Convergence of velocities for the short range communicated discrete-time Cucker−Smale model

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Abstract

Most existing literature about the discrete-time Cucker−Smale model focus on the asymptotic flocking behavior. When the communication weight has a long range, asymptotic flocking holds for any initial data. Actually, the velocity of every agent will exponentially converge to the same limit in this case. However, when the communication weight has a short range, asymptotic flocking does not exist for general initial data. In this note, we will prove the convergence of velocities for any initial data in the short range communication case. We first propose a new strategy about the convergence of velocities, and then show an important inequality about the velocity-position moment, according to which we will successfully prove the convergence of velocities and obtain the convergence rates for two kinds of communication weights. Besides, for some special initial data we show that the limits of velocities can be different from each other. Simulation results are given to validate the theoretical results.

Key words: discrete-time; Cucker−Smale model; velocity convergence; convergence rates.

1 Introduction

The investigation of asymptotic behavior of the Cucker−Smale model has gained increasing interest in recent years. This is due to its wide applications in self-propelled agents, formation of various spacecrafts and so on [1–6]. This model was first proposed in [7] by Cucker and Smale in 2007 on the basis of Vicsek model [8], and then extensively studied in directions such as nonlinear velocity couplings [9–11], collision avoidance [12–15], pattern formation as well as additional deterministic or adding stochastic noises [16–18], to include singular communication functions [20–22] or more general interaction [3, 18, 23] and so on. Moreover, a remarkable application of the Cucker−Smale principle was given in [24] for the spacecrafts of the Darwin mission of the European Space Agency. Its main advantage is that, with rather less fuel expenditure, the spacecraft fleet keep remaining in flight (flock).

The results of asymptotic behavior of the C−S model can be roughly divided into two categories: conditional flocking and unconditional flocking, depending on a parameter β which reflects the decay rate of the communication weight between agents [7, 18, 21–23, 25]. Specifically speaking, when the communication weight has a long range, asymptotic flocking occurs without any restriction on initial configurations. But when the communication weight has a short range, asymptotic flocking appears only for very special initial configurations. However, the short range communication case is more suitable for practical applications since agent's sensory ability is limited. Moreover, most of the existing works on the C−S model concentrated on the continuous-time case [11, 12, 20, 27]. [7] proved that under certain conditions on the initial positions and velocities of the birds, flocking occurs for the discrete-time C−S model. [18] modified this model by adding some random noise and proved that under conditions similar to those mentioned in work [7], (nearly) flocking occurs in finite time with a certain confidence. [4] studied the discrete-time C−S model subject to random failure and proved that the flocking would occur almost surely under some condi-

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[0] A long range communication weight usually means a nonnegative, measurable and decreasing function satisfying $\int_0^\infty \phi(r)dr = \infty$. A short range communication weight is such function satisfying $\int_0^\infty \phi(r)dr < \infty$.

tions on the initial states. By adding some conditions on the time step h and the initial constraints of the agents' positions and velocities, [3] established the flocking behavior of the discrete-time C−S model on general digraphs. [23] studied the flocking behavior in a discrete-time C−S model under hierarchical leadership. Apparently, the above obtained flocking conditions are all dependent upon initial conditions, which have limitations in realistic applications. So, we wonder what other asymptotic behavior can be established when there is no constraint on the initial conditions?

In this technical note, we try to explore further intrinsic asymptotic properties of the discrete-time C−S model with short range communication weights and without any restriction on initial configurations. Work [28] proposed a new asymptotic behavior, based on which it studied the convergence properties of agents' velocities for the short range communicated C−S model. However, it focused on the continuous-time case. It is more difficult and challenging here, since up to now for the discrete-time C−S model with short range communication weights there is very limited work concerning the convergence of velocities, not to mention the convergence rates. In other words, we have to explore new approaches or new perspectives to solve this problem. The main contributions of this work can be summarized as follows. First, we show the relationship between the convergence of velocities and the decay of the second order velocity-position moment. Then, we establish a new inequality about this velocity-position moment, and successfully deduce the convergence of velocities. More importantly, the convergence rates are obtained for the discrete-time C−S model with two kinds of typical short range communication weights.

2 Preliminaries

Consider N agents labeled as $\{1, 2, \ldots, N\}$ moving in $\mathbb{R}^d$ with $d \geq 1$. The discrete-time C−S model of the N agents is described by the following dynamical system:

$$\begin{cases} x_i(n+1) = x_i(n) + hv_i(n), \\ v_i(n+1) = v_i(n) + \dfrac{h}{N} \displaystyle\sum_{j\neq i} \phi_{ij}(n)(v_j(n) - v_i(n)), \end{cases} \tag{1}$$

where $\phi_{ij}(n) = \phi(|x_j(n) - x_i(n)|), n \geq 0, (x_i(n), v_i(n)) \in \mathbb{R}^{2d}$ denotes the position and velocity of ith agent at the time nh, $h > 0$ is the time step. $|\cdot|$ is the standard Euclidean norm of $\mathbb{R}^d$. The weight function $0 \leq \phi \in C^1(\mathbb{R}^+)$ quantifies the influence of agent j over agent i. The initial configuration $\{(x_i(0), v_i(0))\}_{i=1}^N$ is given.

