Destabilization of rotating flows with positive shear by azimuthal magnetic fields

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According to Rayleigh’s criterion, rotating flows are linearly stable when their specific angular momentum increases radially outward [1]. Since this criterion applies to the Keplerian rotation profiles which are typical for low-mass accretion disks, the growth mechanism of central objects, such as protostars and black holes, had been a conundrum for many decades. Nowadays, magnetorotational instability (MRI) [2] is considered the main candidate to explain the turbulence and enhanced angular momentum in accretion disks. The standard version of MRI (SMRI), with a vertical magnetic field \( B_z \) applied to the rotating flow, requires both the rotation period and the Alfvén crossing time to be shorter than the time scale for magnetic diffusion [3]. This implies, for a disk of height \( H \), that both the magnetic Reynolds number \( Rm = \mu_0 \sigma H^2 \Omega \) and the Lundquist number \( S = \mu_0 \sigma H \nu_\Lambda \) must be larger than one. \( \Omega \) is the angular velocity, \( \mu_0 \) is the magnetic permeability constant, \( \sigma \) the conductivity, and \( \nu_\Lambda := B_z / \sqrt{\mu_0 \rho} \) is the Alfvén velocity, with \( \rho \) denoting the density. While these conditions are safely fulfilled in well-conducting parts of accretion disks, the situation is less clear in the “dead zones” of protoplanetary disks, in stellar interiors and liquid cores of planets, because of the small value of the magnetic Prandtl number \( Pm := \nu / \eta [4] \), i.e., the ratio of viscosity \( \nu \) to magnetic diffusivity \( \eta := (\mu_0 \sigma)^{-1} \). This low Prm case is also the subject of intense theoretical and experimental research initiated by Hollerbach and Rüdiger [5]. Adding an azimuthal magnetic field \( B_\phi \) to \( B_z \), the authors found a new version of MRI, now called helical MRI (HMRI). It was proved to work also in the inductionless limit [6], \( Pm = 0 \), and to be governed by the Reynolds number \( Re = \rho m \) and the Hartmann number \( Ha := \rho m \) in contrast to standard SMRI that is governed by \( Rm \) and \( S \).

A somewhat sobering limitation of HMRI was identified by Liu et al. [7] who used a local approximation [also called the short-wavelength, Wentzel-Kramers-Brillouin (WKB), or geometric optics approximation—see Ref. [8]] to find a minimum steepness of the rotation profile \( \Omega(r) \), expressed by the Rossby number \( Ro := r(2\Omega)^{-1/2} \partial \Omega / \partial r, \) of \( Ro_{\text{ULL}} = 2(1 - \sqrt{2}) \approx -0.828. \) This lower Liu limit (LLL) implies that, at least for \( B_\phi(r) \propto 1/r \), HMRI does not extend to the most relevant Keplerian case, characterized by \( Ro_{\text{Kep}} = -3/4 \). Surprisingly, in addition to the LLL, the authors also found a second threshold of \( Ro \), which we call the upper Liu limit (ULL), at \( Ro_{\text{ULL}} = 2(1 + \sqrt{2}) \approx 4.828. \) For \( Ro > Ro_{\text{ULL}} \) one expects a magnetic destabilization of those flows with strongly increasing angular velocity that would be stable even with respect to SMRI.

By relaxing the demand that the azimuthal field is current free in the liquid, i.e., \( B_\phi(r) \propto 1/r \), and allowing fields with arbitrary radial dependence, we have recently shown [8,9] that the LLL and the ULL are just the endpoints of one common instability curve in a plane that is spanned by \( Ro \) and a corresponding steepness of the azimuthal magnetic field, called the magnetic Rossby number, \( Rb := r(2B_\phi(r))^{-1} \partial (B_\phi(r)) / \partial r. \) In the limit of large \( Re \) and \( Ha \), this curve acquires the closed and simple form

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Rb = -\frac{1}{8} \left( \frac{Ro + 2}{Re} \right)^2.
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A nonaxisymmetric “relative” of HMRI, the azimuthal MRI (AMRI) [10], which appears for purely or dominantly \( B_\phi \), has been shown to be governed by basically the same scaling behavior, and the same Liu limits [11]. Actually, the key parameter dependencies of HMRI and AMRI were confirmed in various liquid metal experiments at the PROMISE facility [12,13].

