Estimation and Compensation for Lipschitz Nonlinear Discrete-Time Systems Subjected to Unknown Measurement Delays

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***Abstract*—Unknown measurement delays usually degrade system performance, and even damage the system under output feedback control, which motivates us to develop an effective method to attenuate or offset the adverse effect from the measurement delays. In this paper, an augmented observer is proposed for discrete-time Lipschitz nonlinear systems subjected to unknown measurement delays, enabling a simultaneous estimation for system states and perturbed terms caused by the output delays. On the basis of the estimates, a sensor compensation technique is addressed to remove the influence from the measurement delays to the system performance. Furthermore, an integrated robust estimation and compensation technique is proposed to decouple constant piece-wise disturbances, attenuate other disturbances/noises, and offset the adverse effect caused by the measurement delays. The proposed methods are applied to a two-stage chemical reactor with delayed recycle, and an electro-mechanical servo system, which demonstrates the effectiveness of the present techniques.**

***Index Terms*—Discrete-time systems, measurement delays, nonlinear systems, observers, robustness, signal compensation.**

I. Introduction

LIPSCHITZ nonlinear systems have attracted continuous interest during the last decades, since many nonlinearities in engineering systems can be characterized, at least locally, as Lipschitz form. Fruitful results have been reported for a variety of research issues on Lipscthiz nonlinear systems such as stabilization and control [1-3], estimation and filtering [4, 5], and fault diagnosis [6, 7]. Meanwhile, time delays always exist in practical processes due to the distributed nature of the system, material transport, and communication lag, which may cause the system performance degradation, and even instability [8-11]. Therefore, research on nonlinear systems subjected to delays has received much more attention. In [12], state and output feedback stabilization was discussed for Lipschitz nonlinear systems with delays. In [13], a distributed H_∞ filter was proposed for a class of Markovian jump nonlinear time-delay systems over lossy sensor networks. By using linear matrix inequality technique, a robust observer was presented in [14] for nonlinear discrete-time systems with time-delays. In [15], an adaptive observer was addressed for Lipschitz nonlinear systems with time-varying delays. In addition, a simultaneous state and disturbance estimator was proposed in [16] for time-delay continuous systems with application to fault diagnosis and fault tolerance. It is noted that all the results in [12-16] were focused on state delayed systems rather than output delayed systems.

In many engineering systems, the output measurements are often subjected to non-negligible time delays. As a matter of fact, this type of time-delay occurs when the measured output data are transmitted via a low speed communication system. Moreover, it may be encountered when systems to be monitored, managed or controlled are located far from the computing unit [17]. Moreover, sensor technology may be the source of output data with significant delays [18]. It is evident that the out-of-date outputs may lead to wrong control commands, which may cause the performance degradation of the closed-loop control system, and even the damage of the machine. Therefore, it is of significance to reconstruct the present system states using the delayed measurement data. Recently, an interesting state observer was proposed in [19] for a discrete-time Lipschitz non-linear system with constant delayed output. In [20], a high-gain observer was designed for nonlinear continuous systems with varying delayed measurement. It is noticed that the results of [19, 20] are for state estimates only, and no efforts were paid for the recovery of the system performance caused by measurement delays. As a result, there is a strong motivation to develop an integrated state and delay perturbation estimator, and compensator for recovering the system performance.

Disturbance and noises are inevitable in real-time environments. Very recently, disturbance observers have significantly contributed to the disturbance estimation, which can be further implemented to offset the effects from the disturbance [6, 16, 21, 22]. Along with delay perturbation estimation and compensation, the disturbance removal and

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noises attenuation is another contribution of the paper, which makes the proposed approach possess robustness in practical noise environment. In this paper, a novel augmented observer will be developed to simultaneously estimate system states and the perturbed term caused by output unknown delays. Furthermore, an integrated delay and input disturbance compensator will be addressed to remove the effect from the output delays and input disturbances, leading to a performance recovery of system outputs. To facilitate real-time application, *discrete-time* Lipschitz systems are concentrated on this study. Finally, the proposed methods will be applied to a two-stage chemical reactor with delayed recycle, and an electro-mechanical servo system with real data, in order to demonstrate the effectiveness of the present techniques.

In this paper, the symbols used are standard. R represents the set of all real numbers; M^T stands for the transpose of M; M^- denotes the inverse of M; M^+ denotes the generalized inverse of M; $block-diag(M_1, M_2)$ implies $\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$; $X > 0$ (or $X < 0$) indicates the symmetric matrix X is positive (or negative) definite; $\lambda_{min}(X)$ represents the minimal eigenvalue of X; $|m|$ denotes the modulus or absolute value of the scalar m; $\|.\|$ represents the standard norm symbol; l_2 is the Lebesgue space consisting of all discrete-time vector-valued function that are square-summable over Z_+, where Z_+ denotes the set of all positive integers. $\|z\|_2$ denotes the l_2 norm of a discrete-time signal z, which is defined as $\|z\|_2 = \sqrt{\sum_{k=0}^{\infty} z^T(k) z(k)}$.

II. Estimation and Compensation for output delays

A. *Observer for State and Sensor Delay-perturbation*

Consider a discrete-time nonlinear dynamic system subjected to output delays in the form of

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + \Phi\big(x(k), u(k)\big) \\ y(k) = Cx(k-\Delta) \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state vector, $u(k) \in R^m$ represents the control input vector, $\Phi(x(k), u(k))$ is a nonlinear function vector, $y(k) \in R^p$ is the measured output vector, $x(k-\Delta) = [x_1(k-\Delta_1), x_2(k-\Delta_2) \cdots x_n(k-\Delta_n)]^T$, $\Delta_1, \Delta_2, \cdots, \Delta_n$ are the unknown delays which can be either constant or time-variable delays. As constant time delays can be regarded as the special case of time-variable delays, the unknown delays here are mainly referred to unknown time-variable delays.

Assumption 1. The function $\Phi(x(k), u(k))$ is assumed to be Lipschitz, that is, for any $x_1(k),\ x_2(k) \in R^n$, and $u(k) \in R^m$, there is a constant $\gamma > 0$ such that

$$\begin{aligned} &\|\Phi\big(x_1(k), u(k)\big) - \Phi\big(x_2(k), u(k)\big)\| \\ &\leq \gamma \|x_1(k) - x_2(k)\|. \end{aligned} \tag{2}$$

Suppose $\Phi(x(k),\ u(k)) = 0$ when $x(k) = 0$ for any $u(k) \in R^m$. Therefore, Assumption 1 implies for any $u(k) \in R^m$, there is a constant $\gamma > 0$ such that

$$\|\Phi\big(x(k), u(k)\big)\| \leq \gamma \|x(k)\|. \tag{3}$$

Remark 1. The nonlinear term $\Phi(x(k), u(k))$ is globally Lipschitz from Assumption 1. However, in practical cases, most of the nonlinear systems are locally Lipschitz in a region including the origin with respect to $x(k)$ and uniformly in $u(k)$. It is worthy to point out that all the results derived for a globally Lipschitz system can be applied to a locally Lipschitz system directly.

Lemma 1[23]. Given a symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$. $S < 0$ if and only if $S_{22} < 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

The above lemma is known as the Schur complement lemma, which is useful for the design of the observer and compensator gains in this paper.

Let

$$\omega(k) = C[x(k-\Delta) - x(k)], \tag{4}$$

the output equation in (1) thus becomes

$$y(k) = Cx(k) + \omega(k). \tag{5}$$

Define

$$\begin{aligned} & x_a(k) = \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}, \\ & E_a = \begin{bmatrix} I_n & 0 \\ 0 & 0_{p\times p} \end{bmatrix}, \qquad A_a = \begin{bmatrix} A & 0 \\ 0 & -\alpha I_p \end{bmatrix}, \\ & B_a = \begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix}, \quad N_a = \begin{bmatrix} 0_{n\times p} \\ \alpha I_p \end{bmatrix}, \\ & C_a = [C \quad I_p], \\ & \Phi_a(x(k), u(k)) = \begin{bmatrix} \Phi(x(k), u(k)) \\ 0_{p\times n} \end{bmatrix} \end{aligned} \tag{6}$$

where α is a constant.

In terms of (1), and (4)-(6), we can construct an augmented descriptor system as follows:

$$\begin{cases} E_a x_a(k+1) = A_a x_a(k) + B_a u(k) \\ \qquad\qquad + \Phi_a\big(x(k), u(k)\big) + N_a \omega(k) \\ y(k) = C_a x_a(k). \end{cases} \tag{7}$$

It is noted that the augmented state vector $x_a(k)$ is composed of the original system state $x(k)$, and the perturbed termed $\omega(k)$ caused by output delay. Therefore, the simultaneous estimation of the original system state and the sensor perturbed term can be realized if the state estimation of the extended state $x_a(k)$ can be obtained.

