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Jean Lerbet, Marwa Aldowaji, Noël Challamel,
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GEOMETRIC DEGREE OF NONCONSERVATIVITY

JEAN LERBET, MARWA ALDOWAJI, NOËL CHALLAMEL,
OLEG N. KIRILLOV, FRANÇOIS NICOT AND FÉLIX DARVE

This paper deals with nonconservative mechanical systems subjected to nonconservative positional forces leading to nonsymmetric tangential stiffness matrices. The geometric degree of nonconservativity of such systems is then defined as the minimal number ℓ of kinematic constraints necessary to convert the initial system into a conservative one. Finding this number and describing the set of corresponding kinematic constraints is reduced to a linear algebra problem. This index ℓ of nonconservativity is the half of the rank of the skew-symmetric part K_a of the stiffness matrix K that is always an even number. The set of constraints is extracted from the eigenspaces of the symmetric matrix K_a^2. Several examples including the well-known Ziegler column illustrate the results.

Introduction

This paper is concerned with the statics or dynamics of a discrete or discretized system Σ_{free}. We assume that, after having started with different possible nonlinear settings, convenient assumptions and approximations lead to a dynamic evolution governed by the following equation of motion of the free system Σ_{free}:

$$M\ddot{X} + K(p)X = 0, \tag{1}$$

where $K(p) = K_s(p) + K_a(p)$, with $K_s(p) = \frac{1}{2}(K(p) + K^T(p))$ and $K_a(p) = \frac{1}{2}(K(p) - K^T(p))$; note that $K(p)$ is generally a nonsymmetric matrix ($K_a(p) \neq 0$) because of the nonconservativity of Σ_{free}, whereas M is symmetric positive definite.

The paradoxical effects of mechanical systems linearly governed by (1) with a nonsymmetric stiffness matrix have been known for some time (see for example [Ziegler 1952; Bolotin 1961]) and have been investigated in depth, especially concerning the so-called destabilizing effect of adding friction in the system (see [Kirillov and Verhulst 2010] for recent developments). Bigoni and Noselli [2011] have illustrated through an experimental device calculations starting from dynamics with dry friction and coming to equations like (1). Circulatory forces seem to have

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appeared first in the rotor dynamics works of early 1920s. It is generally agreed that E. L. Nikolai was the first to have found some curious paradoxes induced by the nonconservative aspect of the loading system. Here, we only investigate systems without rotating effects which means that circulatory forces are only nonconservative positional loading. More precisely, by decomposing in the linear case the stiffness matrix $K(p)$ into a symmetric part and a skew-symmetric part, one then decomposes nonpotential forces into a potential component and a circulatory component. Zhuravlev [2007] suggests an extension of this decomposition for the nonlinear case by using the trick of Poincaré in his theorem about exact and closed differential forms. This could be a good way to tackle the nonlinear extension of the present paper.

If p is a loading parameter any norm $\|K_a(p)\|$ of the skew-symmetric part $K_a(p)$ of $K(p)$ is an elementary measure of the nonconservativity of the corresponding nonpotential forces by any norm on the space of matrices. However, this rough measure indicates the amplitude of the nonconservativity and masks another more intrinsic measure of this nonconservativity which is defined in this paper. This is here defined by a lower semicontinuous function with only finite integer values (for an increasing load). This function is then locally independent on the load parameter value, except perhaps for a finite number of singular values $\{p_0^* < p_1^* < \cdots < p_r^*\}$ of p. Obviously $p_0^* = 0$ is such a value, because for $p = 0$ the system is conservative and $K_a(0) = 0$. In all the examples except the so-called Bigoni system, the only value is $p = 0$. Because it is also linked to a dimension of a linear space, we then propose to call this number the geometric degree of nonconservativity of the system (or of the forces).

The genesis of the used approach lies in several papers [Challamel et al. 2009; 2010; Nicot et al. 2011; Lerbet et al. 2012] that investigated the deep rule of the second-order work criterion proposed in [Hill 1958] for solids in the framework of nonassociated plasticity, and independently also proposed for instabilities for systems subjected to nonpotential forces in [Absi and Lerbet 2004]. This criterion performs especially well for nonconservative systems because, contrary to the divergence criterion, it remains "stable" under the action of additional kinematics constraints: if this criterion holds for a free system and for a value p of the load parameter, it still holds for the same value p and for a system subjected to any family of additional kinematic constraints. This property, contrary to a similar well-known consequence of the Rayleigh theorems for conservative systems, is generally no more valid for nonconservative systems. This paradoxical behavior of the mechanical system, or more precisely of the stability of the investigated equilibrium configuration of the mechanical system when adding additional kinematic constraints, is actually a characteristic of nonconservative systems.