In the present Rapid Communication, we focus exclusively on the case of positive \( Ro \), i.e., on flows whose angular velocity (not only the angular frequency) is increasing outward. From a purely hydrodynamic point of view, such flows are linearly stable (while nonlinear instabilities were actually observed in experiments [14]). Flows with positive \( Ro \) are indeed stable (a second threshold of \( Ro \), which we call the upper Liu limit (ULL), at \( Ro_{\text{ULL}} = 2(1 + \sqrt{2}) \approx 4.828. \) For \( Ro > Ro_{\text{ULL}} \) one expects a magnetic destabilization of those flows with strongly increasing angular velocity that would be stable even with respect to SMRI.

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instability for \( Ro > Ro_{ULL} \), a real phenomenon (which would fundamentally modify the stability criteria for rotating flows in general), or just an artifact of the local approximation, and is there any chance to observe it in a TC experiment?

\[
\text{Re}^2 = \frac{1}{4} \left[ (1 + Ha^2 n^2)^2 - 4 Ha^2 \beta (1 + Ha^2 n^2) - 4 Ha^4 n^2 (1 + Ha^2 n^2 - 2 Rb)^2 \right]
\]

for the marginal curves of the instability, where the following definitions for \( \text{Re} \), \( Ha \), and the modified azimuthal wave number \( n \) are used:

\[
\text{Re} = \frac{\alpha}{|k|^2} \Omega(r) \quad \text{and} \quad Ha = \frac{\alpha}{|k|^2} B_\phi(r) \quad \text{with} \quad \alpha = k_y \mu_0 \eta \nu \quad \text{and} \quad |k|^2 = k_x^2 + k_y^2 \quad \text{for} \quad \mu_0 \neq 0.
\]

\[
Re^{(TI)} [18], \text{when going over from} \ Rb = 0 \text{to} \ Rb \neq 0, \text{the original) or just an artifact of the local approximation, and is there any chance to observe it in a TC experiment?}

In order to tackle these problems we restrict our attention here to nonaxisymmetric instabilities, which are the relevant ones for pure \( B_\phi \), and further assume \( P_m = 0 \). Under these assumptions, we had recently [8] derived the closed equation

\[
H_{\text{Re}=0} = 1/\sqrt{n(2-n)}.
\]

\[
H_{\text{Re}=\infty} = \frac{(Ro + 1) + \sqrt{(Ro + 1)(Ro + 2)/n}}{Ro^2 + (Ro + 1)(4 - n^2)}.
\]

![FIG. 1. (Color online) Marginal curves for Rb = −1. (a) Dependence on Ro for n = 1.4. (b) Dependence on n for Ro = 5.5. The inset shows the dependence of the minimum value (with respect to Ha) of the critical Re on n. The arrow points to the optimum n ≈ 1.35 that leads to the lowest critical Re.](image)
In the limit $\text{Ro} \to \infty$ the limit values of $\text{Ha}$ converge slowly to zero according to $\text{Ha}(\text{Re}, \text{Ro}) \to \infty \simeq n^{-1/2} \text{Ro}^{-1/4}$.

In the following, we compare our WKB results with recent findings [19] obtained for a TC flow with inner and outer radii $r_i$ and $r_o$ rotating with the angular velocities $\Omega_i$ and $\Omega_o$, respectively. The corresponding ratios are defined as $\hat{\eta} = r_i/r_o$, and $\hat{\mu} = \Omega_o/\Omega_i$. For this TC configuration, the following modified definitions of the Reynolds and Hartmann number were used: $\hat{\text{Re}} = \Omega_i r_i (r_o - r_i)/\nu$, $\hat{\text{Ha}} = B\phi(r_i) [r_i (r_o - r_i)]^{1/2}/(\mu_0 \rho \nu)^{1/2}$. The nontrivial point is now how to translate the $\hat{\mu}$ of a TC flow, characterized by $\Omega_i(r) = a + b/r^2$, to the Ro of a flow with $\Omega(r) \sim r^2$. An often used correspondence, based on equalizing the corresponding angular velocities at $r_i$ and $r_o$ [20], leads to

$$\text{Ro}^* \simeq -\frac{1}{2} \log\hat{\eta} \hat{\mu},$$

while an alternative, more shear-oriented version leads to

$$\text{Ro}^{**} \simeq \frac{1}{2} \frac{(1 + \hat{\eta})(\hat{\mu} - 1)}{(1 - \hat{\eta})(\hat{\mu} + 1)}.$$  

