A quasi-3D hyperbolic shear deformation theory for functionally graded plates

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Abstract

A quasi-3D hyperbolic shear deformation theory for functionally graded plates is developed. The theory accounts for both shear deformation and thickness stretching effects by a hyperbolic variation of all displacements across the thickness, and satisfies the stress-free boundary conditions on the top and bottom surfaces of the plate without requiring any shear correction factor. The benefit of the present theory is that it contains less number of unknowns and governing equations than the existing quasi-3D theories, but its solutions are compared well with 3D and quasi-3D solutions. Equations of motion are derived from Hamilton principle. Analytical solutions for bending and free vibration problems are obtained for simply supported plates. Numerical examples are presented to verify the accuracy of the present theory.

Keywords: functionally graded plate; higher-order theory; bending; vibration

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1. Introduction

Functionally graded materials (FGMs) are a type of nonhomogeneous composites materials, in which the material properties vary smoothly and continuously from one surface to another. A typical FGM is made from a mixture of two material phases, for example, a ceramic and a metal. An advantage of FGMs over laminated composites is that it eliminates the delamination mode of failure found in the laminated composites. In addition, the material properties of FGMs can be tailored to different applications and working environments. This makes FGMs preferable in many structural applications such as nuclear reactor, aerospace, mechanical, automotive, and civil engineering.

Since the shear deformation effects are more pronounced in advanced composites like FGMs, shear deformation theories such as first-order shear deformation theory (FSDT) and higher-order shear deformation theories (HSDTs) should be used. The FSDT [1-9] gives acceptable prediction, but requires a shear correction factor which is hard to find out consistently because of dependent on many parameters including geometry, boundary conditions, and loading conditions. The HSDTs [10-17] do not require a shear correction factor, but their equations of motion are more complicated than those of the FSDT. It should be noted that the thickness stretching effect (i.e., \( \varepsilon_z = 0 \)) is ignored in both the FSDT and HSDTs by assuming a constant transverse displacement through the thickness of the plate. Although this assumption is appropriate for moderately thick functionally graded (FG) plates, but is inaccurate for thick FG ones [18]. The importance of the thickness stretching effect in FG plates has been pointed out in the work of Carrera et al. [19].

Quasi-3D theories are HSDTs in which the transverse displacement is expanded as a higher-order variation through the thickness of the plate, and hence, thickness stretching
effect is captured. There are several quasi-3D theories proposed in the literature. For example, Kant and Swaminathan [20] proposed a quasi-3D theory with all displacement components expanded as a cubic variation through the thickness. The theories presented by Chen et al. [21], Talha and Singh [22], Reddy [23], and Neves et al. [24] are based on a cubic variation of in-plane displacements and a quadratic variation of transverse displacement. Instead of using polynomial functions, Ferreira et al. [25] employed the sinusoidal functions for all displacement components. Neves et al. [26-27] employed the polynomial and the non-polynomial (sinusoidal [26] and hyperbolic [27]) functions for transverse and in-plane displacements, respectively. It should be noted that the above-mentioned quasi-3D theories are too cumbersome and computationally expensive since they handle many unknowns (e.g., theories by Ref. [20] with twelve unknowns, Refs. [21-23] with eleven unknowns, and Refs. [25-27,24] with nine unknowns). Recently, Mantari and Guedes Soares [28] presented a generalized formulation in which many hybrid quasi-3D theories with six unknowns can be derived. Although the hybrid quasi-3D theories [28] have six unknowns, they are still more complicated than the FSDT. As a consequence, a simple quasi-3D theory proposed in the present work is necessary.

This work aims to develop a simple quasi-3D theory with only five unknowns for bending and free vibration analysis of FG plates. The displacement field is chosen based on a generalized formulation [28] with a hyperbolic variation for all displacements. By dividing the transverse displacement into the bending and shear parts, the number of unknowns of the theory is reduced, and thus saving computational time. Equations of motion derived from Hamilton principle are analytically solved for bending and free vibration problems of a simply supported plate. Numerical examples are presented to verify the accuracy of the present theory.
2. Theoretical formulation