Now we give the commonly used definition of asymptotic flocking of the discrete-time multi-agent systems.

Definition 1 *[7, 12, 29] A discrete-time multi-agent system $\{(x_i, v_i)\}_{i=1}^N$ has asymptotic flocking if and only if*

(i) $\Lambda(n) := \sum_{i=1}^N \sum_{j\neq i} |v_i(n) - v_j(n)|^2$ converges to zero;

(ii) $\Gamma(n) := \sum_{i=1}^N \sum_{j\neq i} |x_i(n) - x_j(n)|^2$ is bounded.

Lemma 1 *[7,19] Let $\{(x_i(n), v_i(n))\}_{i=1}^N$ be a solution to model (1) with the communication weight $\phi(r) = K(1 + r^2)^{-\frac{\beta}{2}}, \beta \in [0, 1], K > 0$. Assume that $h < K^{-1}$, then the model has asymptotic flocking for any initial data, and there exist positive constants λ, C such that $\sum |v_i(n) - v_c|^2 \leq Ce^{-\lambda n}$, where $v_c = \frac{1}{N}\sum v_i(0)$.*

For $0 \leq \beta < 1$, asymptotic flocking was obtained in [7] for both the continuous-time and discrete-time C−S models. Then, it was extended to the case of $\beta = 1$ for the continuous-time and discrete-time models in [20, 29] and [19], respectively. Furthermore, in [20] asymptotic flocking was proved for the continuous-time model with decreasing communication weights ϕ such that $\int_0^\infty \phi(r)dr = \infty$. By Theorem 4 in [19], we know that this extension also holds for the discrete-time model.

However, for $\beta > 1$ or more general short range communication weights $\phi \in L^1(\mathbb{R}^+)$, asymptotic flocking occurs only for very special initial configurations. In these cases, as Cucker and Smale pointed out in [7], asymptotic flocking may fail even for two birds flying on a line. The above discussions prompt us to wonder about the following three questions: (1) Is it possible that each $v_i(n)$ itself converges for any initial data? (2) If each $v_i(n)$ converges, what is the estimation of its convergence rate? (3) What is v_i^* if $v_i(n) \to v_i^*$?

Before answering these questions, we need to show the boundedness of $\Lambda(n)$ and linear growth of $\Gamma(n)$ firstly.

Lemma 2 *Let $\{(x_i(n), v_i(n))\}_{i=1}^N$ be a solution to model (1) with a bounded communication weight ϕ. Assume that $h \leq \|\phi\|_{L^\infty}^{-1}$, then $\Lambda(n+1) \leq \Lambda(n) - \frac{2h}{N} \sum_{i=1}^N \sum_{j\neq i} \phi_{ij}(n)|v_i(n) - v_j(n)|^2$.*

Proof. From model (1), we have that $N^{-1}\sum v_i(n) = v_c$. Then $\Lambda(n) = \sum_{i=1}^N \sum_{j\neq i} |v_i(n) - v_j(n)|^2 = 2N \sum_{i=1}^N |v_i(n)|^2 - 2N^2|v_c|^2$, we only need to prove that $\sum_{i=1}^N |v_i(n+1)|^2 \leq \sum_{i=1}^N |v_i(n)|^2 - \frac{h}{N^2} \sum_{i=1}^N \sum_{j\neq i} |v_i(n) - v_j(n)|^2 \phi_{ij}(n)$. It follows from (1) that

$$\sum_{i=1}^N |v_i(n+1)|^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}|v_i(n)|^2 + \sum_{i=1}^{N}\left|\frac{h}{N}\sum_{j\neq i}\phi_{ij}(n)(v_j(n)-v_i(n))\right|^2 \\
&\quad + \frac{2h}{N}\sum_{i=1}^{N}\sum_{j\neq i}\phi_{ij}(n)(v_j(n)-v_i(n))\cdot v_i(n) \\
&= \sum_{i=1}^{N}|v_i(n)|^2 - \frac{h}{N}\sum_{i=1}^{N}\sum_{j\neq i}|v_i(n)-v_j(n)|^2\phi_{ij}(n) \\
&\quad + \frac{h^2}{N^2}\sum_{i=1}^{N}\left|\sum_{j\neq i}\phi_{ij}(n)(v_j(n)-v_i(n))\right|^2 \\
&\le \sum_{i=1}^{N}|v_i(n)|^2 - \left(1-\frac{N-1}{N}h\|\phi\|_{L^\infty}\right)\cdot \\
&\quad \frac{h}{N}\sum_{i=1}^{N}\sum_{j\neq i}|v_i(n)-v_j(n)|^2\phi_{ij}(n), \qquad (2)
\end{aligned}
$$

where the last inequality is obtained from the Cauchy-Schwarz inequality, i.e.,

$$
\begin{aligned}
&\left|\sum_{j\neq i}\phi_{ij}(n)(v_j-v_i)\right|^2 \\
&\le \left(\sum_{j\neq i}\phi_{ij}(n)\right)\left(\sum_{j\neq i}\phi_{ij}(n)|v_j-v_i|^2\right) \\
&\le (N-1)\|\phi\|_{L^\infty}\sum_{j\neq i}^{N}\phi_{ij}(n)|v_j-v_i|^2. \qquad (3)
\end{aligned}
$$