A discrete-time augmented observer can be constructed in the form of:

$$\begin{cases} S_a \xi(k+1) = (A_a - K_a C_a)\xi(k) + B_a u(k) \\ \qquad\qquad - N_a y(k) + \Phi_a(\hat{x}(k), u(k)) \\ \hat{x}_a(k) = \xi(k) + S_a^{-1} L_a y(k) \end{cases} \tag{8}$$

where $\xi(k) \in R^{n+p}$ is the state vector of the above dynamic system, $\hat{x}_a(k) \in R^{n+p}$ is the estimate of the augment state $x_a(k) \in R^{n+p}$, $\hat{x}(k) = (I_n \quad 0_{n\times p})\hat{x}_a(k)$ is the estimate of the original sate $x(k) \in R^n$, $S_a = E_a + L_a C_a$, and $L_a \in R^{(n+p)\times p}$ and $K_a \in R^{(n+p)\times p}$ are respectively the derivative gain and proportional gain of the observer to be designed.

Let

$$L_a = \begin{bmatrix} 0_{n\times p} \\ L_s \end{bmatrix}, \tag{9}$$

then

$$\begin{aligned} S_a^{-1} &= (E_a + L_a C_a)^{-1} \\ &= \begin{bmatrix} I_n & 0 \\ -C & L_s^{-1} \end{bmatrix} \end{aligned} \tag{10}$$

which implies S_a^{-1} exists provided that L_s is non-singular.

In terms of the (6), (9) and (10), one can obtain

$$A_a S_a^{-1} L_a = -N_a, \quad C_a S_a^{-1} L_a = I_p. \tag{11}$$

Substituting the second equation in (8) into the first equation in (8), and using (11), the augmented observer can be rewritten as

$$\begin{aligned} &S_a \hat{x}_a(k+1) \\ &= (A_a - K_a C_a)\hat{x}_a(k) + B_a u(k) + \Phi_a\big(\hat{x}(k), u(k)\big) \\ &\quad + K_a y(k) + L_a y(k+1) \end{aligned} \tag{12}$$

Noticing that $L_a y(k+1) = L_a C_a x_a(k+1)$, and $K_a y(k) = K_a C_a x_a(k)$, the dynamic equation of the plant in (7) can be rewritten as

$$\begin{aligned} &S_a x_a(k+1) \\ &= (A_a - K_a C_a) x_a(k) + B_a u(k) + N_a \omega(k) \\ &+ \Phi_a\big(x(k), u(k)\big) + K_a y(k) + L_a y(k+1). \end{aligned} \tag{13}$$

Let

$$e_a(k) = x_a(k) - \hat{x}_a(k), \tag{14}$$

$$\Phi_r = \Phi_a(x(k), u(k)) - \Phi_a\big(\hat{x}(k), u(k)\big). \tag{15}$$

Subtracting (12) from (13), one has

$$S_a e_a(k+1) = (A_a - K_a C_a) e_a(k) + \Phi_r + N_a \omega(k) \tag{16}$$

which is equivalent to the following:

$$\begin{aligned} &e_a(k+1) \\ &= S_a^{-1}(A_a - K_a C_a) e_a(k) + S_a^{-1}\Phi_r + N_{am}\omega_m(k) \end{aligned} \tag{17}$$

where

$$N_{am} = \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix}, \quad \omega_m = \alpha L_s^{-1}\omega(k). \tag{18}$$

From (18), one can see the effect from the bounded perturbed term $\omega(k)$ to the estimation error dynamics can be attenuated by selecting reasonably high matrix L_s and small constant α. Therefore, the term $N_{am}\omega_m$ can be ignored if a reasonably high matrix L_s or/and a sufficient small α is chosen. As a result, the estimation error dynamic equation (16) can be simplified as

$$e_a(k+1) = S_a^{-1}(A_a - K_a C_a) e_a(k) + S_a^{-1}\Phi_r. \tag{19}$$

The estimation error dynamics is called asymptotically stable if $e_a(k) \to 0$ when $k \to \infty$. Now it is time to design the proportional gain K_a to ensure the asymptotic stability of the estimation error dynamics in (19).

Theorem 1: The estimation error dynamic system (19) is asymptotically stable if there exist positive constants θ and ϵ_o, a symmetric positive definite matrix P_a, and a matrix Y_a such that for a given positive constant γ

$$\begin{bmatrix} -S_a^T P_a S_a + \epsilon_o I + \theta\gamma^2 I & A_a^T P_a - C_a^T Y_a^T & A_a^T P_a - C_a^T Y_a^T \\ P_a A_a - Y_a C_a & P_a - \theta I & 0 \\ P_a A_a - Y_a C_a & 0 & -P_a \end{bmatrix} < 0 \tag{20}$$

where $S_a = E_a + L_a C_a$ is nonsingular and L_a is in the form of (9). The proportional gain K_a can be calculated as $K_a = P_a^{-1} Y_a$.

Proof.

Define a Lyapunov function as

$$V_o\big(e_a(k)\big) = e_a^T(k) S_a^T P_a S_a e_a(k). \tag{21}$$

In terms of (19) and (21), and (2) in Assumption 1, one has

$$\begin{aligned} &\Delta V_o\big(e_a(k)\big) \\ &= V_o\big(e_a(k+1)\big) - V_o\big(e_a(k)\big) \\ &= e_a^T(k)[(A_a - K_a C_a)^T P_a (A_a - K_a C_a) - S_a^T P_a S_a] e_a(k) \\ &\quad + 2e_a^T(k)(A_a - K_a C_a)^T P_a \Phi_r + \Phi_r^T P_a \Phi_r \\ &\le e_a^T(k)[(A_a - K_a C_a)^T P_a (A_a - K_a C_a) - S_a^T P_a S_a] e_a(k) \\ &\quad + 2e_a^T(k)(A_a - K_a C_a)^T P_a \Phi_r + \Phi_r^T P_a \Phi_r \\ &\quad + \epsilon_o e_a^T e_a - \epsilon_o e_a^T e_a + \theta\gamma^2 e_a^T e_a - \theta \Phi_r^T \Phi_r \\ &= (e_a^T(k) \quad \Phi_r^T)\Omega_a (e_a^T(k) \quad \Phi_r^T)^T - \epsilon_o e_a^T(k) e_a(k) \end{aligned} \tag{22}$$

where

$$\Omega_a = \begin{bmatrix} \Omega_{a11} & (A_a - K_a C_a)^T P_a \\ P_a (A_a - K_a C_a) & P_a - \theta I \end{bmatrix}, \tag{23}$$

$$\begin{aligned} \Omega_{a11} &= (A_a - K_a C_a)^T P_a (A_a - K_a C_a) - S_a^T P_a S_a \\ &\quad + \epsilon_o I + \theta\gamma^2 I. \end{aligned} \tag{24}$$

Applying the Schur complement shown by Lemma 1 to (23), and noticing that $Y_a = P_a K_a$, one can conclude that (20) indicates $\Omega_a < 0$. Therefore, from (22), we have

$$\Delta V_o\big(e_a(k)\big) \le -\epsilon_o \|e_a(k)\|^2. \tag{25}$$

which means $e_a(k) \to 0$ when $k \to \infty$, that is, the estimation error $e_a(k)$ is asymptotically stable. This completes the proof.

Remark 2. If there exists an additional term $\Psi(y(k))$ on the right hand side of the first equation in (1), the observer (8) has to be modified readily by adding $\Psi_a\big(y(k)\big) = \big[\Psi^T\big(y(k)\big), 0_{n\times p}\big]^T$ on the right-hand side of the first equation in (8). In this case, the design methods of the proposed observer gains L_a and K_a above are still valid.

B. Compensation for Sensor Delay-Perturbation

Apply the following feedback law to the plant (1):

$$u(k) = -F\hat{x}(k) = -\underbrace{[F, \ 0_{m\times p}]}_{F_a}\hat{x}_a(k) \tag{26}$$

where $F \in R^{m\times n}$, and $\hat{x}_a(k) = [\hat{x}^T(k), \ \hat{\omega}^T(k)]^T$. Therefore, the closed-loop plant becomes

$$\begin{aligned} x(k+1) &= Ax(k) - BF\hat{x}(k) + \Phi\big(x(k), u(k)\big) \\ &= (A - BF)x(k) + BF_a e_a(k) + \Phi\big(x(k), u(k)\big). \end{aligned} \tag{27}$$

The compensated output can be expressed as

$$\begin{aligned} y_c(k) &= y(k) - \hat{\omega}(k) \\ &= Cx(k) + D_a e_a(k) \end{aligned} \tag{28}$$

where $D_a = [0_{p\times n}, \ I_p]$.