As mentioned above, the displacement field of the present theory is chosen based on the generalized formulation with a hyperbolic variation for all displacement components. In fact, the use of hyperbolic functions was first proposed by Soldatos [29], later used by Xiang et al. [30], Akavci [31], and El Meiche et al. [32], and recently by Neves et al. [27]. According to Refs. [33,28], the displacement field takes the form

\[
\begin{align*}
    u_1(x, y, z, t) &= u(x, y, t) - z \frac{\partial w}{\partial x} + \Psi(z) \varphi_x \\
    u_2(x, y, z, t) &= v(x, y, t) - z \frac{\partial w}{\partial y} + \Psi(z) \varphi_y \\
    u_3(x, y, z, t) &= w(x, y, t) + \Psi'(z) \varphi_z(x, y, t)
\end{align*}
\] (1)

where \( u, v, w, \varphi_x, \varphi_y \) and \( \varphi_z \) are six unknown displacement functions of midplane of the plate; and \( \Psi(z) \) is a shape function representing the distribution of the transverse shear strains and shear stresses through the thickness. In this study, the shape function is chosen based on the hyperbolic function proposed by Soldatos [29] as

\[
\Psi(z) = h \sinh \left( \frac{z}{h} \right) - z \cosh \left( \frac{1}{2} \right)
\] (2)

with \( h \) being the thickness of the plate. By deviding the transverse displacement \( w \) into bending and shear parts (i.e., \( w = w_b + w_s \)) and making further assumptions given by \( \varphi_x = \partial w_b / \partial x \) and \( \varphi_y = \partial w_b / \partial y \), the displacement field of the present theory can be rewritten in simpler form as

\[
\begin{align*}
    u_1(x, y, z, t) &= u(x, y, t) - z \frac{\partial w_b}{\partial x} - f(z) \frac{\partial w_s}{\partial x} \\
    u_2(x, y, z, t) &= v(x, y, t) - z \frac{\partial w_b}{\partial y} - f(z) \frac{\partial w_s}{\partial y} \\
    u_3(x, y, z, t) &= w_b(x, y, t) + w_s(x, y, t) + g(z) \varphi_z(x, y, t)
\end{align*}
\] (3)
where \( f(z) = z - \Psi(z) \) and \( g(z) = \Psi'(z) = 1 - f'(z) = \cosh(z/h) - \cosh(1/2) \).

Clearly seen that the displacement field in Eq. (3) handles only five unknowns, i.e., \( u, v, w, w_z, \varphi \).

The strains associated with the displacement field in Eq. (3) are:

\[
\varepsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} - f(z) \frac{\partial^2 w}{\partial x^2}, \quad (4a)
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y} - z \frac{\partial^2 w}{\partial y^2} - f(z) \frac{\partial^2 w}{\partial y^2}, \quad (4b)
\]

\[
\varepsilon_z = g'(z) \varphi, \quad (4c)
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} - 2f(z) \frac{\partial^2 w}{\partial x \partial y}, \quad (4d)
\]

\[
\gamma_{xz} = g(z) \left( \frac{\partial w}{\partial x} + \frac{\partial \varphi}{\partial x} \right), \quad (4e)
\]

\[
\gamma_{yz} = g(z) \left( \frac{\partial w}{\partial y} + \frac{\partial \varphi}{\partial y} \right), \quad (4f)
\]

It can be seen from Eqs. (4e) and (4f) that the transverse shear strains \( \gamma_{xz}, \gamma_{yz} \) are equal to zero at the top \( z = h/2 \) and bottom \( z = -h/2 \) surfaces of the plate. A shear correction factor is, therefore, not required.

The constitutive relations of a FG plate can be written as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix} \quad (5)
\]

where \( C_{ij} \) are the three-dimensional elastic constants determined by
\[ C_{11} = C_{22} = C_{33} = \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} \quad (6a) \]

\[ C_{12} = C_{13} = C_{23} = \frac{\nu E}{(1-2\nu)(1+\nu)} \quad (6b) \]

\[ C_{44} = C_{55} = C_{66} = \frac{E}{2(1+\nu)} \quad (6c) \]

with \( E \) and \( \nu \) being Young’s modulus and Poisson’s ratio, respectively, of a FG plate. Hamilton’s principle is used herein to derive the equations of motion. The principle can be stated in analytical form as

\[ \int_0^T \left( \delta U + \delta V - \delta K \right) dt = 0 \quad (7) \]

where \( \delta U \) is the variation of strain energy; \( \delta V \) is the variation of work done by external forces; and \( \delta K \) is the variation of kinetic energy.