By the assumption of h we get that $1-\frac{N-1}{N}h\|\phi\|_{L^\infty}\ge \frac{1}{N}$, so from (2) we complete the proof. □

Following from (1) and the above lemma, we have that

$$
\begin{aligned}
\Gamma(n) &= \sum_{i=1}^{N}\sum_{j\neq i}\left|x_i(0)-x_j(0)+h\sum_{k=0}^{n-1}[v_i(k)-v_j(k)]\right|^2 \\
&\le 2\Gamma(0)+2n^2h^2\Lambda(0). \qquad (4)
\end{aligned}
$$

3 Main results

In this section, we mainly focus on the investigation of the convergence of $v_i(n)$ and the convergence rates for the discrete-time C−S model (1) with short range communication weights. As mentioned before, for $0\le\beta\le 1$, $v_i(n)$ tends to converge to the same velocity v_c for any i, and the value of v_c is known. So we are devoted to computing $\sum|v_i(n)-v_c|^2$ or $\Lambda(n)$ to prove this convergence and obtain the convergence rates. However, for $\beta>1$ the limits $\{v_i^*\}_{i=1}^N$ are unknown (if they exist), and can be different from each other. So we need to explore a new strategy to solve this problem for this case.

3.1 Strategy to show the convergence of $v_i(n)$

This subsection mainly deduces a lemma, in which we will show the polynomial convergence of $v_i(n)$ if $v_i(n)-\frac{x_i(n)}{(n+\alpha)h}$ polynomially converges to zero. It means that we can transfer the problem of $v_i(n)\to v_i^*$ to the investigation of $v_i(n)-\frac{x_i(n)}{(n+\alpha)h}\to 0$ in some cases.

Lemma 3 *Let $\{(x_i(n),v_i(n))\}_{i=1}^N$ be a solution to model (1) with given initial data $\{(x_i(0),v_i(0))\}_{i=1}^N$. Assume that there exist $h>0$ and $\alpha>0$ such that $g(n):=v_i(n)-x_i(n)/((n+\alpha)h)\to 0$ and $\sum\frac{g(n)}{n+\alpha+1}<+\infty$. Then, $v_i(n)\to v_i^*$. Furthermore, if there exist constants $c_0>0$ and $\gamma>0$ such that $|v_i(n)-x_i(n)/((n+\alpha)h)|\le c_0n^{-\gamma}$, $\forall\, n\ge 1$, then $|v_i(n)-v_i^*|\le\frac{c_0(2\gamma+1)}{\gamma}n^{-\gamma}$, $\forall n\ge 1$.*

Proof. Let $v_i(n)-x_i(n)/((n+\alpha)h)=g(n)$. From (1) we know that $v_i(n)=\frac{x_i(n+1)-x_i(n)}{h}$, so we have $\frac{x_i(n+1)}{n+\alpha+1}-\frac{x_i(n)}{n+\alpha}=\frac{hg(n)}{n+\alpha+1}$, which means that $\frac{x_i(n)}{n+\alpha}=\sum_{k=0}^{n-1}\frac{hg(k)}{k+\alpha+1}+\frac{x_i(0)}{\alpha}$. Thus, $v_i(n)=g(n)+\sum_{k=0}^{n-1}\frac{g(k)}{k+\alpha+1}+\frac{x_i(0)}{\alpha h}$. Note that $\sum_{k=0}^{\infty}\frac{g(k)}{k+\alpha+1}$ exists, so $v_i(n)$ converges. In this case, define

$$
v_i^*=\sum_{k=0}^{\infty}\frac{g(k)}{k+\alpha+1}+\frac{x_i(0)}{\alpha h}, \qquad (5)
$$

then we can get that $|v_i(n)-v_i^*|\le|g(n)|+\sum_{k=n}^{\infty}\frac{|g(k)|}{k+\alpha+1}$. If we further assume that $|g(n)|\le c_0n^{-\gamma},\forall\, n\ge 1$, then

$$
\begin{aligned}
\sum_{k=n}^{\infty}\frac{|g(k)|}{k+\alpha+1} &\le \frac{c_0}{n^\gamma(n+\alpha+1)}+\sum_{k=n+1}^{\infty}\frac{c_0}{k^{\gamma+1}} \\
&\le \frac{c_0}{n^\gamma}+\int_n^\infty\frac{c_0}{s^{\gamma+1}}ds\le\frac{c_0(\gamma+1)}{\gamma}n^{-\gamma}.
\end{aligned}
$$

Thus, $|v_i(n)-v_i^*|\le\frac{c_0(2\gamma+1)}{\gamma}n^{-\gamma},\forall\, n\ge 1$. □

3.2 A new inequality

Based on Lemma 3, we are devoted to computing $\sum|v_i(n)-x_i(n)/((n+\alpha)h)|^2$ instead of $\sum|v_i(n)-v_c|^2$. For the continuous-time Cucker−Smale model, we have established a key equality about $\sum|v_i(t)-x_i(t)/t|^2$ in Proposition 1 in [28], which is important to obtain the corresponding asymptotic behavior. However, for the discrete-time case here, not an equality but an appropriate and explicit inequality about

$\sum |v_i(n) - x_i(n)/((n+\alpha)h)|^2$ is obtained, which is more complex and takes the role of bridge for this note. For simplicity, we denote $(x_i(n), v_i(n)) = (x_i, v_i)$.

