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Interval Coordination of Multiagent Networks with Antagonistic Interactions

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Abstract—This paper studies interval coordination problems for multiagent systems with antagonistic interactions. For strongly connected signed networks, it is shown that when the intersection of intervals imposed by agents is nonempty: (1) the multiagent system achieves bipartite consensus with structurally balanced network; (2) all agents' states must converge to 0, if the signed network is structurally unbalanced. We establish the consensus conditions for bipartite consensus and zero-value consensus by employing the Gauge Transformation and robust analysis of signed networks. When the signed networks are strongly connected and the intersection of intervals is empty, the system reaches an asymptotically stable and unique equilibrium. Moreover, the equilibrium states are only decided by the network structure and interval constraints, and are not related to initial agents' states. Associating the equilibrium of dynamics with the solution of a system of nonlinear equations, we obtain the uniqueness, stability and continuity of equilibria. Finally, numerical simulations are presented to illustrate the theoretical results.

Index Terms—Multiagent systems, interval constraint, bipartite consensus, unique equilibrium, signed digraph

I. INTRODUCTION

CONSENSUS is the foundation of multiagent coordination, which purpose is to make agents' states converge to a common value. With the development of research, more and more consensus problems are studied, and lots of protocols are proposed [1]–[3].

In above consensus problems, there are only cooperative interactions between agents of MAS. However, in many real-world scenarios, competition is often accompanied by cooperation, such as relationship in social psychology [4]. The concept of antagonistic networks is considered in the area of social networks [5], where communication networks are represented as signed graphs. In consideration of the competition in MASs, the consensus problem with antagonistic interactions has been proposed in [6], [7].

In most consensus problems, agents transmit states without any limitation. But in reality, there are diverse saturation limitations among states' transmissions because of the limited actuators or limited communication bandwidth. The consensus problem with state limitation, denoted as constrained consensus, is studied as a new issue. A distributed projected consensus algorithm has been presented in the discrete-time case, where agents combine their local states with projection on constraint sets [8]. In the continuous-time case, projection-based state constraint has been introduced in [9]. Alternative approaches for solving state constraints of consensus problems have been proposed in [10]. The interval consensus problem has been proposed and studied in [11], where agents transmit states limited in individual interval constraints.

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Although there are diverse constraints among agent states in the real-world scenarios, few people study how to constrain the state of multiagent systems with signed graphs. Hence, studying the constrained consensus for multiagent systems with antagonistic interactions is interesting and challenging. As a novel class of constraints, the main feature of interval constraint is soft control constraint, where the agent states could temporarily cross the constraint set and the assumption that the initial states are in the constraint set is not necessary [11]. For the above analysis, we choose interval constraints as the imposed constraints to constrain the agent states of multiagent systems with antagonistic interactions, i.e., make the final bipartite states fall into the constraint set. More importantly, some real applications (see the following statements of main results and significance) inspire our interest in this problem. It is worth noticing that the “interval bipartite consensus” in [7] is a completely different problem, in which agent states fall into a convex set of root nodes without considering the constraints of states.

In this work, we study this problem in two parts: 1) Depending on the structure of the interaction network, we study consensus conditions with nonempty interval intersection; 2) When the interval intersection is empty, the equilibrium phenomenon emerges where agents' states are non-consensus (defined as *non-consensus equilibrium*). In the second part, we study the properties of equilibrium and prove its existence, uniqueness, stability and continuity.

The main results and related significance are as follows.

- 1) *Consensus Case*: When the interval (or transformed interval) intersection is non-empty, the system will achieve bipartite consensus under structurally balanced graphs (Theorem 1), or converge to 0 for structurally unbalanced graphs (Theorem 2). The multiagent system with antagonistic interactions is commonly used to describe duopolistic markets, rival business cartels, etc. [6]. The interval constraint can be seen as the acceptable interval during a price agreement [11]. Hence, the dynamics studied in this paper can simply model the price agreement in duopolistic markets. Moreover, under signed social networks, some researchers study the opinion dynamics with biased assimilation [12], [13], or susceptibility to persuasion [14], etc. Since the interval constraint can depict the observer effect [11] or the phenomenon of expressed and private opinions [15], the problem studied in this work is common and applicable for opinion dynamics.
- 2) *Equilibrium Case*: When the interval intersection is empty, agents' states will reach a unique (Theorem 3) and asymptotically stable (Theorem 4) equilibrium, and its continuity is studied in Theorem 5. Specifically, if the interval intersection is non-empty, but 0 is not in it, the system also achieves an equilibrium. The systematic equilibrium is studied in many research fields, such as network games [16] and neural networks [17]. In an opinion dynamics context, this equilibrium phenomenon illustrates that opinions evolve to a unique equilibrium under manipulative behaviors (can be approximated by interval constraints) [18]. Furthermore, Theorem 4 shows that consensus can be broken by imposing interval constraints on some communication links, which can act as cyber-attacks. Hence, relative results can deepen our understanding of secure

consensus control to avoid potential security problems [19].

Compared with the works about consensus of networked systems with state constraints, this work has the following contributions.

- 1) The interval coordination problem of multiagent networks with antagonistic interactions is first considered, which extends the interaction network of MAS in [11] to the case with antagonistic interactions. The antagonistic interaction increases the complexity of the original nonlinear system. To obtain the robustness of signed networks, we introduce the idea of a root node on a negative cycle.
- 2) Compared with the dynamics in [11], this work extends the individual interval constraint to interval constraint in transmission, which is more general in reality. Under structurally balanced networks, if one agent sends its state to both co-operators and antagonists, the original interval constraint (in [11]) must have positive upper bounds and negative lower bounds to ensure the intersection of gauge transformed intervals is nonempty. Our extension solves the above problem, and can more accurately constrain the multiagent system with antagonistic interactions.
- 3) As a new type of constraint, interval constraints will cause the system to reach non-consensus equilibrium in some particular cases. The non-consensus equilibrium phenomenon of MAS is an emerging issue, which has only received a small amount of attention and research. We notice this phenomenon and study some properties of equilibrium. The convergence of the equilibrium is studied, where a novel Lyapunov function is constructed via coordinate transformation. To prove the uniqueness of equilibrium, we convert the uniqueness of equilibrium to the uniqueness of a system of nonlinear equations and apply the Perron-Frobenius theorem to obtain a contradiction. Using the solutions of systems of equations to quantify the effect of interval constraints on equilibria, we prove the equilibria are Lipschitz continuous to the interval constraints. To the best of our knowledge, this work is the first one to prove that in an empty interval intersection case, MAS reaches an asymptotically stable equilibrium.

The rest of the paper is organized as follows. We set up the system model and background material in Section II. Part A of Section III is for the nonempty interval intersection case where the MAS will achieve consensus or bipartite consensus. Part B of Section III is for the empty interval intersection case where the MAS will reach non-consensus equilibria, and properties of the equilibrium are studied. Meanwhile, supports of numerical examples are presented in Section IV. Finally, some remnant proofs are given in the Appendix.

Notations: Denote $\mathbf{S} = \text{diag}\{S_1, S_2, \dots, S_n\}$ as the Gauge Transformation matrix, where $S_i \in \{\pm 1\}$ is the Gauge Transformation variable ($i = 1, 2, \dots, n$).

Consider an autonomous system $\dot{\mathbf{z}}(t) = f(\mathbf{z}(t))$, where $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a continuous function. Denote $\Lambda^+(\mathbf{z})$ as the set of all positive limit points of this system (Appendix III in [20]).