Theorem 2: The closed-loop system described by (27) and (28) is asymptotically stable, if there exist a positive constant $\bar{\epsilon}$, symmetric positive definite matrices X and U, and matrix Y such that for a given positive constant γ

$$\begin{bmatrix} -X + \gamma^2 U & XA^T - Y^T B^T & XA^T - Y^T B^T & X \\ AX - BY & -U + X & 0 & 0 \\ AX - BY & 0 & -X & 0 \\ X & 0 & 0 & -\bar{\epsilon} I \end{bmatrix} < 0. \tag{29}$$

Based on the solution to (29), the state-feedback control gain can be calculated as $F = YX^{-1}$.

Proof.

Let

$$V_c(x(k)) = x^T(k) P x(k) \tag{30}$$

where P is symmetric and positive definite matrix.

In terms of (27) and (30), and (3) in Assumption 1, one has

$$\begin{aligned} &\Delta V_c\big(x(k)\big) \\ &= V_c\big(x(k+1)\big) - V_c\big(x(k)\big) \\ &\le \big[(A - BF)x(k) + \Phi\big(x(k), u(k)\big) + BF_a e_a(k)\big]^T P \\ &\quad \times \big[(A - BF)x(k) + \Phi\big(x(k), u(k)\big) + BF_a e_a(k)\big] \\ &\quad - x^T(k) P x(k) + \epsilon_c x^T(k) x(k) - \epsilon_c x^T(k) x(k) \end{aligned}$$

$$+\theta\gamma^2 x^T(k)x(k) - \theta\Phi^T\big(x(k),u(k)\big)\Phi\big(x(k),u(k)\big)$$
$$= [x^T(k), \Phi^T(x(k),u(k))]\Omega[x^T(k), \Phi^T(x(k),u(k))]^T$$
$$-\epsilon_c\, x^T(k)x(k) + 2x^T(k)(A-BF)^T PBF_a e_a(k)$$
$$+2\Phi^T\big(x(k),u(k)\big)PBF_a e_a(k)$$
$$+e_a^T(k)(BF_a)^T PBF_a e_a(k) \quad (31)$$

where

$$\Omega = \begin{bmatrix} (A-BF)^T P(A-BF) - P + \epsilon_c I + \theta\gamma^2 I & (A-BF)^T P \\ P(A-BF) & -\theta I + P \end{bmatrix}. \quad (32)$$

Letting $= P^{-1}$, $U = \theta XX$, $Y = FX$, and $\bar{\epsilon} = \epsilon_c^{-1}$, and pre-multiplying and post-multiplying $block - diag(P,P,P,P)$ on both sides of (29), one has

$$\begin{bmatrix} -P+\gamma^2\theta I & (A-BF)^T P & (A-BF)^T P & P \\ P(A-BF) & -\theta I + P & 0 & 0 \\ P(A-BF) & 0 & -P & 0 \\ P & 0 & 0 & -\epsilon_c^{-1}P^2 \end{bmatrix} < 0. \quad (33)$$

Applying the Schur complement to (33), one can get $\Omega < 0$. Therefore from (31), one has

$$\Delta V_c\big(x(k)\big) \le -\epsilon_c x^T(k)x(k) + 2x^T(k)(A-BF)PBF_a e_a(k) +2\Phi^T\big(x(k),u(k)\big)PBF_a e_a(k) +e_a^T(k)(BF_a)^T PBF_a e_a(k). \quad (34)$$

Let

$$V_{co}(x(k), e_a(k)) = V_c\big(x(k)\big) + gV_o(e_a(k)). \quad (35)$$

From (25), (34) and (35), one has

$$\Delta V_{co}\big(x(k)\big) = \Delta V_c\big(x(k)\big) + g\Delta V_o\big(e_a(k)\big) \le -\epsilon_c\|x(k)\|^2 + \epsilon_{xe}\|x(k)\|\|e_a(k)\| +\epsilon_a\|e_a(k)\|^2 - g\epsilon_o\|e_a(k)\|^2 \quad (36)$$

where

$$\epsilon_{xe} = 2\|(A-BF)^T PBF_a\| + 2\gamma\|PBF_a\| \quad (37)$$
$$\epsilon_a = \|P\|\|BF_a\|^2 \quad (38)$$

Selecting

$$g \ge \frac{\epsilon_{xe}^2 + \epsilon_c\epsilon_e}{\epsilon_c\epsilon_o}, \quad (39)$$

and using (36), one has

$$\Delta V_{co}\big(x(k), e_a(k)\big) \le -\frac{\epsilon_c}{2}\|x(k)\|^2 - \frac{g}{2}\left(\epsilon_o - \frac{\epsilon_a}{g}\right)\|e_a(k)\|^2 \quad (40)$$

which indicates $e_a(k) \to 0$ and $x(k) \to 0$ as $k \to \infty$. Therefore the closed-loop system described by (27) and (28) is asymptotically stable. This completes the proof.

C. Design Procedure for Estimation and Compensation

The integrated design of the simultaneous observer for state and delay-perturbation estimation, and compensator is summarized as follows.

(*a*). Construct the augmented plant in the form of (7), where the augmented matrices E_a, A_a, B_a, C_a and N_a, and the augmented vectors $x_a(k)$ and $\Phi_a(x(k),u(k))$ are defined by (6). Here, α is chosen as a small constant in order to reduce the effect from the perturbed term $\omega(k)$.

(*b*). Select the derivative gain matrix L of the observer in the form of (9), where L_s is chosen as a nonsingular high-gain matrix so that $S_a = E_a + L_a C_a$ is nonsingular and the effect from the perturbed term $\omega(k)$ can be further attenuated by L_s^{-1}.

(*c*). Solve the linear matrix inequality (20) to give the matrices P_a and Y_a, leading to the proportional gain $K_a = P_a^{-1}Y_a$.

(*d*). Implement the augmented observer in the form of (8) to the plant (7), and produce the simultaneous state and delay-perturbation estimates as follows:

$$\hat{x}(k) = (I_n \quad 0_{n\times p})\hat{x}_a(k)$$
$$\hat{\omega}(k) = (0_{p\times n} \quad I_p)\hat{x}_a(k)$$

where $\hat{x}_a(k)$ is the estimate of the augmented state $x_a(k)$.

(*e*). Solve the linear matrix inequality (29) to yield the state-feedback gain $F = YX^{-1}$. Apply the feedback control $u(k) = -F\hat{x}(k)$ to the plant (1), and implement the sensor compensation as $y_c(k) = y(k) - \hat{\omega}(k)$. As a result, the compensated system is asymptotically stable and the effect from the delay to the output has been completely removed.

III. Robust Estimation and Compensation

A. Robust Simultaneous Estimation

Consider a discrete-time nonlinear dynamic system subjected to output delay, process disturbance and measurement noise in the form of

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + \Phi\big(x(k),u(k)\big) + B_d n_d(k) \\ y(k) = Cx(k-\Delta) + n_\omega(k) \end{cases} \quad (41)$$

where $n_d(k) \in R^l$ is the process disturbance, $n_\omega(k) \in R^p$ is the measurement noise, and other terms are the same as in (1).

Assume

$$n_d(k) = n_{dc}(k) + n_{dh}(k) \quad (42)$$

where $n_{dc}(k)$ is a piecewise-constant disturbance signal, and $n_{dh}(k)$ is a l_2 norm bounded disturbance signal.