Theorem 1 *Let $\alpha \geq 1$. Let $\{(x_i(n), v_i(n))\}_{i=1}^N$ be a solution to model (1). Assume that $0 \leq \phi \in C^1$ satisfies*

$$\phi(r) \leq \frac{K}{(1+r^2)^{\frac{\beta}{2}}} \quad \text{and} \quad |\phi'(r)| \leq \frac{K}{(1+r^2)^{\frac{\beta+1}{2}}} \tag{6}$$

for some $K > 0$ and $\beta \in (1, 2]$. If $h < (16K)^{-1}$, then there exists $C(K, N, \beta, \alpha) > 0$ depending only upon K, N, β, α and the initial data such that for any $n \geq 0$,

$$\begin{aligned}
&\sum_{i=1}^N \frac{|x_i - (n+\alpha)hv_i|^2}{(n+\alpha)h} + \frac{1}{N}\sum_{i=1}^N \sum_{j\neq i} \Phi(|x_i - x_j|) \\
\leq &\sum_{i=1}^N \frac{|x_i(0) - \alpha h v_i(0)|^2}{\alpha h} + \frac{1}{N}\sum_{i=1}^N \sum_{j\neq i} \Phi(|x_i(0) - x_j(0)|) \\
&+ C(K, N, \beta, \alpha),
\end{aligned} \tag{7}$$

where $\Phi'(r) = -r\phi(r)$. In particular, $C(K, N, \beta, \alpha) = 4NK/\alpha$ when $\beta = 2$.

For the proof of the above theorem, we need two lemmas.

Lemma 4 *Let $\alpha \geq 1$. Let $\{(x_i(n), v_i(n))\}_{i=1}^N$ be a solution to model (1). If $h < \frac{1}{2\|\phi\|_{L^\infty}}$, then for any $n \geq 0$,*

$$\begin{aligned}
&\sum_{i=1}^N \frac{|x_i(n+1) - (n+\alpha+1)hv_i(n+1)|^2}{(n+\alpha+1)h} \\
&- \sum_{i=1}^N \frac{|x_i - (n+\alpha)hv_i|^2}{(n+\alpha)h} \\
\leq &\frac{h}{N}\sum_{i=1}^N \sum_{j\neq i} \phi_{ij}(n)(v_j - v_i)\cdot(x_j - x_i) \\
&- \sum_{i=1}^N \frac{|x_i - (n+\alpha)hv_i|^2}{(n+\alpha)(n+\alpha+1)h}.
\end{aligned} \tag{8}$$

Proof. Following from (1) we have that for any $n \geq 0$,

$$\begin{aligned}
&\sum_{i=1}^N \frac{|x_i(n+1) - (n+\alpha+1)hv_i(n+1)|^2}{(n+\alpha+1)h} \\
= &\sum_{i=1}^N \frac{\left| x_i - (n+\alpha)hv_i - \frac{(n+\alpha+1)h^2}{N}\sum_{j\neq i}\phi_{ij}(n)(v_j - v_i)\right|^2}{(n+\alpha+1)h} \\
= &\sum_{i=1}^N \frac{|x_i - (n+\alpha)hv_i|^2}{(n+\alpha+1)h} \\
&+ \frac{(n+\alpha+1)h^3}{N^2}\sum_{i=1}^N \left|\sum_{j\neq i}\phi_{ij}(n)(v_j - v_i)\right|^2 \\
&- \frac{2h}{N}\sum_{i=1}^N \sum_{j\neq i}\phi_{ij}(n)(v_j - v_i)\cdot(x_i - (n+\alpha)hv_i).
\end{aligned}$$

Since $\phi_{ij}(n) = \phi(|x_i(n) - x_j(n)|) = \phi_{ji}(n)$, the last term of the right hand side can be rewritten as

$$\begin{aligned}
\mathcal{T}_3 = &\frac{h}{N}\sum_{i=1}^N \sum_{j\neq i}\phi_{ij}(n)(v_j - v_i) \\
&\cdot(x_j - x_i + (n+\alpha)hv_i - (n+\alpha)hv_j) \\
= &-\frac{(n+\alpha)h^2}{N}\sum_{i=1}^N \sum_{j\neq i}\phi_{ij}(n)|v_j - v_i|^2 \\
&+ \frac{h}{N}\sum_{i=1}^N \sum_{j\neq i}\phi_{ij}(n)(v_j - v_i)\cdot(x_j - x_i).
\end{aligned}$$

Using (3) again, the second term satisfies

$$\begin{aligned}
\mathcal{T}_2 = &\frac{(n+\alpha+1)h^3}{N^2}\sum_{i=1}^N \left|\sum_{j\neq i}\phi_{ij}(n)(v_j - v_i)\right|^2 \\
\leq &\frac{(n+\alpha+1)h^3(N-1)\|\phi\|_{L^\infty}}{N^2}\sum_{i=1}^N \sum_{j\neq i}\phi_{ij}(n)|v_j - v_i|^2 \\
\leq &\frac{(n+\alpha)h^2}{N}\sum_{i=1}^N \sum_{j\neq i}\phi_{ij}(n)|v_j - v_i|^2
\end{aligned}$$

since $h\|\phi\|_{L^\infty} \leq 1/2 \leq \frac{n+\alpha}{n+\alpha+1}$. Combining the above three estimates we complete the proof. □