Denote L_∞ -norm $\|\mathbf{a}\|_\infty = \max_i |a_i|$ for a vector $\mathbf{a} = [a_1, \dots, a_N]^T$. Giving a measurable function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^N$ which is defined on $[0, +\infty)$, $\|f\|_{[0, \infty)}$ denotes the essential supremum of $\{|f(t)|, t \in [0, +\infty)\}$. Denote $\mathcal{F}_1 = \{f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^N : f \text{ is piecewise continuous, } \|f\|_{[0, \infty)} < \infty \text{ and } \lim_{t \rightarrow \infty} f(t) = 0\}$ [21].

Let $W = [w_{ij}] \in M_{m,n}$, and the rank of W is denoted by $R(W)$. $\text{sgn}(a)$ is the sign function of a scalar $a \in \mathbb{R}$. Denote $d^+Y(t)$ as the upper Dini derivative of $Y(t)$:

$$d^+Y(t) = \limsup_{s \rightarrow 0^+} \frac{Y(t+s) - Y(t)}{s}.$$

II. PRELIMINARIES

A. Graph Theory

We study a MAS which consists of n agents and denote $\mathbf{N} = \{1, 2, \dots, n\}$. A (weighted) signed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is a triple, in which $\mathcal{V} = \{v_1, \dots, v_n\}$ is a finite vertex set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set, and $\mathcal{A} \in \mathbb{R}^{n \times n}$ is an adjacency weight matrix with $a_{ij} \neq 0 \iff (v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. In this work, assume that \mathcal{G} has no self-loops, i.e., $a_{ii} = 0, \forall i \in \mathbf{N}$. (v_i, v_j) is defined as the directed edge from v_i to v_j , and $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$ denotes the neighbor set of v_i . $\text{Card}(\mathbf{N})$ means the cardinality of set \mathbf{N} . Let $\aleph = \max |a_{ij}|$.

In graph \mathcal{G} , a (directed) path \mathcal{P} (of length $p-1$) is a sequence of (directed) edges: $\mathcal{P} = \{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_{p-1}}, v_{i_p})\} \subseteq \mathcal{E}$, where v_{i_1}, \dots, v_{i_p} are distinct. The distance $\text{dis}(k, l)$ denotes $\max\{\text{length of } \mathcal{P} : \mathcal{P} \text{ is a path from } v_k \text{ to } v_l\}$; $\max\{\text{dis}(k, l) : k, l \in \mathbf{N}\}$ denotes the diameter of \mathcal{G} .

A closed path in which the end node coincides with the start node is a cycle, such as $(v_{s_0}, v_{s_1}), (v_{s_1}, v_{s_2}), \dots, (v_{s_{m-1}}, v_{s_0})$. And a directed cycle is negative if $a_{s_1 s_0} a_{s_2 s_1} \dots a_{s_0 s_{m-1}} < 0$. A graph \mathcal{G} is strongly connected if $\forall v_i, v_j \in \mathcal{V}, \exists \mathcal{P} \subseteq \mathcal{E}$ concatenating v_i and v_j .

If not specifically pointed out, a **strongly connected** graph \mathcal{G} is our standing assumption throughout the statements of all theorems.

The following is the definition about the connectivity of \mathcal{G} .

Definition 1: A signed graph \mathcal{G} is structurally balanced if there exists a bipartition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of the nodes, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $a_{ij} \geq 0 \forall v_i, v_j \in \mathcal{V}_m$ ($m \in \{1, 2\}$) and $a_{ij} \leq 0 \forall v_i \in \mathcal{V}_m, v_j \in \mathcal{V}_l, m \neq l$ ($m, l \in \{1, 2\}$). And the signed graph \mathcal{G} is structurally unbalanced otherwise.

B. Problem Statement

Let $\mathbf{x}_i(t) \in \mathbb{R}$ be the state of $v_i, \forall i \in \mathbf{N}$. Differing from the general agent evolution, the transmission of state $\mathbf{x}_i(t)$ will be limited in an interval $\mathcal{I}_{ij} := [p_{ij}, q_{ij}], i \in \mathcal{N}_j$, and the value is denoted by $\psi_{ij}(\mathbf{x}_i)$:

$$\psi_{ij}(\mathbf{x}_i) = \begin{cases} p_{ij} & \text{if } \mathbf{x}_i < p_{ij}, \\ \mathbf{x}_i & \text{if } p_{ij} \leq \mathbf{x}_i \leq q_{ij}, \\ q_{ij} & \text{if } \mathbf{x}_i > q_{ij}. \end{cases} \quad i \in \mathcal{N}_j, \quad (1)$$

The continuous-time dynamics of agents is as follows,

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \sum_{j \in \mathcal{N}_i} |a_{ij}| \left[\text{sgn}(a_{ij}) \cdot \psi_{ji}(\mathbf{x}_j(t)) - \mathbf{x}_i(t) \right] \\ &= \sum_{j \in \mathcal{N}_i} a_{ij} \left[\psi_{ji}(\mathbf{x}_j(t)) - \text{sgn}(a_{ij}) \cdot \mathbf{x}_i(t) \right]. \end{aligned} \quad (2)$$

Obviously, the dynamics of MAS (2) is nonlinear. The problem of interval coordination of multiagent networks with antagonistic interactions described above is the following: for any initial states $\mathbf{x}_i(0) \in \mathbb{R}$ and a prior given constraint set $\Omega \in \mathbb{R}$, construct interval constraints \mathcal{I}_{ij} for each agent i under structurally balanced or unbalanced digraphs, such that there exists $c^* \in \Omega$, and

$$\lim_{t \rightarrow \infty} |\mathbf{x}_i(t)| = |c^*|, \quad \forall i \in \mathbf{N}.$$

Definition 2: For all $j \in \mathbf{N}, i \in \mathcal{N}_j$, if \mathbf{e}_i is an interior point of $[p_{ij}, q_{ij}]$, i.e., $\mathbf{e}_i \in \bigcap_{j: i \in \mathcal{N}_j} [p_{ij}, q_{ij}]$, the equilibrium $\mathbf{e} = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n]^T$ is an **equi-unconstrained equilibrium**; if an equilibrium \mathbf{e} is not an equi-unconstrained equilibrium, then it is a **non-equi-unconstrained equilibrium**.

Assumption 1: (Assumption 1 in [21]) $\forall i \in \mathbf{N}, \gamma_i(t)$ is continuous with $t \in [0, \infty)$ except for at most a set with measure zero.

C. Noises

Consider antagonistic interactions and noise disturbance in the dynamics of MAS, one has

$$\dot{\mathbf{x}}_i(t) = \sum_{j=1}^N |a_{ij}| (\text{sgn}(a_{ij}) \cdot \mathbf{x}_j(t) - \mathbf{x}_i(t)) + \gamma_i(t), \quad (3)$$

where $\gamma_i(t)$ is a noise disturbance function.

Denote $\mathbf{x}(t) = [\mathbf{x}_1(t), \dots, \mathbf{x}_N(t)]^T$ and consider the MAS (3) with initial state $\mathbf{x}(t_0) = \mathbf{x}^0 \in \mathbb{R}^N$. We introduce two different definitions based on the structure of \mathcal{G} .