The variation of strain energy is given explicitly by

\[
\delta U = \int_{-h/2}^{h/2} \left( \sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \sigma_z \delta \varepsilon_z + \sigma_{xy} \delta \gamma_{xy} + \sigma_{xz} \delta \gamma_{xz} + \sigma_{yz} \delta \gamma_{yz} \right) dAdz
\]

\[
= \int_A \left[ N_x \frac{\partial \delta u}{\partial x} - M_x \frac{\partial^2 \delta w_b}{\partial x^2} - P_x \frac{\partial^2 \delta w_b}{\partial y^2} + N_y \frac{\partial \delta v}{\partial y} - M_y \frac{\partial^2 \delta w_b}{\partial y^2} - P_y \frac{\partial^2 \delta w_b}{\partial y^2} \right. \\
\left. + R \delta \varphi_x + N_{xy} \left( \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) - 2M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} - 2P_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} \\
+ Q_x \left( \frac{\partial \delta w}{\partial x} + \frac{\partial \delta \varphi_x}{\partial x} \right) + Q_y \left( \frac{\partial \delta w}{\partial y} + \frac{\partial \delta \varphi_x}{\partial y} \right) \right] dA
\]

(8)

where \( N, M, P, Q, \) and \( R \) are the stress resultants defined by

\[
(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} \left( \sigma_x, \sigma_y, \sigma_{xy} \right) dz 
\quad (9a)
\]

\[
(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} \left( \sigma_x, \sigma_y, \sigma_{xy} \right) zdz 
\quad (9b)
\]
\[
\begin{align*}
(P_x, P_y, P_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) f(z) dz \\
(Q_x, Q_y) &= \int_{-h/2}^{h/2} (\sigma_{xz}, \sigma_{yz}) g(z) dz \\
R_z &= \int_{-h/2}^{h/2} \sigma_z g'(z) dz
\end{align*}
\] (9c)

Substituting Eq. (4) into Eq. (5) and the subsequent results into Eq. (9), the stress resultants can be expressed in terms of generalized displacements \((u, v, w, \varphi_x, \varphi_y)\) as

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_{xy} \\
P_x \\
P_y \\
P_{xy} \\
R_z
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & B_{11}^e & B_{12}^e & 0 & X_{13} \\
A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 & B_{12}^e & B_{22}^e & 0 & X_{23} \\
0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0 & B_{66}^e & 0 \\
B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & D_{11}^e & D_{12}^e & 0 & Y_{13} \\
B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 & D_{12}^e & D_{22}^e & 0 & Y_{23} \\
0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0 & D_{66}^e & 0 \\
B_{11}^e & B_{12}^e & 0 & D_{11}^e & D_{12}^e & 0 & H_{11} & H_{12} & 0 & Y_{13}^e \\
B_{12}^e & B_{22}^e & 0 & D_{12}^e & D_{22}^e & 0 & H_{12} & H_{22} & 0 & Y_{23}^e \\
0 & 0 & B_{66}^e & 0 & 0 & D_{66}^e & 0 & 0 & H_{66} & 0 \\
X_{13} & X_{23} & 0 & Y_{13} & Y_{23} & 0 & Y_{13}^e & Y_{23}^e & 0 & Z_{33}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial y} \\
\frac{\partial \varphi_x}{\partial x} \\
\frac{\partial \varphi_x}{\partial y} \\
\frac{\partial \varphi_y}{\partial x} \\
\frac{\partial \varphi_y}{\partial y}
\end{bmatrix}
\] (10a)

\[
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix} =
\begin{bmatrix}
A_{55} & 0 \\
0 & A_{44}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial y} \\
\frac{\partial \varphi_x}{\partial x} \\
\frac{\partial \varphi_x}{\partial y} \\
\frac{\partial \varphi_y}{\partial x} \\
\frac{\partial \varphi_y}{\partial y}
\end{bmatrix}
\] (10b)

where

\[
\begin{align*}
(A_y, A_y', B_y, B_y', D_y, D_y', H_y) &= \int_{-h/2}^{h/2} \left(1, g^2, z, f, z^2, f z, f^2\right) C_y dz \\
(X_y, Y_y, Y_y', Z_y) &= \int_{-h/2}^{h/2} \left(1, z, f, g'\right) g' C_y dz
\end{align*}
\] (11a)

