Long-time saturation of the Loschmidt echo in quantum chaotic billiards

Martha Gutiérrez[1] and Arseni Goussev[2]
[1] *Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany*
[2] *School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom*
(Dated: April 1, 2009)

The Loschmidt echo (LE) (or fidelity) quantifies the sensitivity of the time evolution of a quantum system with respect to a perturbation of the Hamiltonian. In a typical chaotic system the LE has been previously argued to exhibit a long-time saturation at a value inversely proportional to the effective size of the Hilbert space of the system. However, until now no quantitative results have been known and, in particular, no explicit expression for the proportionality constant has been proposed. In this paper we perform a quantitative analysis of the phenomenon of the LE saturation and provide the analytical expression for its long-time saturation value for a semiclassical particle in a two-dimensional chaotic billiard. We further perform extensive (fully quantum mechanical) numerical calculations of the LE saturation value and find the numerical results to support the semiclassical theory.

I. INTRODUCTION

In his seminal 1984 paper [1], Peres studied the stability of motion of quantum systems with respect to small perturbations of the Hamiltonian. He discovered that the quantum motion of a system, whose underlying classical dynamics is chaotic, is more unstable than that of a system, whose dynamics is regular in the classical limit. The quantity introduced by Peres, presently known as the *Loschmidt echo* (LE) or *fidelity*, has been the subject of thorough theoretical and experimental research in the fields of quantum chaos and quantum information [2, 3].

The LE, defined as

$$M(t) = |O(t)|^2 \tag{1}$$

with the amplitude

$$O(t) = \langle \phi_0 | e^{i\tilde{H}t/\hbar} e^{-iHt/\hbar} | \phi_0 \rangle \,, \tag{2}$$

quantifies the "distance" (in the Hilbert space) between the state $e^{-iHt/\hbar}|\phi_0\rangle$, resulting form the initial state $|\phi_0\rangle$ in the course of evolution through a time t under the Hamiltonian H, and the state $e^{-i\tilde{H}t/\hbar}|\phi_0\rangle$ obtained by evolving the same initial state through the same time t, but under a slightly different, perturbed Hamiltonian $\tilde{H}$. The LE, by construction, equals unity for $t = 0$ and typically decays further in time. A variety of different decay regimes – the most prominent ones being the *Lyapunov* [4], *Fermi-Golden-Rule* [4, 5], and the *perturbative* [5, 6, 7] regime – have been found in chaotic systems with various Hamiltonians and Hamiltonian perturbations. In this paper, however, we address the property of the LE generally shared by all (Hermitian) chaotic systems: the saturation of the decay at long times.

Peres provided in his original work [1] a *qualitative* (order-of-magnitude) estimate for the value M_∞ of the LE saturation in chaotic systems. He argued that for small enough perturbations

$$M_\infty \sim N^{-1} \,, \tag{3}$$

where N is the number of eigenstates (of the unperturbed Hamiltonian H) that are significantly represented in the initial state $|\phi_0\rangle$. In other words, N is the size of the effective Hilbert space that is required for a reasonable description of the time evolution of the initial state.

The phenomenon of the LE saturation has been previously addressed in the literature from numerical [8] and analytical [9] perspective, and the validity of the Peres' argument, Eq. (3), has been verified. However, no explicit expression for the proportionality constant in Eq. (3) has been proposed. Our work complements the theory of the LE in chaotic systems by providing the (previously missing) proportionality constant.

In this paper we present the semiclassical analysis of the LE at long times, and derive an expression for the LE saturation value. Our result, while in agreement with Eq. (3), constitutes a *quantitative* estimate of M_∞. The system treated in this paper is a two-dimensional, quantum billiard that exhibits chaotic dynamics in the classical limit. The key method underlying our analytical calculation, however, is not restricted to billiards and can be generalized to a wider range of chaotic systems. We further perform numerical simulations of the time evolution of an initially localized, Gaussian wave packet in a chaotic billiard, and compute the saturation of the LE due to a perturbation caused by a deformation of the boundary. The results of the numerical simulation strongly support our analytical predictions. Finally, we conclude the paper with a discussion and final remarks.