Let

$$\omega(k) = C[x(k-\Delta) - x(k)] + n_\omega(k), \quad (43)$$

the output equation in (41) thus becomes

$$y(k) = Cx(k) + \omega(k). \quad (44)$$

Define

$$x_{ar}(k) = \begin{bmatrix} x(k) \\ n_{dc}(k) \\ \omega(k) \end{bmatrix},\ n_{ad}(k) = \begin{bmatrix} n_{dh}(k) \\ \omega(k) \end{bmatrix},$$
$$E_{ar} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_l & 0 \\ 0 & 0 & 0_{p\times p} \end{bmatrix}, \quad A_{ar} = \begin{bmatrix} A & B_d & 0 \\ 0 & I_l & 0 \\ 0 & 0 & -\alpha I_p \end{bmatrix},$$
$$B_{ar} = \begin{bmatrix} B \\ 0_{l\times m} \\ 0_{p\times m} \end{bmatrix},\ B_{ad} = \begin{bmatrix} B_d & 0 \\ 0 & 0_{l\times p} \\ 0 & \alpha I_p \end{bmatrix},\ N_{ar} = \begin{bmatrix} 0_{n\times p} \\ 0_{l\times p} \\ \alpha I_p \end{bmatrix},$$
$$C_{ar} = [C \quad 0_{p\times l} \quad I_p],$$
$$\Phi_{ar}\big(x(k),u(k)\big) = \begin{bmatrix} \Phi\big(x(k),u(k)\big) \\ 0_{l\times n} \\ 0_{p\times n} \end{bmatrix}. \quad (45)$$

In terms of (41)-(45), we can construct an augmented descriptor system as follows:

$$\begin{cases} E_{ar}x_{ar}(k+1) = A_{ar}x_{ar}(k) + B_{ar}u(k) \\ \qquad\qquad +\Phi_{ar}\big(x(k),u(k)\big) + B_{ad}n_{ad}(k) \\ y(k) = C_{ar}x_{ar}(k). \end{cases} \quad (46)$$

A discrete-time augmented observer can be constructed in the form of:

$$\begin{cases} S_{ar}\eta(k+1) = (A_{ar} - K_{ar}C_{ar})\eta(k) + B_{ar}u(k) \\ \qquad\qquad -N_{ar}y(k) + \Phi_{ar}(\hat{x}(k),u(k)) \\ \hat{x}_{ar}(k) = \eta(k) + S_{ar}^{-1}L_{ar}y(k) \end{cases} \quad (47)$$

where $\eta(k) \in R^{n+l+p}$ is the state vector of the above dynamic system, $\hat{x}_a(k) \in R^{n+l+p}$ is the estimate of the augmented state $x_{ar}(k) \in R^{n+l+p}$, $\hat{x}(k) = (I_n \quad 0_{n\times(l+p)})\hat{x}_{ar}(k)$ is the estimate of the original sate $x(k) \in R^n$, $S_{ar} = E_{ar} + L_{ar}C_{ar}$, and $L_{ar} \in R^{(n+l+p)\times p}$ and $K_{ar} \in R^{(n+l+p)\times p}$ are respectively the derivative gain and proportional gain of the observer to be designed.

Let

$$L_{ar} = \begin{bmatrix} 0_{n\times p} \\ 0_{l\times p} \\ L_t \end{bmatrix}, \quad (48)$$

then

$$\begin{aligned} S_{ar}^{-1} &= (E_{ar} + L_{ar}C_{ar})^{-1} \\ &= \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_l & 0 \\ -C & 0 & L_t^{-1} \end{bmatrix} \end{aligned} \quad (49)$$

which implies S_{ar}^{-1} exists provided that L_r is non-singular.

In terms of the (45), (48) and (49), one can obtain

$$A_{ar}S_{ar}^{-1}L_a = -N_{ar}, \quad C_{ar}S_{ar}^{-1}L_{ar} = I_p. \quad (50)$$

Let

$$e_{ar}(k) = x_{ar}(k) - \hat{x}_{ar}(k), \quad (51)$$

$$\widetilde{\Phi}_{ar} = \Phi_{ar}(x(k),u(k)) - \Phi_{ar}\big(\hat{x}(k),u(k)\big). \quad (52)$$

Using (46), (47), (50)-(52) and the similar manner to derive (16), one can obtain

$$\begin{aligned} &S_{ar}e_{ar}(k+1) \\ &\quad = (A_{ar} - K_{ar}C_{ar})e_{ar}(k) + \widetilde{\Phi}_{ar} + B_{ad}n_{ad}(k) \end{aligned} \quad (53)$$

or the equivalent formula

$$\begin{aligned} e_{ar}(k+1) = S_{ar}^{-1}&(A_{ar} - K_{ar}C_{ar})e_{ar}(k) \\ &+S_{ar}^{-1}\widetilde{\Phi}_{ar} + S_{ar}^{-1}B_{ad}n_{ad}(k) \end{aligned} \quad (54)$$

In the light of (45), one can compute

$$B_{ad}n_{ad}(k) = \underbrace{\begin{bmatrix} B_d & 0 \\ 0 & 0_{l\times p} \\ 0 & I_p \end{bmatrix}}_{B_{md}} \underbrace{\begin{bmatrix} n_{dh}(k) \\ \alpha\omega(k) \end{bmatrix}}_{n_{md}(k)}. \quad (55)$$

From (55), one can see that the effect from $\omega(k)$ to the error dynamics can be effectively attenuated by selecting a small scalar α.

Substitution (55) into (54) yields

$$\begin{aligned} e_{ar}(k+1) = S_{ar}^{-1}&(A_{ar} - K_{ar}C_{ar})e_{ar}(k) \\ &+S_{ar}^{-1}\widetilde{\Phi}_{ar} + S_{ar}^{-1}B_{md}n_{md}(k). \end{aligned} \quad (56)$$

If the estimation error $e_{ar}(k)$ is asymptotically stable when the disturbance $n_{md}(k)$ is null, and satisfies $\|e_{ar}\|_2 \le \epsilon_{rd}\|n_{md}\|_2$ for a positive scalar ϵ_{rd}, the error dynamics is called robustly stable. Based on (56), the next step is to design the proportional observer gain K_{ar} to ensure the system (56) to be robustly stable.

Theorem 3: The estimation error dynamic system (56) is robustly stable and satisfies the robust performance index

$$\|e_{ar}\|_2{}^2 \le \epsilon_r^{-1}\epsilon_d\|n_{md}\|_2{}^2 \quad (57)$$

if there exist positive constants θ, ϵ_r, and ϵ_d, a symmetric positive definite matrix P_{ar}, and a matrix Y_{ar} such that for a given positive constant γ

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} < 0 \quad (58)$$

where

$$\Gamma_{11} = \begin{bmatrix} -S_{ar}^TP_{ar}S_{ar} + \epsilon_rI + \theta\gamma^2I & A_{ar}^TP_{ar} - C_{ar}^TY_{ar}^T & A_{ar}^TP_{ar} - C_{ar}^TY_{ar}^T \\ P_{ar}A_{ar} - Y_{ar}C_{ar} & P_{ar} - \theta I & 0 \\ P_{ar}A_{ar} - Y_{ar}C_{ar} & 0 & -P_{ar} \end{bmatrix}$$

$$\Gamma_{12} = \begin{bmatrix} A_{ar}^TP_{ar}B_{md} - C_{ar}^TY_{ar}^TB_{md} \\ P_{ar}B_{md} \\ 0 \end{bmatrix},$$

$$\Gamma_{21} = \Gamma_{12}^T,$$

$$\Gamma_{22} = B_{md}^TB_{md} - \epsilon_dI. \quad (59)$$

Moreover, $S_{ar} = E_{ar} + L_{ar}C_{ar}$ is nonsingular and L_{ar} is in the form of (48). The proportional gain K_{ar} can be calculated as $K_{ar} = P_{ar}^{-1}Y_{ar}$.

Proof.

Define a Lyapunov function as

$$V_o\big(e_{ar}(k)\big) = e_{ar}^T(k)S_{ar}^TP_{ar}S_{ar}e_{ar}(k). \quad (60)$$

In terms of (56) and (60), and (2) in Assumption 1, one has

$$\begin{aligned} &\Delta V_o\big(e_{ar}(k)\big) \\ &= V_o\big(e_{ar}(k+1)\big) - V_o\big(e_{ar}(k)\big) \\ &\le e_{ar}^T(k)[(A_{ar} - K_{ar}C_{ar})^TP_{ar}(A_{ar} - K_{ar}C_{ar}) \\ &\qquad - S_{ar}^TP_{ar}S_{ar}]e_{ar}(k) + \widetilde{\Phi}_{ar}^TP_{ar}\widetilde{\Phi}_{ar} \\ &\quad +2e_{ar}^T(k)(A_{ar} - K_{ar}C_{ar})^TP_{ar}\widetilde{\Phi}_{\mathrm{ar}} \\ &+2e_{ar}^T(k)(A_{ar} - K_{ar}C_{ar})^TP_{ar}B_{md}n_{md}(k) \\ &+2\widetilde{\Phi}_{ar}^TP_{ar}B_{md}n_{md}(k) + n_{md}^T(k)B_{md}^TB_{md}n_{md}(k) \\ &\quad +\epsilon_re_{ar}^T(k)e_{ar}(\mathrm{k}) - \epsilon_re_{ar}^T(k)e_{ar}(\mathrm{k}) + \theta\gamma^2e_{ar}^T(k)e_{ar}(k) \\ &\quad -\theta\widetilde{\Phi}_{ar}^T\widetilde{\Phi}_{ar} + \epsilon_dn_{md}(k)n_{md}(k) - \epsilon_dn_{md}(k)n_{md}(k) \\ &= (e_{ar}^T(k) \quad \widetilde{\Phi}_{ar}^T \quad n_{md}^T(k))\Omega_{ar}(e_{ar}^T(k) \quad \widetilde{\Phi}_{ar}^T \quad n_{md}^T(k))^T \\ &\qquad -\epsilon_re_a^T(k)e_a(k) + \epsilon_dn_{md}^T(k)n_{md}(k) \end{aligned} \quad (61)$$