Lemma 5 *Let $\alpha \geq 1$. Let $\{(x_i(n), v_i(n))\}_{i=1}^N$ be a solution to model (1). If $0 \leq \phi \in C^1$ satisfies (6), then*

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \sum_{j\neq i}[\Phi(|x_i(n+1) - x_j(n+1)|) - \Phi(|x_i - x_j|)] \\
\leq &-\frac{h}{N}\sum_{i=1}^N \sum_{j\neq i}\phi_{ij}(n)(x_i - x_j)\cdot(v_i - v_j) \\
&+ 4h^2\|\Phi''\|_{L^\infty}\sum_{i=1}^N \left|v_i - \frac{x_i}{(n+\alpha)h}\right|^2 \\
&+ \frac{N}{(n+\alpha)^2}\sup_{r\in[0, C_0(n+\alpha)]}\{r^2|\Phi''(r)|\},
\end{aligned} \tag{9}$$

where $\Phi'(r) = -r\phi(r)$ and $C_0 = \max\{2h\sqrt{\Lambda(0)}, 2\sqrt{\Gamma(0)}\}$.

Proof. For any fixed i, j, denote $x_i - x_j = \Delta x$, $v_i - v_j = \Delta v$, then $x_i(n+1) - x_j(n+1) = \Delta x + h\Delta v$. By Taylor's theorem, there exists a constant $\theta \in (0,h)$ such that

$$
\begin{aligned}
&\Phi(|x_i(n+1)-x_j(n+1)|)-\Phi(|x_i-x_j|)\\
&=\Phi(|\Delta x+h\Delta v|)-\Phi(|\Delta x|)\\
&=\Phi(|\Delta x|)+h\frac{d}{dr}\Phi(|\Delta x+r\Delta v|)\big|_{r=0}\\
&\quad+\frac{1}{2}h^2\frac{d^2}{dr^2}\Phi(|\Delta x+r\Delta v|)\big|_{r=\theta}-\Phi(|\Delta x|)\\
&=h\Phi'(|\Delta x|)\frac{\Delta x\cdot\Delta v}{|\Delta x|}\\
&\quad+\frac{1}{2}h^2\Bigg[\Phi''(|\Delta x+\theta\Delta v|)\left(\frac{(\Delta x+\theta\Delta v)\cdot\Delta v}{|\Delta x+\theta\Delta v|}\right)^2\\
&\quad+\Phi'(|\Delta x+\theta\Delta v|)\cdot\\
&\quad\frac{|\Delta x+\theta\Delta v|^2|\Delta v|^2-((\Delta x+\theta\Delta v)\cdot\Delta v)^2}{|\Delta x+\theta\Delta v|^3}\Bigg].
\end{aligned}
$$

Considering $\Phi'(r)=-r\phi(r)\le 0$ and $|\Delta x+\theta\Delta v||\Delta v|-|(\Delta x+\theta\Delta v)\cdot\Delta v|\ge 0$, we have that

$$
\begin{aligned}
&\Phi(|x_i(n+1)-x_j(n+1)|)-\Phi(|x_i-x_j|)\\
&\le h\Phi'(|\Delta x|)\frac{\Delta x\cdot\Delta v}{|\Delta x|}+\frac{h^2}{2}|\Phi''(|\Delta x+\theta\Delta v|)||\Delta v|^2. \quad (10)
\end{aligned}
$$

Moreover, following from $v_i=\frac{(n+\alpha)h}{(n+\alpha)h+\theta}\left(v_i-\frac{x_i}{(n+\alpha)h}\right)+\frac{x_i+\theta v_i}{(n+\alpha)h+\theta}$ we obtain that

$$
\begin{aligned}
&|\Phi''(|\Delta x+\theta\Delta v|)||\Delta v|^2=|\Phi''(|\Delta x+\theta\Delta v|)||v_i-v_j|^2\\
&=|\Phi''(|\Delta x+\theta\Delta v|)|\Bigg|\frac{(n+\alpha)h}{(n+\alpha)h+\theta}\left(v_i-\frac{x_i}{(n+\alpha)h}\right)\\
&\quad-\frac{(n+\alpha)h}{(n+\alpha)h+\theta}\left(v_j-\frac{x_j}{(n+\alpha)h}\right)\\
&\quad+\frac{1}{(n+\alpha)h+\theta}(x_i-x_j+\theta(v_i-v_j))\Bigg|^2\\
&\le|\Phi''(|\Delta x+\theta\Delta v|)|\Bigg(\left|v_i-\frac{x_i}{(n+\alpha)h}\right|+\left|v_j-\frac{x_j}{(n+\alpha)h}\right|\\
&\quad+\frac{1}{(n+\alpha)h}|x_i-x_j+\theta(v_i-v_j)|\Bigg)^2\\
&\le 4\|\Phi''\|_{L^\infty}\left(\left|v_i-\frac{x_i}{(n+\alpha)h}\right|^2+\left|v_j-\frac{x_j}{(n+\alpha)h}\right|^2\right)\\
&\quad+\frac{2}{(n+\alpha)^2h^2}|\Phi''(|\Delta x+\theta\Delta v|)||\Delta x+\theta\Delta v|^2, \quad (11)
\end{aligned}
$$

where the last inequality is obtained from the following basic inequality $(a+b+c)^2\le 4a^2+4b^2+2c^2, \forall\, a,b,c\ge 0$. By (4), we also have from the definition of $\Delta x, \Delta v$ that

$$
\begin{aligned}
|\Delta x+\theta\Delta v|&=\left|\frac{\theta}{h}(\Delta x+h\Delta v)+\left(1-\frac{\theta}{h}\right)\Delta x\right|\\
&\le\sup\{|\Delta x+h\Delta v|,|\Delta x|\}\\
&\le\sqrt{2\Gamma(0)+2(n+1)^2h^2\Lambda(0)}.
\end{aligned}
$$