Thus, extending the above-mentioned works concerning the effects of additional constraints on such n-DOF (degree of freedom) nonconservative mechanical systems [Challamel et al. 2010; Lerbet et al. 2012; 2013], we focus on families of kinematic constraints that could convert Σ_{free} into a conservative system. More precisely, we address both the problems of the existence of a minimal family (according to the number of constraints) of such constraints and that of building the set of such families. The minimal number of constraints required to convert the nonconservative system Σ_{free} into a conservative system is then a measure of the nonconservativity of Σ_{free} and will be called the geometric degree of nonconservativity of Σ_{free}.

The paper is organized as follows: in Section 1, the mechanical problem is reduced to a linear algebra problem. In Section 2, the solution is developed leading to the concept of the geometric degree of nonconservativity of a mechanical system. In Section 3, several examples illustrate the mathematical results.

1. Modeling of the mechanical problem

Let Σ_{free} be a n-DOF discrete mechanical system and suppose, as above, that the dynamic evolution of Σ_{free} is governed by (1). X is the vector of kinematic unknowns ($X^T = (x_1, \dots, x_n) \in \mathcal{M}_{1n}(\mathbb{R})$), M is the mass matrix (symmetric definite positive), and $K = K(p)$ is the stiffness matrix. The latter is any square matrix because of the nonconservativity of Σ_{free}. Let p be a (loading) parameter. Suppose that m (independent) additional kinematic constraints $C^1, \dots, C^m$ are set up on Σ_{free}. The linear framework leads us to model each kinematic constraint C^j by a linear relationship $\sum_{k=1}^{n} \alpha_k^j x_k = 0$. Thus C^j is represented by and identified with a vector $\alpha^j = (\alpha_1^j, \dots, \alpha_n^j)$ of $\mathbb{R}^n$ (actually it is a linear form on $\mathbb{R}^n$ but by the canonical scalar product we may identify both spaces). The family of m constraints $\{\alpha^1, \dots, \alpha^m\}$ may be considered as an element of an nm-dimensional vector space — for instance as an $n \times m$ matrix $A = \begin{pmatrix} \alpha^1 & \cdots & \alpha^m \end{pmatrix}$ in $\mathcal{M}_{nm}(\mathbb{R})$, or more precisely in $\mathcal{G}_{nm}(\mathbb{R})$, the open subset of matrices of $\mathcal{M}_{nm}(\mathbb{R})$ with rank m, because of the independence of the constraints. If m is fixed (it will have to be found in a first step), we then have to find the set $C_m(\Sigma_{\text{free}}) \subset \mathcal{G}_{nm}(\mathbb{R})$ such that if $A \in C_m(\Sigma_{\text{free}})$ then the constrained mechanical system $\Sigma_{\text{cons}} = \Sigma_{\text{cons}}(A)$ becomes conservative. Thus

$$A^T = \begin{pmatrix} \alpha^{1T} \\ \vdots \\ \alpha^{mT} \end{pmatrix} = \begin{pmatrix} \alpha_1^1 & \cdots & \alpha_n^1 \\ \vdots & \ddots & \vdots \\ \alpha_1^m & \cdots & \alpha_n^m \end{pmatrix}$$

(every vector $\alpha^{iT} = (\alpha_1^i, \dots, \alpha_n^i)$ could be normalized $\alpha^{iT}\alpha^i = 1$).

Let $\Lambda \in \mathcal{M}_{m1}(\mathbb{R})$, with $\Lambda^T = (\lambda_1 \ \dots \ \lambda_m)$, be the Lagrange multiplier attached to the constraints. The equation of motion of the constrained system $\Sigma_{\text{cons}}(A)$ is

$$A^T X = 0, \tag{2}$$

$$M\ddot{X} + K(p)X + A\Lambda = 0, \tag{3}$$

Let $T(A) = \mathrm{Vect}\{\alpha^1, \ldots, \alpha^m\}$ and let $H(A) = T(A)^{\perp}$ be the orthogonal to $T(A)$ in $\mathbb{R}^n$ identified with $\mathcal{M}_{n1}(\mathbb{R})$. Thus $\dim T(A) = m$ and $\dim H(A) = n - m$.