(i) \mathcal{G} is structurally balanced: because the underlying graph \mathcal{G} is structurally balanced, we have $\text{sgn}(a_{ij}) = S_i \cdot S_j = S_j \cdot S_i$ for all $a_{ij} \neq 0$. Let $y_i(t) = S_i \cdot \mathbf{x}_i(t)$ and

$$h(t) = \max_{i \in \mathbf{N}} \{y_i(t)\}, \quad l(t) = \min_{i \in \mathbf{N}} \{y_i(t)\}.$$

(ii) \mathcal{G} is structurally unbalanced: let

$$h(t) = \max_{i \in \mathbf{N}} \{|\mathbf{x}_i(t)|\}, \quad l(t) = \min_{i \in \mathbf{N}} \{-|\mathbf{x}_i(t)|\} = -h(t). \quad (4)$$

Denote $\Delta(\mathbf{x}(t)) = h(t) - l(t)$ and the definition of robust consensus is the same as that in Definition 3.4 in [21].

Lemma 1: Let t_* be the initial time and $h(t) = \max_{i \in \mathbf{N}} \{|\mathbf{x}_i(t)|\}$.

Then along MAS (3), one has $d^+ h(t) \leq \|\gamma(t)\|_\infty, \forall t \geq t_* \geq 0$.

Proof: There are two cases for $h(t)$:

$$(1) h(t) = \max_{i \in \mathbf{N}} \{|\mathbf{x}_i(t)|\} = \max_{i \in \mathbf{N}} \{\mathbf{x}_i(t)\}.$$

Let $l_m := \{j : \mathbf{x}_j(t) = \max_{i \in \mathbf{N}} \{\mathbf{x}_i(t)\}\}$. From Lemma 2.2 in [22], we can conclude that

$$\begin{aligned} d^+ h &= \max_{i \in l_m} \left[\sum_{j \in \mathcal{N}_i} |a_{ij}| (\text{sgn}(a_{ij}) \cdot \mathbf{x}_j(t) - \mathbf{x}_i(t)) + \gamma_i(t) \right] \\ &\leq \max_{i \in l_m} \gamma_i(t) \leq \|\gamma(t)\|_\infty. \end{aligned}$$

$$(2) h(t) = \max_{i \in \mathbf{N}} \{|\mathbf{x}_i(t)|\} = \max_{i \in \mathbf{N}} \{-\mathbf{x}_i(t)\}. \text{ By symmetry, we have } d^+ h \leq \max_{i \in l_m} \gamma_i(t) \leq \|\gamma(t)\|_\infty. \quad \blacksquare$$

Lemma 2: Suppose the Assumption 1 holds. When \mathcal{G} is structurally unbalanced,

(1) MAS (3) achieves global robust consensus, if and only if there exist at least one center (root) node is on a negative and directed cycle and \mathcal{G} has a directed spanning tree;

(2) along the MAS (3), agents' states converge to zero for any $\gamma \in \mathcal{F}_1$, if there exist at least one center (root) node is on a negative and directed cycle and \mathcal{G} has a directed spanning tree.

The proof of Lemma 2 is provided in the Appendix.

Remark 1: Lemma 2 can be trivially extended to time-varying graphs by following its proof method. For the readability of this paper, we just give a version of fixed graphs.

III. MAIN RESULTS

A. Nonempty Interval Intersection: Interval Consensus

In this part, depending on the structure of interaction network, we study consensus conditions with nonempty interval intersection.

Denote $p'_{ij} = \min\{S_i p_{ij}, S_i q_{ij}\}$, and $q'_{ij} = \max\{S_i p_{ij}, S_i q_{ij}\}$. Then the interval is denoted by $\mathcal{I}'_{ij} := [p'_{ij}, q'_{ij}]$. To distinguish \mathcal{I}'_{ij} from the original interval \mathcal{I}_{ij} , we name it **shadow interval**.

The following theorem indicates that if the intersection of shadow intervals is nonempty, agents' states will be enforced to fall into the intersection under a strongly connected and structurally balanced \mathcal{G} , then MAS (2) will achieve bipartite consensus.

Theorem 1: Suppose the signed digraph \mathcal{G} is structurally balanced, $\bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}'_{ij} \neq \emptyset$. Then in MAS (2), for any initial state $\mathbf{x}_* \in \mathbb{R}^n$, $\exists c^*(\mathbf{x}_*) \in \bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}'_{ij}$ such that $\lim_{t \rightarrow \infty} \mathbf{x}_i(t) = S_i \cdot c^*, \forall i \in \mathbf{N}$.

Proof: Since the signed digraph \mathcal{G} is structurally balanced, the Gauge Transformation $\mathbf{x} = \mathbf{S}\mathbf{z}$ can be used to rewrite the dynamics (2),

$$\dot{\mathbf{z}}_i(t) = \sum_{j \in \mathcal{N}_i} |a_{ij}| \left[\psi'_{ji}(\mathbf{z}_j(t)) - \mathbf{z}_i(t) \right],$$

where

$$\psi'_{ij}(\mathbf{z}) = \begin{cases} p'_{ij} & \text{if } \mathbf{z} < p'_{ij}, \\ \mathbf{z} & \text{if } p'_{ij} \leq \mathbf{z} \leq q'_{ij}, \\ q'_{ij} & \text{if } \mathbf{z} > q'_{ij}. \end{cases} \quad (5)$$

Then, this proof can be completed by applying existing results in [11]. \blacksquare

Lemma 3: Suppose all equilibria are equi-unconstrained equilibria, and the directed graph \mathcal{G} is strongly connected and structurally balanced. MAS (2) achieves bipartite consensus if and only if $\bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}'_{ij} \neq \emptyset$.

Proof: The proof for sufficiency statement can be obtained from Theorem 1.

We complete the proof of necessity statement by contradiction. Suppose MAS (2) converges to bipartite consensus and $\bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}'_{ij} = \emptyset$. Then it is obvious that there can not exist equi-unconstrained equilibrium. \blacksquare

The following theorem obtains the condition for MAS to achieve consensus with structurally unbalanced interaction networks.

Theorem 2: Suppose the signed digraph \mathcal{G} is structurally unbalanced, $\bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}_{ij} \neq \emptyset$ and $0 \in \bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}_{ij}$, then $\lim_{t \rightarrow \infty} \mathbf{x}(t) = 0$, for any initial state $\mathbf{x}_* \in \mathbb{R}^n$.