where

$$\Omega_{ar} = \begin{bmatrix} \Omega_{ar1} & \Omega_{ar12} \\ \Omega_{ar12}^T & \Omega_{ar2} \end{bmatrix}, \quad (62)$$

with the followngs:

$$\Omega_{ar1} = \begin{bmatrix} \Omega_{ar11} & (A_{ar} - K_{ar}C_{ar})^TP_{ar} \\ P_{ar}(A_{ar} - K_{ar}C_{ar}) & P_{ar} - \theta I \end{bmatrix},$$

$$\begin{aligned} \Omega_{ar11} = &(A_{ar} - K_{ar}C_{ar})^TP_{ar}(A_{ar} - K_{ar}C_{ar}) \\ &-S_{ar}^TP_{ar}S_{ar} + \epsilon_rI + \theta\gamma^2I, \end{aligned}$$

$$\Omega_{\mathrm{ar}12} = \begin{bmatrix} (A_{ar} - K_{ar}C_{ar})^TP_{ar}B_{md} \\ P_{ar}B_{md} \end{bmatrix},$$

$$\Omega_{\mathrm{ar}2} = B_{md}^TB_{md} - \epsilon_dI.$$

Applying the Schur complement to (62), and noticing that $Y_{ar} = P_{ar}K_{ar}$, one can conclude that (58) indicates $\Omega_{ar} < 0$. Therefore, from (61), we have

$$\Delta V_o\big(e_{ar}(k)\big) \le -\epsilon_re_{ar}^T(k)e_{ar}(k) + \epsilon_dn_{md}^T(k)n_{md}(k). \quad (63)$$

When the perturbed term $n_{md}(k)$ is zero or can be ignored, the estimation error dynamics is asymptotically stable according to (63).

Now let us look at the robust performance index when $n_{md}(k)$ cannot be ignored.

Under zero initial conditions, it is followed from (63)

$$0 \le V_o\big(e_{ar}(n+1)\big) \le -\epsilon_r \sum_{k=0}^{n} e_{ar}^T(k)e_{ar}(k) + \epsilon_d \sum_{k=0}^{n} n_{md}^T(k)n_{md}(k) \quad (64)$$

which implies (57) . This completes the proof.

B. Compensation for Delay-Perturbation and Disturbance

Apply the following feedback law to the plant (41):

$$\begin{aligned} u(k) &= -F_x\hat{x}(k) - F_d\hat{n}_{dc}(k) \\ &= -\underbrace{[F_x \;\; F_d \;\; 0_{m\times p}]}_{F_{ar}}\hat{x}_{ar}(k), \end{aligned} \quad (65)$$

where $F_x \in R^{m\times n}$, $F_d \in R^{m\times l}$, and $\hat{x}_{ar}(k) = [\hat{x}^T(k),\ \hat{n}_{dc}(k),\ \hat{\omega}^T(k)]^T$. Therefore, the closed-loop plant becomes

$$\begin{aligned} &x(k+1) \\ &= Ax(k) - BF_{ar}\hat{x}_{ar}(k) + \Phi\big(x(k),u(k)\big) + B_d n_d(k) \\ &= (A - BF_x)x(k) + (B_d - BF_d)n_{dc}(k) + BF_{ar}e_{ar}(k) \\ &\quad +\Phi\big(x(k),u(k)\big) + B_d n_{dh}(k). \end{aligned} \quad (66)$$

Select

$$F_d = B^+B_d, \quad (67)$$

which indicates

$$B_d - BF_d = 0 \quad (68)$$

provided that

$$rank\ [B_d \quad B] = rank(B). \quad (69)$$

Therefore, the system (66) becomes readily

$$\begin{aligned} x(k+1) &= (A - BF_x)x(k) + BF_{ar}e_{ar}(k) \\ &\quad +\Phi\big(x(k),u(k)\big) + B_d n_{dh}(k). \end{aligned} \quad (70)$$

The compensated output can be expressed as

$$\begin{aligned} y_c(k) &= y(k) - \hat{\omega}(k) \\ &= Cx(k) + D_{ar}e_{ar}(k) \end{aligned} \quad (71)$$

where $D_{ar} = [0_{p\times(n+l)},\ I_p]$.

Let

$$\begin{aligned} e_{Fa}(k) &= \begin{bmatrix} e_{ar}(k) \\ F_x e_x(k) \end{bmatrix} \\ e_x(k) &= [I_n,\ 0_{n\times l}, 0_{n\times p}]e_{ar}(k) \\ B_{Fa} &= [0_{n\times n}\ BF_d\ 0_{n\times p}\ B] \\ D_{Fa} &= [D_{ar}\ 0_{p\times m}]. \end{aligned} \quad (72)$$

The plant described by (70) and (71) can be modified as

$$\begin{cases} x(k+1) = (A - BF_x)x(k) + \Phi\big(x(k),u(k)\big) \\ \qquad +B_d n_{dh}(k) + B_{Fa}e_{Fa}(k) \\ y_c(k) = Cx(k) + D_{Fa}e_{Fa}(k) \end{cases} \quad (73)$$

Now it is ready to design the gain F_x to ensure the stability of (73) and satisfy a robust performance index.

Theorem 4: The closed-loop system (73) is robustly stable, and satisfies the robust performance index:

$$\|y_c\|_2 \le \gamma_{cd}\|n_{dh}\|_2 + \gamma_{ce}\|e_{Fa}\|_2 \quad (74)$$

if there exist positive constants γ_{cd} and γ_{ce}, symmetric positive definite matrices X and U, and matrix Y such that for a given positive constant γ

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} < 0 \quad (75)$$

where

$$\Psi_{11} = \begin{bmatrix} -X + \gamma^2 U & XA^T - Y^T B^T & 0 & 0 \\ AX - BY & X - U & B_d & B_{Fa} \\ 0 & B_d^T & -\gamma_{cd}^2 I & 0 \\ 0 & B_{Fa}^T & 0 & -\gamma_{ce}^2 I \end{bmatrix}$$

$$\begin{aligned} \Psi_{21} &= \begin{bmatrix} AX - BY & 0 & B_d & B_{Fa} \\ CX & 0 & 0 & D_{Fa} \end{bmatrix} \\ \Psi_{12} &= \Psi_{21}^T \\ \Psi_{22} &= \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix} \end{aligned} \quad (76)$$

Based on the solution to (76), the state-feedback control gain can be calculated as $F_x = YX^{-1}$.

Proof.

Let $X = P^{-1}, U = \theta XX$, and $Y = F_x X$. Pre-multiplying and post-multiplying $block - diag(P,P,I,I,P,I)$ on both sides of (75), and using the Schur complement to (75), one can conclude that $\Psi < 0$ in (75) implies $\Omega_{ac} < 0$, that is,

$$\Omega_{ac} = \begin{bmatrix} \Omega_{ac1} & \Omega_{ac12} \\ \Omega_{ac21} & \Omega_{ac2} \end{bmatrix} < 0 \quad (77)$$

where

$$\Omega_{ac1} = \begin{bmatrix} (A - BF_x)^T P(A - BF_x) - P + \theta\gamma^2 I + C^T C & (A - BF_x)^T P \\ P(A - BF_x) & P - \theta I \end{bmatrix}$$

$$\Omega_{ac12} = \begin{bmatrix} (A - BF_x)^T PB_d & (A - BF_x)^T PB_{Fa} + C^T D_{Fa} \\ PB_d & PB_{Fa} \end{bmatrix}$$

$$\Omega_{ac21} = \Omega_{ac12}^T$$

$$\Omega_{ac2} = \begin{bmatrix} B_d^T PB_d - \gamma_{cd}^2 I & B_d^T PB_{Fa} \\ B_{Fa}^T PB_d & B_{Fa}^T PB_{Fa} + D_{Fa}^T D_{Fa} - \gamma_{ce}^2 I \end{bmatrix}.$$

(a). Stability.

Firstly, let us to consider the stability when $n_{dh}(k) = 0$.

Let

$$V_c(x(k)) = x^T(k)Px(k) \quad (78)$$

where P is symmetric and positive definite matrix.