II. SEMICLASSICAL APPROACH

We consider the time evolution of a quantum particle moving inside a two-dimensional ballistic cavity – a quantum billiard. In this paper we only consider hard-wall billiards whose underlying classical dynamics is fully hyperbolic [10]. The initial state of the particle is assumed to be the coherent state

$$\phi_0(\mathbf{r}) = \frac{1}{\sqrt{\pi}\sigma} \exp\left[\frac{i}{\hbar} \mathbf{p}_0 \cdot (\mathbf{r} - \mathbf{r}_0) - \frac{(\mathbf{r} - \mathbf{r}_0)^2}{2\sigma^2} \right] \,, \tag{4}$$

with σ quantifying the dispersion of the wave packet, and $\mathbf{r}_0$ and $\mathbf{p}_0$ representing respectively the average position and momentum of the particle. The dispersion σ is assumed to be small compared to the linear size of the billiard for the wave function to be normalizable to unity. We further define the de Broglie wavelength of the particle as

$$\lambda = \frac{2\pi\hbar}{p_0}, \tag{5}$$

where $p_0 = |\mathbf{p}_0|$ is the magnitude of the particle's momentum. (Hereinafter we denote the magnitude of a vector by its corresponding symbol in italics.)

The time evolution of the initial state in the unperturbed system with the Hamiltonian H is given by $\phi_t(\mathbf{r}) = \int d\mathbf{r}' K_t(\mathbf{r},\mathbf{r}')\phi_0(\mathbf{r}')$. In the (short-wavelength) *semiclassical* approximation the propagator $K_t(\mathbf{r},\mathbf{r}')$, for a two-dimensional system, can be written as [11]

$$K_t(\mathbf{r},\mathbf{r}') \approx \frac{1}{2\pi i\hbar} \sum_{\gamma'(\mathbf{r}'\to\mathbf{r},t)} D_{\gamma'}\, e^{iS_{\gamma'}/\hbar}. \tag{6}$$

Here, $S_{\gamma'}$ denotes the action integral along the *classical* path γ' leading from the position $\mathbf{r}'$ to $\mathbf{r}$ in time t, and $D_{\gamma'} = \left|\det(-\partial^2 S_{\gamma'}/\partial\mathbf{r}\partial\mathbf{r}')\right|^{1/2} \mathrm{e}^{-i\pi\mu_{\gamma'}/2}$ with the Maslov index $\mu_{\gamma'}$. Then, in the limit (see Appendix A of Ref. [12])

$$\lambda \ll 2\pi\sigma \ll \sqrt{2\pi\lambda l_{\mathrm{L}}}, \tag{7}$$

with l_{L} being the Lyapunov length of the billiard, the action integral $S_{\gamma'}$ can be linearized about the trajectory $\gamma(\mathbf{r}_0 \to \mathbf{r}, t)$ connecting the wave packet center $\mathbf{r}_0$ and the point $\mathbf{r}$ in time t: $S_{\gamma'} \approx S_\gamma - \mathbf{p}_\gamma^{(\mathrm{i})}\cdot(\mathbf{r}'-\mathbf{r}_0)$, where $\mathbf{p}_\gamma^{(\mathrm{i})}$ is the initial momentum on the trajectory γ. Using this action linearization and performing a Gaussian integration over the initial point $\mathbf{r}'$ one obtains the semiclassical expression for the time-dependent wave function evolving under H [4]:

$$\begin{aligned} \phi_t(\mathbf{r}) \;\approx\; & \frac{\sigma}{\sqrt{\pi} i\hbar} \sum_{\gamma(\mathbf{r}_0\to\mathbf{r},t)} D_\gamma \\ & \times \exp\left[\frac{i}{\hbar}S_\gamma - \frac{\sigma^2}{2\hbar^2}(\mathbf{p}_\gamma^{(\mathrm{i})} - \mathbf{p}_0)^2\right]. \end{aligned} \tag{8}$$

The wave function $\tilde{\phi}_t(\mathbf{r})$ corresponding to the time evolution under the Hamiltonian $\tilde{H}$ of the perturbed system is given by an equation analogous to Eq. (8) with the trajectories $\gamma(\mathbf{r}_0 \to \mathbf{r}, t)$ replaced by $\tilde{\gamma}(\mathbf{r}_0 \to \mathbf{r}, t)$ satisfying the classical evolution corresponding to $\tilde{H}$.

