Fault Reconstruction and Resillient Control for Discrete-time Stochastic Systems with Brownian Perturbations

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Abstract—In this paper, an integrated resilient control technique is proposed for discrete-time stochastic Brownian systems with simultaneous unknown inputs and faults. Prior to previous work, the stochastic Brownian systems under consideration is quite general, where stochastic perturbations exist in states, control inputs, uncertainties and faults. Moreover, the unknown inputs concerned cannot be fully decoupled. Integrated observer by employing augmented approach, decomposition observer, and optimization algorithms is developed to achieve simultaneous state and fault estimates. Then, fault reconstruction-based signal compensation is formulated to eliminate the influences of actuator faults and sensor faults, while a state estimator-based controller is constructed to enhance the stability and robustness of the closed-loop dynamic system. The integrated resilient control technique can ensure the system has reliable output even under faults. The systems under investigation can be linear or Lipschitz nonlinear, and the design procedures are provided respectively. Finally, the proposed resilient control techniques are validated via an electromechanical servosystem and a flight control system.

Keywords—General Brownian systems, discrete-time systems, state and fault estimation, integrated faut tolerant control

1. Introduction

Reliability and safety play significant roles in engineering fields. Fault diagnosis algorithms can monitor the health condition of dynamic systems， and generate indication and reacting time to prevent serious situations due to faults. Fault tolerant control (resilient control) aims at reducing faults from the system, providing reliable system output even when the system is under faulty scenarios. Therefore, fruitful results about fault diagnosis and resilient control have been recorded for engineering system (see [1-4]).

One of the fault diagnosis technique, which is known as fault estimation, has been paid much attention, due to its capability to provide rich information of faults (such as magnitude, time, shape, type, etc.). Advanced fault estimation approaches, such as reduced-order observer [5], k-step induction observer [6], Augmented observer [7, 8] have been investigated recently. Augmented observer can achieve simultaneous fault and state estimations, which means state estimates can be achieved as by-products when we do fault estimation. The estimation of fault provides online information to design active fault tolerant control, and the estimation of system state meets the demand of available state value to establish state-feedback control, which take advantages over output-feedback control. Based on the estimation of faults, signal compensation is an advanced fault tolerant control technique, which can eliminate the influences of faults without replacing the pre-existing controller of the system (e.g. [9-11]).

The effectiveness of observer/estimator-based control depends on the accuracy of fault and/or state estimation. Unknown inputs, including systems uncertainties, modeling errors, external disturbances are unavoidable in practical process. Unlike faults which break down the system, unknown inputs are acceptable by real dynamic systems. Therefore, robustness against unknown inputs is essential for fault and state estimation. Unknown inputs can be eliminated by utilizing decomposition techniques, such as unknown input observer (UIO) [12], differential geometric approach [13], or optimization algorithms [14]. A jointly unknown input observer-based fault & state estimator has been developed in our previous work [15] by combining both decomposition and optimization schemes to mitigate the influence of partially decoupled unknown inputs.

Many practical systems in engineering, physics, biology etc. are subjected with stochastic perturbations. These stochastic perturbations can exist in states, inputs, disturbances and faults, making the system trajectory non-deterministic. Stochastic systems can capture the random natures of a real process, hence has been a hot research topic recently [16-21]. Due to Brownian perturbations, fault reconstruction and resilient controller cannot be designed separately in stochastic Brownian systems. Integrated fault tolerant control scheme has been developed for continuous-time stochastic Brownian systems for linear, Lipschitz nonlinear, and quadratic-inner-bounded nonlinear cases in [22], for Takagi-Sugeno nonlinear case in recent work [23], and for uncertain systems in [24]. In traditional investigation of Brownian system, only state and input diffusion have been under consideration. However, diffusion of disturbances and faults can also exist in real systems. Therefore, research of general form of Brownian system, where stochastic perturbations exist in states, inputs, disturbances and faults, is in stringent requirement. At the end of [23], general form of Brownian distributions is considered, that is, the Brownian perturbations not only exist in states, but also exist in inputs, uncertainties, and faults. With the development of digital control, discrete-time stochastic system has been a crucial research topic. Sampling process makes the discrete-time dynamic different from continuous-time dynamic, hence the technique developed for continuous-time stochastic system is not valid in discrete-time stochastic system. The investigation of discrete-time stochastic Brownian systems can be found in [16-17, 25, 26], where Brownian perturbations in both state and control input were considered in [18]. So far, fault estimation and fault tolerant control of discrete-time Brownian system is not fully studied. According to the authors' survey, no result on fault reconstruction and resilient control of discrete-time stochastic systems in presence of unknown inputs and general form of Brownian perturbations has been recorded.

This paper is then motivated to develop integrated resilient control of discrete-time stochastic Brownian systems, which aims to obtain robust estimation of actuator and sensor faults, mitigate the influence of faults, and enhance the stability of the overall closed-loop system. Specifically, the stochastic system under investigation is in presence of unknown inputs that cannot be fully decoupled, which is general but bring challenges for the design of robust estimator. The contribution of the paper includes：1) This work is a start to investigate fault tolerant control design for general discrete-time stochastic Brownian systems, where stochastic perturbations exist in states, inputs, faults, and uncertainties. 2) By using an optimized decomposition observer, the mean estimates for fault and state are generated simultaneously, where the influence of partially decoupled unknown inputs can be reduced. 3) The integrated fault tolerant controller is composed of a robust fault estimator-based signal compensator, and a robust state estimator-based feedback controller. As a result, the actuator and sensor faults can be mitigated from system outputs; the stochastic system and estimation error trajectories are exponentially bounded in mean square and satisfy desired robustness performance.

The paper is organized as follows: the construction of optimized decoupling observer-based fault & state estimation and estimator-based fault tolerant controller is introduced for linear system in Section 2, while for Lipschitz nonlinear system in Section 3; Section 4 demonstrates simulation work on an electromechanical servosystem and a flight control system; the paper is ended with a conclusion in section 5.

Notation. The superscript “T” represents the transpose of matrices or vectors. $\mathcal{R}^n$ and $\mathcal{R}^{n\times m}$ stand for the n-dimentional Euclidean space and the set of $n \times m$ real matrices, respectively. $X < 0$ indicates the symmetric matrix X is negative definite, while the notation $X > Y$ means that $X - Y$ is positive definite. I_n denotes the identity matrix with the dimension of $n \times n$, while 0 is a scalar zero or a zero matrix with appropriate zero entries. $\|A\| = \sqrt{\lambda_{max}(A^T A)}$; $\|x\|$ refers to the Euclidean norm of x, and

$\|x\|_2 = \left(\sum_{k=0}^{\infty} x^T(k)\, x(k)\right)^{1/2}$. $\mathbb{E}[x]$ is expectation value of x, and $\mathbb{E}[x|y]$ is the expectation value of x conditional on y. For brevity, $\begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix} \overset{\Delta}{\Leftrightarrow} \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix}$.

2. Fault reconstruction and fault tolerant control for linear stochastic system

The linear discrete-time stochastic system under consideration is influenced by actuator and/or sensor faults, unknown inputs and Brownian motions. The system dynamic is described as follows:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_f f(k) + B_d d(k) \\ \quad +\left[Wx(k) + Gu(k) + G_f f(k) + Md_p(k)\right]\omega(k) \\ y(k) = Cx(k) + D_f f(k) \end{cases} \tag{1}$$

where $x(k) \in \mathcal{R}^n$ is system state; $u(k) \in \mathcal{R}^m$ is control input; $y(k) \in \mathcal{R}^p$ represents measurement output; $d(k) \in \mathcal{R}^{l_d}$ includes $\mathcal{L}_2$ bounded deterministic unknown input; $f(k) \in \mathcal{R}^{l_f}$ is composed of the means of actuator and/or sensor faults; $d_p(k) \in \mathcal{R}^{l_{d_p}}$ represents unknown input uncertainty vector in the multiplicative term of the stochastic Brownian noise; the discrete-time instant k is a simplified representation of kT_S, where T_S is the sampling period. $\omega(k)$ represents Brownian motion defined on the probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathcal{P}\right)$. $\omega(k)$ satisfies : $\mathbb{E}[\omega(k)] = 0$, $\mathbb{E}[\omega^2(k)] = 1$, and $\mathbb{E}\left[\omega_i(k)\omega_j(k)\right] = 0$ $(i \neq j)$. A, B, C, B_d, B_f, W, G, G_f, M and D_f are known constant coefficient matrices with appropriate dimensions.

Remark 1. Through defining coefficient B_f and D_f accordingly, $f(k)$ represents different scenario of fault. Specifically, when $f(t) = f_a(t)$, where $f_a(t)$ is actuator fault, then $B_f = B, D_f = 0^{p \times l_f}$; when $f(t) = f_s(t)$, where $f_s(t)$ is sensor fault, then $B_f = 0^{n \times l_f}, D_f = I_p$; when $f(t) = [f_a^T(t) \quad f_s^T(t)]^T$, then $B_f = [B_{fa} \quad B_{fs}], D_f = [D_{fa} \quad D_{fs}]$, where B_{fa} and D_{fa} represent the coefficients of actuator fault, B_{fs} and D_{fs} are coefficients of sensor fault [23].

Without loss of generality, the sampling interval T_S of the considered discrete-time stochastic system is sufficiently small, such that the faults do not vary too much between two consecutive sample instances. Then, we presume $\forall\ k$,

$$f(k+1) = f(k) + \Delta f(k) \tag{2}$$

where $\Delta f(k)$ is supposed to be $\mathcal{L}_2$ bounded.

We assume that $d(k) = [d_1^T(k) \;\; d_2^T(k)]^T$, where $d_1(k) \in \mathcal{R}^{l_{d1}}$ can be decoupled and $d_2(k) \in \mathcal{R}^{l_{d2}}$ cannot. Accordingly, $B_d = [B_{d1} \;\; B_{d2}]$. We also assume that B_{d1} is of full column rank.

In order to generate simultaneous estimates of state and faults, an augmented state vector can be constructed as $\bar{x}(k) = [x^T(k) \quad f^T(k)]^T \in \mathcal{R}^{\bar{n}}$, where $\bar{n} = n + l_f$. And define $d_f(k) = [d^T(k) \quad \Delta f^T(k)]^T$. Then we can construct the following augmented system for model (1)

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{B}_{df}d_f(k) + [\bar{W}x(k) \\ \qquad +\bar{G}u(k) + \bar{G}_f f(k) + \bar{M}d_p(k)]\omega(k) \\ y(k) = \bar{C}\bar{x}(k) \end{cases} \tag{3}$$

where

$\bar{A} = \begin{bmatrix} A & B_f \\ 0_{l_f \times n} & I_{l_f} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times\bar{n}}$, $\bar{B} = \begin{bmatrix} B \\ 0_{l_f \times m} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times m}$, $\bar{B}_{df} = \begin{bmatrix} B_d & 0_{n \times l_f} \\ 0_{l_f \times l_d} & I_{l_f} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times(l_d + l_f)}$, $\bar{W} = \begin{bmatrix} W \\ 0_{l_f \times n} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times n}$, $\bar{G} = \begin{bmatrix} G \\ 0_{l_f \times m} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times m}$, $\bar{G}_f = \begin{bmatrix} G_f \\ 0_{l_f \times l_f} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times l_f}$, $\bar{M} = \begin{bmatrix} M \\ 0_{l_f \times l_{d_p}} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times l_{d_p}}$, $\bar{C} = [C \quad D_f] \in \mathcal{R}^{p\times\bar{n}}$.

Then, the simultaneous mean estimations of fault and state can be obtained through establishing an estimator for model (3) to estimate the mean of augmented state vector $\bar{x}(k)$. It can be noted that the existence of unknown inputs makes adverse effect on the accuracy of estimation. In order to reduce influences of unknown inputs, the following UIO is employed:

$$\begin{cases} \bar{z}(k+1) = R\bar{z}(k) + T\bar{B}u(k) + (L_1 + L_2)y(k) \\ \hat{\bar{x}}(k) = \bar{z}(k) + Hy(k) \end{cases} \tag{4}$$

where $\bar{z}(k) \in \mathcal{R}^{\bar{n}}$ represents a middle variable; $\hat{\bar{x}}(k) \in \mathcal{R}^{\bar{n}}$ represents the estimate of $\bar{x}(k)$, while $R \in \mathcal{R}^{\bar{n}\times\bar{n}}$, $L_1 \in \mathcal{R}^{\bar{n}\times p}$, $L_2 \in \mathcal{R}^{\bar{n}\times p}$, $T \in \mathcal{R}^{\bar{n}\times m}$ and $H \in \mathcal{R}^{\bar{n}\times p}$ are unknown matrices to be determined.