By the definition of C_0 and $\alpha\ge 1$, we get that $|\Delta x+\theta\Delta v|\le C_0(n+\alpha)$. Thus, following from (10) (11) and $\Phi'(r)=-r\phi(r)$ we complete the proof. □

Proof of Theorem 1. Firstly, it follows from (6) that

$$
\begin{aligned}
&(1+r^2)^{\beta/2}|\Phi''(r)|=(1+r^2)^{\beta/2}|[-r\phi(r)]'|\\
&\le(1+r^2)^{\beta/2}\phi(r)+r(1+r^2)^{\beta/2}|\phi'(r)|\le 2K. \quad (12)
\end{aligned}
$$

Combining Lemma 4, Lemma 5 with (12), we get that

$$
\begin{aligned}
&\sum_{i=1}^N\frac{|x_i(n+1)-(n+\alpha+1)hv_i(n+1)|^2}{(n+\alpha+1)h}\\
&\quad+\frac{1}{N}\sum_{i=1}^N\sum_{j\ne i}\Phi(|x_i(n+1)-x_j(n+1)|)\\
&\le\sum_{i=1}^N\frac{|x_i-(n+\alpha)hv_i|^2}{(n+\alpha)h}+\frac{1}{N}\sum_{i=1}^N\sum_{j\ne i}\Phi(|x_i-x_j|)\\
&\quad-\left(\frac{(n+\alpha)h}{n+\alpha+1}-8Kh^2\right)\sum_{i=1}^N\left|v_i-\frac{x_i}{(n+\alpha)h}\right|^2\\
&\quad+\frac{N}{(n+\alpha)^2}\|r^\beta\Phi''(r)\|_{L^\infty}[C_0(n+\alpha)]^{2-\beta}\\
&\le\sum_{i=1}^N\frac{|x_i-(n+\alpha)hv_i|^2}{(n+\alpha)h}+\frac{1}{N}\sum_{i=1}^N\sum_{j\ne i}\Phi(|x_i-x_j|)\\
&\quad+\frac{2KNC_0^{2-\beta}}{(n+\alpha)^\beta},
\end{aligned}
$$

since $h\le(16K)^{-1}$. By iteration we obtain (7) since $\sum_{n=0}^\infty\frac{1}{(n+\alpha)^\beta}\le\frac{1}{\alpha^\beta}+\int_0^\infty\frac{1}{(s+\alpha)^\beta}ds\le\frac{\beta\alpha^{1-\beta}}{\beta-1}$. □

3.3 Asymptotic behavior

Now we are ready to get the convergence of velocities and the convergence rates.

Theorem 2 *Let $\{(x_i(n),v_i(n))\}_{i=1}^N$ be a solution to model (1). Assume that $0\le\phi\in C^1$ satisfies (6) and $h<(6K)^{-1}$. Then, $v_i^*:=\lim_{n\to\infty}v_i(n)$ exists for any i.*

(i) When $\beta=2$, there exists a constant $C>0$ such that $|v_i(n)-v_i^*|\le C\sqrt{\frac{1+\log n}{n}}$ for any $n\ge 1$. When $\beta\in(1,2)$, there exists a constant $C>0$ such that $|v_i(n)-v_i^*|\le Cn^{-\frac{\beta-1}{2}}$ for any $n\ge 1$.

(ii) Assume that ϕ further satisfies that $r\phi(r)\in L^1(\mathbb{R}^+)$. Then, there exists a constant $C>0$ such that $|v_i(n)-v_i^*|\le C\frac{1}{\sqrt{n}}$ for any $n\ge 1$.

Proof. We first prove (ii). By choosing $\Phi(r) = \int_r^\infty s\phi(s)ds$, we know that $0 \le \Phi \le \|r\phi(r)\|_{L^1}$. From inequality (7) we have that

$$(n+\alpha)\sum_{i=1}^{N}\left|v_i(n) - \frac{x_i(n)}{(n+\alpha)h}\right|^2 \le C_{\alpha,h}, \qquad (13)$$

where $C_{\alpha,h} = 4NK/\alpha + \sum |x_i(0) - \alpha h v_i(0)|^2/(\alpha h) + \frac{1}{N}\sum\sum \Phi(|x_i(0) - x_j(0)|)$. By Lemma 3 we get the convergence of $v_i(n)$ and $|v_i(n) - v_i^*| \le C/\sqrt{n}$.