The LE amplitude, $O(t) = \langle\tilde{\phi}_t|\phi_t\rangle$, is given by

$$\begin{aligned} O(t) \;\approx\; & \frac{\sigma^2}{\pi\hbar^2} \int d\mathbf{r} \sum_{\gamma,\tilde{\gamma}(\mathbf{r}_0\to\mathbf{r},t)} D_\gamma D_{\tilde{\gamma}}^* \exp\left[\frac{i}{\hbar}(S_\gamma - S_{\tilde{\gamma}})\right] \\ & \times \exp\left[-\frac{\sigma^2}{2\hbar^2}\left((\mathbf{p}_\gamma^{(\mathrm{i})} - \mathbf{p}_0)^2 + (\mathbf{p}_{\tilde{\gamma}}^{(\mathrm{i})} - \mathbf{p}_0)^2\right)\right]. \end{aligned} \tag{9}$$

The expression for the LE, then, being the product $O^*(t)O(t)$, with the asterisk denoting the complex conjugation, involves two integrals over the final points, say $\mathbf{r}$ and $\tilde{\mathbf{r}}$, over four sums over trajectories, two corresponding to the perturbed system and two to the unperturbed one. The integrand, in general, is a rapidly oscillating function of $\mathbf{r}$ and $\tilde{\mathbf{r}}$; therefore, only the trajectories with the overall phase difference smaller than $\hbar$ give a finite contribution to the integral. Considering trajectories such that $S_\gamma \approx S_{\tilde{\gamma}}$ leads to exponentially decaying regimes of the LE [4]. Therefore, non-decaying contributions to the LE (responsible for the LE saturation) can only result from trajectories that are close in action and belong to the same Hamiltonian. This imposes a restriction on the possible configurations of the trajectories of interest, namely $\mathbf{r} \approx \tilde{\mathbf{r}}$. This makes it convenient to make the following transformation to the new integration coordinates: $\mathbf{Q} = (\mathbf{r}+\tilde{\mathbf{r}})/2$ and $\mathbf{q} = \mathbf{r} - \tilde{\mathbf{r}}$. Then, following the procedure above, we linearize the four trajectories entering the expression for the LE about the same final point $\mathbf{Q}$ to obtain

$$\begin{aligned} M(t) \approx & \frac{\sigma^4}{\pi^2\hbar^4} \int d\mathbf{Q} \int d\mathbf{q} \sum_{\gamma,\gamma',\tilde{\gamma},\tilde{\gamma}'} D_\gamma D_{\gamma'}^* D_{\tilde{\gamma}}^* D_{\tilde{\gamma}'} \\ & \times \exp\left\{\frac{i}{\hbar}\Delta S - \frac{\sigma^2}{2\hbar^2} \sum_{\gamma_i=\{\gamma,\gamma',\tilde{\gamma},\tilde{\gamma}'\}} (\mathbf{p}_{\gamma_i}^{(\mathrm{i})} - \mathbf{p}_0)^2\right\}, \end{aligned} \tag{10}$$

where $\Delta S = (S_\gamma - S_{\gamma'} - S_{\tilde{\gamma}} + S_{\tilde{\gamma}'}) + (\mathbf{p}_\gamma^{(\mathrm{f})} + \mathbf{p}_{\gamma'}^{(\mathrm{f})} - \mathbf{p}_{\tilde{\gamma}}^{(\mathrm{f})} - \mathbf{p}_{\tilde{\gamma}'}^{(\mathrm{f})})\cdot\mathbf{q}/2$, all the four paths ($\gamma$, γ', $\tilde{\gamma}$ and $\tilde{\gamma}'$) connect $\mathbf{r}_0$ and $\mathbf{Q}$ in time t, with two of them (γ and γ') corresponding to the unperturbed Hamiltonian H and the other two ($\tilde{\gamma}$ and $\tilde{\gamma}'$) to the perturbed Hamiltonian $\tilde{H}$; here $\mathbf{p}^{(\mathrm{f})}$ denotes the final momentum (at the end point $\mathbf{Q}$) on the corresponding classical path. The integrand in Eq. (10) is still a rapidly oscillating function of $\mathbf{Q}$ – ΔS is generally much greater than $\hbar$ and is sensitive to $\mathbf{Q}$ – unless the paths γ, γ', $\tilde{\gamma}$ and $\tilde{\gamma}'$ are correlated. In the *diagonal approximation* [13] the main contribution to the $\mathbf{Q}$-integral comes from such terms in the sum that $S_\gamma - S_{\gamma'} - S_{\tilde{\gamma}} + S_{\tilde{\gamma}'} = 0$. One group of such terms, defined by the identification $\gamma = \tilde{\gamma}$ and $\gamma' = \tilde{\gamma}'$, is responsible for the (generally exponential) time-decay of the LE [4], and leads to a vanishing contribution at long times. The other group, defined by the identification $\gamma = \gamma'$ and $\tilde{\gamma} = \tilde{\gamma}'$, gives rise to a term surviving in the limit $t \to \infty$ and, therefore, provides the leading order contribution to the LE saturation value. Thus, identifying the trajectories of the unperturbed ($\gamma = \gamma'$) and perturbed ($\tilde{\gamma} = \tilde{\gamma}'$) Hamiltonian we obtain