Define an estimation error vector as

$$e(k) = \bar{x}(k) - \hat{\bar{x}}(k) \tag{5}$$

Then by using (3) to (5), we can calculate the error dynamic as

$$e(k+1) = [(I_{\bar{n}} - H\bar{C})\bar{A} - L_1\bar{C}]\bar{x}(k) - R\hat{\bar{x}}(k) + [(I_{\bar{n}} - H\bar{C}) - T]\bar{B}u(k) + (I_{\bar{n}} - H\bar{C})\bar{B}_{d1}d_1(k)$$

$$+(I_{\bar{n}} - H\bar{C})\bar{B}_{d2f}d_{2f}(k) + (I_{\bar{n}} - H\bar{C})\bar{W}x(k)\omega(k) + (I_{\bar{n}} - H\bar{C})\bar{G}u(k)\omega(k)$$

$$+(I_{\bar{n}} - H\bar{C})\bar{G}_f f(k)\omega(k) + (I_{\bar{n}} - H\bar{C})\bar{M}d_p(k)\omega(k) + (RH - L_2)y(k) \tag{6}$$

where $[\bar{B}_{d1}\ \bar{B}_{d2f}] = \bar{B}_{df}$, $\bar{B}_{d1} = \begin{bmatrix} B_{d1} \\ 0_{l_f \times l_{d1}} \end{bmatrix} \in \mathcal{R}^{\bar{n} \times l_{d1}}$, $\bar{B}_{d2f} = \begin{bmatrix} B_{d2} & 0_{n \times l_f} \\ 0_{l_f \times l_{d2}} & I_{l_f} \end{bmatrix} \in \mathcal{R}^{\bar{n} \times (l_{d2} + l_f)}$ and $d_{2f}(k) = [d_2^T(k) \quad \Delta f^T(k)]^T$.

If the observer gains satisfy the following equations:

$$(I_{\bar{n}} - H\bar{C})\bar{B}_{d1} = 0 \tag{7}$$

$$R = \bar{A} - H\bar{C}\bar{A} - L_1\bar{C} \tag{8}$$

$$T = I_{\bar{n}} - H\bar{C} \tag{9}$$

$$L_2 = RH \tag{10}$$

the estimation error dynamic (6) can be simplified to

$$e(k+1) = Re(k) + T\bar{B}_{d2f}d_{2f}(k) + T\bar{W}x(k)\omega(k)$$

$$+T\bar{G}u(k)\omega(k) + T\bar{G}_f f(k)\omega(k) + T\bar{M}d_p(k)\omega(k) \tag{11}$$

Conditions (7) - (10) can be satisfied under the following assumptions:

Assumption 1. $\text{rank}(CB_{d1}) = \text{rank}(B_{d1})$;

Assumption 2. $\begin{bmatrix} A & B_f & B_{d1} \\ C & D_f & 0 \end{bmatrix}$ is of full column rank;

Assumption 3. $\text{rank}\begin{bmatrix} zI_n - A & B_{d1} \\ C & 0 \end{bmatrix} = n + l_{d1}$, for all z with $|z| \geq 1$;

Assumption 4. $\text{rank}\begin{bmatrix} B & B_f \\ G & G_f \end{bmatrix} = \text{rank}\begin{bmatrix} B \\ G \end{bmatrix}$.

Assumptions 1 through 4 are reasonable in fault reconstruction and fault tolerant control. *Assumptions* 1 to 3 are universal conditions of unknown input observer-based fault estimation [13, 15]. Specifically, *Assumption 1* ensures that equation (7) is solvable, in other words, unknown input d_1 can be decoupled. Moreover, a special solution of (7) is

$$H^* = \bar{B}_{d1}[(\bar{C}\bar{B}_{d1})^T(\bar{C}\bar{B}_{d1})]^{-1}(\bar{C}\bar{B}_{d1})^T \tag{12}$$

Assumption 2 and 3 indicate that the augmented system model is observable. *Assumption 4* is the existing condition of estimator-based signal compensator. To be precise, $\text{rank}[B \quad B_f] = \text{rank}\,[B]$ is the condition of signal compensation without stochastic perturbations on fault, e.g. [9-11]. Assumption 4 is a revised condition such that diffusion of fault can also be compensated.

It can be noticed that by selecting observer gain H to make (7) hold, $d_1(k)$ has been decoupled from error dynamic. However, the error dynamic is still subjected to $d_{2f}(k)$ and Brownian perturbations. As a result, the other observer gains should be determined to guarantee boundness of error trajectory and eliminate the influences of $d_{2f}(k)$ on estimation error $e(k)$.

Brownian perturbations make the estimation error dependent on the trajectories of system under investigation. In other words, it is hard to design fault estimation observer and fault tolerant controller separately. Therefore, the estimator gains cannot be determined at this stage, and a fault tolerant control scheme should be constructed first.

Remark 2: In addition to state and input, the Brownian perturbations on fault and disturbance are also under consideration. Therefore, the investigated system is more general but brings challenges to achieve convergence of the estimation error. Specifically, the objectives of fault tolerant control strategy include mitigation of both drift and distribution caused by faults, whereas only drift induced by fault have been reduced in existing work. Moreover, disturbance also leads to distribution of the estimation, namely, besides $d_{2f}(k)$, robustness against $d_p(k)$ should be achieved.

Simultaneous estimations of system state and fault are achieved via the reconstruction of the augmented state $\hat{\bar{x}}$, that is

$$\hat{x}(k) = J_1\hat{\bar{x}}(k) \tag{13}$$

and

$$\hat{f}(k) = J_2\hat{\bar{x}}(k) \tag{14}$$

where $\hat{x}(k)$ is the estimation of state $x(k)$, and $\hat{f}(k)$ is the estimation of fault $f(k)$. $J_1 = \begin{bmatrix} I_n & 0_{n\times l_f} \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0_{l_f\times n} & I_{l_f} \end{bmatrix}$.

Based on $\hat{\bar{x}}(k)$, the following fault tolerant controllers are constructed:

$$u = \bar{K}\hat{\bar{x}}(k) = [K \quad K_f]\begin{bmatrix} \hat{x}(k) \\ \hat{f}(k) \end{bmatrix} = K\hat{x}(k) + K_f\hat{f}(k) \tag{15}$$

and

$$y_c(k) = y(k) - D_f J_2\hat{\bar{x}}(k) \tag{16}$$

Then substituting controller (15) into system (1) and replacing $y(k)$ by $y_c(k)$, we have

$$\begin{aligned} x(k+1) &= (A+BK)x(k) - BKJ_1e(k) + \left(BK_f + B_f\right)f(k) - BK_fJ_2e(k) + B_dd(k) \\ &\quad +[(W+GK)x(k) - GKJ_1e(k) + \left(GK_f + G_f\right)f(k) - GK_fJ_2e(k) + Md_p(k)]\omega(k) \end{aligned} \tag{17}$$

and

$$y_c(k) = Cx(k) + D_fJ_2e(k) \tag{18}$$

As mentioned above, the fault tolerant controller will make influences on estimation error, then we substitute controller (15) into UIO (4), the error dynamic after feedback control can be reconstructed as

$$e(k+1) = Re(k) + T\bar{B}_{d2f}d_{2f}(k) + T(\bar{W} + \bar{G}K)x(k)\omega(k) - T\bar{G}KJ_1e(k)\omega(k)$$

$$+T\left(\bar{G}K_f+\bar{G}_f\right)f(k)\omega(k)\ -T\bar{G}K_fJ_2e(k)\omega(k)+T\bar{M}d_p(k)\omega(k) \tag{19}$$

According to *Assumption 4*, namely

$$\text{rank}\begin{bmatrix} B & B_f \\ G & G_f \end{bmatrix} = \text{rank}\begin{bmatrix} B \\ G \end{bmatrix} \tag{20}$$

and K_f is designed as

$$K_f = -\begin{bmatrix} B \\ G \end{bmatrix}^+ \begin{bmatrix} B_f \\ G_f \end{bmatrix} \tag{21}$$

we have

$$B_f + BK_f = 0 \tag{22}$$

and

$$G_f + GK_f = 0 \tag{23}$$

We can also derive that

$$\bar{G}_f + \bar{G} = \begin{bmatrix} G_f + GK_f \\ 0_{l_f\times l_f} \end{bmatrix} = 0 \tag{24}$$

Then system (17) and (19) can be reduced to

$$x(k+1) = (A+BK)x(k) + B_e e(k) + B_d d(k)$$

$$+[(W+GK)x(k) + G_e e(k) + Md_p(k)]\omega(k) \tag{25}$$

and

$$e(k+1) = Re(k) + T\bar{B}_{d2f}d_{2f}(k) + T(\bar{W}+\bar{G}K)x(k)\omega(k)$$

$$+T\bar{G}_e e(k)\omega(k) + T\bar{M}d_p(k)\omega(k) \tag{26}$$

where $B_e = -BKJ_1 - B_fJ_2$, $G_e = -GKJ_1 - G_fJ_2$, $\bar{G}_e = -\bar{G}KJ_1 - \bar{G}_fJ_2$.

Then, combining (16), (25) and (26), we can establish the following closed-loop system:

$$
\begin{cases}
x(k+1) = (A+BK)x(k) + B_d d(k) + B_e e(k) \\
\qquad\qquad + \left[(W+GK)x(k) + G_e e(k) + M d_p(k)\right]\omega(k) \\
e(k+1) = Re(k) + T\bar{B}_{d2f} d_{2f}(k) + T[(\bar{W}+\bar{G}K)x(k) \\
\qquad\qquad + \bar{G}_e e(k) + \bar{M} d_p(k)]\omega(k) \\
y_c(k) = Cx(k) + D_f J_2 e(k)
\end{cases} \tag{27}
$$

It can be noticed that system (27) is composed of the original system and estimation error system after signal compensation and state-estimator-based feedback control, moreover, a measurement compensator is implemented on output. Due to Brownian motions, it is difficult to separate system dynamic and error dynamic. In other words, the observer gains and controller gains are interacted, which bring challenges of the integrated fault tolerant control scheme. A typical way to reduce the complication is to select a controller gain K to make the modulus of all eigenvalues of $A+BK$ and $W+GK$ less than 1. Then the problem is to find proper observer gains to guarantee the boundness and robustness of the overall closed-loop system. Before designing observer gains, the following lemmas are introduced.

Lemma 1 [26]**:** Assume there is a stochastic process $V_n(\zeta_n)$, as well as real number $\underline{v}$, $\overline{v}$, $\mu > 0$ and $0 < \alpha \leq 1$, such that

$$
\underline{v}\|\zeta_n\|^2 \leq V_n(\zeta_n) \leq \overline{v}\|\zeta_n\|^2 \tag{28}
$$

and

$$
\mathbb{E}\{V_{n+1}(\zeta_{n+1})|\zeta_n\} - V_n(\zeta_n) \leq \mu - \alpha V_n(\zeta_n) \tag{29}
$$

are fulfilled for every solution of (27). Then the stochastic process is exponentially bounded in mean square.

Lemma 2 [27]. For any matrices $X \in \mathcal{R}^{s\times t}$, $Y \in \mathcal{R}^{t\times s}$, a time-varying matrix $F(t) \in \mathcal{R}^{t\times t}$ with $\|F(t)\| \leq 1$ and any scalar $\varepsilon > 0$, we have:

$$
XF(t)Y + Y^T F^T(t) X^T \leq \varepsilon^{-1} XX^T + \varepsilon Y^T Y. \tag{30}
$$

Lemma 3 (Schur complement**)** [28]. Let $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$ to be a symmetric matrix, then the LMI $S < 0$ is equivalent to $S_{22} < 0$ and $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$.

Then the following theorem is proposed to make system (27) exponentially bounded in mean square, and satisfy robust performance as follows:

$$\mathbb{E}(\|y_c\|_2^2) \leq \mathbb{E}(\gamma_d^2\|d\|_2^2) + \mathbb{E}(\gamma_{d2f}^2\left\|d_{2f}\right\|_2^2) + \mathbb{E}(\gamma_{d_p}^2\left\|d_p\right\|_2^2) \tag{31}$$

where γ_d, γ_{d2f} and γ_{d_p} are robustness performance indices.