Let us choose an orthonormal basis of $T(A)$ (by Gram–Schmidt from $(\alpha^1, \ldots, \alpha^m)$, for example) $(t_1(A), \ldots, t_m(A))$ and another $(h_{m+1}(A), \ldots, h_n(A))$ of $H(A)$ such that $b(A) = (t_1(A), \ldots, t_m(A), h_{m+1}(A), \ldots, h_n(A))$ is an orthonormal basis of $\mathbb{R}^n$ and let $P = P(A) \in O_n(\mathbb{R})$ be the orthogonal matrix passing from the canonical basis of $\mathbb{R}^n$ to $b(A)$:

$$P = P(A) = \mathrm{mat}(t_1(A), \ldots, t_m(A), h_{m+1}(A), \ldots, h_n(A)).$$

Let Y be defined by $X = P(A)Y$. The previous system reads:

$$(P(A)^T A)^T Y = 0, \tag{4}$$

$$P^T(A)MP(A)\ddot{Y} + P^T(A)K(p)P(A)Y + P(A)^T A\Lambda = 0, \tag{5}$$

Considering $M_{\mathrm{cons}}(A)$ (resp. $K_{\mathrm{cons}}(A, p)$) the square submatrix of $P^T(A)MP(A)$ (resp. $P^T(A)K(p)P(A)$) built by suppressing the first m rows and the first m columns of $P^T(A)MP(A)$ (resp. of $P^T(A)K(p)P(A)$), we get the following equations of the constrained system without the Lagrange multipliers:

$$M_{\mathrm{cons}}(A)\ddot{Y}_{\mathrm{cons}} + K_{\mathrm{cons}}(A, p)Y_{\mathrm{cons}} = 0, \tag{6}$$

where $Y_{\mathrm{cons}}^T = (y_{m+1}, \ldots, y_n) \in \mathcal{M}_{1\,n-m}(\mathbb{R})$.

We are then led to investigate when $K_{\mathrm{cons}}(A, p)$, a $(n-m) \times (n-m)$ square submatrix of $P^T(A)K(p)P(A)$, is symmetric. Note that, in the standard case of structural mechanics, $K(p) = K_{\mathrm{el}} - pK_{\mathrm{ext}}$ with K_{el} the symmetric definite-positive stiffness matrix relative to elastic actions and K_{ext} the nonsymmetric matrix relative to external actions (circulatory force). $K_{\mathrm{cons}}(A, p)$ reads:

$$K_{\mathrm{cons}}(A, p) = \begin{pmatrix} h_{m+1}^T(A)K(p)h_{m+1}(A) & \cdots & h_{m+1}^T(A)K(p)h_n(A) \\ \vdots & \ddots & \vdots \\ h_n^T(A)K(p)h_{m+1}(A) & \cdots & h_n^T(A)K(p)h_n(A) \end{pmatrix}.$$

The condition for the constraints defined by A to convert the free nonconservative system into a conservative one is then

$$h_i^T(A)K(p)h_j(A) = h_i^T(A)K^T(p)h_j(A) \quad \text{for all } i, j,$$

or, in a more geometrical phrasing: for every pair u and v of two orthogonal vectors of $H(A)$ orthogonal to $T(A) = \mathrm{Vect}\{\alpha^1, \ldots, \alpha^m\}$, $u^T K(p)v = u^T K^T(p)v$. This is equivalent to $u^T K_a(p)v = 0$ for every pair u and v of any two vectors of $H(A)$ with $K_a(p)$ the skew-symmetric part of $K(p)$. This is obviously right for $m = n-1$ because for any $n-1$ independent constraints the submatrix becomes a scalar, the skew-symmetric part of which is always nil!

Let $\phi_a(p)$ be the linear map whose matrix is in the canonical basis of $\mathbb{R}^n$ is $K_a(p)$. Geometrically, the condition means that for every vector u of $H(A)$, $\phi_a(p)(u)$ is orthogonal to $H(A)$ or equivalently belongs to $T(A)$:

$$\phi_a(p)(u) \in T(A) \quad \text{for all } u \in H(A). \tag{7}$$

The initial mechanical problem has then been modeled into the following original problem of linear algebra and more precisely of Euclidean spaces: Does there exist an $(n-m)$-dimensional subspace of $\mathbb{R}^n$ which is sent onto its orthogonal by $\phi_a(p)$? We denote by $(\cdot \mid \cdot)$ the scalar product of $\mathbb{R}^n$ ($(u \mid v) = u^T v$ with the identification of $\mathbb{R}^n$ with $\mathcal{M}_{n1}(\mathbb{R})$). In the following section, this problem is solved.