The variation of work done by externally transverse loads \(q\) can be expressed as
\[ \delta V = -\int_A q\delta (w_b + w'_z + g\phi_z) \, dA \]  

(12)

The variation of kinetic energy is

\[ \delta K = \int_A \int_{\frac{h}{2}}^{\frac{h}{2}} \rho (\dot{u} \delta \ddot{u} + \ddot{u} \delta \dot{u} + \dot{v} \delta \ddot{v} + \ddot{v} \delta \dot{v} + \delta \dot{w}_b + \dot{w}_b \delta \ddot{w}_b + \ddot{w}_b \delta \dot{w}_b + \delta \dot{w}_r + \dot{w}_r \delta \ddot{w}_r + \ddot{w}_r \delta \dot{w}_r ) \, dAdz \]

\[ = \int_A \left\{ I_0 \left[ \dot{u} \delta \ddot{u} + \ddot{u} \delta \dot{u} + \dot{v} \delta \ddot{v} + \ddot{v} \delta \dot{v} \right] + J_0 \left[ (\dot{w}_b + \dot{w}_r) \delta \phi_z + \phi_z \delta (\dot{w}_b + \dot{w}_r) \right] \right\} \, dA \]

\[ - I_1 \left( \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \delta \dot{w}_b}{\partial y} + \frac{\partial \delta \dot{w}_b}{\partial \dot{y}} \right) - J_1 \left( \frac{\partial \delta \dot{w}_r}{\partial x} + \frac{\partial \delta \dot{w}_r}{\partial y} + \frac{\partial \delta \dot{w}_r}{\partial \dot{y}} \right) + K_2 \left( \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \delta \dot{w}_b}{\partial y} + \frac{\partial \delta \dot{w}_r}{\partial \dot{y}} \right) \]

\[ + J_2 \left( \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \delta \dot{w}_b}{\partial y} + \frac{\partial \delta \dot{w}_r}{\partial \dot{y}} \right) + K_0 \phi_z \delta \phi_z \right\} \, dA \]  

(13)

where dot-superscript convention indicates the differentiation with respect to the time variable \( t \); \( \rho \) is the mass density; and \( (I_i, J_i, K_i) \) are the mass moments of inertia defined by

\[ (I_0, I_1, I_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, z^2) \rho dz \]  

(14a)

\[ (J_0, J_1, J_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (g, f, fz) \rho dz \]  

(14b)

\[ (K_0, K_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (g^2, f^2) \rho dz \]  

(14c)

The equations of motion can be obtained by substituting the expressions for \( \delta U \), \( \delta V \), and \( \delta K \) from Eqs. (8), (12), and (13) into Eq. (7), integrating by parts, and collecting the coefficients of \( \delta u \), \( \delta v \), \( \delta w_b \), \( \delta w_r \), and \( \delta \phi_z \).

\[ \delta u : \frac{\partial N_u}{\partial x} + \frac{\partial N_v}{\partial y} = I_0 \ddot{u} - I_1 \ddot{w}_b - J_1 \ddot{w}_r \]  

(15a)

\[ \delta v : \frac{\partial N_u}{\partial x} + \frac{\partial N_v}{\partial y} = I_0 \ddot{v} - I_1 \ddot{w}_b - J_1 \ddot{w}_r \]  

(15b)
\[ \delta w_b : \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = I_0 \left( \ddot{w}_b + \dddot{w}_b \right) + J_0 \dddot{\phi}_2 + I_1 \left( \frac{\partial \ddot{u}}{\partial x} + \frac{\partial \dddot{v}}{\partial y} \right) - I_2 \nabla^2 \ddot{w}_b - J_2 \nabla^2 \dddot{w}_b \] (15c)

\[ \delta w_s : \frac{\partial^2 P}{\partial x^2} + 2 \frac{\partial^2 P_{sz}}{\partial x \partial y} + \frac{\partial^2 P_{sz}}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_0 \left( \ddot{w}_b + \dddot{w}_b \right) + J_0 \dddot{\phi}_2 + J_1 \left( \frac{\partial \ddot{u}}{\partial x} + \frac{\partial \dddot{v}}{\partial y} \right) - J_2 \nabla^2 \ddot{w}_b - K_2 \nabla^2 \dddot{w}_b \] (15d)

\[ \delta \phi_z : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - R_z + gq = J_0 \left( \ddot{w}_b + \dddot{w}_b \right) + K_0 \dddot{\phi}_2 \] (15e)