Theorem 1. The closed-loop system (27) is exponentially bounded in mean square, and satisfies robust performance (31), if there exist positive definite matrices $P$and $\tilde{P}$, and matrix Y, such that

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 & (A+BK)^T\tilde{P}B_d & \Omega_{15} & 0 \\ * & \Omega_{22} & 0 & B_e^T\tilde{P}B_d & \Omega_{25} & \Omega_{26} \\ * & * & -\gamma_{d2f}^2 I_{l_{d2f}} & 0 & 0 & \Omega_{36} \\ * & * & * & B_d^T\tilde{P}B_d - \gamma_d^2 I_{l_d} & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 \\ * & * & * & * & * & -P \end{bmatrix} < 0 \tag{32}$$

where

$$\Omega_{11} = (A+BK)^T\tilde{P}(A+BK) + (W+GK)^T\tilde{P}(W+GK) + (\bar{W}+\bar{G}K)^T T^T PT(\bar{W}+\bar{G}K) + \alpha\tilde{P} - \tilde{P} + C^T C$$

$$\Omega_{12} = (A+BK)^T\tilde{P}B_e + (W+GK)^T\tilde{P}G_e + (\bar{W}+\bar{G}K)^T T^T PT\bar{G}_e + C^T D_f J_2$$

$$\Omega_{22} = B_e^T\tilde{P}B_e + G_e^T\tilde{P}G_e + \bar{G}_e^T T^T PT\bar{G}_e - P + \alpha P + J_2^T\, D_f^T D_f J_2$$

$$\Omega_{15} = (W+GK)^T\tilde{P}M + (\bar{W}+\bar{G}K)^T T^T PT\bar{M}$$

$$\Omega_{25} = G_e^T\tilde{P}M + \bar{G}_e^T T^T PT\bar{M}$$

$$\Omega_{55} = M^T\tilde{P}M + \bar{M}^T T^T PT\bar{M} - \gamma_{dp}^2 I_{l_{d_p}}$$

$$\Omega_{26} = \bar{A}^T T^T P - \bar{C}^T Y^T$$

$$\Omega_{36} = \bar{B}_{d2f}^T T^T P$$

$Y = PL_1$, α is a positive scalar that $0 < \alpha \leq 1$. γ_{d2f}, γ_d and γ_{d_p} are given robustness performance indices. Then we have $L_1 = P^{-1}Y$.

Proof. A Lyapunov function candidate in the following form can be established for error dynamic system (27):

$$\tilde{V}(\tilde{x}_e(k)) = V_1(x(k)) + V_2(e(k)) = x^T(k)\tilde{P}x(k) + e^T(k)Pe(k) \tag{33}$$

where $\tilde{P}$ and P are positive matrices. And let $\tilde{x}_e(k) = [x^T(k) \;\; e^T(k)]^T$

Then $\tilde{V}(\tilde{x}_e(k))$ satisfy (28), where

$$\underline{v} = \min\{\lambda_{min}(\tilde{P}), \lambda_{min}(P)\},$$

and

$$\overline{v} = \max\{\lambda_{max}(\tilde{P}), \lambda_{max}(P)\}$$

Then we move to validate that $\tilde{V}(\tilde{x}_e(k))$ satisfy (29). We can calculate that

$$\begin{aligned}
\Delta V_1(x(k)) &= \mathbb{E}[V_1(x(k+1))|x(k)] - V_1(x(k)) \\
&= x^T(k)(A+BK)^T\tilde{P}(A+BK)x(k) + 2x^T(k)(A+BK)^T\tilde{P}B_d d(k) \\
&\quad + d^T(k)B_d^T\tilde{P}B_d d(k) + 2e^T(k)B_e^T\tilde{P}B_d d(k) + 2x^T(k)(A+BK)^T\tilde{P}B_e e(k) \\
&\quad + e^T(k)B_e^T\tilde{P}B_e e(k) + x^T(k)(W+GK)^T\tilde{P}(W+GK)x(k) \\
&\quad + 2e^T(k)G_e^T\tilde{P}(W+GK)x(k) + e^T(k)G_e^T\tilde{P}G_e e(k) \\
&\quad + 2x^T(k)(W+GK)^T\tilde{P}Md_p(k) + 2e^T(k)G_e^T\tilde{P}Md_p(k) \\
&\quad + d_p^T(k)M^T\tilde{P}Md_p(k) - x^T(k)\tilde{P}x(k)
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\Delta V_2(e(k)) &= \mathbb{E}[V_2(e(k+1))|e(k)] - V_2(e(k)) \\
&= e^T(k)R^TPRe(k) + 2e^T(k)R^TPT\bar{B}_{d2f}d_{2f}(k) + d_{2f}^T(k)\bar{B}_{d2f}^TT^TPT\bar{B}_{d2f}d_{2f}(k)
\end{aligned}$$

$$+x^T(k)(\overline{W}+\bar{G}K)^T T^T PT(\overline{W}+\bar{G}K)x(k)+2e^T(k)\bar{G}_e^T T^T PT(\overline{W}+\bar{G}K)x(k)$$

$$+e^T(k)\bar{G}_e^T T^T PT\bar{G}_e e(k)+2e^T(k)\bar{G}_e^T T^T PT\overline{M}d_p(k)+2x^T(k)(\overline{W}+\bar{G}K)^T T^T PT\overline{M}d_p(k)$$

$$+d_p^T(k)\overline{M}^T T^T PT\overline{M}d_p(k)-e(k)Pe(k) \tag{35}$$

Substituting (34), (35) into (33), then adding and subtracting

$$-\gamma_d^2 d^T(k)d(k)-\gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k)-\gamma_{dp}^2 d_p^T(k)d_p(k)+\alpha\tilde{V}\big(\tilde{x}_e(k)\big) \tag{36}$$

to $\Delta\tilde{V}\big(\tilde{x}_e(k)\big)$ yields:

$$\Delta\tilde{V}\big(\tilde{x}_e(k)\big)<\begin{bmatrix}x^T(k) & e^T(k) & d_{2f}^T(k) & d^T(k) & d_p^T(k)\end{bmatrix}\Theta\begin{bmatrix}x(k)\\ e(k)\\ d_{2f}(k)\\ d(k)\\ d_p(k)\end{bmatrix}$$

$$+\gamma_d^2 d^T(k)d(k)+\gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k)+\gamma_{dp}^2 d_p^T(k)d_p(k)-\alpha\tilde{V}\big(\tilde{x}_e(k)\big) \tag{37}$$

where

$$\Theta=\begin{bmatrix}\Theta_{11} & \Theta_{12} & 0 & \Theta_{14} & \Theta_{15}\\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} & \Theta_{25}\\ * & * & \Theta_{33} & 0 & 0\\ * & * & * & \Theta_{44} & 0\\ * & * & * & * & \Theta_{55}\end{bmatrix} \tag{38}$$

$$\Theta_{11}=(A+BK)^T\tilde{P}(A+BK)+(W+GK)^T\tilde{P}(W+GK)+(\overline{W}+\bar{G}K)^T T^T PT(\overline{W}+\bar{G}K)+\alpha\tilde{P}-\tilde{P}$$

$$\Theta_{12}=(A+BK)^T\tilde{P}B_e+(W+GK)^T\tilde{P}G_e+(\overline{W}+\bar{G}K)^T T^T PT\bar{G}_e$$

$$\Theta_{22}=B_e^T\tilde{P}B_e+G_e^T\tilde{P}G_e+R^T PR+\bar{G}_e^T T^T PT\bar{G}_e-P+\alpha P$$

$$\Theta_{23}=R^T PT\overline{B}_{d2f}$$

$$\Theta_{33}=\overline{B}_{d2f}^T T^T PT\overline{B}_{d2f}-\gamma_{d2f}^2 I_{l_{d2f}}$$

$$\Theta_{14}=(A+BK)^T\tilde{P}B_d$$

$$\Theta_{24}=B_e^T\tilde{P}B_d$$

$$\Theta_{44}=B_d^T\tilde{P}B_d-\gamma_d^2 I_{l_d}$$

$$\Theta_{15} = (W + GK)^T \tilde{P} M + (\bar{W} + \bar{G}K)^T T^T P T \bar{M}$$

$$\Theta_{25} = \bar{G}_e^T T^T P T \bar{M} + G_e^T \tilde{P} M$$

$$\Theta_{55} = M^T \tilde{P} M + \bar{M}^T T^T P T \bar{M} - \gamma_{dp}^2 I_{l_{d_p}}$$

From (8), we know $PR = PT\bar{A} - Y\bar{C}$, where

$$Y = PL_1 \tag{39}$$

LMI (32) indicates

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & (A+BK)^T \tilde{P} B_d & \Phi_{15} & 0 \\ * & \Phi_{22} & 0 & B_e^T \tilde{P} B_d & \Phi_{25} & \Phi_{26} \\ * & * & -\gamma_{d2f}^2 I_{l_{d2f}} & 0 & 0 & \Phi_{36} \\ * & * & * & B_d^T \tilde{P} B_d - \gamma_d^2 I_{l_d} & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 \\ * & * & * & * & * & -P \end{bmatrix} < 0 \tag{40}$$

where

$$\Phi_{11} = (A + BK)^T \tilde{P} (A + BK) + (W + GK)^T \tilde{P} (W + GK) + (\bar{W} + \bar{G}K)^T T^T P T (\bar{W} + \bar{G}K) + \alpha \tilde{P} - \tilde{P}$$

$$\Phi_{12} = (A + BK)^T \tilde{P} B_e + (W + GK)^T \tilde{P} G_e + (\bar{W} + \bar{G}K)^T T^T P T \bar{G}_e$$

$$\Phi_{22} = B_e^T \tilde{P} B_e + G_e^T \tilde{P} G_e + \bar{G}_e^T T^T P T \bar{G}_e - P + \alpha P$$

$$\Phi_{15} = (W + GK)^T \tilde{P} M + (\bar{W} + \bar{G}K)^T T^T P T \bar{M}$$

$$\Phi_{25} = G_e^T \tilde{P} M + \bar{G}_e^T T^T P T \bar{M}$$

$$\Phi_{55} = M^T \tilde{P} M + \bar{M}^T T^T P T \bar{M} - \gamma_{dp}^2 I_{l_{d_p}}$$

$$\Phi_{26} = \bar{A}^T T^T P - \bar{C}^T Y^T$$

$$\Phi_{36} = \bar{B}_{d2f}^T T^T P$$

Pre-multiplying and post-multiplying the block $diag\{I, I, I, I, I, P^{-1}\}$ on both sides of inequality (40), we have

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & (A+BK)^T\tilde{P}B_d & \Phi_{15} & 0 \\ * & \Phi_{22} & 0 & B_e^T\tilde{P}B_d & \Phi_{25} & R^T \\ * & * & -\gamma_{d2f}^2 I_{l_{d2f}} & 0 & 0 & \bar{B}_{d2f}^T T^T \\ * & * & * & B_d^T\tilde{P}B_d - \gamma_d^2 I_{l_d} & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 \\ * & * & * & * & * & -P^{-1} \end{bmatrix} < 0 \tag{41}$$

According to Schur Lemma, inequality (41) is equivalent with

$$\Theta < 0 \tag{42}$$

leading to

$$\Delta\tilde{V} < \gamma_d^2 d^T(k)d(k) + \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) + \gamma_{dp}^2 d_p^T(k)d_p(k) - \alpha\tilde{V}\big(\tilde{x}_e(k)\big) \tag{43}$$

As $d(k)$, $d_{2f}(k)$ and $d_p(k)$ are bounded, a positive scalar μ can be found to satisfy the following inequality

$$\gamma_d^2 d^T(k)d(k) + \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) + \gamma_{dp}^2 d_p^T(k)d_p(k) \le \mu \tag{44}$$

Then we have

$$\Delta\tilde{V}(\tilde{x}_e(k)) < \mu - \alpha\tilde{V}\big(\tilde{x}_e(k)\big) \tag{45}$$

which indicate the closed-loop dynamic (27) is exponentially bounded in mean square.