In terms of (73) and (78), and (3) in Assumption 1, one has

$$\begin{aligned} &\Delta V_c\big(x(k)\big) \\ &= V_c\big(x(k+1)\big) - V_c\big(x(k)\big) \\ &\le [(A - BF_x)x(k) + \Phi\big(x(k),u(k)\big) + B_{Fa}e_{Fa}(k)]^T \\ &\quad \times P[(A - BF_x)x(k) + \Phi\big(x(k),u(k)\big) + B_{Fa}e_{Fa}(k)] \\ &\quad -x^T(k)Px(k) + \theta\gamma^2 x^T(k)x(k) \\ &\quad -\theta\Phi^T\big(x(k),u(k)\big)\Phi\big(x(k),u(k)\big) \\ &= \big(x^T(k),\Phi^T\big(x(k),u(k)\big)\big)\Omega_{ax}\big(x^T(k),\Phi^T\big(x(k),u(k)\big)\big)^T \\ &\quad +2x^T(k)(A - BF_x)^T PB_{Fa}e_{Fa}(k) \\ &\quad +2\Phi^T\big(x(k),u(k)\big)PB_{Fa}e_{Fa}(k) \\ &\quad +e_{Fa}^T(k)(B_{Fa}^T PB_{Fa})e_{Fa}(k) \end{aligned} \quad (79)$$

where

$$\Omega_{\phi x} = \begin{bmatrix} (A - BF_x)^T P(A - BF_x) - P + \theta\gamma^2 I & (A - BF_x)^T P \\ P(A - BF_x) & P - \theta I \end{bmatrix}. \quad (80)$$

Since $\Omega_{ac} < 0$ in (77), it is evident that $\Omega_{\phi x}$ in (80) is negative definite, that is,

$$\Omega_{\phi x} < 0. \quad (81)$$

Substitution (81) into (79) yields

$$\begin{aligned} &\Delta V_c\big(x(k)\big) \\ &\le -\epsilon_{\phi x}\|x_\phi(k)\|^2 + 2x^T(k)(A - BF_x)^T PB_{Fa}e_{Fa}(k) \\ &\quad +2\Phi^T\big(x(k),u(k)\big)PB_{Fa}e_{Fa}(k) \\ &\quad +e_{Fa}^T(k)(B_{Fa}^T PB_{Fa})e_{Fa}(k) \end{aligned} \quad (82)$$

where

$$\epsilon_{\phi x} = \lambda_{min}(-\Omega_{\phi x}) \quad (83)$$

$$x_{\phi}(k)=\begin{bmatrix} x(k) \\ \Phi(x(k),u(k)) \end{bmatrix}. \tag{84}$$

Define

$$V_{co}(x(k),e_{ar}(k))=V_c\big(x(k)\big)+gV_o(e_{ar}(k)). \tag{85}$$

According to (63), (82) and (85), one has

$$\begin{aligned}
&\Delta V_{co}\big(x(k),e_{ar}(k)\big)\\
&=\Delta V_c\big(x(k)\big)+g\Delta V_o(e_{ar}(k))\\
&\le -\epsilon_{\phi x}\|x_{\phi}(k)\|^2+\epsilon_{\phi e}\|x_{\phi}(k)\|\|e_{Fa}(k)\|\\
&\quad +\epsilon_{ee}\|e_{Fa}(k)\|^2-g\epsilon_r\|e_{Fa}(k)\|^2.
\end{aligned} \tag{86}$$

where

$$\epsilon_{\phi e}=\|2(A-BF_x)^TPB_{Fa}\|+2\gamma\|PB_{Fa}\| \tag{87}$$

$$\epsilon_{ee}=\|B_{Fa}^TPB_{Fa}\|. \tag{88}$$

In the derivation of (86), $n_{md}(k)=\left[n_{dh}^T(k),\ \big(\alpha\omega(k)\big)^T\right]^T$ is ignored as $n_{dh}(k)=0$, and $\alpha\omega(k)$ is small by selecting a sufficient small α.

Selecting

$$g\ge \frac{\epsilon_{\phi e}^2+\epsilon_{ee}\epsilon_{\phi x}}{\epsilon_{\phi x}\epsilon_r}, \tag{89}$$

and using (86), one has

$$\begin{aligned}
&\Delta V_{co}\big(x(k),e_{ar}(k)\big)\\
&\le -\frac{\epsilon_{\phi x}}{2}\left\|x_{\phi(k)}\right\|^2-\frac{g}{2}\left(\epsilon_r-\frac{\epsilon_{ee}}{g}\right)\|e_{Fa}(k)\|^2\\
&\le 0
\end{aligned} \tag{90}$$

which indicates $e_{Fa}(k)\to 0$ and $x(k)\to 0$ as $k\to\infty$.

(b). Robust performance index.

Consider then case when $n_{dh}(k)\neq 0$. In terms of (73), (77) and (78), one has

$$\begin{aligned}
&\Delta\, V_c\big(x(k)\big)\\
&=V_c\big(x(k+1)\big)-V_c\big(x(k)\big)\\
&\le [(A-BF_x)x(k)+\Phi\big(x(k),u(k)\big)+B_dn_{dh}(k)\\
&\quad +B_{Fa}e_{Fa}(k)]^T\,P[(A-BF_x)x(k)+\Phi\big(x(k),u(k)\big)\\
&\quad +B_dn_{dh}(k)+B_{Fa}e_{Fa}(k)]-x^T(k)Px(k)\\
&\quad +\theta\gamma^2x^T(k)x(k)-\theta\Phi^{\mathrm{T}}\big(x(k),u(k)\big)\Phi\big(x(k),u(k)\big)\\
&\quad +y_c^T(k)y_c(k)-y_c^T(k)y_c(k)-\gamma_{cd}^2n_{dh}^T(k)n_{dh}(k)\\
&\quad +\gamma_{cd}^2n_{dh}^T(k)n_{dh}(k)-\gamma_{ce}^2e_{ar}^T(k)e_{ar}(k)\\
&\quad +\gamma_{ce}^2e_{ar}^T(k)e_{ar}(k)\\
&=\big(x^T(k),\Phi^{\mathrm{T}}\big(x(k),u(k)\big),n_{dh}^T(k),e_{Fa}^T(k)\,\big)\Omega_{ac}\\
&\qquad\times\big(x^T(k),\Phi^{\mathrm{T}}\big(x(k),u(k)\big),n_{dh}^T(k),e_{Fa}^T(k)\,\big)^T\\
&\quad -y_c^T(k)y_c(k)+\gamma_{cd}^2n_{dh}^T(k)n_{dh}(k)\\
&\quad +\gamma_{ce}^2e_{Fa}^T(k)e_{Fa}(k)\\
&\le -y_c^T(k)y_c(k)+\gamma_{cd}^2n_{dh}^T(k)n_{dh}(k)\\
&\qquad +\gamma_{ce}^2e_{Fa}^T(k)e_{Fa}(k).
\end{aligned} \tag{91}$$

Under zero initial conditions, it is followed from (91)

$$\begin{aligned}
0&\le V_c\big(x(n+1)\big)\\
&\le -\textstyle\sum_{k=0}^n y_c^T(k)y_c(k)+\gamma_{ce}^2\sum_{k=0}^n e_{Fa}^T(k)e_{Fa}(k)\\
&\quad +\gamma_{cd}^2\textstyle\sum_{k=0}^n n_{dh}^T(k)n_{dh}\,(k)
\end{aligned} \tag{92}$$

which implies (74) . This completes the proof.

Remark 3.

From (68) and (69), one can see the condition of $B_d-BF_d=0$ is $\mathrm{rank}\,[B_d\quad B]=\mathrm{rank}(B)$, which often meets particularly for disturbances acting on the actuators. However, even if $B_d-BF_d\neq 0$, but $F_d=B^+B_d$ can still minimize $\|B_d-BF_d\|$ so that the effect from $n_{dc}(k)$ is reduced. Furthermore, the disturbance term in equation (70) can be modified as $\underbrace{[(B_d\quad B_d-BF_d)]}_{B_{dt}}\begin{bmatrix} n_{dh}(k) \\ n_{dc}(k) \end{bmatrix}$. Therefore, by replacing B_d by B_{dt} in Theorem 4, the influences from both $n_{dh}(k)$ and $n_{dc}(k)$ can be attenuated by using the obtained feedback gain F_x.

C. Procedure for Robust Estimation and Compensation

Now we can summarize the integrated design of the simultaneous robust observer and compensator as follows.

(*a*). Construct the augmented plant in the form of (46), where the augmented matrices $E_{ar},A_{ar},B_{ar},C_{ar},B_{ad}$ and N_{ar}, and the augmented vectors $x_{ar}(k)$, $\Phi_{ar}(x(k),u(k))$ and $n_{ad}(k)$ are defined by (45). Here, α is chosen as a small constant in order to reduce the effect from the perturbed term $\omega(k)$.