Substituting Eq. (10) into Eq. (15), the equations of motion of the present theory can be expressed in terms of displacements \((u, v, w_b, w_s, \phi_z)\) as

\[ A_{11} \frac{\partial^2 u}{\partial x^2} + A_{66} \frac{\partial^2 v}{\partial y^2} + (A_{12} + A_{66}) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \right) - B_{11} \frac{\partial^3 w_b}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w_b}{\partial x \partial y^2} \] (16a)

\[-B_{11} \frac{\partial^3 w_s}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w_s}{\partial x \partial y^2} + X_{13} \frac{\partial \phi_z}{\partial x} = I_0 \ddot{u} - I_1 \dddot{w}_b - J_1 \dddot{w}_s \]

\[ A_{22} \frac{\partial^2 u}{\partial x^2} + A_{66} \frac{\partial^2 v}{\partial y^2} + (A_{12} + A_{66}) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \right) - B_{22} \frac{\partial^3 w_b}{\partial y^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w_b}{\partial x^2 \partial y} \] (16b)

\[-B_{22} \frac{\partial^3 w_s}{\partial y^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w_s}{\partial x^2 \partial y} + X_{23} \frac{\partial \phi_z}{\partial y} = I_0 \ddot{v} - I_1 \dddot{w}_b - J_1 \dddot{w}_s \]

\[ B_{11} \frac{\partial^3 u}{\partial x^3} + (B_{12} + 2B_{66}) \left( \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \right) + B_{22} \frac{\partial^3 v}{\partial y^3} - D_{11} \frac{\partial^4 w_b}{\partial x^4} - D_{22} \frac{\partial^4 w_s}{\partial y^4} \] (16c)

\[-2 \left( D_{12} + 2D_{66} \right) \frac{\partial^4 w_b}{\partial x^2 \partial y^2} - D_{11} \frac{\partial^4 w_s}{\partial x^4} - D_{22} \frac{\partial^4 w_s}{\partial y^4} - 2 \left( D_{12} + 2D_{66} \right) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} \]

\[+Y_{13} \frac{\partial^2 \phi_z}{\partial x^2} + Y_{23} \frac{\partial^2 \phi_z}{\partial y^2} + q = I_0 \left( \ddot{w}_b + \dddot{w}_b \right) + J_0 \dddot{\phi}_2 + J_1 \left( \frac{\partial \ddot{u}}{\partial x} + \frac{\partial \dddot{v}}{\partial y} \right) - I_2 \nabla^2 \ddot{w}_b - J_2 \nabla^2 \dddot{w}_b \]
Consider a simply supported rectangular plate with length $a$ and width $b$ under transverse load $q$. Based on Navier solution method, the following expansions of displacements $(u,v,w_b,w_z,\varphi_z)$ are assumed as

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} e^{i\omega t} \cos \alpha x \sin \beta y$$

$$v(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} e^{i\omega t} \sin \alpha x \cos \beta y$$

$$w_b(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} e^{i\omega t} \sin \alpha x \sin \beta y$$

$$w_z(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} e^{i\omega t} \sin \alpha x \cos \beta y$$

$$\varphi_z(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} e^{i\omega t} \sin \alpha x \sin \beta y$$

where $i = \sqrt{-1}$, $\alpha = m\pi / a$, $\beta = n\pi / b$, $(U_{mn}, V_{mn}, W_{mn}, W_{mn}, \phi_{mn})$ are the unknown maximum displacement coefficients, and $\omega$ is the vibration frequency. The transverse load $q$ is also expanded as
\[ q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y \] (18)