2. Solution of the mathematical modeling

We forget the p-dependency of all the quantities. In the introduction, we already noted that the loading interval $I = [0, +\infty[$ may be decomposed as $\{0\} \cup]0, p_1^*] \cup]p_1^*, p_2^*] \cup \cdots \cup]p_r^*, +\infty[$ with $p_1^*, \ldots, p_r^*$ nonzero singular values of loading. Forgetting the p-dependency means that $p \notin \{0, p_1^*, \ldots, p_r^*\}$. In the examples excepted for Bigoni's system, 0 is the only singular value. The singular problem is not investigated in this paper.

Let $F_a = \mathrm{Im}(\phi_a)$ and $G_a = \mathrm{Ker}(\phi_a)$. We know that (as every skew-symmetric linear map) ϕ_a has an even rank, $r = 2\ell$, and that its kernel and its image are orthogonal spaces. Let $G_a = \mathrm{Ker}(\phi_a)$. Thus $\mathbb{R}^n = F_a \overset{\perp}{\oplus} G_a$. We set the following:

Definition. The integer ℓ is called the geometric degree of nonconservativity of Σ_{free}.

As ϕ_a^2 is a symmetric linear mapping it is diagonalizable in an orthonormal basis. Moreover $G_a = \mathrm{Ker}(\phi_a) = \mathrm{Ker}(\phi_a^2)$, the nonzero eigenvalues of ϕ_a^2 are negative, and the associated eigenspaces are two-dimensional and mutually orthogonal. Note these values $-\mu_1^2, \ldots, -\mu_\ell^2$ and $E_{-\mu_i^2}$, the associated eigenspaces for $i = 1, \ldots, \ell$. Each of these spaces are ϕ_a-stable. Because of the ϕ_a-stability of each of the spaces of the decomposition

$$\mathbb{R}^n = G_a \overset{\perp}{\oplus} E_{-\mu_1^2} \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} E_{-\mu_\ell^2},$$

we deduce (by Cartan's theorem) the existence of an orthonormal basis b' of $\mathbb{R}^n$

such that the matrix of ϕ_a in b' is

$$\begin{pmatrix} 0 & \cdots & & & & & & \cdots & 0 \\ \vdots & \ddots & & & & & & & \vdots \\ & & 0 & & & & & \cdots & \\ & & & 0 & -\mu_1 & & & & \\ & & & \mu_1 & 0 & & & & \\ & & & & & \ddots & & & \\ \vdots & & & & & & & 0 & -\mu_\ell \\ 0 & & & & & & & \mu_\ell & 0 \end{pmatrix}.$$

Thus

$$\mathbb{R}^n = G_a \overset{\perp}{\oplus} E_{-\mu_1^2} \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} E_{-\mu_\ell^2} = F_a \overset{\perp}{\oplus} G_a = H(A) \overset{\perp}{\oplus} T(A).$$

Proposition. *Equation* (7) *holds if and only if $m = \ell$ and the family A of constraints must be built by choosing $\alpha^i \in E_{-\mu_i^2}$ for $i = 1, \dots, \ell = m$, being, however, careful that the property fails if two constraints are chosen in the same eigenspace $E_{-\mu_i^2}$.*

Proof. Suppose first that $m = \ell = \frac{1}{2}\operatorname{rank}(\phi_a)$ and A built as proposed in the proposition. Let $u \in H(A)$. Complete the basis $(\alpha^1, \dots, \alpha^m)$ by $n - 2m$ vectors $\beta^1, \dots, \beta^{n-2m}$ of G_a and the m other vectors $(\phi_a(\alpha^1), \dots, \phi_a(\alpha^m))$ so that the family $(\beta^1, \dots \beta^{n-2m}, \phi_a(\alpha^1), \dots, \phi_a(\alpha^m))$ is an orthogonal basis of $T(A)$, as may be easily checked. By definition,

$$u = \sum_{k=1}^{m} u_k \alpha^k,$$

and then

$$\phi_a(p)(u) = \sum_{k=1}^{m} u_k \phi(\alpha^k) \in T(A),$$

which is exactly (7).