To prove the robustness of (27), Let

$$\Gamma = \mathbb{E}\{\textstyle\sum_{k=0}^{K}[{y_c}^T(k)y_c(k) - \gamma_d^2 d^T(k)d(k) - \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) - \gamma_{d_p}^2 d_p^T(k)d_p(k)]\} \tag{46}$$

Then adding and subtracting $\mathbb{E}(\sum_{k=0}^{K}\Delta\tilde{V})$ to Γ, we can calculate

$$\Gamma = \mathbb{E}\{\textstyle\sum_{k=0}^{K}[x^T(k)C^TCx(k) + e^T(k)J_2^T\, D_f^T D_f J_2 e(k) + 2x^T(k)C^T D_f J_2 e(k) - \gamma_d^2 d^T(k)d(k)$$

$$-\gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) - \gamma_{d_p}^2 d_p^T(k)d_p(k) + \Delta\tilde{V}]\} - \mathbb{E}\{\textstyle\sum_{k=0}^{K}\Delta\tilde{V}\}$$

$$< \mathbb{E}\{\sum_{k=0}^{K}\left[x^T(k)\ \ e^T(k)\ \ d_{2f}^T(k)\ \ d^T(k)\ \ d_p^T(k)\right]\Pi\begin{bmatrix} x(k) \\ e(k) \\ d_{2f}(k) \\ d(k) \\ d_p(k) \end{bmatrix} - \mathbb{E}(\sum_{k=0}^{K}\Delta\tilde{V})$$

(47)

where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & 0 & (A+BK)^T\tilde{P}B_d & \Pi_{15} \\ * & \Pi_{22} & 0 & B_e^T\tilde{P}B_d & \Pi_{25} \\ * & * & -\gamma_{d2f}^2 I_{l_{d2f}} & 0 & 0 \\ * & * & * & B_d^T\tilde{P}B_d - \gamma_d^2 I_{l_d} & 0 \\ * & * & * & * & \Pi_{55} \end{bmatrix}$$

$$\Pi_{11} = \Theta_{11} - \alpha\tilde{P} + C^T C$$

$$\Pi_{12} = \Theta_{12} + C^T D_f J_2$$

$$\Pi_{22} = \Theta_{22} - \alpha P + J_2^T\ D_f^T D_f J_2$$

$$\Pi_{15} = \Theta_{15}$$

$$\Pi_{25} = \Theta_{25}$$

$$\Pi_{55} = \Theta_{55}$$

It can be noticed that under zero condition

$$\mathbb{E}\left(\sum_{k=0}^{K}\Delta\tilde{V}\right) = \mathbb{E}\left(\tilde{V}\right) > 0 \tag{48}$$

Pre-multiplying and post-multiplying the block $diag\{I, I, I, I, I, P^{-1}\}$ on both sides of inequality (32), and based on Lemma 3, we can derive

$$\Pi < 0 \tag{49}$$

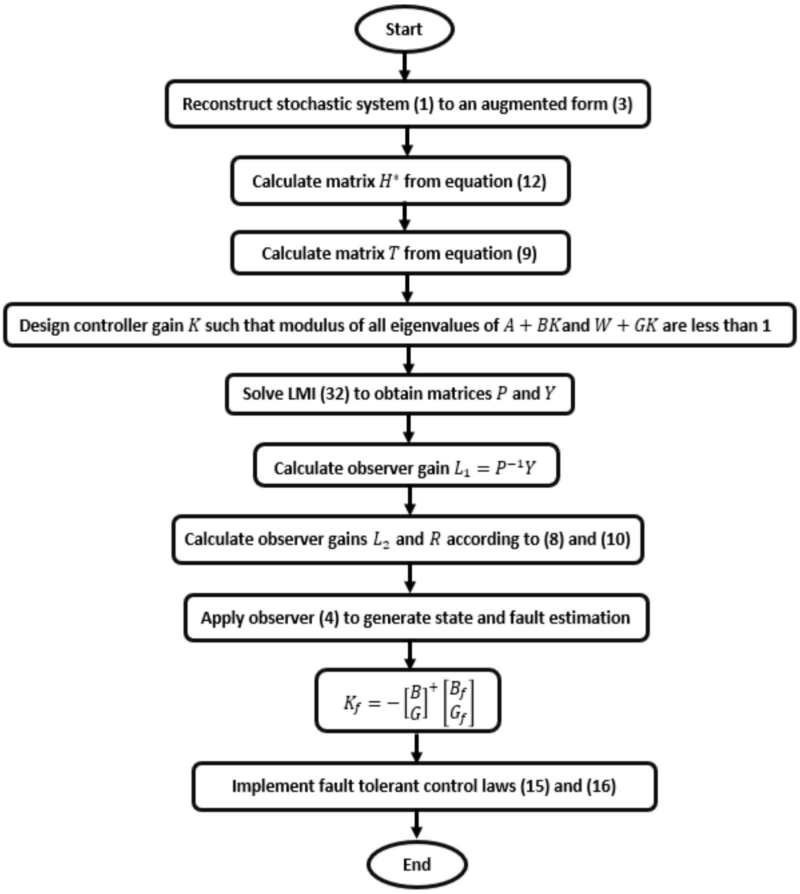
which means

$$\mathbb{E}(\|y_c\|_2^2) \leq \mathbb{E}(\gamma_d^2\|d\|_2^2) + \mathbb{E}(\gamma_{d2f}^2\left\|d_{2f}\right\|_2^2) + \mathbb{E}(\gamma_{d_p}^2\left\|d_p\right\|_2^2) \tag{50}$$

Therefore, we can prove Theorem 1.

Now the Algorithm of integrated fault tolerant control can be given as follows:

Algorithm 1.



3. Fault reconstruction and fault tolerant control for Lipschitz nonlinear stochastic system

In section 2, fault reconstruction-based resilient controller has been proposed for discrete-time linear stochastic system. In this section, integrated fault tolerant control technique will be designed for a class of nonlinear stochastic system, that is, Lipschitz nonlinear stochastic systems. Moreover, the influence of measurement noise is also under investigation. The considered system is in the form of:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + \Phi\big(x(k), u(k)\big) + B_f f(k) + B_d d(k) \\ \qquad\qquad +[Wx(k) + Gu(k) + G_f f(k) + M d_p(k)]\omega(k) \\ y(k) = Cx(k) + D_f f(k) + D_d d_m(k) \end{cases} \tag{51}$$

where $\Phi(x(k), u(k))$ is a nonlinear function vectors, which is assumed to be Lipschitz, i.e. $\forall\ x(k)$, $\hat{x}(k) \in \mathcal{R}^n$, and $u(k) \in \mathcal{R}^m$, there is a constant $\theta > 0$ such that

$$\left|\Phi\left(x(k),u(k)\right)-\Phi\left(\hat{x}(k),u(k)\right)\right| \leq \theta|x(k)-\hat{x}(k)| \tag{52}$$

$d_m(k) \in \mathcal{R}^{l_{dm}}$ represents measurement noise, where D_d is the coefficient matrix with appropriate dimension. $d_m(k)$ is assumed to be $\mathcal{L}_2$ bounded. The other symbols are the same as defined in (1). The nonlinearities in many practical systems follows satisfy Lipschitz condition (52), therefore, the fault tolerant control scheme develop in this section is applicable to many real plants.

An augmented system can be reconstructed through the employment of the auxiliary state vector $\bar{x}(k) = [x^T(k) \quad f^T(k)]^T \in \mathcal{R}^{\bar{n}}$, that is

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{B}_{df}d_f(k) + \bar{\Phi}(x(k),u(k)) \\ \qquad +[\bar{W}x(k) + \bar{G}u(k) + \bar{G}_f f(k) + \bar{M}d_p(k)]\omega(k) \\ \qquad y(k) = \bar{C}\bar{x}(k) + D_d d_m(k) \end{cases} \tag{53}$$

where $\bar{\Phi}\left(x(k),u(k)\right) = \left[\Phi^{\mathrm{T}}(x(k),u(k)) \quad 0_{1\times lf}\right]^T \in \mathcal{R}^{\bar{n}}$. The definitions of other symbols are the same with (3).

Then a nonlinear UIO can be established for (53)

$$\begin{cases} \bar{z}(k+1) = R\bar{z}(k) + T\bar{B}u(k) + T\bar{\Phi}\left(\hat{x}(k),u(k)\right) \\ \qquad +(L_1+L_2)y(k) \\ \hat{\bar{x}}(k) = \bar{z}(k) + Hy(k) \end{cases} \tag{54}$$

where R, T, L_1, L_2 and H are observer gains satisfying (7) - (10), and L_1 is to be designed.

The estimation error which is defined as (5) can be calculated as

$$e(k+1) = Re(k) + T\bar{B}_{d2f}d_{2f}(k) + T\,\tilde{\Phi}(k) - L_1D_d d_m(k) - HD_d d_m(k+1)$$

$$+T\bar{W}x(k)\omega(k) + T\bar{G}u(k)\omega(k) + T\bar{G}_f f(k)\omega(k) + T\bar{M}d_p(k)\omega(k) \tag{55}$$

where $\tilde{\Phi}(k) = \bar{\Phi}\left(x(k),u(k)\right) - \bar{\Phi}\left(\hat{x}(k),u(k)\right)$.

Under estimator-based controllers (15) and (16), the following closed-loop dynamic can be derived

$$\begin{cases} x(k+1) = (A+BK)x(k) + B_d d(k) + B_e e(k) + \Phi\big(x(k),u(k)\big) \\ \qquad\qquad +\big[(W+GK)x(k) + G_e e(k) + M d_p(k)\big]\omega(k) \\ e(k+1) = Re(k) + T\bar{B}_{d2f} d_{2f}(k) + T\,\tilde{\Phi}(k) - L_1 D_d d_m(k) - H D_d d_m(k+1) \\ \qquad\qquad +T[(\bar{W}+\bar{G}K)x(k) + \bar{G}_e e(k) + \bar{M} d_p(k)]\omega(k) \\ y_c(k) = Cx(k) + D_d d_m(k) + D_f J_2 e(k) \end{cases} \tag{56}$$

Similarly, the controller gain K is designed to assign matrices the eigenvalue of $A+BK$ and $W+GK$ inside unit circle. However, the existence of nonlinear component $\Phi\big(x(k),u(k)\big)$ and $\tilde{\Phi}(k)$ on the closed-loop system (56) makes the methodology presented in Section 2 invalid. Therefore, additional techniques will be required in the design of robust estimator for Lipschitz nonlinear stochastic systems.

The following theorem then presented to determine observer gain L_1, such that model (56) above is exponentially bounded in mean square and satisfy robust performance (31b):

$$\mathbb{E}(\|y_c\|_2^2) \le \mathbb{E}(\gamma_d^2\|d\|_2^2) + \mathbb{E}\left(\gamma_{d2f}^2\left\|d_{2f}\right\|_2^2\right) + \mathbb{E}\left(\gamma_{d_p}^2\left\|d_p\right\|_2^2\right)$$

$$+\,\mathbb{E}[(\gamma_{m1}^2+\gamma_{m2}^2)\|d_m\|_2^2] \tag{31b}$$

Theorem 2. The closed-loop system (56) is exponentially bounded in mean square, and satisfies robust performance (31b), if there exist a positive definite matrix P and $\tilde{P}$, and matrix Y, such that