(*b*). Select the derivative gain matrix L_{ar} of the observer in the form of (48), where L_t is chosen as a nonsingular matrix so that $S_{ar}=E_{ar}+L_{ar}C_{ar}$ is nonsingular. The disturbance distribution matrix B_{md} and disturbance $n_{md}(k)$ is constructed as in (55).

(*c*). Solve the linear matrix inequality (58) to give the matrices P_{ar} and Y_{ar}, leading to the proportional gain $K_{ar}=P_{ar}^{-1}Y_{ar}$.

(*d*). Implement the augmented observer in the form of (47) to the plant (46), and produce the simultaneous state, input disturbance and delay-perturbation estimates as follows:

$$\hat{x}(k)=(I_n\quad 0_{n\times(l+p)})\hat{x}_{ar}\,(k)$$
$$\hat{n}_{dc}(k)=(0_{l\times n}\quad I_l\quad 0_{l\times p})\hat{x}_{ar}\,(k)$$
$$\hat{\omega}(k)=(0_{p\times(n+l)}\quad I_p)\hat{x}_{ar}\,(k)$$

where $\hat{x}_{ar}\,(k)$ is the estimate of the augmented state $x_{ar}(k)$.

(*e*). Select $F_d=B^+B_d$. Solve the linear matrix inequality (75) to yield the state-feedback gain $F_x=YX^{-1}$. Apply the feedback control $u(k)=-F_x\hat{x}(k)-F_d\hat{n}_{dc}(k)$ to the plant (41), and implement the sensor compensation as $y_c(k)=y(k)-\hat{\omega}(k)$. As a result, the compensated system is robustly stable and the effect from the delays to the output and the input disturbance to the system dynamics can be attenuated.

IV. SIMULATION STUDY

A. Chemical reactor with delayed recycle streams

A two-stage chemical reactor with delayed recycle streams can be described as follows:

$$\begin{cases}
\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}-\frac{1}{s_1}-r_1 & \frac{1-R_2}{V_1}\\ 0 & -\frac{1}{s_2}-r_2\end{bmatrix}\underbrace{\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}}_{x(t)}+\begin{bmatrix}0\\ \frac{W}{V_2}\end{bmatrix}u(t)\\
\qquad+\begin{bmatrix}0 & 0\\ \frac{R_1}{V_2} & \frac{R_2}{V_2}\end{bmatrix}y(t)+\Phi(x(t),u(t))\\
y(t)=\begin{bmatrix}x_1(t-\tau_1(t))\\ x_2(t-\tau_2(t))\end{bmatrix}
\end{cases} \tag{93}$$

where $x_1(t)$ and $x_2(t)$ are the compositions, s_1 and s_2 are the reactor residence times, R_1 and R_2 are the recycle flow rate, W

is the feed rate, V_1 and V_2 are the reactor volumes, r_1 and r_2 are the reaction constants, $\Phi(x(t),u(t)) = [0,\ \sin(x_1(t))]^T$ is a nonlinear perturbed term, and $y(t)$ is the output subjected to time-varying delays.

The parameters, $s_1 = s_2 = 1$, $r_1 = r_2 = 1$, $R_1 = R_2 = 0.5$, and $V_1 = V_2 = 1$, are taken from [24], and $u(t) = 6\,sin(t)$, $\tau_1(t) = 20 + 5\,sin(0.1t)$, and $\tau_2(t) = 10 + 10\,sin(0.2t)$ are used in this simulation study.

(a) Taking the sampling as $0.025s$, the system (93) can be discretized using the Euler discretization leading to

$$\begin{cases} x(k+1) = \underbrace{\begin{bmatrix} 0.9500 & 0.0125 \\ 0 & 0.9500 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 0 \\ 0.0125 \end{bmatrix}}_{B} u(k) \\ \qquad + \underbrace{\begin{bmatrix} 0 & 0 \\ 0.0125 & 0.0125 \end{bmatrix}}_{B_y} y(k) + \underbrace{\begin{bmatrix} 0 \\ 0.025\sin(x_1(k)) \end{bmatrix}}_{\Phi(x(k),u(k))} \\ y(k) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1(k-\tau_1(k)) \\ x_2(k-\tau_2(k)) \end{bmatrix} \end{cases} \tag{94}$$

(b) Let $\omega_1(k) = x_1(k-\tau_1(k)) - x_1(k)$ and $\omega_2(k) = x_2(k-\tau_2(k)) - x_2(k)$. Based on the discrete-time model (94), we can construct the augmented matrices E_a, A_a, B_a, C_a and N_a, and the augmented vectors $x_a(k)$ and $\Phi_a(x(k),u(k))$ in terms of (6), where $\alpha = 0.001$. In addition, we also can have the augmented matrix $B_{ya} = \begin{bmatrix} B_y \\ 0_{2\times 2} \end{bmatrix}$.

Select the derivative gain

$$L_a = \begin{bmatrix} 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}^T.$$

Solve the linear matrix inequality (20), one can then obtain the proportional gain:

$$K_a = \begin{bmatrix} 0.4766 & 0.0066 \\ 0.0003 & 0.4764 \\ -0.0005 & 0.0000 \\ 0.0000 & -0.0005 \end{bmatrix}.$$

Therefore, we have obtained the observer in the form of

$$\begin{cases} S_a\eta(k+1) \\ \quad = (A_a - K_aC_a)\eta(k) + B_au(k) + B_{ya}y(k) \\ \quad\ -N_ay(k) + \Phi_a(\hat{x}(k),u(k)) \\ \hat{x}_a(k) = \eta(k) + S_a^{-1}L_ay(k) \end{cases} \tag{95}$$

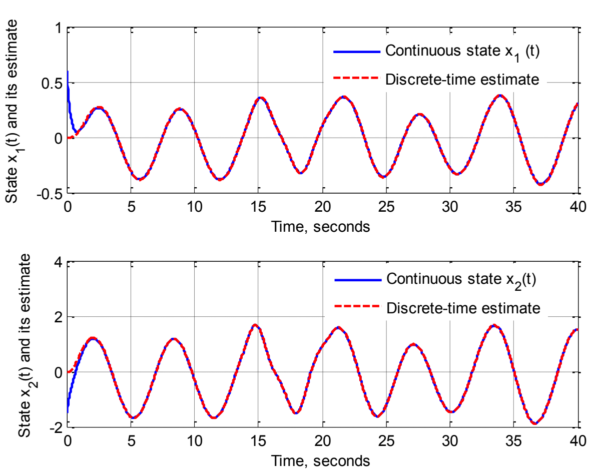

Fig. 1. States and their estimates: reactor.

Implement the observer (95) to the plant (93), we can have the estimates for the states and delayed perturbed terms. Fig. 1 and Fig. 2 show the continuous states $x_1(t)$ and $x_2(t)$, the delay perturbed terms $\omega_1(t)$ and $\omega_2(t)$ and their estimates by using the discrete-time observer (95), which have shown excellent tracking performance.

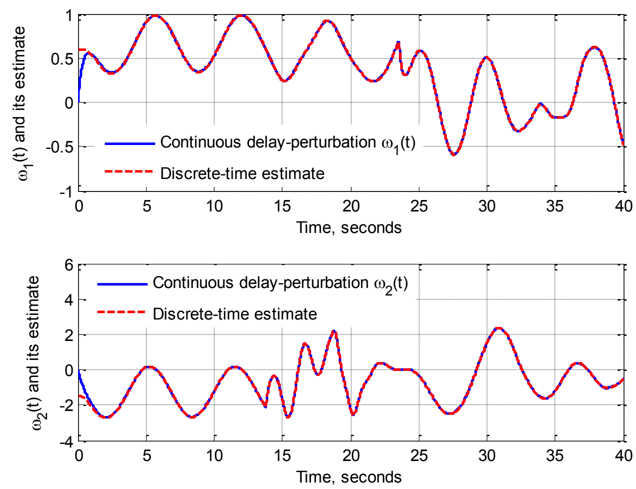

Fig. 2. Delay perturbation estimates: reactor.

(c) Subtracting the estimates of the delay perturbed terms from the system outputs, we can realize the sensor signal compensation. From Fig. 3, one can see the delayed outputs are seriously distorted compared with the outputs without delays. It is more interesting to see the outputs have been successfully recovered after the sensor signal compensation.

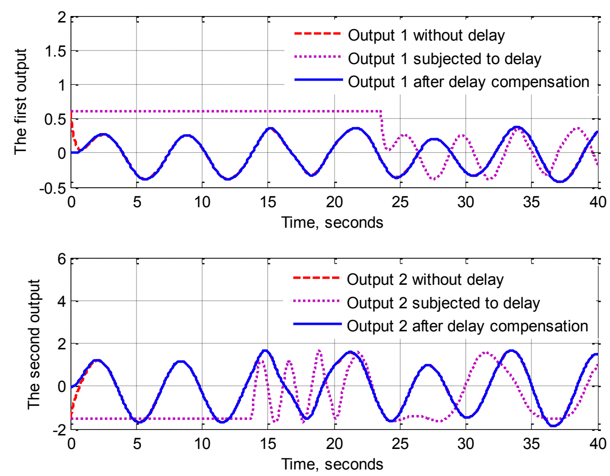

Fig. 3. Output compensation: reactor.