Proof: Along MAS (2), a Lyapunov function is constructed. Define $M(\mathbf{x}(t)) = \max_{i \in \mathbf{N}} \{|\mathbf{x}_i|\}$. And it is obvious that M is locally Lipschitz continuous. Next, we discuss $d^+ M(\mathbf{x}(t))$ in two cases.

$$(1) \max_{i \in \mathbf{N}} |\mathbf{x}_i| = \max_{i \in \mathbf{N}} \mathbf{x}_i.$$

$$\text{Let } l_m(t) := \{j : \mathbf{x}_j(t) = \max_{i \in \mathbf{N}} \mathbf{x}_i(t)\}.$$

$$\begin{aligned} d^+ M(\mathbf{x}(t)) &= d^+ \max_{i \in \mathbf{N}} \{\mathbf{x}_i(t)\} = \max_{i \in l_m(t)} \{\dot{\mathbf{x}}_i(t)\} \\ &= \max_{i \in l_m(t)} \sum_{j \in \mathcal{N}_i} |a_{ij}| \left(\text{sgn}(a_{ij}) \cdot \psi_{ji}(\mathbf{x}_j(t)) - \mathbf{x}_i(t) \right). \quad (6) \end{aligned}$$

Let $i_m \in l_m(t)$, we have $\max_{i \in \mathbf{N}} |\mathbf{x}_i| = \max_{i \in \mathbf{N}} \mathbf{x}_i = \mathbf{x}_{i_m} > 0$. Since $0 \in \bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}_{ij}$, it follows that $p_{ij} \leq 0 \leq q_{ij}, \forall i, j \in \mathbf{N}$. Then $\mathbf{x}_{i_m} \geq |\mathbf{x}_j| \geq |\text{sgn}(a_{ij}) \cdot \psi_{ji}(\mathbf{x}_j(t))| \geq \text{sgn}(a_{ij}) \cdot \psi_{ji}(\mathbf{x}_j(t))$ for all j . So it can be concluded that $d^+ M(\mathbf{x}(t)) \leq 0$ since $\mathbf{x}_{i_m} - \text{sgn}(a_{ij}) \cdot \psi_{ji}(\mathbf{x}_j(t)) \geq 0$.

$$(2) \max_{i \in \mathbf{N}} |\mathbf{x}_i| = -\min_{i \in \mathbf{N}} \mathbf{x}_i, \text{ it also turns out that } d^+ M(\mathbf{x}(t)) \leq 0.$$

On the contrary, the function $m(\mathbf{x}(t)) = \min_{i \in \mathbf{N}} \{|\mathbf{x}_i(t)|\}$ is non-decreasing, i.e., $d^+ m(\mathbf{x}(t)) \geq 0$. Hence, for $V(\mathbf{x}(t)) = M(\mathbf{x}(t)) - m(\mathbf{x}(t))$, we can conclude that $d^+ V(\mathbf{x}(t)) \leq 0$.

Denote $p_A = \max_{j \in \mathbf{N}, i \in \mathcal{N}_j} p_{ij}$, $q_A = \min_{j \in \mathbf{N}, i \in \mathcal{N}_j} q_{ij}$, $p_B = \max\{p_A, -q_A\}$, $q_B = \min\{q_A, -p_A\}$ and $\Theta = \{\mathbf{x} : d^+ V(\mathbf{x}) = 0\}$. Then, we will show that $\Theta \subseteq [p_B, q_B]^n$ if the signed graph \mathcal{G} is strongly connected.

Suppose a solution $\mathbf{x}^* \in \Theta$ with $\mathbf{x}^* \notin [p_B, q_B]^n$, which means there exists one agent i satisfying $\mathbf{x}_i^* \notin [p_B, q_B]$, i.e., $|\mathbf{x}_i^*| > |p_B| = |q_B|$. By symmetry we assume $\mathbf{x}_i^* > q_B$.

Along MAS (2), let the initial solution $\mathbf{x}(t_*) = \mathbf{x}^*$. Denote $l_m = \{j : \mathbf{x}_j^* = \max_{k \in \mathbf{N}} \mathbf{x}_k^*\}$, and let the agent $i \in l_m$, then $\forall j \in l_m$, $\mathbf{x}_j^* = \mathbf{x}_i^*$. It is obvious that there are two possibilities.

(1) $\mathbf{N} \setminus l_m \neq \emptyset$. Because \mathcal{G} is strongly connected, then \mathbf{x}_j^* ($j \in l_m$) will be attracted by the agents in $\mathbf{N} \setminus l_m$. Therefore, there exists $\varepsilon > 0$ and $\forall j \in l_m$ $\mathbf{x}_j(\varepsilon) < \mathbf{x}_j^* = \mathbf{x}_i^*$, which means $M(\mathbf{x}(t_* + \varepsilon)) < M(\mathbf{x}(t_*))$. Hence, $d^+ M(\mathbf{x}) \neq 0$ and the solution \mathbf{x}^* is not in Θ .
(2) $\mathbf{N} \setminus l_m = \emptyset$, i.e., $\forall i, j \in \mathbf{N}$, $\mathbf{x}_i^*(t_*) = \mathbf{x}_j^*(t_*)$. Because the signed graph \mathcal{G} is structurally unbalanced, then $\exists i, j \in \mathbf{N}$, $\text{sgn}(a_{ji}) = -1$. From the dynamics of agent i , we can conclude that $\dot{\mathbf{x}}_i < 0$ and there exists $\mathbf{x}_i(t_* + \varepsilon) < \mathbf{x}_i^*(t_*) = \mathbf{x}_j^*(t_*)$, i.e., $i \notin l_m$, and $\mathbf{N} \setminus l_m \neq \emptyset$. Hence, we get a contradiction.

Combining the two possibilities, we have proved that $\Theta \subseteq [p_B, q_B]^n$. According to Theorem 3.2 in [20], $\Lambda^+(\mathbf{x})$ is contained in Θ and it is clearly that $\Lambda^+(\mathbf{x}) \subseteq [p_B, q_B]^n$. Noting that $[p_B, q_B]^n \subseteq [p_A, q_A]^n$, and applying Theorems 4 and 5 in Appendix III of [20], it can be concluded that: $\mathbf{x}(t) \rightarrow [p_A, q_A]^n$, as $t \rightarrow \infty$. Finally, applying Lemma 2, we can prove Theorem 2. \blacksquare

B. Empty Interval Intersection: Existence, Stability, Uniqueness and Continuity of Equilibria

In Part A, the case where MAS achieves (monopartite or bipartite) consensus has been discussed and it turns out that a nonempty intersection is one of the essential conditions. However, as the agents' number continues to increase, this condition becomes more difficult to guarantee, which causes MAS to achieve a non-consensus equilibrium solution. In this part, we study the existence of equilibria when the system (2) cannot achieve consensus. Furthermore, the condition which leads to an asymptotically stable and unique equilibrium has been obtained. Finally, we prove that the unique equilibrium is Lipschitz continuous with respect to the configuration of interval constraints.

The following lemma points out the existence of equilibrium. Furthermore, the distribution range of equilibria is obtained under strongly connected interaction networks. Denote $p_m = \min_{j \in \mathbf{N}, i \in \mathcal{N}_j} \{p_{ij}, -q_{ij}\}$ and $q_m = \max_{j \in \mathbf{N}, i \in \mathcal{N}_j} \{-p_{ij}, q_{ij}\}$.

Lemma 4: MAS (2) has at least one equilibrium. If signed graph \mathcal{G} is strongly connected, then all equilibria are in $[p_m, q_m]^n$.

Proof: This proof method is similar to the proof of Theorem 2 in [11]. We omit it here. \blacksquare

According to the structure of interaction networks, the following theorem states the conditions for MAS to have a unique equilibrium.

Theorem 3: If any of the following conditions holds:

- (1) \mathcal{G} is structurally unbalanced,
- (2) \mathcal{G} is structurally balanced and $\bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}'_{ij} = \emptyset$,

then MAS (2) has a unique equilibrium.

Proof: Theorem 3 is proven step by step.

Step 1: In this step, we will prove that, along system (2), every equilibrium \mathbf{e} is in \mathcal{Z} , where \mathcal{Z} is a union of solution sets for systems of nonlinear equations.