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 & \Psi_{14} & \Psi_{15} & \Psi_{16} & 0 & \Psi_{18} & 0 & 0 \\ * & \Psi_{22} & 0 & \Psi_{24} & \Psi_{25} & \Psi_{26} & 0 & \Psi_{28} & 0 & \Psi_{210} \\ * & * & \Psi_{33} & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{310} \\ * & * & * & \Psi_{44} & 0 & \Psi_{46} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Psi_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Psi_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Psi_{77} & 0 & 0 & \Psi_{710} \\ * & * & * & * & * & * & * & \Psi_{88} & 0 & \Psi_{810} \\ * & * & * & * & * & * & * & * & \Psi_{99} & \Psi_{910} \\ * & * & * & * & * & * & * & * & * & \Psi_{1010} \end{bmatrix} < 0 \tag{57}$$

where

$$\Psi_{11} = (A+BK)^T\tilde{P}(A+BK) + (W+GK)^T\tilde{P}(W+GK) + (\overline{W}+\bar{G}K)^T T^T P T(\overline{W}+\bar{G}K) + \alpha\tilde{P} - \tilde{P}$$

$$+ C^T C + \gamma_1^2\theta^2 I_n$$

$$\Psi_{12} = (A+BK)^T\tilde{P}B_e + (W+GK)^T\tilde{P}G_e + (\overline{W}+\bar{G}K)^T T^T P T\bar{G}_e + C^T D_f J_2$$

$$\Psi_{22} = B_e^T\tilde{P}B_e + G_e^T\tilde{P}G_e + \bar{G}_e^T T^T P T\bar{G}_e - P + \alpha P + J_2^T\, D_f^T D_f J_2$$

$$+\gamma_2^2\theta^2 I_{\bar{n}}$$

$$\Psi_{33} = -\gamma_{d2f}^2 I_{l_{d2f}}$$

$$\Psi_{14} = (A+BK)^T\tilde{P}B_d$$

$$\Psi_{24} = B_e^T\tilde{P}B_d$$

$$\Psi_{44} = B_d^T\tilde{P}B_d - \gamma_d^2 I_{l_d}$$

$$\Psi_{15} = (W+GK)^T\tilde{P}M + (\overline{W}+\bar{G}K)^T T^T P T\overline{M}$$

$$\Psi_{25} = G_e^T\tilde{P}M + \bar{G}_e^T T^T P T\overline{M}$$

$$\Psi_{55} = M^T\tilde{P}M + \overline{M}^T T^T P T\overline{M} - \gamma_{dp}^2 I_{l_{d_p}}$$

$$\Psi_{16} = (A+BK)^T\tilde{P}$$

$$\Psi_{26} = B_e^T\tilde{P}$$

$$\Psi_{46} = B_d^T\tilde{P}$$

$$\Psi_{66} = \tilde{P} - \gamma_1^2 I_n$$

$$\Psi_{77} = -\gamma_2^2 I_{\bar{n}}$$

$$\Psi_{210} = \bar{A}^T T^T P - \bar{C}^T Y^T$$

$$\Psi_{310} = \bar{B}_{d2f}^T T^T P$$

$$\Psi_{710} = T^T P$$

$$\Psi_{810} = -D_d^T Y^T$$

$$\Psi_{910} = -D_d^T H^T P$$

$$\Psi_{88} = D_d^T D_d - \gamma_{m1}^2 I_{l_{dm}}$$

$$\Psi_{99} = -\gamma_{m2}^2 I_{l_{dm}}$$

$$\Psi_{1010} = -P$$

$$\Psi_{18} = C^T D_d$$

$$\Psi_{28} = J_2^T D_f^T D_d$$

$Y = PL_1$, α is a positive scalar that $0 < \alpha \leq 1$. $\gamma_{d2f}, \gamma_d, \gamma_{d_p}, \gamma_1, \gamma_2, \gamma_{m1}, \gamma_{m2}$, are given robustness performance indices. Then we have $L_1 = P^{-1}Y$.

Proof. Lyapunov function candidate (33) can be chosen for error dynamic system (55). Then it can be noticed that $\tilde{V}\left(\tilde{x}_e(k)\right)$ satisfies condition (28) in *Lemma 1*.

Based on (33) and (56), we can calculate $\Delta\tilde{V}\left(\tilde{x}_e(k)\right)$. Adding and subtracting $-\gamma_d^2 d^T(k)d(k) - \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) - \gamma_{dp}^2 d_p^T(k)d_p(k) - \gamma_1^2\Phi^{\mathrm{T}}(k)\Phi(k) - \gamma_2^2\widetilde{\Phi}^T(k)\widetilde{\Phi}(k) - \gamma_{m1}^2 d_m^T(k)d_m(k) - \gamma_{m2}^2 d_m^T(k+1)d_m(k+1) + \alpha\tilde{V}\left(\tilde{x}_e(k)\right)$ to $\Delta\tilde{V}\left(\tilde{x}_e(k)\right)$, where γ_1 and γ_2 are positive scalars, and using Lemma 2, one has

$$\Delta\tilde{V}\left(\tilde{x}_e(k)\right)$$

$$\leq \left[x^T(k) \;\; e^T(k) \;\; d_{2f}^T(k) \;\; d^T(k) \;\; d_p^T(k) \;\; \Phi^T(k) \;\; \widetilde{\Phi}^T(k) \;\; d_m^T(k) \;\; d_m^T(k+1)\right]\Lambda \begin{bmatrix} x(k) \\ e(k) \\ d_{2f}(k) \\ d(k) \\ d_p(k) \\ \Phi(k) \\ \widetilde{\Phi}(k) \\ d_m(k) \\ d_m(k+1) \end{bmatrix}$$

$$+\gamma_d^2 d^T(k)d(k) + \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) + \gamma_{dp}^2 d_p^T(k)d_p(k) + \gamma_{m1}^2 d_m^T(k)d_m(k) + \gamma_{m2}^2 d_m^T(k+1)d_m(k+1) - \alpha\tilde{V}\left(\tilde{x}_e(k)\right) \tag{58}$$

where

$$\Lambda=\begin{bmatrix}\Lambda_{11} & \Lambda_{12} & 0 & \Lambda_{14} & \Lambda_{15} & \Lambda_{16} & 0 & 0 & 0\\ * & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} & \Lambda_{25} & \Lambda_{26} & \Lambda_{27} & \Lambda_{28} & \Lambda_{29}\\ * & * & \Lambda_{33} & 0 & 0 & 0 & \Lambda_{37} & \Lambda_{38} & \Lambda_{39}\\ * & * & * & \Lambda_{44} & 0 & \Lambda_{46} & 0 & 0 & 0\\ * & * & * & * & \Lambda_{55} & 0 & 0 & 0 & 0\\ * & * & * & * & * & \Lambda_{66} & 0 & 0 & 0\\ * & * & * & * & * & * & \Lambda_{77} & \Lambda_{78} & \Lambda_{79}\\ * & * & * & * & * & * & * & \Lambda_{88} & \Lambda_{89}\\ * & * & * & * & * & * & * & * & \Lambda_{99}\end{bmatrix} \tag{59}$$

$$\Lambda_{11}=(A+BK)^T\tilde{P}(A+BK)+(W+GK)^T\tilde{P}(W+GK)+(\bar{W}+\bar{G}K)^TT^TPT(\bar{W}+\bar{G}K)+\alpha\tilde{P}-\tilde{P}$$

$$+\gamma_1^2\theta^2I_n$$

$$\Lambda_{12}=(A+BK)^T\tilde{P}B_e+(W+GK)^T\tilde{P}G_e+(\bar{W}+\bar{G}K)^TT^TPT\bar{G}_e$$

$$\Lambda_{22}=B_e^T\tilde{P}B_e+G_e^T\tilde{P}G_e+R^TPR+\bar{G}_e^TT^TPT\bar{G}_e-P+\alpha P+\gamma_2^2\theta^2I_{\bar{n}}$$

$$\Lambda_{23}=R^TPT\bar{B}_{d2f}$$

$$\Lambda_{33}=\bar{B}_{d2f}^TT^TPT\bar{B}_{d2f}-\gamma_{d2f}^2I_{l_{d2f}}$$

$$\Lambda_{14}=(A+BK)^T\tilde{P}B_d$$

$$\Lambda_{24}=B_e^T\tilde{P}B_d$$

$$\Lambda_{44}=B_d^T\tilde{P}B_d-\gamma_d^2I_{l_d}$$

$$\Lambda_{15}=(W+GK)^T\tilde{P}M+(\bar{W}+\bar{G}K)^TT^TPT\bar{M}$$

$$\Lambda_{25}=\bar{G}_e^TT^TPT\bar{M}+G_e^T\tilde{P}M$$

$$\Lambda_{55}=M^T\tilde{P}M+\bar{M}^TT^TPT\bar{M}-\gamma_{dp}^2I_{l_{d_p}}$$

$$\Lambda_{16}=(A+BK)^T\tilde{P}$$

$$\Lambda_{26}=B_e^T\tilde{P}$$

$$\Lambda_{46}=B_d^T\tilde{P}$$

$$\Lambda_{66}=\tilde{P}-\gamma_1^2I_n$$

$$\Lambda_{27}=\bar{A}^TT^TPT-\bar{C}^TY^TT$$

$$\Lambda_{37}=\bar{B}_{d2f}^TT^TPT$$

$$\Lambda_{77}=T^TPT-\gamma_2^2I_{\bar{n}}$$

$$\Lambda_{28} = -R^T P L_1 D_d$$

$$\Lambda_{29} = -R^T P H D_d$$

$$\Lambda_{38} = -\bar{B}_{d2f}^T T^T Y D_d$$

$$\Lambda_{39} = -\bar{B}_{d2f}^T T^T P H D_d$$

$$\Lambda_{78} = -T^T Y D_d$$

$$\Lambda_{79} = -T^T P H D_d$$

$$\Lambda_{88} = D_d^T L_1^T P L_1 D_d - \gamma_{m1}^2 I_{l_{dm}}$$

$$\Lambda_{89} = D_d^T L_1^T P H D_d$$

$$\Lambda_{99} = D_d^T H^T P H D_d - \gamma_{m2}^2 I_{l_{dm}}$$

LMI (57) indicates

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & 0 & \Psi_{14} & \Psi_{15} & \Psi_{16} & 0 & 0 & 0 & 0 \\ * & \Sigma_{22} & 0 & B_e^T \tilde{P} B_d & \Psi_{25} & \Psi_{26} & 0 & 0 & 0 & \Psi_{210} \\ * & * & \Psi_{33} & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{310} \\ * & * & * & \Psi_{44} & 0 & \Psi_{46} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Psi_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Psi_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Psi_{77} & 0 & 0 & \Psi_{710} \\ * & * & * & * & * & * & * & \Sigma_{88} & 0 & \Psi_{810} \\ * & * & * & * & * & * & * & * & \Psi_{99} & \Psi_{910} \\ * & * & * & * & * & * & * & * & * & \Psi_{1010} \end{bmatrix} < 0 \tag{60}$$

where

$$\Sigma_{11} = \Psi_{11} - C^T C$$

$$\Sigma_{11} = \Psi_{12} - C^T D_f J_2$$

$$\Sigma_{22} = \Psi_{22} - J_2^T D_f^T D_f J_2$$

$$\Sigma_{88} = \Psi_{88} - D_d^T D_d$$

Pre-multiplying and post-multiplying the block $diag\{I, I, I, I, I, I, I, I, I, P^{-1}\}$ on both sides of inequality (60), and according to Schur Lemma, inequality (60) is equivalent with

$$\Lambda < 0 \tag{61}$$

leading to

$$\Delta\tilde{V} < \gamma_d^2 d^T(k)d(k) + \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) + \gamma_{dp}^2 d_p^T(k)d_p(k) + \gamma_{m1}^2 d_m^T(k)d_m(k) + \gamma_{m2}^2 d_m^T(k+1)d_m(k+1) - \alpha\tilde{V}\big(\tilde{x}_e(k)\big) \quad (62)$$

As $d(k)$, $d_{2f}(k)$, $d_p(k)$, $d_m(k)$ are bounded, we can find a positive scalar μ such that

$$\gamma_d^2 d^T(k)d(k) + \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) + \gamma_{dp}^2 d_p^T(k)d_p(k) + \gamma_{m1}^2 d_m^T(k)d_m(k) \le \mu \quad (63)$$

Then we have

$$\Delta\tilde{V}(\tilde{x}_e(k)) < \mu - \alpha\tilde{V}\big(\tilde{x}_e(k)\big) \quad (64)$$

which indicate dynamic (56) is exponentially bounded in mean square.