For the case of sinusoidal load, coefficient \( Q_{mn} = q_0 \) represents the intensity of the load at the plate center. Substituting Eqs. (17) and (18) into Eq. (16), the analytical solutions can be obtained by

\[
\begin{bmatrix}
  k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\
  k_{12} & k_{22} & k_{23} & k_{24} & k_{25} \\
  k_{13} & k_{23} & k_{33} & k_{34} & k_{35} \\
  k_{14} & k_{24} & k_{34} & k_{44} & k_{45} \\
  k_{15} & k_{25} & k_{35} & k_{45} & k_{55}
\end{bmatrix}
- \omega^2
\begin{bmatrix}
  m_{11} & 0 & m_{13} & m_{14} & 0 \\
  0 & m_{22} & m_{23} & m_{24} & 0 \\
  m_{13} & m_{23} & m_{33} & m_{34} & m_{35} \\
  m_{14} & m_{24} & m_{34} & m_{44} & m_{45} \\
  0 & 0 & m_{35} & m_{45} & m_{55}
\end{bmatrix}
\begin{bmatrix}
  U_{mn} \\
  V_{mn} \\
  W_{mn} \\
  \phi_{mn}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  Q_{mn} \\
  0
\end{bmatrix} \tag{19}
\]

where

\[
\begin{align*}
  k_{11} &= A_1 \alpha^2 + A_{66} \beta^2, \\
  k_{12} &= A_6 \alpha^2 + A_{22} \beta^2, \\
  k_{13} &= -B_{11} \alpha^3 - (B_{12} + 2B_{66}) \alpha \beta^2, \\
  k_{14} &= -B_{11} \alpha^3 - (B_{12} + 2B_{66}) \alpha \beta^2, \\
  k_{15} &= -X_{13} \alpha \\
  k_{23} &= -B_2 \beta^3 - (B_{12} + 2B_{66}) \alpha^2 \beta, \\
  k_{24} &= -B_2 \beta^3 - (B_{12} + 2B_{66}) \alpha^2 \beta, \\
  k_{25} &= -X_{23} \beta \\
  k_{33} &= D_1 \alpha^4 + 2(D_{12} + 2D_{66}) \alpha^2 \beta^2 + D_{22} \beta^4, \\
  k_{34} &= D_1 \alpha^4 + 2(D_{12} + 2D_{66}) \alpha^2 \beta^2 + D_{22} \beta^4, \\
  k_{35} &= A_{11} \alpha^2 + A_{44} \beta^2 + Z_{33} \tag{20}
\end{align*}
\]

\[
\begin{align*}
  k_{44} &= Y_{11} \alpha^2 + Y_{22} \beta^2, \\
  k_{45} &= H_{11} \alpha^4 + 2(H_{12} + 2H_{66}) \alpha^2 \beta^2 + H_{22} \beta^4 + A_{22} \alpha^2 + A_{44} \beta^2 \\
  m_{11} &= I_0, \\
  m_{13} &= -\alpha I_1, \\
  m_{14} &= -\alpha J_1, \\
  m_{22} &= I_0, \\
  m_{23} &= -\beta I_1, \\
  m_{24} &= -\beta J_1, \\
  m_{33} &= I_0 + I_2 (\alpha^2 + \beta^2), \\
  m_{34} &= I_0 + J_1 (\alpha^2 + \beta^2), \\
  m_{35} &= J_0 \\
  m_{44} &= I_0 + K_2 (\alpha^2 + \beta^2), \\
  m_{45} &= J_0, \\
  m_{55} &= K_0
\end{align*}
\]

4. Numerical results

4.1. Results for bending analysis

Consider a simply supported FG plate subjected to sinusoidal loads. The effective Young’s modulus \( E(z) \) is assumed to vary exponentially through the thickness of the plate as [33]

\[
E(z) = E_0 \bar{f}(z), \quad \bar{f}(z) = e^{\rho(0.5 + z/h)} \tag{21}
\]
where \( E_b = E_0 \) and \( E_t = E_0 e^p \) denote Young’s modulus of the bottom and top surfaces of the FG plate, respectively; \( E_0 \) is Young’s modulus of the homogeneous plate; and \( p \) is a parameter that indicates the material variation through the thickness and takes values greater than or equal to zero. The variation of the exponential function \( \bar{f}(z) \) through the thickness of the plate is presented in Fig. 1 for different values of \( p \).