$$\begin{aligned} M_\infty \;\approx\; & \frac{\sigma^4}{\pi^2\hbar^4} \int d\mathbf{Q} \int d\mathbf{q} \sum_{\gamma,\tilde{\gamma}} |D_\gamma|^2 |D_{\tilde{\gamma}}|^2 \\ & \times \exp\left\{\frac{i}{\hbar}(\mathbf{p}_\gamma^{(\mathrm{f})} - \mathbf{p}_{\tilde{\gamma}}^{(\mathrm{f})})\cdot\mathbf{q} \right. \\ & \left. -\frac{\sigma^2}{\hbar^2}\left[(\mathbf{p}_\gamma^{(\mathrm{i})} - \mathbf{p}_0)^2 + (\mathbf{p}_{\tilde{\gamma}}^{(\mathrm{i})} - \mathbf{p}_0)^2\right]\right\}. \end{aligned} \tag{11}$$

In order to evaluate the double sum in the right hand side of Eq. (11) we utilize the sum rule [14]

$$\sum_{\gamma(\mathbf{r}\to\mathbf{r}',t)} |D_\gamma|^2 f(\mathbf{r},\mathbf{p}_\gamma^{(\mathrm{i})};\mathbf{r}',\mathbf{p}_\gamma^{(\mathrm{f})}) = \int d\mathbf{p}\int d\mathbf{p}'\;\mathcal{P}_t(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}')f(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}')\,, \quad (12)$$

where $\mathcal{P}_t(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}') = \delta(\mathbf{r}_\gamma(t)-\mathbf{r}')\delta(\mathbf{p}_\gamma(t)-\mathbf{p}')$ is the classical phase-space probability density for a trajectory $\gamma = \{(\mathbf{r}_\gamma(\tau),\mathbf{p}_\gamma(\tau)),\tau\in[0,t]\}$ starting from the phase-space point $(\mathbf{r}_\gamma(0),\mathbf{p}_\gamma(0)) = (\mathbf{r},\mathbf{p})$ to end at the point $(\mathbf{r}_\gamma(t),\mathbf{p}_\gamma(t)) = (\mathbf{r}',\mathbf{p}')$ while evolving under the Hamiltonian H through time t. Then, since dealing with chaotic Hamiltonians and long times, we replace the probability distribution $\mathcal{P}_t$ by its phase-space average

$$\overline{\mathcal{P}}(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}') = \frac{\delta(H(\mathbf{r}',\mathbf{p}')-H(\mathbf{r},\mathbf{p}))}{\Omega(H(\mathbf{r},\mathbf{p}))}, \quad (13)$$

where $\Omega(E)$ is the phase-space volume of the energy shell $H(\mathbf{r},\mathbf{p}) = E$. For the case of two-dimensional billiards $\Omega(E) = 2\pi mA$, with m being the mass of the particle and A the billiard area, so that in view of Eqs. (12) and (13) the long time ($t\to\infty$) limit of Eq. (11) reads