For (i), we only consider the case of $\beta = 2$. By choosing $\Phi(r) = -\int_0^r s\phi(s)ds \le 0$, it follows from (4) that $-\Phi(|x_i(n) - x_j(n)|) \le \int_0^{|x_i(n)-x_j(n)|} Ks(1+s^2)^{-1}ds \le C(1+\log n)$. Thus, $\left|v_i(n) - \frac{x_i(n)}{(n+\alpha)h}\right| \le C\sqrt{\frac{1+\log n}{n+\alpha}}$. By the proof of Lemma 3, we get the desired convergence rate. The case of $\beta \in (1,2)$ is similar. □

Remark 1 *When ϕ satisfies (6) with $\beta > 2$, we have that $r\phi(r) \in L^1$, so it is contained in case (ii). In particular, any smooth function with compact support is contained in case (ii).*

Remark 2 *Now the first and second questions proposed in Section 2 have been answered in Theorem 2. Although the precise values of the limits $\{v_i^*\}_{i=1}^N$ are not obtained, we can use Theorem 2 and equality (5) to briefly discuss the multi-cluster phenomenon. In fact, multi-cluster has been observed in numerical simulations for the discrete-time (and the continuous-time) C−S model with various short range communication weights in [26, 30].*

As an example, we consider case (ii) of Theorem 2. From (5) we have that $v_i^* = \sum_{k=0}^{\infty} \frac{v_i(k) - x_i(k)/(k+\alpha)}{k+\alpha+1} + \frac{x_i(0)}{\alpha h}$. So,

$$|v_i^* - v_j^*| \ge \frac{|x_i(0) - x_j(0)|}{\alpha h} - \sum_{k=0}^{\infty} \frac{|v_i(k) - x_i(k)/(k+\alpha)| + |v_j(k) - x_j(k)/(k+\alpha)|}{k+\alpha+1}.$$

By (13) we have that $\left|v_i(k) - \frac{x_i(k)}{k+\alpha}\right| + \left|v_j(k) - \frac{x_j(k)}{k+\alpha}\right| \le \sqrt{\frac{2C_{\alpha,h}}{(k+\alpha)h}}$. Combining the above two inequalities, it can be obtained that

$$|v_i^* - v_j^*| \ge \frac{|x_i(0) - x_j(0)| - 2\sqrt{2\alpha h C_{\alpha,h}}}{\alpha h}, \qquad (14)$$

since $\sum_{k\ge 0} \frac{1}{\sqrt{k+\alpha}(k+\alpha+1)} < \sum_{k\ge 0}\left(\frac{2}{\sqrt{k+\alpha}} - \frac{2}{\sqrt{k+1+\alpha}}\right) = \frac{2}{\sqrt{\alpha}}$. And then, by the definition of $C_{\alpha,h}$ we know that $\alpha h C_{\alpha,h} = \sum_{i=1}^{N} |x_i(0) - \alpha h v_i(0)|^2 + \alpha h \frac{1}{N}\sum_{i=1}^{N}\sum_{j\ne i} \Phi(|x_i(0) - x_j(0)|) + 4hNK$. Choose $\alpha = \frac{\|x^{in}\|\cos\vartheta}{h\|v^{in}\|}$, we know that

$$\alpha h C_{\alpha,h} = 4hNK + \|x^{in}\|^2\sin^2\vartheta + \frac{\|x^{in}\|\cos\vartheta}{\|v^{in}\|}\frac{1}{N}\sum_{i=1}^{N}\sum_{j\ne i}\Phi(|x_i(0) - x_j(0)|),$$

where $x^{in} = (x_1(0), \cdots, x_N(0)), v^{in} = (v_1(0), \cdots, v_N(0))$, $\vartheta \in [0, \pi/2)$ is the included angle of x^{in} and v^{in}.

Thus, for a large number of initial data we can get $|v_i^* - v_j^*| > 0$. In view of (14) and the above inequality, the key is small ϑ, $\|v^{in}\| \ne 0$ and large $|x_i(0) - x_j(0)|$.

Remark 3 *In particular, for any fixed initial non-collisional positions $\{x_i(0)\}_{i=1}^N$, we can design h and initial velocities $\{v_i(0)\}_{i=1}^N$ such that the group will be separated into N-clusters.*

4 Simulations

In this section, we consider a multi-agent system with ten agents to validate the theoretical results. According to Theorem 2 and Remark 1, we demonstrate the results in three cases with three different communication weights:
(1)$\phi(r) = (1+r^2)^{-1}$, $h = 0.2$;
(2) $\phi(r) = (1+r^2)^{-\frac{\beta}{2}}$ with $\beta = 3$ and $h = 0.02$;
(3) $\phi(r) = \frac{1+\cos(\pi r)}{2}$ if $0 \le r \le 3$, and $\phi(r) = 0$ if $r > 3$. (ϕ has a compact support.) $h = 0.2$.

The evolutions of $v_i(n)$, the dynamic changes of $v_i(n) - x_i(n)/((n+1)h)$ for the three cases are shown in Figs. 1, 2 and 3, respectively. We can see that there is no asymptotic flocking in all these cases according to Definition 1, but $|v_i(n) - x_i(n)/((n+1)h)|^2 \to 0$ as $n \to \infty$. Moreover, we plot the dynamic changes of $\frac{n}{1+\log n}\sum |v_i(n) - x_i(n)/(n+1)h|^2$ for Case 1 in Fig. 1 and $n\sum |v_i(n) - x_i(n)/(n+1)h|^2$ for Cases 2 and 3 in Figs. 2 and 3. They are all bounded as time grows. So the estimations of convergence rates are $\sqrt{\frac{1+\log n}{n}}$ and $\frac{1}{\sqrt{n}}$ for Case 1 and Cases 2, 3, respectively.