Poisson’s ratio is assumed to be constant \( \nu = 0.3 \). For convenience, the following dimensionless forms are used:

\[
\bar{u}(z) = \frac{10E_0 h^3}{q_0 a^4} u\left(0, \frac{b}{2}, z\right), \quad \bar{w}(z) = \frac{10E_0 h^3}{q_0 a^4} w\left(0, \frac{b}{2}, z\right)
\]

\[
\bar{\sigma}_{x,y}(z) = \frac{h^2}{q_0 a^2} \sigma_{x,y}\left(0, \frac{b}{2}, z\right), \quad \bar{\sigma}_{x,y}(z) = \frac{10h^2}{q_0 a^2} \sigma_{x,y}\left(0, 0, z\right)
\]

\[
\bar{\sigma}_{x,z}(z) = \frac{h}{q_0 a} \sigma_{x,z}\left(0, \frac{b}{2}, z\right), \quad \bar{\sigma}_{x,z}(z) = \frac{h}{q_0 a} \sigma_{x,z}\left(0, 0, z\right)
\]

The dimensionless displacement and stress are presented in Tables 1-4 for various values of aspect ratio \( b/a \), thickness ratio \( a/h \), and material parameter \( p \). The through thickness variations of the dimensionless displacements and stresses are also given in Fig. 2 for a thick FG plates with \( a/h = 4 \) and \( p = 0.5 \). The obtained results are compared with the exact 3D [33] and quasi-3D solutions [28,33-34]. It should be noted that the quasi-3D solutions [33-34] are derived based on a trigonometric variation of both in-plane and transverse displacements, while the quasi-3D solutions [28] are computed based on a cubic variation of in-plane displacements and a parabolic variation of transverse displacement across the thickness. In addition, the results of HSDT [35] are also provided to show the importance of including the thickness stretching effect.

The HSDT solution [35] is computed based on a trigonometric variation of in-plane displacements and a constant transverse displacement across the thickness (i.e.,
thickness stretching effect is omitted, \( \varepsilon_z = 0 \).

It can be observed that the obtained results are in excellent agreement with 3D and quasi-3D solutions, particularly with those reported by Mantari and Guedes Soares \([28,34]\). The present quasi-3D theory is even more accurate than the quasi-3D sinusoidal theory \([33]\). Since the present quasi-3D theory and other quasi-3D theories include the thickness stretching effect, their solutions are very close to each other. Meanwhile, the HSDT \([35]\), which omits this effect, gives inaccurate prediction and slightly overestimates the deflection especially for very thick plates (i.e., \( a/h = 2 \), see Tables 1 and 3). The errors in the HSDT are reduced with increasing the thickness ratio \( a/h \). In general, the present quasi-3D theory is highly accurate and comparable to 3D solution even in the case of very thick plates, e.g., \( a/h = 2 \). It is worth noting that the developed theory consists of five unknowns, while the number of unknowns in the HSDT \([35]\) and other quasi-3D theories \([28,33-34]\) is five and six, respectively. Consequently, it may be concluded that the present quasi-3D theory is not only more accurate than the HSDT having the same five unknowns, but also comparable with the quasi-3D theories having more number of unknowns.

4.2. Results for free vibration analysis

The accuracy of the proposed quasi-3D theory is also verified through the free vibration analysis. Consider a simply supported Al/ZrO\(_2\) plate made from a mixture of a metal (Al) and a ceramic (ZrO\(_2\)). Young’s modulus and density of the metal are \( E_m = 70 \) GPa and \( \rho_m = 2702 \) kg/m\(^3\), respectively, and that of ceramic are \( E_c = 380 \) GPa and \( \rho_c = 3800 \) kg/m\(^3\), respectively. Poisson’s ratio is assumed to be constant and equal to 0.3. The effective Young’s modulus is estimated using the power-law distribution with Mori-Tanaka scheme. According to the power-law distribution with Mori-Tanaka
scheme, the bulk modulus \( K(z) \) is given by [36]

\[
K(z) = K_m + (K_c - K_m) \frac{V_c}{1 + V_m \frac{K_c - K_m}{K_m + 4/3 G_m}}
\]  \( \text{(23)} \)

where subscripts \( m \) and \( c \) represent the metal and ceramic constituents, respectively; \( G \) is the shear modulus; and the volume fractions of the metal phase \( V_m \) and ceramic phase \( V_c \) are given by

\[
V_m = 1 - V_c \quad \text{and} \quad V_c = (0.5 + z/h)^p
\]  \( \text{(24)} \)

with \( p \) being the power law index. The variation of the volume fraction \( V_c \) through the thickness of the plate is given in Fig. 3 for various values of the power law index \( p \).