$$\begin{aligned} M_\infty \approx{}& \frac{\sigma^4}{\pi^2\hbar^4\Omega^2}\int d\mathbf{Q}\int d\mathbf{q} \\ &\iiiint d\mathbf{p}d\mathbf{p}'d\tilde{\mathbf{p}}d\tilde{\mathbf{p}}'\,\delta(H(\mathbf{r}_0,\mathbf{p})-H(\mathbf{Q},\mathbf{p}')) \\ &\times\delta(\tilde{H}(\mathbf{r}_0,\tilde{\mathbf{p}})-\tilde{H}(\mathbf{Q},\tilde{\mathbf{p}}'))\exp\Big\{\frac{i}{\hbar}(\mathbf{p}'-\tilde{\mathbf{p}}')\cdot\mathbf{q} \\ &-\frac{\sigma^2}{\hbar^2}\left[(\mathbf{p}-\mathbf{p}_0)^2+(\tilde{\mathbf{p}}-\mathbf{p}_0)^2\right]\Big\}\,. \end{aligned} \quad (14)$$

We now assume that both Hamiltonians can be written as $\mathbf{p}^2/2m+V(\mathbf{r})$ and perform the integration in the right hand side of Eq. (14) as follows. The $\mathbf{q}$-integration, with the integration limits extended to $\mathbb{R}^2$, results in $(2\pi\hbar)^2\delta(\mathbf{p}'-\tilde{\mathbf{p}}')$. Consequently integrating over $\mathbf{p}'$ and $\tilde{\mathbf{p}}'$ we obtain

$$\begin{aligned} M_\infty \approx{}& \frac{8\pi m\sigma^4}{\hbar^2\Omega^2}\int d\mathbf{Q}\iint d\mathbf{p}d\tilde{\mathbf{p}} \\ &\times\exp\left(-\frac{\sigma^2}{\hbar^2}\left[(\mathbf{p}-\mathbf{p}_0)^2+(\tilde{\mathbf{p}}-\mathbf{p}_0)^2\right]\right) \\ &\times\delta(\Sigma(\mathbf{r}_0)-\Sigma(\mathbf{Q})+(\mathbf{p}^2/2m-\tilde{\mathbf{p}}'^2/2m))\,, \end{aligned} \quad (15)$$

where $\Sigma(\mathbf{r}) = V(\mathbf{r})-\tilde{V}(\mathbf{r})$. Now we assume that the perturbation is small compared to the kinetic part of the Hamiltonian. Alternatively, one may consider perturbations of the Hamiltonian produced by deformations of the billiard boundary [12, 15]; it is the perturbation of the latter type that we use in our numerical experiments of the following section. Thus, assuming $\Sigma = 0$, we have

$$\begin{aligned} M_\infty \approx{}& \frac{4\sigma^2}{\pi\hbar^2 A}\iint d\mathbf{p}d\tilde{\mathbf{p}}\,\delta(p^2-\tilde{p}^2) \\ &\times\exp\left\{-\frac{\sigma^2}{\hbar^2}\left[(\mathbf{p}-\mathbf{p}_0)^2+(\tilde{\mathbf{p}}-\mathbf{p}_0)^2\right]\right\}\,. \end{aligned} \quad (16)$$

Now, we use the integral representation of the δ-function, $\delta(p^2-\tilde{p}^2) = (2\pi)^{-1}\int d\xi\exp(i\xi p^2-i\xi\tilde{p}^2)$, and perform the Gaussian integration over $\mathbf{p}$ and $\tilde{\mathbf{p}}$ (eventually doing the variable change $x = \xi\hbar^2/\sigma^2$) to get

$$\begin{aligned} M_\infty \approx{}& \frac{2\sigma^2}{A}\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}\exp\left(-2a\frac{x^2}{1+x^2}\right) \\ ={}& \frac{2\pi\sigma^2}{A}I_0(a)\exp(-a)\,, \end{aligned} \quad (17)$$

where $a = (p_0\sigma/\hbar)^2 = (2\pi\sigma/\lambda)^2$, and I_0 is the zeroth order modified Bessel function of the first kind. In the limit $a\gg 1$ (or $\lambda\ll\sigma$), which is in agreement with Eq. (7), the asymptotic form $I_0(a)\approx(2\pi a)^{-1/2}\exp(a)$ yields

$$M_\infty \approx \frac{1}{\sqrt{2\pi}}\frac{\lambda\sigma}{A}\,. \quad (18)$$

Equation (18) constitutes the central analytical result of our paper.