Reciprocally, suppose now $m < l$ or $m = l$ but A is not built as proposed in the proposition. Thus there is some $i \in \{1, \dots, l\}$ such that any α^j belongs to $E_{-\mu_i^2}$ meaning geometrically that $T(A) \cap E_{-\mu_i^2} = \{0\}$. Choose now $u \neq 0$ in $H(A) \cap E_{-\mu_i^2}$. Thus $(u, \phi_a(u))$ is an orthogonal basis of $E_{-\mu_i^2}$, meaning that $\phi_a(u)(\neq 0) \in E_{-\mu_i^2}$, implying $\phi_a(u) \notin T(A)$. □

Thus, coming back to the mechanical problem, *a free nonconservative mechanical system Σ_{free} can be made conservative by means of m constraints if and only if $K_a(p)$ has rank $2m$ and the matrix A is formed by by m vectors $\alpha^1, \dots, \alpha^m$, each α^i being chosen in the eigenspace $E_{-\mu_i^2}$ of $K_a(p)^2$; and $C_m(\Sigma) = \{A \in \mathcal{M}_{nm}(\mathbb{R}) \mid$* $\operatorname{col}_i(A) \in E_{-\mu_i^2} \setminus \{0\}\}$ (*with obvious notations*) *is an open $2m$-dimensional cone of $\mathcal{M}_{nm}(\mathbb{R})$.*

As stated previously, any constrained conservative system is still a conservative system. Thus if there are $k \geq m$ constraints, and m of the k constraints are chosen as above, the constrained system is still conservative. If $\operatorname{rank}(K_a(p)) = 2m$, then m is the minimum number of constraints needed to convert the system into a conservative one. The nonconservativity of the free system Σ_{free} is then characterized by two measures of nonconservativity. The first is the norm of the skew-symmetric part $K_a(p)$, which indicates the amplitude of the nonconservativity, while the second is the rank 2ℓ of $K_a(p)$, which acts as a geometric measure of the nonconservativity or a sort of dimension (ℓ) of the nonconservativity. This is the reason for the above definition of the geometric degree of nonconservativity of Σ_{free}. Moreover we may localize this nonconservativity because we may build families of ℓ vectors (or constraints) allowing us to convert the initial nonconservative system into a conservative one. Note also that the proof is constructive because it builds the kinematic constraints A converting the system Σ_{free} into a conservative one ($\Sigma_{\text{cons}}(A)$). There are 2^m different independent systems A of constraints converting the nonconservative Σ_{free} into a conservative $\Sigma_{\text{cons}}(A)$. This result may be considered as a sort of dual to the result about the destabilizing effect of adding kinematic constraints in nonconservative systems (see again [Challamel et al. 2009; 2010; Nicot et al. 2011; Lerbet et al. 2012]): by adding a suitable constraint in a suitable eigenspace of K_s, one can destabilize a stable nonconservative system. Here, by choosing appropriate constraints in suitable eigenspaces of K_a^2, one can convert a nonconservative system into a conservative one. In the following section, several examples issued from different mechanical systems illustrate these results.

3. Examples

In this section, we propose a collection of examples consisting in variations on the paradigmatic Ziegler column. The degree of freedom (parameter n) and the nature of the follower force (partial or complete follower force parameter γ) may change. In the most general case, the system Σ consists of n bars $OA_1, A_1A_2, \ldots, A_{n-1}A_n$ with $OA_1 = A_1A_2 = \cdots = A_{n-1}A_n = h$ linked with n elastic springs with the same stiffness k. $\vec{P}$ is the follower nonconservative load acting on A_n. Adopting a dimensionless format, we use $p = \|\vec{P}\|h/k$ as a loading parameter. To investigate how the algebraic method is performing, we conduct the complete calculation only for the three-DOF Ziegler column.