Then the robustness of (56) is to be discussed. Let

$$\Gamma = \mathbb{E}\{\textstyle\sum_{k=0}^{K}[{y_c}^T(k)y_c(k) - \gamma_d^2 d^T(k)d(k) - \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) - \gamma_{d_p}^2 d_p^T(k)d_p(k) - \gamma_{m1}^2 d_m^T(k)d_m(k)]\} \quad (65)$$

Then adding and subtracting $\mathbb{E}(\sum_{k=0}^{K}\Delta\tilde{V})$ to Γ, we can calculate

$$y_c(k) = Cx(k) + D_d d_m(k) + D_f J_2 e(k)$$

$$\begin{aligned}\Gamma = \mathbb{E}\{\sum_{k=0}^{K}[&x^T(k)C^TCx(k) + e^T(k)J_2^T D_f^T D_f J_2 e(k) + d_m^T(k)D_d^T D_d d_m(k) + 2x^T(k)C^T D_f J_2 e(k)\\ &+ 2x^T(k)C^T D_d d_m(k) + 2e^T(k)J_2^T D_f^T D_d d_m(k) - \gamma_d^2 d^T(k)d(k)\\ &- \gamma_{d2f}^2 d_{2f}^T(k)d_{2f}(k) - \gamma_{d_p}^2 d_p^T(k)d_p(k) - \gamma_{m1}^2 d_m^T(k)d_m(k) + \Delta\tilde{V}]\} - \mathbb{E}\left\{\sum_{k=0}^{K}\Delta\tilde{V}\right\}\end{aligned}$$

$$< \mathbb{E}\{\sum_{k=0}^{K}\left[x^T(k)\ \ e^T(k)\ \ d_{2f}^T(k)\ \ d^T(k)\ \ d_p^T(k)\ \ \Phi^T(k)\ \ \tilde{\Phi}^T(k)\ \ d_m^T(k)\ \ d_m^T(k+1)\right]\Omega\begin{bmatrix} x(k) \\ e(k) \\ d_{2f}(k) \\ d(k) \\ d_p(k) \\ \Phi(k) \\ \tilde{\Phi}(k) \\ d_m(k) \\ d_m(k+1) \end{bmatrix}$$

$$-\mathbb{E}(\textstyle\sum_{k=0}^{K}\Delta\tilde{V}) \tag{66}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 & \Lambda_{14} & \Lambda_{15} & \Lambda_{16} & 0 & \Omega_{18} & 0 \\ * & \Omega_{22} & \Lambda_{23} & \Lambda_{24} & \Lambda_{25} & \Lambda_{26} & \Lambda_{27} & \Omega_{28} & \Lambda_{29} \\ * & * & \Lambda_{33} & 0 & 0 & 0 & \Lambda_{37} & \Lambda_{38} & \Lambda_{39} \\ * & * & * & \Lambda_{44} & 0 & \Lambda_{46} & 0 & 0 & 0 \\ * & * & * & * & \Lambda_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Lambda_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Lambda_{77} & \Lambda_{78} & \Lambda_{79} \\ * & * & * & * & * & * & * & \Omega_{88} & 0 \\ * & * & * & * & * & * & * & * & \Lambda_{99} \end{bmatrix}$$

$$\Omega_{11} = \Lambda_{11} - \alpha\tilde{P} + C^T C$$

$$\Omega_{12} = \Lambda_{12} + C^T D_f J_2$$

$$\Omega_{22} = \Lambda_{22} - \alpha P + J_2^T\, D_f^T D_f J_2$$

$$\Omega_{18} = C^T D_d$$

$$\Omega_{28} = \Lambda_{28} + J_2^T D_f D_d$$

$$\Omega_{88} = \Lambda_{88} + D_d^T D_d$$

It can be noticed that under zero condition

$$\mathbb{E}\left(\textstyle\sum_{k=0}^{K}\Delta\tilde{V}\right) = \mathbb{E}\left(\tilde{V}\right) > 0 \tag{67}$$

Pre-multiplying and post-multiplying the block $diag\{I, I, I, I, I, I, I, I, P^{-1}\}$ on both sides of inequality (57), and using Schur Lemma, we have

$$\Omega < 0 \tag{68}$$

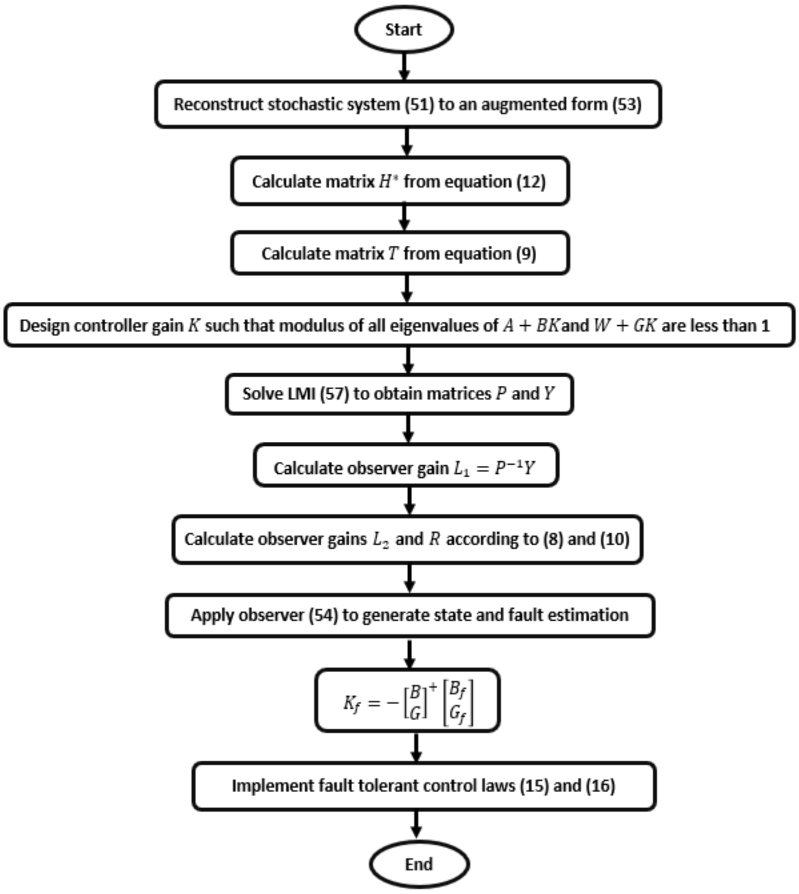
which means

$$\mathbb{E}(|y_c|_2^2) \leq \mathbb{E}(\gamma_d^2|d|_2^2) + \mathbb{E}(\gamma_{d2f}^2\left|d_{2f}\right|_2^2) + \mathbb{E}(\gamma_{d_p}^2\left|d_p\right|_2^2) \tag{69}$$

Therefore, we can prove Theorem 2.

Now the Algorithm of integrated fault tolerant control can be given as follows:

Algorithm 2.



4. Simulation

This section introduces two case studies on an Electromechanical servosystem and a flight control plant to validate the effectiveness of the designed resilient control technique.

4.1. Electromechanical servosystem

An electromechanical servosystem is identified by a discrete-time model in the following form [29]:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{70}$$

where $x(k) = [x_1^T(k) \quad x_2^T(k)]^T$, $x_1(k)$ represents the load angular position, $x_2(k)$ denotes the shaft speed, $u(k)$ stands for the input voltage. The sampling time is 0.1s, and the system coefficients are given as

$$A = \begin{bmatrix} 0.0468 & 0.1564 \\ 0.2083 & 0.8154 \end{bmatrix}, B = \begin{bmatrix} 39.2076 \\ 11.5299 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Considering the influence of faults, unknown uncertainties, and Brownian motions, the system can be described by stochastic discrete-time model (1). The unknown inputs are random signals taking value from -0.1 to 0.1, and the coefficient is $B_d = \begin{bmatrix} 0.3 & 0.1 & -0.05 \\ -0.2 & 0.3 & -0.15 \end{bmatrix}$, where $B_{d1} = \begin{bmatrix} 0.3 \\ -0.2 \end{bmatrix}$ and $B_{d2} = \begin{bmatrix} 0.1 & -0.05 \\ 0.3 & -0.15 \end{bmatrix}$. The uncertainties in stochastic item d_p is random noises from -0.01 to 0.01. The fault under concern is 20% loss of actuation effectiveness from 20 second to 30 second. Hence, $B_f = B$, $D_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The stochastic distribution coefficients are supposed as $W = 0.05A$, $G = 0.1B$, $G_f = 0.1B_f$ and $M = \begin{bmatrix} 0.02 \\ 0.01 \end{bmatrix}$.

According to *Algorithm 1*, we can observer gains $H = \begin{bmatrix} 0.6923 & -0.4615 \\ -0.4615 & 0.3007 \\ 0 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 0.3007 & 0.4615 & 0 \\ 0.4615 & 0.6923 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The eigenvalue of system matrices A and G are already inside the unit circle, therefore, we select the control gain $K = [0 \quad 0]$. Choose $r_d = 50$, $r_{d2f} = 500$, $r_{dp} = 5$, $a = 0.1$, the observer gain L_1 can be solved from (31) as

$$L_1 = \begin{bmatrix} 0.4479 & 1.1056 \\ 0.6662 & 1.6622 \\ 0.0192 & 0.0392 \end{bmatrix}$$

Then we can calculate gain matrices L_2 and R from (8) and (10), respectively. Moreover, $L = L_1 + L_2$. Controller gain K_f can be obtained from (21) as $K_f = -1$.

The fault reconstruction and resilient control technique can be then implemented to the electromechanical servosystem. Euler-Maruyama method [31] is employed to simulate the standard Brownian motions (with 5 Brownian paths), simultaneous estimations of the 5-path states and fault can be obtained. The means of the 5 paths are compared with the original signals to show the results in Fig. 1-3. The comparisons of the outputs under three scenarios, which is fault-free, faulty with fault tolerant control, and faulty without fault tolerant control, are shown in Fig. 4-5.