Consider a system of nonlinear equations:

$$\begin{cases} -\sum_{i \in \mathbf{N}} |a_{1i}| y_1 + a_{12} \psi_{21}(y_2) + \cdots + a_{1n} \psi_{n1}(y_n) = 0, \\ a_{21} \psi_{12}(y_1) - \sum_{i \in \mathbf{N}} |a_{2i}| y_2 + \cdots + a_{2n} \psi_{n2}(y_n) = 0, \\ \vdots \\ a_{n1} \psi_{1n}(y_1) + a_{n2} \psi_{2n}(y_2) + \cdots - \sum_{i \in \mathbf{N}} |a_{ni}| y_n = 0. \end{cases} \quad (7)$$

Comparing (7) with agents' dynamics (2), it turns out that the value of equilibrium \mathbf{e} is a solution of the system of nonlinear equations (7) (y^* denotes the solution, i.e., $y^* = \mathbf{e}$).

For any y^* , $\psi_{ij}(y_i^*)$ is a specific formation, which is one of $\{p_{ij}, y_i^*, q_{ij}\}$. Since there are $n(n-1)$ interval constraint functions ψ_{ij} , the system (7) has $3^{n(n-1)}$ different structures, and all of them are systems of linear equations, such as:

- (1) Suppose $\bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}_{ij} \neq \emptyset$. If $y_i^* \in \bigcap_{j=1}^n \mathcal{I}_{ij} \forall i \in \mathbf{N}$, system (7) can be rewritten as $\mathbf{A}_1 y + \mathbf{b}_1 = 0$, \mathcal{Z}_1 denotes the solution set: $\mathcal{Z}_1 = \{y^* : \mathbf{A}_1 y^* + \mathbf{b}_1 = 0\}$, where \mathbf{A}_1 is an $n \times n$ matrix, and $\mathbf{b}_1 = [0, \dots, 0]^T$:

$$\mathbf{A}_1 = \begin{bmatrix} -\sum_{i \in \mathbf{N}} |a_{1i}| & a_{12} & \cdots & a_{1n} \\ a_{21} & -\sum_{i \in \mathbf{N}} |a_{2i}| & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & -\sum_{i \in \mathbf{N}} |a_{ni}| \end{bmatrix}.$$

- (2) If $y_1^* \geq q_{1n}$, $y_1^* \in \bigcap_{j=1}^{n-1} \mathcal{I}_{1j}$, and $y_i^* \in \bigcap_{j=1}^n \mathcal{I}_{ij}$, $\forall i \in \mathbf{N}$, $i \neq 1$, then system (7) can be rewritten as $\mathbf{A}_2 y + \mathbf{b}_2 = 0$, and $\mathcal{Z}_2 = \{y^* : \mathbf{A}_2 y^* + \mathbf{b}_2 = 0\}$, where $\mathbf{b}_2 = [0, \dots, 0, a_{n1} q_{1n}]^T$,

$$\mathbf{A}_2 = \begin{bmatrix} -\sum_{i \in \mathbf{N}} |a_{1i}| & a_{12} & \cdots & a_{1n} \\ a_{21} & -\sum_{i \in \mathbf{N}} |a_{2i}| & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ 0 & a_{n2} & \cdots & -\sum_{i \in \mathbf{N}} |a_{ni}| \end{bmatrix}.$$

Similarly, we can denote $\mathcal{Z}_3, \mathcal{Z}_4, \dots, \mathcal{Z}_{3^{n(n-1)}}$ and $\mathcal{Z} = \bigcup_{i=1}^{3^{n(n-1)}} \mathcal{Z}_i$. We can conclude that every equilibrium $\mathbf{e} \in \mathcal{Z}$.

Step 2: In this step, we will prove that $\text{Card}\{\mathcal{Z}_i\} = 1$, $i = 2, 3, \dots, 3^{n(n-1)}$, i.e., for any non-equi-unconstrained equilibrium, the corresponding system of linear equations has only one solution.

Since \mathbf{A}_1 is a diagonally-dominant matrix, it turns out that $\lambda^{\mathbf{A}_1} \leq 0$ (which means all real parts of the eigenvalues are non-negative). By the Perron-Frobenius Theorem, it can be concluded that $\lambda^{\mathbf{A}_2} < 0 \implies \det(\mathbf{A}_2) \neq 0$. Applying Cramers Rule, we have proved that the system $\mathbf{A}_2 y + \mathbf{b}_2 = 0$ has only one solution, i.e., $\text{Card}\{\mathcal{Z}_2\} = 1$.

Then, repeating the above analysis, we have completed the proof that $\text{Card}\{\mathcal{Z}_i\} = 1$, $i = 2, 3, \dots, 3^{n(n-1)}$.

Step 3: Denote $\bar{\mathcal{Z}}_i = \{z^* : z^* \text{ is a feasible solution and } z^* \in \mathcal{Z}_i\}$, where $i = 1, 2, \dots, 3^{n(n-1)}$, and $\bar{\mathcal{Z}} = \bigcup_{i=1}^{3^{n(n-1)}} \bar{\mathcal{Z}}_i$. It is clearly that $\bar{\mathcal{Z}} \subseteq \mathcal{Z}$ and $\bar{\mathcal{Z}}_i \subseteq \mathcal{Z}_i$, $i = 1, 2, \dots, 3^{n(n-1)}$. In this step, we will prove that if there exists one solution set $\bar{\mathcal{Z}}_i \neq \emptyset$, then $\bar{\mathcal{Z}}_j = \emptyset$, $j \neq i$.

Let $Y_1^* = [\alpha_1, \dots, \alpha_n]^T$ be the solution of $\mathbf{A}_2 y + \mathbf{b}_2 = 0$ which matches the interval constraints, i.e., Y_1^* is a feasible solution and there exists one solution set $\bar{\mathcal{Z}}_i \neq \emptyset$.

Meanwhile, let $Y_2^* = [\beta_1, \dots, \beta_n]^T$ be the feasible solution of $\mathbf{A}_3 y + \mathbf{b}_3 = 0$, where $\mathbf{b}_3 = [0, \dots, 0, a_{n1} q_{1n}]^T$,

$$\mathbf{A}_3 = \begin{bmatrix} -\sum_{i \in \mathbf{N}} |a_{1i}| & a_{12} & \cdots & a_{1n} \\ a_{21} & -\sum_{i \in \mathbf{N}} |a_{2i}| & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ 0 & a_{n2} & \cdots & -\sum_{i \in \mathbf{N}} |a_{ni}| \end{bmatrix}.$$

arbitrary conditions. Theorems 1 and 2 give the conditions for agents' states to converge to a common value (a special equilibrium). Notice that when the interaction network is structurally unbalanced and $0 \in \bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}_{ij}$, as Theorem 2 saying, agents' states will converge to 0, where 0 is a special unique equilibrium. However, suppose the conditions mentioned in Theorem 1 hold, MAS (2) will admit a bipartite consensus solution, which is affected by initial agents' states. In that case, this solution is not a unique equilibrium. Therefore, we conjecture that with strongly connected interaction networks, except for the case shown in Theorem 1, MAS (2) has a unique equilibrium. Theorem 3 proves this conjecture. Furthermore, Theorem 4 shows that the equilibrium is asymptotically stable.

Remark 3: In the conclusion of [11], there is an unproven conjecture which is the empty interval intersection case leads to a single equilibrium. Since the standard interval consensus problem is a particular case of the problem studied in this work, Corollary 1 confirms the conjecture in [11].