It is easy to see that the original argument by Peres, see Eq. (3), is in perfect agreement with Eq. (18) derived in the semiclassical approximation. Indeed, the number of Hamiltonian eigenstate required to properly describe the time evolution of the initial wave packet, given by Eq. (4), can be evaluated as $N = \Omega(E)\Delta E/(2\pi\hbar)^2$. Here, as above, $\Omega(E) = 2\pi mA$ is the phase-space volume of the energy shell at the average energy $E = p_0^2/2m$ of the particle, and $\Delta E = p_0\Delta p/m$ is the energy dispersion of the initial state. Estimating the momentum dispersion as $\Delta p\approx 2\sqrt{2}\hbar/\sigma$ we obtain the following expression for the number of the eigenstates: $N\approx 2\sqrt{2}A/\lambda\sigma$ (and therefore $M_\infty\approx 2\pi^{-1/2}N^{-1}$). In fact, due to certain arbitrariness in determination of Δp the size of the effective Hilbert space N is not properly defined. This difficulty points to a drawback of the original formulation of Eq. (3). On the contrary, Eq. (18) gives the LE saturation value in terms of well defined system parameters, λ, σ, and A, and, therefore, provides a quantitative estimate for M_∞.

In the following section we demonstrate that the semiclassical predictions of Eq. (18) are in agreement with the time saturation of the LE observed in numerical experiments.

III. NUMERICAL SIMULATIONS

In order to support our semiclassical calculations we have performed numerical simulations of a quantum particle moving inside a desymmetrized diamond billiard (DDB). The DDB is defined as a fundamental domain of the area confined by four intersecting disks centered at the vertices of a square. The billiard is fully chaotic [16] and has been previously considered for studying various aspects of quantum chaos [12, 15, 17]. In our numerical experiments we used the piston-like boundary deformation [12] as the perturbation of the Hamiltonian.

The numerical method that we used for propagating the particle's wave function in time is the Trotter-Suzuki algorithm [18]; Reference [12] provides further details on the billiard systems, Hamiltonian perturbation and wave function time propagation.

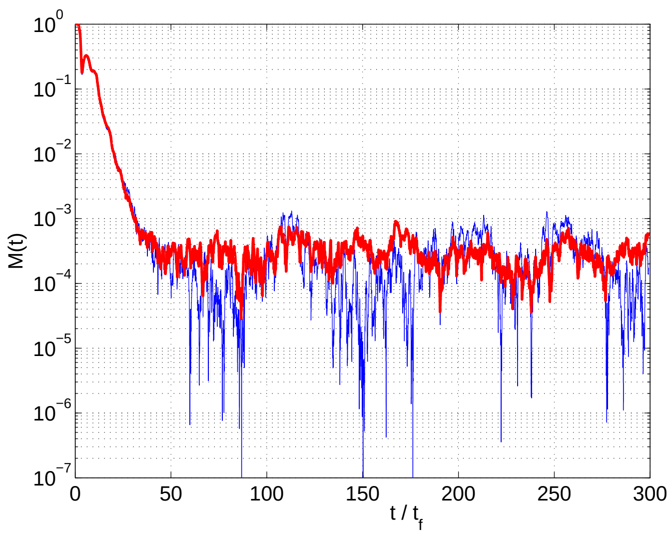

FIG. 1: (Color online) Time decay of the Loschmidt echo in the desymmetrized diamond billiard with a boundary deformation for initial wave packets of de Broglie wavelength $\lambda = 4\pi$ and dispersion $\sigma = 9$. Time is given in units of the free flight time t_{f} of the corresponding classical particle. The thin (blue) line shows an individual LE decay curve resulted from a single numerical experiment. The thick (red) line represents the result of an averaging over 3 individual decay curves obtained for different positions $\mathbf{r}_0$ of the initial wave packet.