5 Conclusion

As we know, for the short range communicated C−S model, asymptotic flocking is conditional, which depends on the initial constraints. In this article, we mainly investigated the convergence properties of agents' velocities for the discrete-time C−S model with short range communicated weights and without any initial constraints, which is more natural in practice. A new strategy or a new asymptotic behavior $v_i(n) - \frac{x_i(n)}{(n+\alpha)h} \to 0$ was established based on the relationship between it and the convergence properties of $v_i(n)$. The convergence

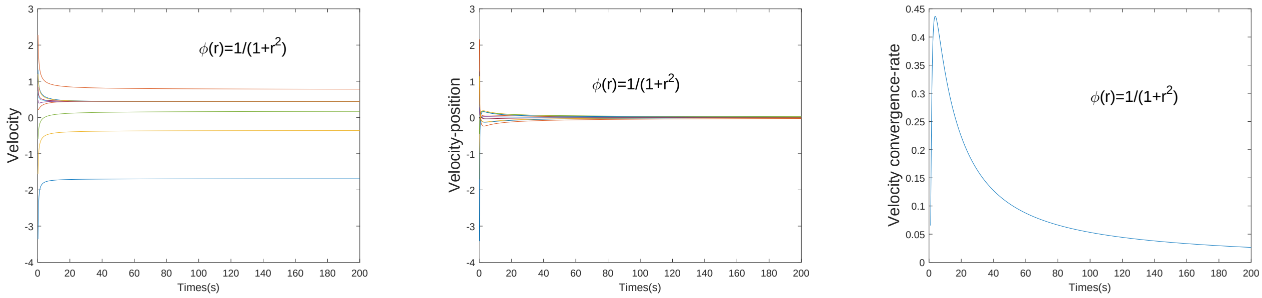

Fig. 1. Evolutions of $v_i(n)$, $v_i(n) - x_i(n)/((n+1)h)$ and estimation of velocities convergence rate for Case 1

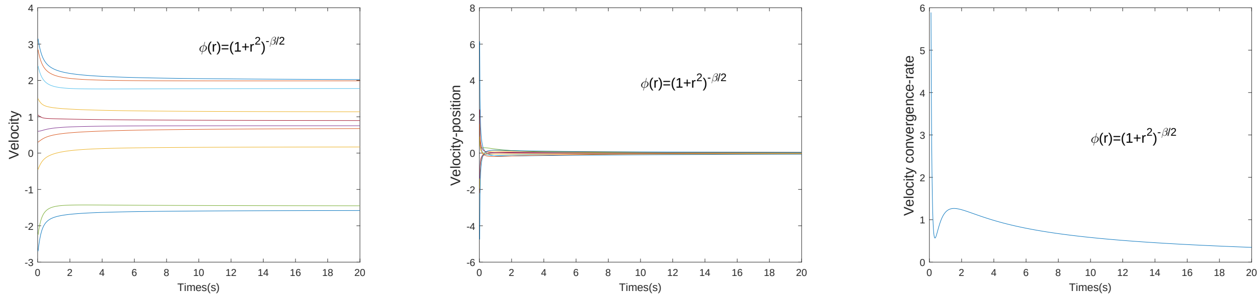

Fig. 2. Evolutions of $v_i(n)$, $v_i(n) - x_i(n)/((n+1)h)$ and estimation of velocities convergence rate for Case 2

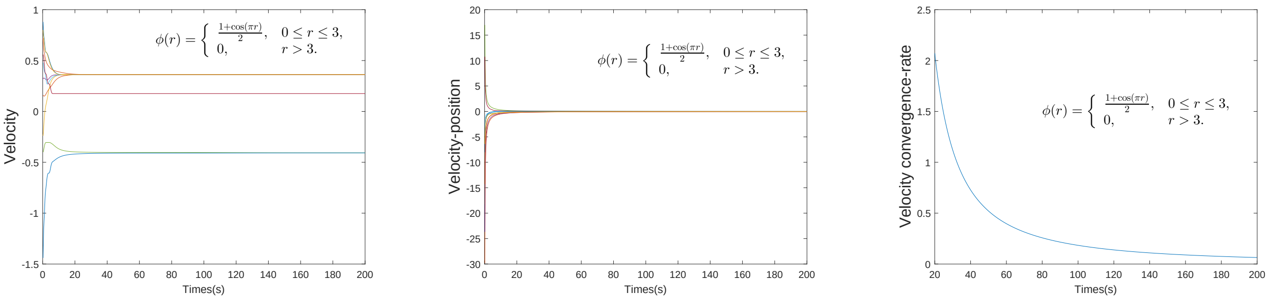

Fig. 3. Evolutions of $v_i(n)$, $v_i(n) - x_i(n)/((n+1)h)$ and estimation of velocities convergence rate for Case 3

rates were also estimated, which could help us a better and deeper understanding of the asymptotic behavior of the discrete-time C−S model.

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