Denote the error between two equilibria \mathbf{e} and \mathbf{e}^* by

$$\varepsilon_{[\mathbf{e}, \mathbf{e}^*]} = \|\mathbf{e} - \mathbf{e}^*\|_\infty,$$

and denote the error between interval configurations \mathcal{I} and \mathcal{I}^* by

$$\varepsilon_{[\mathcal{I}, \mathcal{I}^*]} = \max_{i,j} \{ \|(p_{ij}, q_{ij})^T - (p_{ij}^*, q_{ij}^*)^T\|_\infty \}.$$

Theorem 5: Suppose \mathcal{I} and \mathcal{I}^* are two different interval configurations with unique equilibria \mathbf{e} and \mathbf{e}^* respectively. $\varepsilon_{[\mathbf{e}, \mathbf{e}^*]}$ is Lipschitz continuous with respect to $\varepsilon_{[\mathcal{I}, \mathcal{I}^*]}$.

Proof: From the analysis in Theorem 3, the unique equilibrium \mathbf{e}^* of \mathcal{I}^* can be computed by the following equation,

$$\mathbf{A}_* \mathbf{y}^* + \mathbf{b}_* = 0 \implies \mathbf{e}^* = \mathbf{y}^* = -\mathbf{A}_*^{-1} \mathbf{b}_*,$$

and the existence of \mathbf{A}_*^{-1} is discussed in the proof of Theorem 3.

While the \mathbf{e} of \mathcal{I} can be obtained similarly,

$$\mathbf{A} \mathbf{y} + \mathbf{b} = 0 \implies \mathbf{e} = \mathbf{y} = -\mathbf{A}^{-1} \mathbf{b}.$$

In the above two equations, \mathbf{A}_* , \mathbf{b}_* , \mathbf{A} and \mathbf{b} are decided by \mathcal{I}^* and \mathcal{I} , respectively.

If $\varepsilon_{[\mathcal{I}, \mathcal{I}^*]}$ is small enough such that $\mathbf{A} = \mathbf{A}_*$ and $\mathbf{b} = \mathbf{b}_* + \epsilon$, where $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)^T$, then $\mathbf{y} = -\mathbf{A}_*^{-1}(\mathbf{b}_* + \epsilon) = \mathbf{y}_* - \mathbf{A}_*^{-1}\epsilon$,

$$\begin{aligned} \varepsilon_{[\mathbf{e}, \mathbf{e}^*]} &= \|\mathbf{e} - \mathbf{e}^*\|_\infty = \|\mathbf{y} - \mathbf{y}_*\|_\infty \\ &= \|\mathbf{A}_*^{-1}\epsilon\|_\infty \leq \|\mathbf{A}_*^{-1}\|_\infty \|\epsilon\|_\infty \leq \|\mathbf{A}_*^{-1}\|_\infty \cdot \varepsilon_{[\mathcal{I}, \mathcal{I}^*]}. \end{aligned}$$

If $\mathbf{A} \neq \mathbf{A}_*$, then there exists a finite sequence $\{\mathbf{A} = \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_m = \mathbf{A}_*\}$, which describes a process of $\mathcal{I} \rightarrow \mathcal{I}_*$. By the above analysis, it can be concluded that

$$\varepsilon_{[\mathbf{e}, \mathbf{e}^*]} = \|\mathbf{y} - \mathbf{y}_*\|_\infty \leq \max_i \{\|\mathbf{A}_i^{-1}\|_\infty\} \cdot \varepsilon_{[\mathcal{I}, \mathcal{I}^*]},$$

and it shows that $\varepsilon_{[\mathbf{e}, \mathbf{e}^*]}$ is Lipschitz continuous with respect to $\varepsilon_{[\mathcal{I}, \mathcal{I}^*]}$. ■

Remark 4: From Theorem 3, it is easy to get that the unique equilibrium is decided by the adjacency weight matrix \mathcal{A} and imposed interval constraints \mathcal{I} . Theorem 5 reveals the relation between the unique equilibrium and interval constraints. For two similar configurations of interval constraints, it can conclude that the corresponding equilibria are similar. Therefore, if we obtain the error between two configurations of interval constraints, we can estimate the range of the unknown equilibrium by the known equilibrium.

IV. NUMERICAL EXAMPLE

In this section, numerical examples are presented to illustrate theoretical results. First, we consider the case in which the interaction network is structurally balanced and the intersection of shadow intervals is nonempty. Example 1 illustrates the result in Theorem 1. Then, we simulate a special example where agents' states cannot converge to consensus and reveal that the equilibrium point is unique and asymptotically stable. Denote

$$\begin{aligned} \Omega &= \bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}_{ij}, \quad \Omega' = \bigcap_{j=1}^n \bigcap_{i \in \mathcal{N}_j} \mathcal{I}'_{ij}, \\ p_i &= \max_{j: i \in \mathcal{N}_j} \{p_{ij}\}, \quad q_i = \min_{j: i \in \mathcal{N}_j} \{q_{ij}\}. \end{aligned}$$

Fig. 1 shows that different structures of interaction network and the disturbances of intervals lead to diverse results, which are expounded in Theorems 1-5.

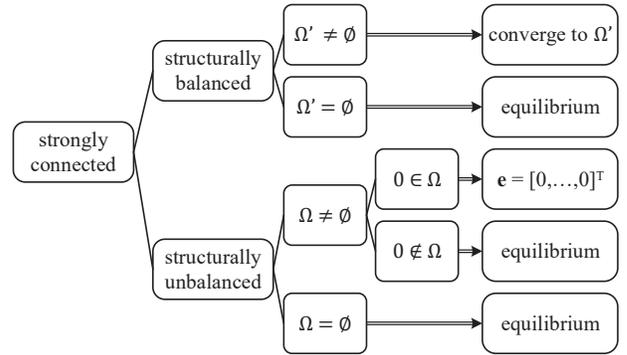


Fig. 1. Diverse results under different system configurations.

Example 1: Figures 2 and 3 show an example of bipartite consensus with $n = 5$ in which the strongly connected graph is structurally balanced and $\Omega' \neq \emptyset$. The interaction network \mathcal{G}_1 is shown in Fig. 2(a). The configuration of interval constraints is shown in Table I. It is worth noting that the intervals \mathcal{I}_{ij} , $i = 2, 4$ and $\forall j : i \in \mathcal{N}_j$ do not have positive upper bounds or negative lower bounds, which illuminates the contribution 2).

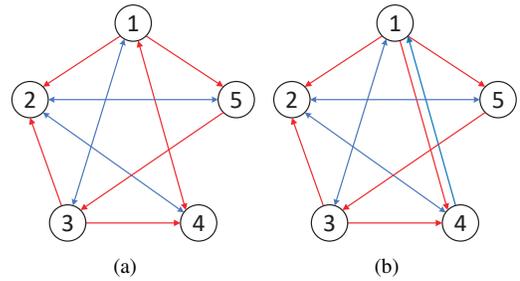


Fig. 2. Interaction Networks \mathcal{G}_1 ((a), structurally balanced) and \mathcal{G}_2 ((b), structurally unbalanced). Blue lines denote cooperative interactions while red ones are antagonistic. Compared with \mathcal{G}_1 , the relation between nodes 1 and 4 in \mathcal{G}_2 is heterogeneous.

Fig. 3(a) shows that agents' states converge to bipartite consensus. In Fig. 3(b), with the help of auxiliary lines, it is clear that consensus value c^* is strictly in the intersection of shadow intervals Ω' .