B. *Electromechanical servo system*

An electromechanical servo system is described by the following discrete-time model with the sampling time 0.1s:

$$\begin{cases} x(k+1) = \underbrace{\begin{bmatrix} 0.0468 & 0.1564 \\ 0.2083 & 0.8154 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 39.2076 \\ 11.5299 \end{bmatrix}}_{B} u(k) \\ \qquad + \Phi(x(k),u(k)) + B_dn_d(k) \\ y(k) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1(k) \\ x_2(k-\tau(k)) \end{bmatrix} + n_\omega(k) \end{cases}$$

(96)

where $x_1(k)$ is the load angular position, $x_2(k)$ is the shaft speed, $u(k)$ is the input voltage, $\Phi\big(x(k),u(k)\big) = [0, 0.005\sin(x_1(k))]^T$ is the perturbed nonlinear term, $\tau(k) = 2 + 0.5\sin(0.02k)$ is the time-varying delay, $n_\omega(k)$ is the measurement random noise vector with values between -20 and 20. In addition, $n_d(k) = n_{dc}(k) + n_{dh}(k)$ where $n_{dh}(k) = 0.02\sin(50k)$, and $n_d(k)$ is a step signal at $30s$, jumping from 0 to 2, and $B_d = B$.

(a) Select $\alpha = \mathrm{diag}(1,\ 0.015)$ and construct the augmented matrices in the form of (45) and (55). Choose the derivative gain as

$$L_{ar} = \begin{bmatrix} 0 & 0 & 0 & 500 & 0 \\ 0 & 0 & 0 & 0 & 500 \end{bmatrix}^T.$$

Solving the linear matrix equality (58), one can thus obtain the proportional gain, that is,

$$K_{ar} = \begin{bmatrix} 356.8383 & 3.8149 \\ 388.1853 & 5.3232 \\ 7.5665 & 0.0749 \\ 305.3800 & 3.4334 \\ 4.2370 & 0.0540 \end{bmatrix}.$$

Implementing the observer in the form of (47) to the plant (96) and using real data, the estimates of the states, delay-perturbed term and the input disturbance are obtained. The curves in Figs. 4 and 5 show excellent tracking performance.

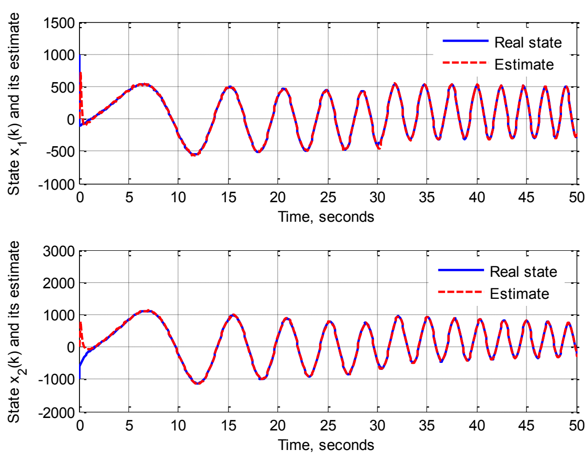

Fig. 4. States and estimates: servo system.

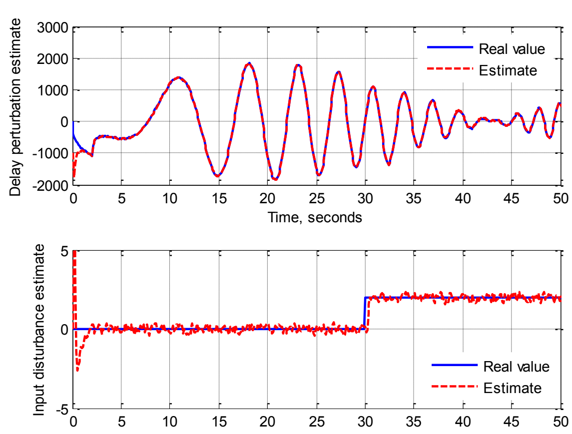

Fig. 5. Estimates of delay perturbed term and input disturbance: servo system.

(b) One can compute $F_d = B^+B_d = 1$. Apply $u(k) = -F_d\hat{n}_{dc}(k)$ to the plant (96) for disturbance compensation, and implement $y_c(k) = y(k) - \hat{\omega}(k)$ for delay compensation. In Fig. 6, the upper sub-figure shows the shaft speed has been seriously distorted by the measurement delay and the input disturbance before compensation, while the sub-figure on the bottom shows the shaft speed is successfully recovered after the signal compensation.

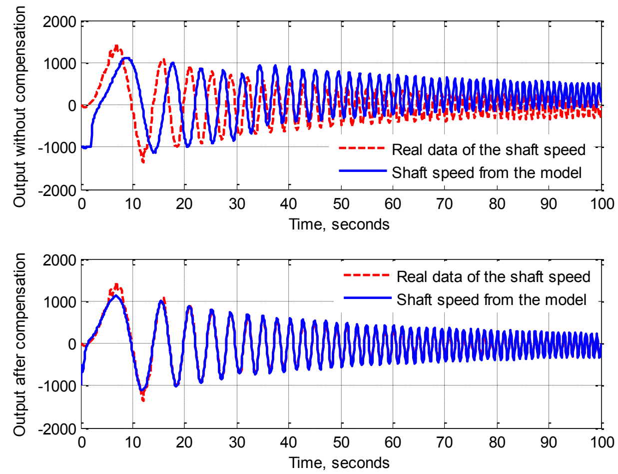

Fig. 6. Output with and without compensation: case 1

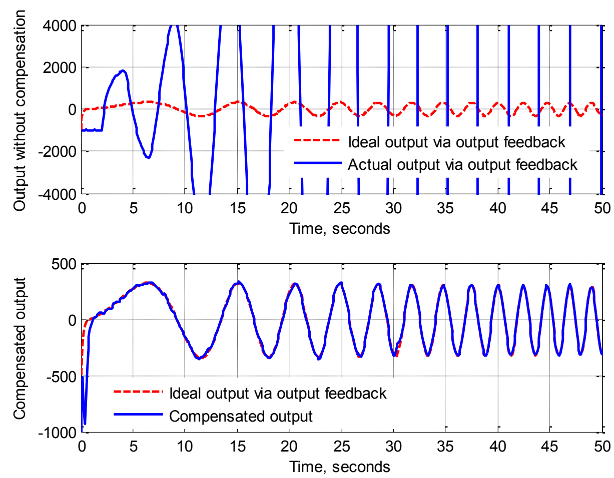

Fig. 7. Output with and without compensation: case 2

(c) In order to improve the robustness against the disturbances and estimation errors, the feedback gain F_x can be obtained as $F_x = [0.0046,\ \ 0.0173]$ by solving the linear matrix inequality (75). Along with the delay compensation, the disturbance compensation and attenuation can be done by applying $u(k) = -F_x\hat{x}(k) - F_d\hat{n}_{dc}(k)$ to the plant (96). For comparison, we also do the simulation for the system via the output feedback $u(k) = -F_x y(k)$ under the scenarios with and without output delays and disturbances. In the upper subfigure of Fig. 7, one can see the shaft speed is divergent via the direct output feedback, mainly caused by the measurement delays. In the subfigure on the bottom, the output curve after the disturbance and delay compensation is consistent with the output under ideal status (i.e, no delays and disturbances). As a result, the compensation has recovered the system performance successfully.

V. CONCLUSION

For a Lipschitz nonlinear discrete-time system subjected to unknown output delays, the novel estimation technique has been proposed to estimate the unknown delay-perturbed term, and the sensor signal compensation has been employed to remove the effect from the delay to the system output. The robustness has been addressed by decoupling the constant piece-wise input disturbance and further attenuating other disturbances/noises via the linear matrix inequality technique. The proposed methods have been demonstrated via the simulation study for two engineering-oriented examples. It worthy to point out that no prior knowledge is needed for the types of the output delays, the results developed in this paper would have a wide scope of applications in a variety of industrial systems.

In the future, it would of interest to extend the proposed techniques to a system with unknown delays acting on states/inputs, or a system subjected to both unknown measurement delays and missing measurements (see [25] about missing measurements), or a system simultaneously corrupted by unknown faults [26] and unknown measurement delays.

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