Actually, for comparably small (positive or negative) values of Ro, the differences are not very significant, but they increase for steeper profiles. This is a key point for the adequateness of TC flows to "emulate" steep power function flows. In Ref. [19], the destabilizing effect of positive shear had been studied for TC flows (with $\text{Rb} = 0$ only), both for a wide gap with $\hat{\eta} = 0.5$ as well as a narrow gap with $\hat{\eta} = 0.95$. In either case, for
large values of $\hat{\mu}$, the critical $Ha$ converged to some nonzero constant, which is not compatible with the translation to $Ro^*$ since the latter should lead to a zero critical $Ha$ (according to $Ha_{k,Re,Ro} \rightarrow \infty \simeq n^{-1/2} Ro^{-1/2}$—see above). It turns out that the translation to $Ro^*$ is physically more adequate.

With the reasonable choice $k_x = k_y = \pi/(r_i - r_o)$ we obtain the translations $Re = \pi^2 2\pi/2 \hat{\mu} \hat{\eta}/((1 + \hat{\mu})(1 - \hat{\eta})) Re$ and $Ha = \pi^2 (1 + \hat{\eta})^2/((2\hat{\eta})^1/2)(1 - \hat{\eta})^3/2) Ha$. For $\hat{\eta} = 0.95$ this amounts to $Re = 1061/(1 + (1/\hat{\mu}) Re$ and $Ha = 2435 Ha$. Figure 4 shows the corresponding WKB results, both for assuming a translation to $Ro^*$ (dashed lines) and to $Ro^*$ (solid lines). For $Re = 0$ our result $Ha = 2670$ agrees reasonably well with the exact value $Ha = 3060$ of the modal stability analysis [19]. What is more, the typical bend of the marginal curve to the left for increasing $Re$, and the limit values of $Ha$ for large $Re$, are also confirmed. Yet, subtle differences show up for the two ways of translation: The use of $Ro^*$ confirms the existence of a finite limit value for the critical $Ha$, as typical for TC flows, while the use of $Ro^*$ would ultimately lead to a zero limit value.

This encouraging consistency of the local approximation and the modal stability analysis, evidenced for $Re = 0$, brings us back to the point whether, for $Re = -1$, the ULL can be confirmed in a TC experiment. Assuming $Ro^*$ as more physical than $Ro^*$, in the limit $\hat{\mu} \rightarrow \infty$ we obtain $Ro_{\hat{\mu} \rightarrow \infty} = 1/2(1 + \hat{\eta})/(1 - \hat{\eta})$. This means, in turn, that to emulate some Ro in a TC flow, $\hat{\eta}$ has to fulfill the relation $\hat{\eta} = (2Ro - 1)/(2Ro + 1)$. With a view on the ULL, this implies that for $Ro = 6$, say, a minimum value of $\hat{\eta} = 11/13 = 0.846$ is needed. For TC flows with wider gaps, such as $\hat{\eta} = 1/2$, the necessary shear could simply not be realized.

What are, then, the prospects for a corresponding experiment? Evidently, we need a rather narrow gap flow. Let us stick, for a first estimate, to the safe value $\hat{\eta} = 0.95$, and take the typical values $Ro = 6$, $Ha = 2$, and $Re = 12$ as read off from Fig. 1(a). This translates to $\hat{\eta} = 1.89$, $Re = 8324$, and $Ha = 4870$. For a prospective TC experiment with Na at 150°C, with $\rho = 910 kg/m^3$, $v = 5.94 \times 10^{-7} m^2/s$, $\alpha = 9 \times 10^5 S/m$, and an outer diameter of $r_o = 0.25 m$, this would amount to a rather moderate rotation frequency of $\Omega_\phi/(2\pi) = 0.26 Hz$, yet a huge magnetic field $B_\phi(r_i) = 0.69 T$ that requires a central current of $I = 8.6 \times 10^7 A$. Exhausting the shear resources, by choosing $\hat{\mu} \rightarrow \infty$ and $\hat{\eta} = 0.85 \approx 11/13$, those values would drop to $Re = 3796, Ha = 892$, or, physically, to $\Omega_\phi/(2\pi) = 0.044 Hz, B_\phi(r_i) = 77 mT, and I = 8.2 \times 10^6 A$. Any real TC experiment, however, would need more detailed simulations with a 1D marginal stability code to confirm and optimize the parameters.

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