Example 2: This example shows that if 'the strongly connected graph is structurally balanced (Fig. 2(a)) and $\Omega' = \emptyset$ ' or 'the strongly connected graph is structurally unbalanced (Fig. 2(b)) and $\Omega = \emptyset$ ', then the system (2) achieves a fixed equilibrium point which is unrelated to initial states of agents (see Fig. 4) but related

TABLE I

CONFIGURATION OF INTERVAL CONSTRAINTS IN EXAMPLE 1

\mathcal{I}_{ij} \begin{matrix} j \\ i \end{matrix}	1	2	3	4	5
1	—	[-4, -0.3]	[-4, -0.7]	[-5, -1]	[-3.3, -1]
2	—	—	—	[0, 2.3]	[-0.5, 2]
3	[-5, 1.2]	[-5.4, 1.2]	—	[-7, 1.4]	—
4	[0.2, 3.5]	[1, 3.7]	—	—	—
5	—	[-2, 5]	[-2.4, 5]	—	—

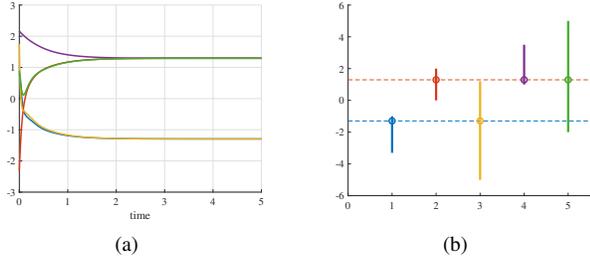


Fig. 3. (a) The trajectories of $\mathbf{x}(t)$ in Example 1; (b) Consensus value c^* (circles), intervals $[p_i, q_i]$ for every agent and auxiliary lines (dotted). The colormap are the same in the above two subfigures.

to the configurations of network and interval constraints. And it can be concluded that MAS (2) has a unique, asymptotically stable equilibrium point which illustrates the obtained results in Theorem 3. It is worth to notice that there is only one difference between \mathcal{G}_1 and \mathcal{G}_2 (the sign of a_{14} is changed), but which causes a shift in each equilibrium.

V. CONCLUSION

This paper focuses on the interval coordination problem with antagonistic interactions. It can be concluded that different combinations of network structures and interval constraints would lead to different convergences of states. Applying the gauge transformation to structurally balanced graph, we prove that agents' states achieve bipartite consensus when the intersection of shadow intervals is non-empty and signed networks are strongly connected. Meanwhile, applying the robust analysis for signed networks, it shows that agents' states will converge to zero if the signed graph is structurally unbalanced. Further, when the intersection of intervals is empty, it is proven that the system achieves a unique and asymptotically stable equilibrium with signed network. Converting the equilibrium to the solution of a system of nonlinear equations, we obtain the uniqueness, stability and continuity of equilibria.

APPENDIX

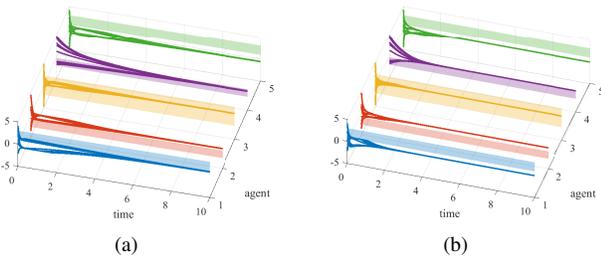


Fig. 4. The trajectories of $\mathbf{x}(t)$ under the graph \mathcal{G}_1 (a) or graph \mathcal{G}_2 (b) in Example 2. Shaded region: intervals $[p_i, q_i]$ for every agent.

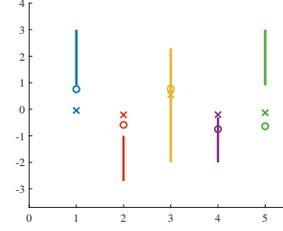


Fig. 5. The equilibrium e (circles for \mathcal{G}_1 and crosses for \mathcal{G}_2) and intervals $[p_i, q_i]$ for every agent in example 2.

A. The proof of Lemma 2

Proof: (Necessity) Since part (1) of Lemma 2 includes two conditions, those are 'a directed spanning tree' and 'at least one center is in a negative cycle', we discuss them in two cases separately to prove the necessity.

(1) We prove this case by a contradiction argument. Suppose \mathcal{G} has a directed spanning tree, but the centers (roots) are not in any negative cycles. Let $\mathbf{N}_1 = \{i : i \text{ is a center node}\}$, $\mathbf{N}_2 = \mathbf{N} \setminus \mathbf{N}_1$. And suppose $\forall i \in \mathbf{N}_1, j \in \mathbf{N}_2$, there are no arcs from j to i . The rest proof is similar to the proof of Theorem 4.1 in [21]. Hence, we omit it.

(2) If the signed graph \mathcal{G} does not have a directed spanning tree, then the condition, i.e., at least one center is in a negative cycle, cannot be met. The following argument is symmetric to Case (1).

Finally, Cases (1) and (2) complete the proof for the necessary condition of Part (1) of Lemma 2.

(Sufficiency) Suppose the signed graph \mathcal{G} has a directed spanning tree, and let d_0 be the diameter of \mathcal{G} . Since there is at least one negative cycle in \mathcal{G} , define \mathcal{C}_1 be one shortest negative cycle of \mathcal{G} , and let d_1 be the length of \mathcal{C}_1 . Let the initial time $t_0 = 0$.

According to Lemma 1, it can be concluded that $\forall t \in [sK_0, (s+1)K_0]$, $s = 1, 2, \dots$,

$$h(t) \leq h(sK_0) + \|\gamma\|_{[0, \infty)} K_0; \quad l(t) \geq l(sK_0) - \|\gamma\|_{[0, \infty)} K_0, \quad (12)$$

where $K_0 = [(d_0 + d_1 - 1)N + 1]T$, $N = \text{Card}(\mathbf{N})$.

Define an index set $\mathbf{N}_C = \{i : i \text{ is the node in } \mathcal{C}_1\}$, and sort the nodes clockwise: $i_0, i_1, \dots, i_{d_C-1}$. Assume that

$$\mathbf{x}_{i_0}(sK_0) \leq \frac{1}{2}l(sK_0) + \frac{1}{2}h(sK_0) = 0. \quad (13)$$

Divide $[sK_0, (s+1)K_0]$ into d_0 time intervals, i.e., $[jmT, (j_m + 1)T] \subseteq [sK_0, (s+1)K_0]$, $m = 1, 2, \dots, d_0$. In this step, we bound $|\mathbf{x}_{i_0}(t)|$ on time interval $[(j_{d_C} + 1)T, (s+1)K_0]$. Denote $y_i = \sum_{j=1}^N |a_{ij}|$, $\forall i \in \mathbf{N}$. With (12), it turns out that

$$\frac{d}{dt} \mathbf{x}_{i_0}(t) \leq -y_{i_0}(\mathbf{x}_{i_0}(t) - h(sK_0) - K_0 \|\gamma\|_{[0, \infty)}) + \|\gamma(t)\|_{\infty}, \quad (14)$$

where $t \in [sK_0, (s+1)K_0]$, which implies

$$\begin{aligned} \mathbf{x}_{i_0}(t) &\leq [1 - e^{-\int_{sK_0}^t y_{i_0} d\tau}] (h(sK_0) + K_0 \|\gamma\|_{[0, \infty)}) \\ &\quad + e^{-\int_{sK_0}^t y_{i_0} d\tau} \mathbf{x}_{i_0}(sK_0) + K_0 \|\gamma\|_{[0, \infty)} \\ &\leq \xi_0 l(sK_0) + (1 - \xi_0) h(sK_0) + 2K_0 \|\gamma\|_{[0, \infty)}, \end{aligned} \quad (15)$$

where $t \in [sK_0, (s+1)K_0]$ and $\chi = e^{-(N-1)K_0 \aleph}$, $\xi_0 = \chi/2$ with \aleph . Based on Gronwall's inequality, the first inequality of (15) is concluded, and from the fact that $l(t) \leq h(t)$, $\forall t \geq 0$, we get the last inequality.

