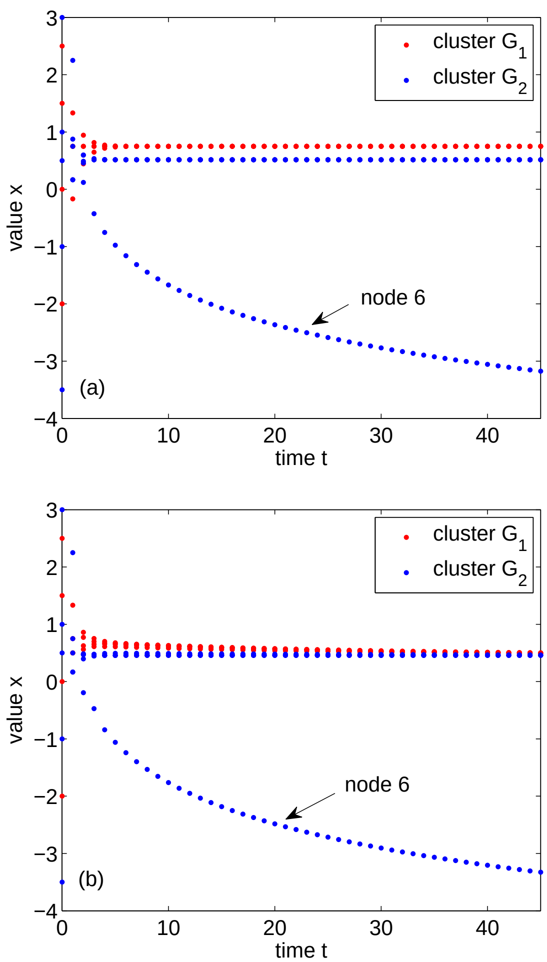

Fig. 9. (a) Resilient cluster consensus over digraph G of Example 3 in the presence of Byzantine node $6 \in G_2$ for discrete-time multi-agent system (2). (b) Resilient global consensus over the entire G applying W-MSR in [38].

models form $V_1 = \{1, \cdots, 4\}$ and those work in mathematical ecology form $V_2 = \{5, \cdots, 9\}$. It is direct to check that G_1 is 1-robust and G_2 is 3-robust. If we set the node 6 as the Byzantine node, the network formed by the normal nodes is $(2, 2)$-robust according to [7]. The initial states are set as $x_1(0) = 2.5$, $x_2(0) = -2$, $x_3(0) = 0$, $x_4(0) = 1.5$, $x_5(0) = 3$, $x_6(0) = -3.5$, $x_7(0) = -1$, $x_8(0) = 0.5$, and $x_9(0) = 1$.

We choose weights $w_{ij}(t) = a_{ij} \cdot \left(\sum_{j\in(\mathcal{N}_i\cup\{i\})\backslash\mathcal{R}_i(t)} a_{ij}\right)^{-1}$ in the discrete-time subsystem (2). The Byzantine node 6 in G_2 is assumed to have its own dynamics given by $x_6(t+1) = -\ln(t) + (x_4(t) + x_5(t))/2$. In Fig. 9(a), we displayed the state evolution for all agents. As predicted in Theorem 3, resilient cluster consensus is reached fairly quickly at $t < 10$. As a comparison, in Fig. 9(b) we adopted the resilient consensus protocol in [38] and the global consensus is observed around $t = 40$. This demonstrates the usefulness of our resilient cluster consensus strategies allowing effectively for different consistent values among clusters, which are not available previously.

V. Conclusion

In this paper, we have considered resilient cluster consensus of three archetypal classes of multi-agent systems over digraphs, including discrete-time, continuous-time and switched multi-agent systems. Resilient cluster censoring strategies are designed to guarantee cluster consensus in a purely distributed manner against locally bounded Byzantine nodes. Explicit robustness conditions on the network topology are proposed to ensure cluster consensus, where no complicated eigenvalue conditions or inter-cluster balance conditions are involved. Furthermore, the results are generalized to accommodate heterogeneous cluster robustness as well as resilient scaled cluster consensus problems. In the future, it would be interesting to adjust the algorithms to accommodate practical applications with more demanding requirements such as the cooperative control of gantry crane systems [39] and micro aerial vehicle control problems [40].

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