In our simulations the initial state of the quantum particle is given by Eq. (4). The blue line in Fig. 1 shows a typical LE decay curve obtained in an individual numerical experiment with the initial wave packet of the dispersion $\sigma = 9$ and de Broglie wavelength $\lambda = 4\pi$; the area of the billiard $A \approx 1.51 \times 10^5$. Time is given in units of the free flight time t_{f} of the counterpart classical billiard, i.e. t/t_{f} is the number of bounces of the corresponding classical particle. The red line in Fig. 1 is the result of the averaging of the LE over 3 individual decay curves, each of which was obtained by propagating a wave packet centered about a different spatial point $\mathbf{r}_0$ inside the billiard domain. (The wave packet centers were chosen such that the three initial states had negligible overlap with one another). Those were the average LE decay curves that we used to determine the LE saturation value and standard deviation – red dots and error bars in Fig. 2 – for the initial quantum state with particular values of the dispersion and de Broglie wavelength.

In Fig. 2 we compare the semiclassical estimate for M_∞, given by Eq. (18), to the saturation values obtained from the numerical simulations. The top (bottom) figure shows the dependence of M_∞ on the dispersion σ (de

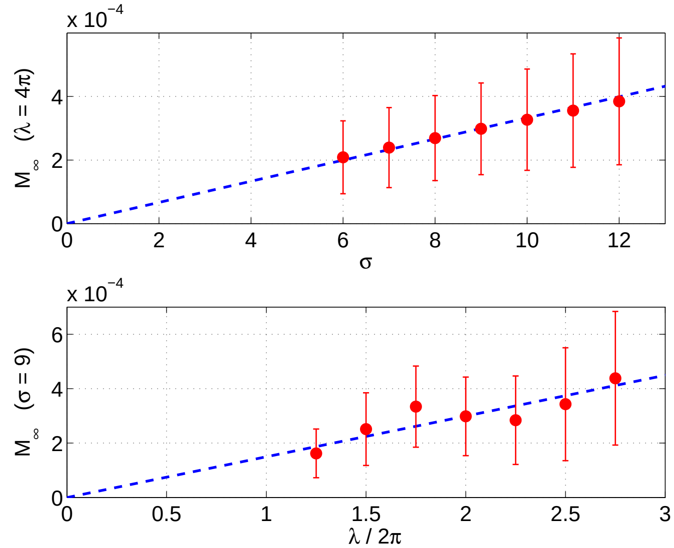

FIG. 2: (Color online) Top figure: The Loschmidt echo saturation value M_∞ as a function of the dispersion σ of the initial wave packet for a fixed de Broglie wavelength $\lambda = 4\pi$. Bottom figure: M_∞ as a function of λ for $\sigma = 9$. In both figures the billiard area $A \approx 1.51 \times 10^5$, and the blue dashed line represents the LE saturation value as predicted by Eq. (18).

Broglie wavelength λ) for $\lambda = 4\pi$ ($\sigma = 9$) in the billiard of the area $A \approx 1.51 \times 10^5$. The red dots together with the error bars represent the numerically observed values of M_∞; the blue dashed lines are plotted in accordance with Eq. (18). We stress here that no free (fitting) parameters have been used in producing the theoretical lines: the slopes of the lines are entirely fixed by Eq. (18).

Finally, to give an idea of the scale of the numerical simulations of this section we note that obtaining an individual LE decay curve, such as the blue curve in Fig. 1, requires more than 8 days of computational time on a high-end (2.8GHz, 2GB RAM) computer. Each data point in Fig. 2 is a result of the averaging over 3 such individual decay curves. Therefore, 39 individual decay curves were obtained to produce the numerical data presented in Fig. 2, amounting to approximately 312 days of (single-processor) computational time.

IV. CONCLUDING REMARKS

In this paper we have used the methods of the semiclassical theory to derive an explicit expression for the value of the long-time saturation of the LE, M_∞, in two-dimensional chaotic billiards. Our quantitative result agrees with the early qualitative argument [1] that the LE saturates at a value inversely proportional to the effective size of the Hilbert space of the system; our calculation provides the previously missing proportionality factor.

In order to support our analytical predictions we have performed careful numerical simulations of a quantum particle moving in a chaotic billiard. In these simulations a deformation of the billiard boundary played the role a Hamiltonian perturbation. The decay of the LE was observed until times long enough to reliably determine M_∞, and a proper ensemble averaging (over the initial position of the quantum particle) was performed to improve the accuracy. The numerically obtained values of the LE saturation were found in a good agreement with the theory.