Because there exists a path from i_0 to i_1 , we discuss the following two cases:

(1) $\text{sgn}(a_{i_0 i_1}) = 1$. Define $\hat{y}_{i_1} = y_{i_1} - |a_{i_1 i_0}|$, $B_1 = h(sK_0) + K_0 \|\gamma\|_{[0, \infty)}$ and $B_2 = \xi_0 l(sK_0) + (1 - \xi_0)h(sK_0) + 2K_0 \|\gamma\|_{[0, \infty)}$. For $t \in [j_1 T, (j_1 + 1)T)$, it turns out that

$$\frac{d}{dt} \mathbf{x}_{i_1}(t) \leq \hat{y}_{i_1} (B_1 - \mathbf{x}_{i_1}(t)) + a_{i_1 i_0} (B_2 - \mathbf{x}_{i_1}(t)) + \gamma_{i_1}(t).$$

Repeating the analysis in step 2 of the proof of Theorem 4.1 in [21], we can get that

$$\mathbf{x}_{i_1}(t) \leq \xi_1 l(sK_0) + (1 - \xi_1)h(sK_0) + 4K_0 \|\gamma\|_{[0, \infty)}, \quad (16)$$

where $t \in [(j_1 + 1)T, (s + 1)K_0]$ and $\xi_1 = \zeta \chi^2 / 2$.

(2) $\text{sgn}(a_{i_0 i_1}) = -1$. Similar to (16), it turns out that

$$\mathbf{x}_{i_1}(t) \geq \xi_1 h(sK_0) + (1 - \xi_1)l(sK_0) - 4K_0 \|\gamma\|_{[0, \infty)}, \quad (17)$$

where $t \in [(j_1 + 1)T, (s + 1)K_0]$ and $\xi_1 = \zeta \chi^2 / 2$.

We can get the following inequations by repeating the analysis about node i_1 , when the time interval is $[j_2 T, (j_2 + 1)T)$,

$$\mathbf{x}_{i_2}(t) \leq \xi_2 l(sK_0) + (1 - \xi_2)h(sK_0) + 6K_0 \|\gamma\|_{[0, \infty)},$$

or $\mathbf{x}_{i_2}(t) \geq \xi_2 h(sK_0) + (1 - \xi_2)l(sK_0) - 6K_0 \|\gamma\|_{[0, \infty)}$, where $t \in [(j_2 + 1)T, (s + 1)K_0]$ and $\xi_2 = \zeta^2 \chi^3 / 2$.

Then we can continue our analysis on time intervals $[j_m T, (j_m + 1)T)$ for $m = 3, \dots, d_C$. Since C_1 is a negative cycle, we have

$$\mathbf{x}_{i_0}(t) \geq \xi_{d_C} h(sK_0) + (1 - \xi_{d_C})l(sK_0) - 2(d_C + 1)K_0 \|\gamma\|_{[0, \infty)}, \quad (18)$$

where $t \in [(j_{d_C} + 1)T, (s + 1)K_0]$ and $\xi_{d_C} = \zeta^{d_C} \chi^{d_C + 1} / 2$.

Combining inequalities (15) and (18), it can be concluded that

$$|\mathbf{x}_{i_0}(t)| \leq \xi_{d_C} l(sK_0) + (1 - \xi_{d_C})h(sK_0) + 2(d_C + 1)K_0 \|\gamma\|_{[0, \infty)}, \quad (19)$$

where $t \in [(j_{d_C} + 1)T, (s + 1)K_0]$.

Hence, we have proved that $|\mathbf{x}_{i_0}(t)|$ is bounded on time interval $[(j_{d_C} + 1)T, (s + 1)K_0]$.

Because there exists a path from i_0 to others, we define $\mathbf{N}_1 = \{k : \text{there is an arc from } i_0 \text{ to } k \text{ for } \mathcal{G}\}$.

Define $y_m = \sum_{n=1}^m a_{mn}$, $\hat{y}_{m_1} = y_{m_1} - a_{m_1 i_0}$, $C_1 = h(sK_0) + K_0 \|\gamma\|_{[0, \infty)}$ and $C_2 = \xi_{d_C} l(sK_0) + (1 - \xi_{d_C})h(sK_0) + 2(d_C + 1)K_0 \|\gamma\|_{[0, \infty)}$, for $t \in [j_{d_C + 1} T, (j_{d_C + 1} + 1)T)$. And we can get the following inequation by repeating the analysis in Step 1,

$$\begin{aligned} \frac{d}{dt} \mathbf{x}_{m_1}(t) &\leq \hat{y}_{m_1} (C_1 - \mathbf{x}_{m_1}(t)) \\ &\quad + |a_{m_1 i_0}| (C_2 - \mathbf{x}_{m_1}(t)) + \gamma_{i_1}(t). \end{aligned} \quad (20)$$

Denote $\hat{C}_1 = l(sK_0) - K_0 \|\gamma\|_{[0, \infty)} = -C_1$, it turns out that

$$\begin{aligned} \frac{d}{dt} \mathbf{x}_{m_1}(t) &\geq \hat{y}_{m_1} (\hat{C}_1 - \mathbf{x}_{m_1}(t)) \\ &\quad + |a_{m_1 i_0}| (-C_2 - \mathbf{x}_{m_1}(t)) - \gamma_{i_1}(t). \end{aligned} \quad (21)$$

Combining inequalities (20) and (21), it can be concluded that

$$|\mathbf{x}_{i_0}(t)| \leq \xi_{d_C} l(sK_0) + (1 - \xi_{d_C})h(sK_0) + 2(d_C + 2)K_0 \|\gamma\|_{[0, \infty)}, \quad (22)$$

where $t \in [(j_{d_C + 1} + 1)T, (s + 1)K_0]$.

The rest proof is similar to the proof of Theorem 4.1 in [21], and we omit it. Finally, we get the desired robust consensus inequality,

$$\begin{aligned} \eta(\Delta(\mathbf{x}^0), t) &= (1 - \zeta_{d_C + d_0})^{\lfloor \frac{t}{K_0} \rfloor} \Delta(\mathbf{x}^0), \\ \alpha(\|\gamma\|_{[0, \infty)}) &= \left(2 + \frac{2d_C + 2d_0 + 3}{\zeta_{d_0}}\right) K_0 \cdot \|\gamma\|_{[0, \infty)}, \end{aligned} \quad (23)$$

where $\lfloor \frac{t}{K_0} \rfloor$ means round-down. Part (1) of Lemma 2 has been proved. Part (2) is similar to the proof (i) of Proposition 4.10 in [21], and we omit it. ■

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