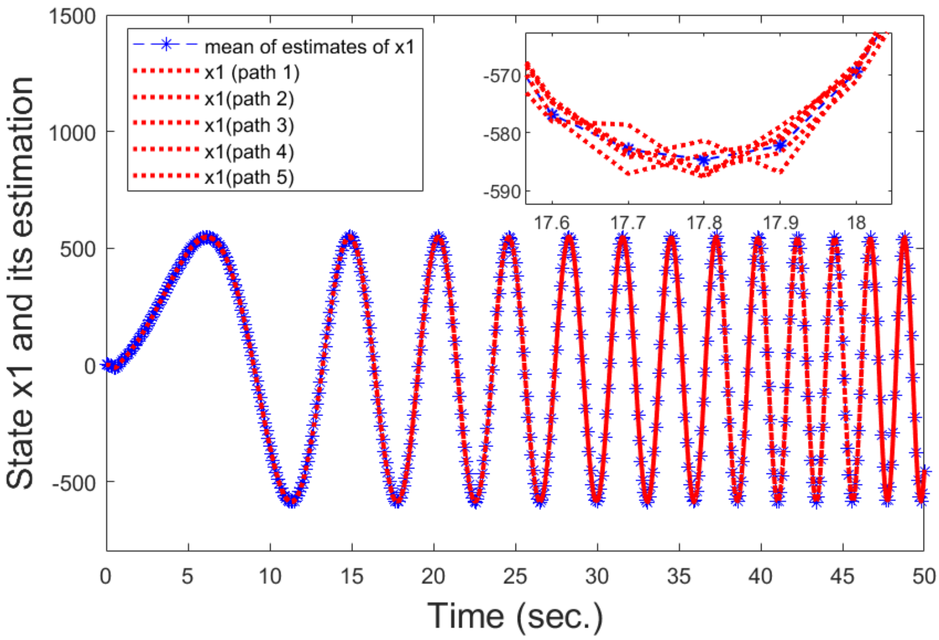


Fig. 1. Load angular positionand its estimation

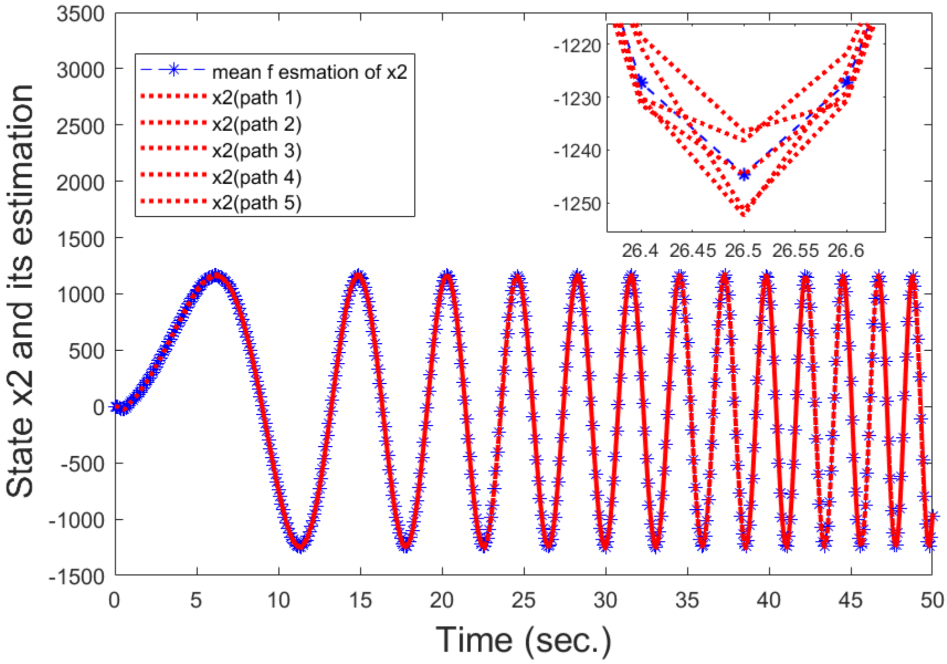


Fig. 2. Shaft speed and its estimation

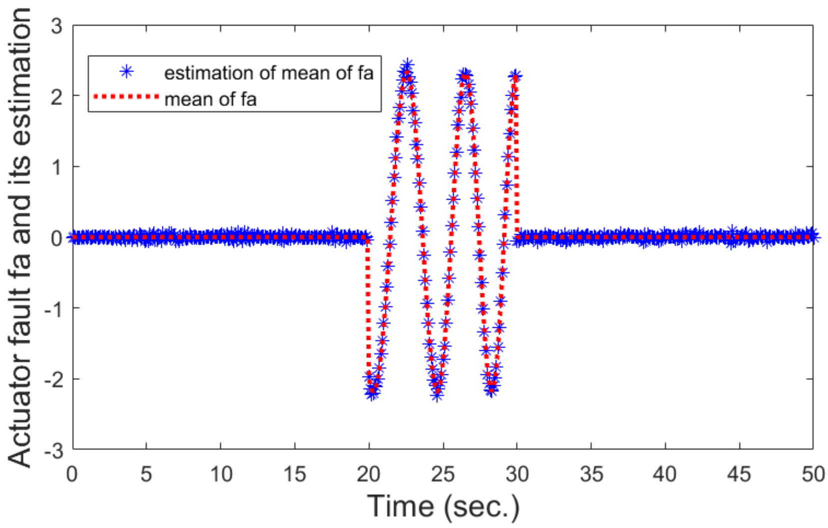


Fig. 3. Actuator fault and its estimation

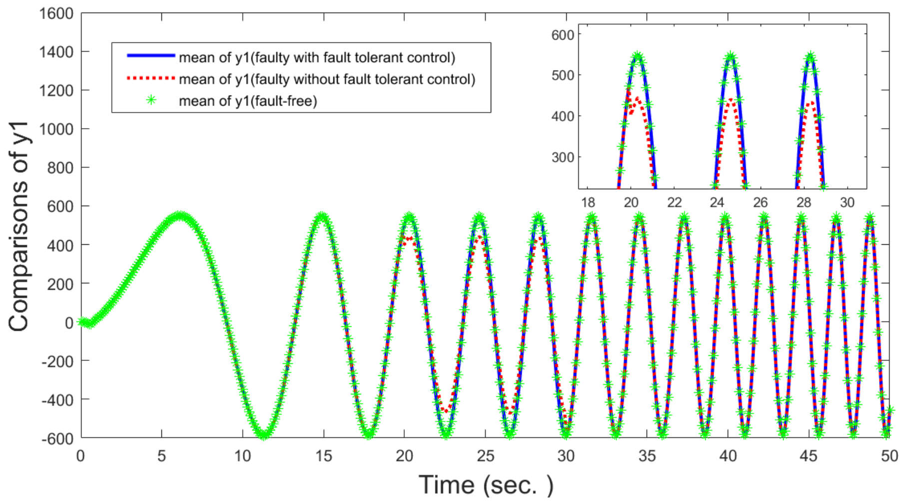


Fig. 4. $y_1(k)$ in different scenarios

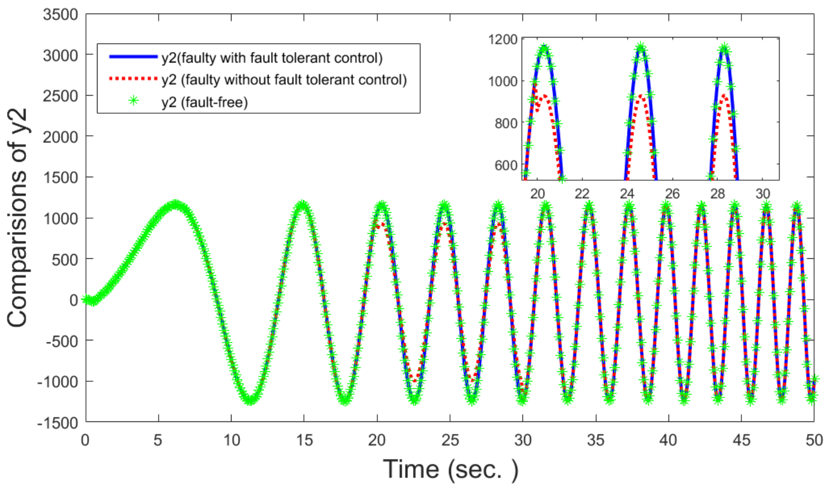


Fig. 5. $y_2(k)$ in different scenarios

It is noticed that we can obtain the estimations system state and fault with satisfactory accuracy, and the unknown inputs can be reduced successfully. Furthermore, the actuator fault makes the system output have bias with fault-free case, however, the designed fault tolerant control technique can compensate the influence of faults successfully. Therefore, this case study illustrates the effectiveness of the presented fault tolerant technique.

4.2. Flight control system

Now, it is time to implement the developed resilient control technique for a Lipschitz nonlinear system. Considering a simplified longitudinal flight control system subjected to unknown inputs, faults, Brownian motions can be represented by system (51). The system state is $x(k) = [\eta_y(k) \quad \omega_z(k) \quad \delta_z(k)]^T$, where η_y is normal velocity, ω_z is pitch rate, and δ_z is pitch angle. The initial condition $x_0 = [1 \quad 0.5 \quad 2]^T$. $u(k)$ is elevator control signal , $\Phi\big(x(k), u(k)\big) = \big[0 \quad 0.01\sin\big(x_3(k)\big) \quad 0\big]^T$. The system matrix can be found in [9,30], that is

$$A = \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The unknown input disturbances in deterministic item $d = [d_1 \quad d_2 \quad d_3]^T$ are random noises taking values from -0.1 to 0.1, where d_1 can be decoupled and d_2, d_3 cannot. The uncertainties in stochastic item are random noises from -0.01 to 0.01 . Measurement noise $d_m = [d_{m1} \quad d_{m2} \quad d_{m3}]^T$ are random noises taking values from -0.1 to 0.1 . The coefficients of uncertainties are given as follows:

$$B_{d1} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.01 \end{bmatrix}, B_{d2} = \begin{bmatrix} 0.1 & -0.15 \\ -0.2 & 0.3 \\ 0.08 & -0.12 \end{bmatrix}, M = \begin{bmatrix} 0.01 \\ 0.05 \\ -0.04 \end{bmatrix}, D_d = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0 & -0.3 & 0 \\ 0.1 & 0 & -0.2 \end{bmatrix}.$$

The actuator fault under consideration $f_a = -2 + 0.2\sin(0.2T_s)$ from 40s to 60s, and the sensor fault is $f_s = -3$ from 70s to 100s, which exists in the third sensor. In this case, $B_{fa} = B$ and $D_{fs} = [0 \quad 0 \quad 1]^T$.

Consequently, the fault vector considered is $f = [f_a \quad f_s]^T$ with $B_f = [B_{fa} \quad 0_{3\times1}]$ and $D_f = [0_{3\times1} \quad D_{fs}]$. The stochastic distribution coefficients are supposed as $W = 0.01A$, $G = 0.1B$, $G_f = 0.1B_f$.

According to *Algorithm 2*, we can solve the observer gains

$$H = \begin{bmatrix} 0.4975 & 0.4975 & 0.0498 \\ 0.4975 & 0.4975 & 0.0498 \\ 0.0498 & 0.0498 & 0.0050 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$T = \begin{bmatrix} 0.5025 & -0.4975 & -0.0498 & 0 & -0.0498 \\ -0.4975 & 0.5025 & -0.0498 & 0 & -0.0498 \\ -0.0498 & -0.0498 & 0.9950 & 0 & -0.0050 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The control gain is chosen as $K = [-2.608 \quad -4.0433 \quad -1.4276]$ to assign the eigenvalues of $A + BK$ to be $\{-0.9, \quad 0.75, \quad 0.8\}$ and the eigenvalues of $W + GK$ to be $\{-0.1276, \quad 0.0055, \quad 0.0085\}$. Then, choose $r_d = 100$, $r_{d2f} = 200$, $r_{dp} = 50$, $a = 0.01$, $r_1 = 50$, $r_2 = 100$, $r_{m1} = 20$, $r_{m2} = 50$, the observer gain L_1 can be solved from (57) as

$$L_1 = \begin{bmatrix} 0.7091 & -0.4087 & 0.0200 \\ -0.6613 & 0.3655 & -0.0496 \\ -0.3911 & 0.2593 & -0.0943 \\ 0.3963 & -0.2925 & 0.0484 \\ -0.0863 & 0.1725 & 0.3908 \end{bmatrix}$$

Then we can calculate gain matrices L_2 and R from (8) and (10), respectively. Moreover, $L = L_1 + L_2$. Controller gain K_f can be obtained from (21) as $K_f = [-1 \quad 0]$. The reference of control input is given to be $u_r(k) = 2$, then $u(k) = u_r(k) + \bar{K}\hat{\bar{x}}(k)$, where $\bar{K} = [K \quad K_f]$.

The fault reconstruction and resilient control technique can be then implemented to the electromechanical servosystem. Euler-Maruyama method [31] is employed to simulate the standard Brownian motions (with 3 Brownian paths), simultaneous estimations of the 3-path states and fault can be obtained. The means of the 3 paths are compared with the original signals to show the results in Fig. 6-10. The comparisons of the outputs under three scenarios, which is fault-free, faulty with fault tolerant control, and faulty without fault tolerant control, are shown in Fig. 11-13.