The central aspect of our semiclassical calculation is the pairing (in the sense of the diagonal approximation) of trajectories that belong to the same (perturbed or unperturbed) Hamiltonian. Those are these trajectory pairs that render the time-independent contribution to the LE in addition to other, exponentially decaying contributions resulting from different trajectory pairs. Here we note that the pairing of trajectories considered in this work has been previously studied in the context of the fidelity fluctuations [9] and the survival probability decay in open chaotic systems [19].

We also note that although the phenomenon of the long-time saturation of the LE has been previously discussed in the literature [8, 9] it has never been subject to a thorough analytical and/or numerical study. In particular, the numerical simulations of the quantum Lorentz gas [8] correctly demonstrated the inverse proportionality of M_∞ to the billiard area, A, while misleadingly suggesting its linear dependence on the square of the wave packet dispersion, σ^2, along with independence of the de Broglie wavelength λ. Reference [9], on the other hand, correctly outlined the semiclassical derivation of the direct proportionality of M_∞ to the effective Plank constant, but did not present an explicit form of the proportionality coefficient. Thus the present paper bridges the gap by providing a quantitative analytical expression for the LE saturation value and, consequently, verifying the expression by means of extensive numerical simulations.

As the final remark we would like to point out that the present semiclassical approach to the phenomenon of the LE saturation is only valid in the long time limit and in the regime of weak Hamiltonian perturbations. In general, however, the LE saturation value will depend on a (properly defined) perturbation strength. It is not yet clear to us how this dependence can be described by the semiclassical theory.

Acknowledgments

The authors are thankful to Rodolfo Jalabert and Klaus Richter for helpful discussions. MG acknowledges financial support from Deutsche Forschungsgemeinschaft (Grant No. FG 760). AG acknowledges EPSRC for support under Grant No. EP/E024629/1.

[1] A. Peres, Phys. Rev. A **30**, 1610 (1984).
[2] For recent reviews see, T. Gorin, T. Prosen, T. H. Seligman, M. Znidaric, Phys. Rep. **435**, 33 (2006); Ph. Jacquod, C. Petitjean, e-print: arXiv:0806.0987.
[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press, 2000).
[4] R. A. Jalabert and H. M. Pastawski, Phys. Rev. Lett. **86**, 2490 (2001).
[5] Ph. Jacquod, P. G. Silvestrov and C. W. J. Beenakker, Phys. Rev. E **64** 055203 (2001).
[6] N. R. Cerruti and S. Tomsovic, Phys. Rev. Lett. **88**, 054103 (2002); J. Phys. A: Math. Gen. **36**, 3451 (2003).
[7] T. Prosen and M. Znidaric, J. Phys. A: Math. Gen. **35**, 1455 (2002).
[8] F. M. Cucchietti, H. M. Pastawski, and D. A. Wisniacki, Phys. Rev. E **65**, 045206(R) (2002); F. M. Cucchietti, H. M. Pastawski, and R. A. Jalabert, Phys. Rev. B **70**, 035311 (2004).
[9] C. Petitjean and Ph. Jacquod, Phys. Rev. E **71**, 036223 (2005); Phys. Rev. Lett. **97**, 194103 (2006); C. Petitjean, Ph.D. thesis, University of Geneva, Geneva, 2007.
[10] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, New York, 1993).
[11] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).
[12] A. Goussev, D. Waltner, K. Richter, and R. A. Jalabert, New J. Phys. 10, 093010 (2008).
[13] M. V. Berry, Proc. R. Soc. Lond. A **400**, 229 (1985).
[14] N. Argaman, Phys. Rev. B **53**, 7035 (1996).
[15] A. Goussev and K. Richter, Phys. Rev. E 75, 015201(R) (2007).
[16] A. Krámli, N. Simányi, and D. Szász, Commun. Math. Phys. **125**, 439 (1989).
[17] S. Müller, Eur. Phys. J. B **34**, 305 (2003); O. Zaitsev, D. Frustaglia, and K. Richter, Phys. Rev. Lett. **94**, 026809 (2005).
[18] H. De Raedt, Annu. Rev. Comput. Phys. **4**, 107 (1996).
[19] M. Gutiérrez, D. Waltner, J. Kuipers, and K. Richter, arXiv:0811.1884.