Recall that the bulk modulus and the shear modulus are related to Young’s modulus \( E \) and Poisson ratio \( \nu \) by \( K = E/3(1-2\nu) \) and \( G = E/2(1+\nu) \). Thus, by rewriting Eq. (23) in terms of \( E \) and \( \nu \), the effective Young’s modulus \( E(z) \) is rewritten by

\[
E(z) = E_m + (E_c - E_m) \frac{V_c}{1 + V_m \left( \frac{E_c}{E_m} - 1 \right)^{1+\nu} \frac{1+\nu}{3-4\nu}}
\]  \( \text{(25)} \)

The effective density \( \rho(z) \) is estimated using the power-law distribution with Voigt’s rule of mixtures as [10]

\[
\rho(z) = \rho_m + (\rho_c - \rho_m)V_c
\]  \( \text{(26)} \)

Table 5 contains the dimensionless fundamental frequency \( \bar{\omega} \) of square plates for different values of thickness ratio and power law index. The dimensionless frequency is defined by \( \bar{\omega} = \omega h \sqrt{\frac{\rho_m}{E_m}} \). The calculated frequencies are compared with 3D solutions of Vel and Batra [37], quasi-3D solutions of Neves et al. [27,26,24], and third-order shear deformation (TSDT) solutions of Ferreira et al. [12]. It should be noted that
the quasi-3D solutions are derived based on the sinusoidal [26], hyperbolic [27], and cubic [24] variations of the in-plane displacements, and a quadratic variation of the transverse displacement across the thickness. Since the proposed and quasi-3D theories include the thickness stretching effect, they lead to solutions close to each other, and their solutions match well with 3D solution [37]. Whereas, the TSDT solutions [12] slightly underestimates frequency due to ignoring the thickness stretching effect. Again, it should be noted that the number of unknowns of the proposed theory is only five as against nine in the case of the quasi-3D theories of Neves et al. [27,26,24].

5. Conclusions

A quasi-3D hyperbolic shear deformation theory is developed for bending and vibration analysis of FG plates. The approach contains five unknowns, but accounts for both shear deformation and thickness stretching effects without the need for any shear correction factor. Equations of motion derived from Hamilton principle are analytically solved for bending and free vibration problems of a simply supported plate. By dividing the transverse displacement into the bending and shear parts, the number of unknowns of the theory is reduced, and the computational time is thus saved. The following main points may be drawn from the present study:

1. The results predicted by the proposed theory are in an excellent agreement with 3D solutions even for the case of very thick plates with $a/h = 2$.

2. The present quasi-3D theory has five unknowns, but gives results comparable with those predicted by the existing quasi-3D theories having more number of unknowns.

3. The proposed theory is even more accurate than the quasi-3D sinusoidal theory when compared to 3D solution.

4. The thickness stretching effect is more pronounced for thick plates and it needs to
be taken in consideration in the modeling.

Acknowledgements

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References


16. Hosseini-Hashemi, S., Fadaee, M., Atashipour, S.R.: Study on the free vibration of thick functionally graded rectangular plates according to a new exact closed-form


Figure Captions

Fig. 1. Variation of exponential function $\bar{f}(z)$ through the thickness of a FG plate for various values of parameter $p$

Fig. 2. Variation of dimensionless displacement and stresses through the thickness of plates ($a/h = 4$, $p = 0.5$)

Fig. 3. Variation of volume fraction $V_c$ through the thickness of the plate for various values of the power law index $p$
Table Captions

Table 1. Dimensionless deflection $\bar{w}(0)$ of plates $(a/h = 2)$

Table 2. Dimensionless deflection $\bar{w}(0)$ of plates $(a/h = 4)$

Table 3. Dimensionless stress $\bar{\sigma}_y(h/2)$ of plates $(a/h = 2)$

Table 4. Dimensionless stress $\bar{\sigma}_y(h/2)$ of plates $(a/h = 4)$

Table 5. Dimensionless fundamental frequency $\bar{\omega}$ of square plates
Fig. 1. Variation of exponential function $\tilde{f}(z)$ through the thickness of a FG plate for various values of parameter $p$. 
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Fig. 3. Variation of volume fraction $V_c$ through the thickness of the plate for various values of the power law index $p$. 
Table 1. Dimensionless deflection $\bar{w}(0)$ of plates ($a/h = 2$)

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Table 2. Dimensionless deflection $\overline{w}(0)$ of plates ($a/h = 4$)

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Table 5. Dimensionless fundamental frequency $\bar{\omega}$ of square plates.