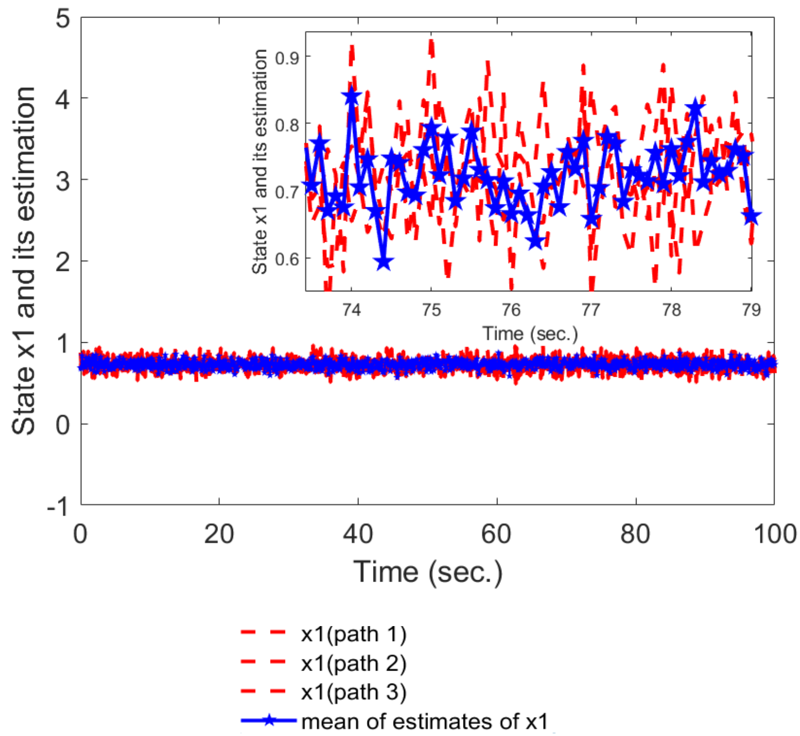


Fig. 6. velocity η_y and its estimation

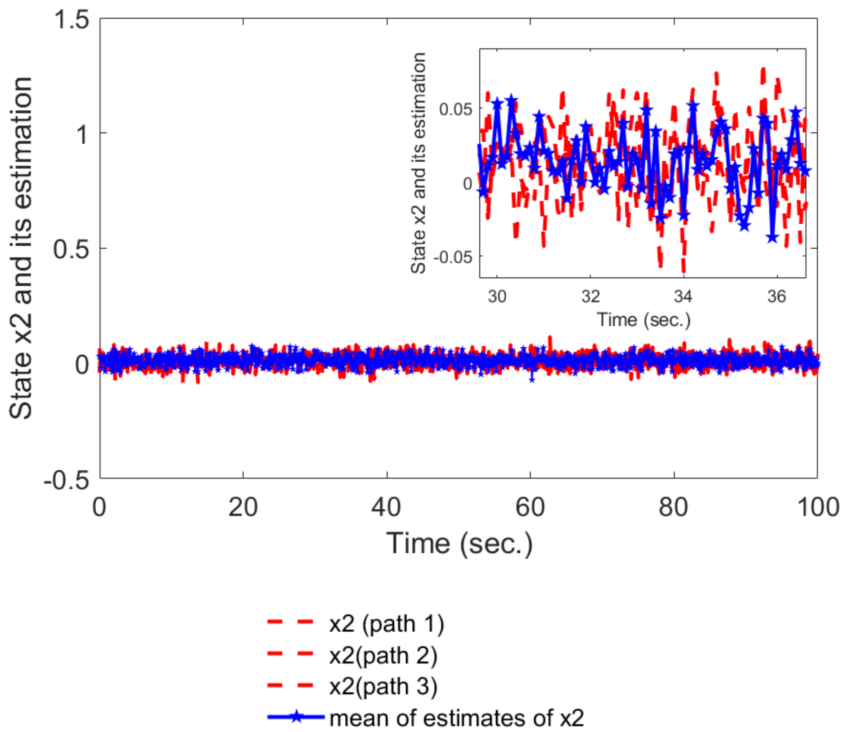


Fig. 7. Pitch rate ω_z and its estimation

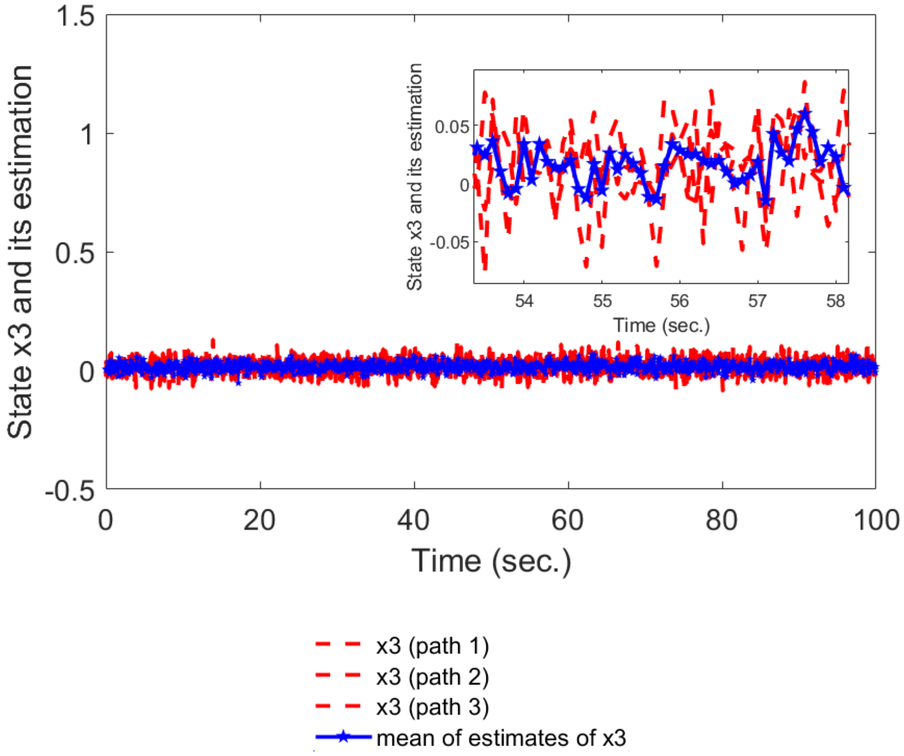


Fig. 8. Pitch angle δ_z and its estimation

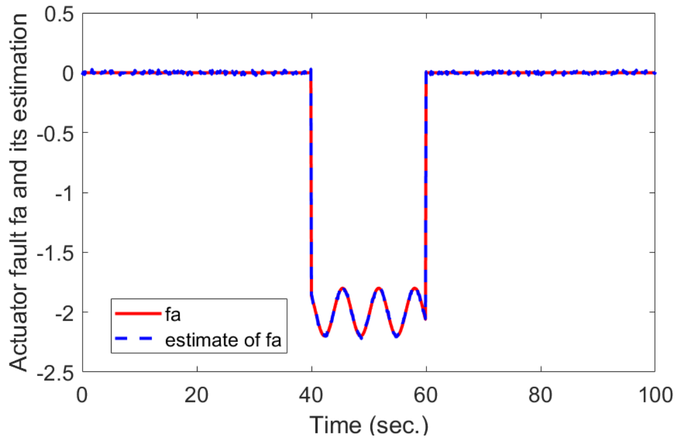


Fig. 9. Actuator fault and its estimation

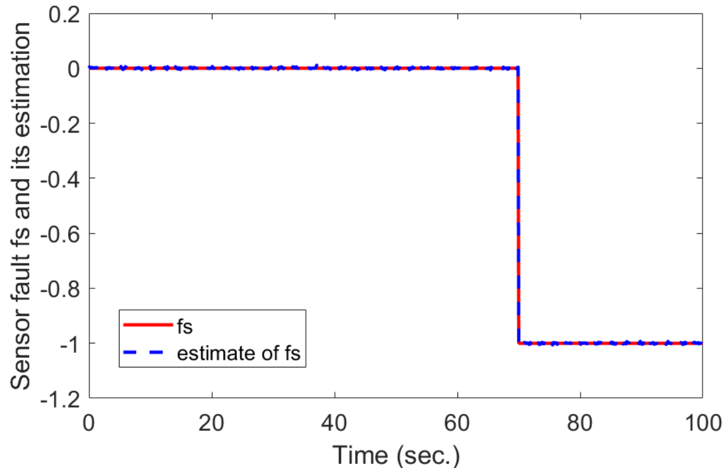


Fig. 10. Sensor fault and its estimation

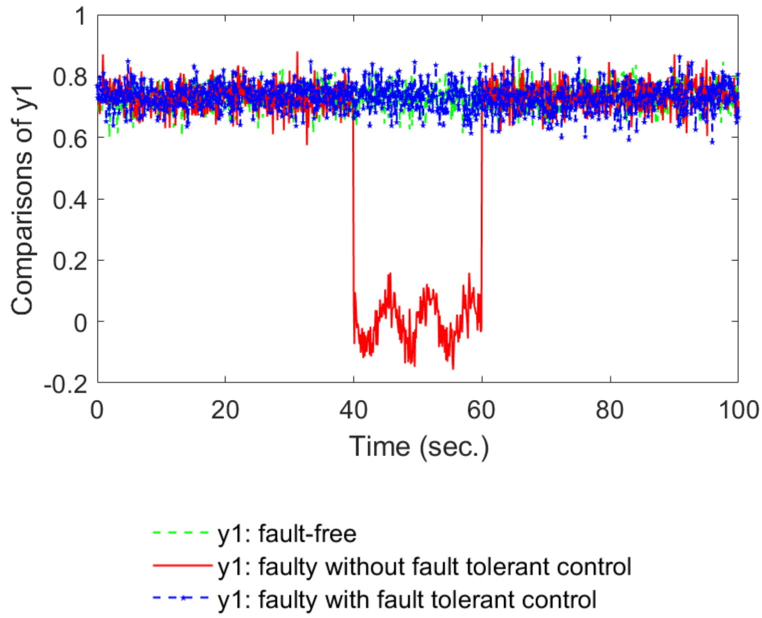
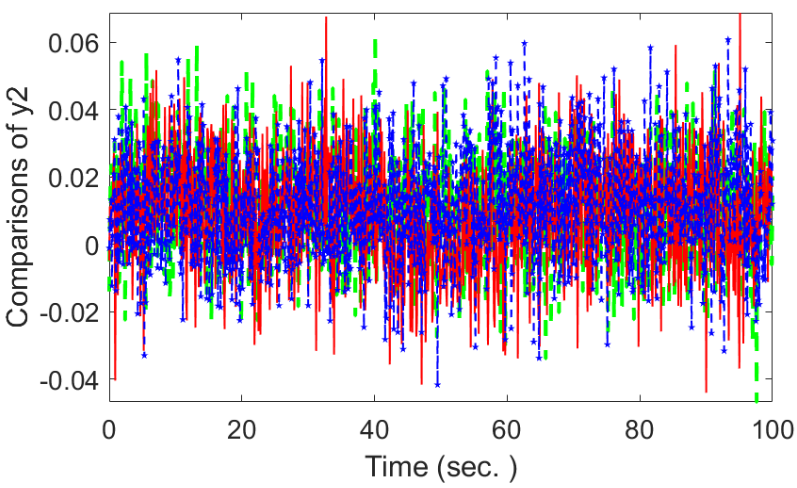


Fig. 11. $y_1(k)$ in different scenarios



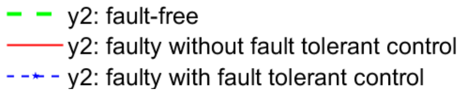


Fig. 12. $y_2(k)$ in different scenarios

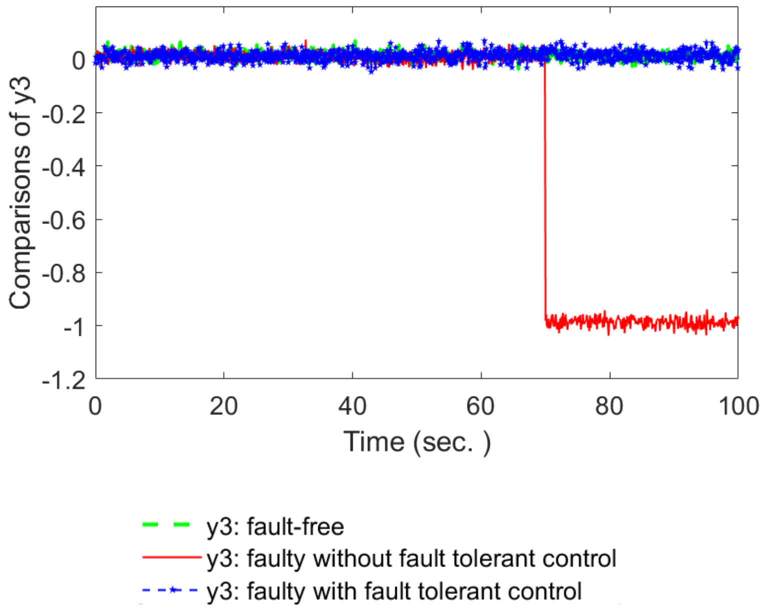


Fig. 13. $y_3(k)$ in different scenarios

Figs. 6-10 illustrate that the presented fault reconstruction-based resilient control scheme can obtain robust estimates of both system state and faults. From Figs. 11-13, we can notice that actuator fault and sensor fault make the system outputs have differences from fault-free outputs. In this example, the faults have significant influences on the first and third outputs. The designed resilient control technique can reduce the influence of faults, providing reliable outputs even under faulty scenarios.

Many systems have high nonlinearities, and fuzzy modelling is an effective technique to handle nonlinearities, see [32, 33]. Therefore, fault estimation and resilient control of Takagi-Sugeno fuzzy systems with general form of stochastic perturbation will be a topic of our further research.

5. Conclusion

A fault reconstruction-based resilient control strategy has been proposed for stochastic discrete-time linear systems and Lipschitz nonlinear systems in this paper. The system under investigation is subjected to unknown inputs, faults, and Brownian perturbations. The unknown inputs under consideration cannot be fully decoupled, and the Brownian motions concerned exist in state, control, fault and uncertainties, simultaneously. The investigation of this type of system is challenging but generally exist in real industrial field. Several advanced techniques, including augmented approach, unknown input observer, optimization algorithms, observer-based control, and signal compensation, have been integrated. Implementing the presented reconstruction and resilient control technique, the system state and fault can be estimated satisfactorily, and the influences of faults have been mitigated successfully. Therefore, the reliability and safety of stochastic discrete-time systems can be enhanced.

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