In Section 3.1, we investigate the pure Ziegler system and we notice that the geometric degree of nonconservativity is one for any number of rigid bars, meaning for any degree of freedom. Increasing the number of bars or the degree of freedom does not change its geometric degree of nonconservativity: from the geometric point of view, the Ziegler system is weakly nonconservative. In Section 3.2, we investigate what we call the multiple-DOF Bigoni system, because it involves device like that of [Bigoni and Noselli 2011] at each joint. This system appears as a generalization of the n-DOF Ziegler column where the load parameter is itself distributed on the system and may vary on each joint. It also may be considered as a discretized Leipholz column [Leipholz 1987]. In this case, the geometric degree of nonconservativity increases with the number of bars and the degree of freedom. Calculations are made only for $n = 2$ and $n = 4$. From a geometric point of view, the Bigoni system is essentially more strongly nonconservative than the Ziegler system.

3.1. ***Ziegler systems.***

3.1.1. *Two-DOF Ziegler column with complete follower force.* The geometric stiffness matrix is

$$K_{\text{ext}} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Its skew-symmetric part is

$$K_{a,\text{ext}} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

The square of $K_{a,\text{ext}}$ is

$$K^2_{a,\text{ext}} = -\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\mu_1^2 = -\frac{1}{4}$, $K^2_{a,\text{ext}}$ is spheric, and $E_{\mu_1}(\phi^2_{a,\text{ext}}) = \mathbb{R}^2$. Then α is any vector in $\mathbb{R}^2$. Obviously any constraint converts the free system into a conservative one as a one-DOF (elastic) system is always conservative because any continuous function has a primitive.

3.1.2. *Two-DOF Ziegler column with partial follower force.* The geometric stiffness matrix is

$$K_{\text{ext}} = \begin{pmatrix} 1 & -\gamma \\ 0 & 1-\gamma \end{pmatrix}.$$

Its skew-symmetric part is

$$K_{a,\text{ext}} = \begin{pmatrix} 0 & -\dfrac{\gamma}{2} \\ \dfrac{\gamma}{2} & 0 \end{pmatrix}.$$

The square of $K_{a,\text{ext}}$ is

$$K^2_{a,\text{ext}} = -\frac{\gamma^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Our conclusions are similar to those above.

3.1.3. *Three-DOF Ziegler column with complete follower force.* The geometric stiffness matrix is

$$K_{\text{ext}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Its skew-symmetric part is

$$K_{a,\text{ext}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Obviously $\text{rank}(K_{a,\text{ext}}) = 2$. The square of $K_{a,\text{ext}}$ is:

$$K^2_{a,\text{ext}} = -\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Calculations give $-\mu_1^2 = -\frac{1}{2}$ and

$$E_{-\frac{1}{2}}(K^2_{a,\text{ext}}) = \text{Vec}\left\{ \alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ K_{a,\text{ext}}\alpha = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\},$$

leading to two generic constraints converting the system into a conservative one: $\theta_3 = 0$ and $\theta_1 + \theta_2 = 0$. In practice, any linear combination of these two constraints lies in the corresponding plane and may be chosen as a possible constraint converting the system into a conservative one. We now propose to check for this case the results coming from our algebraic method with respect to the direct approach to the problem.

The virtual power of the follower force reads:

$$\mathcal{P}^*(P) = Q_1\theta_1^* + Q_2\theta_2^* + Q_3\theta_3^* = Ph(\sin(\theta_3 - \theta_1)\theta_1^* + \sin(\theta_3 - \theta_2)\theta_2^*). \tag{8}$$

The complete nonlinear condition in order to have a conservative system is that there is a function $\theta = (\theta_1, \theta_2, \theta_3) \mapsto U(\theta_1, \theta_2, \theta_3) = U(\theta)$ such that $Q_k = -\partial U/\partial\theta_k$, which is here obviously impossible without an additional constraint: that the free system is nonconservative!

Suppose now the system is subjected to a kinematic constraint $\phi(\theta) = 0$, which leads to the following condition on the virtual parameters:

$$\frac{\partial\phi}{\partial\theta_1}\theta_1^* + \frac{\partial\phi}{\partial\theta_2}\theta_2^* + \frac{\partial\phi}{\partial\theta_3}\theta_3^* = 0. \tag{9}$$

Supposing the problem is resolvable with respect to the variable θ_3, meaning that $\partial\phi/\partial\theta_3 \neq 0$, we deduce from the implicit functions theorem that (locally in the neighborhood of $\theta = 0$), $\theta_3 = \theta_3(\theta_1, \theta_2)$ meaning that, to first order,

$$\theta_3 = \theta_3(\theta_1, \theta_2) \approx \frac{\partial\theta_3}{\partial\theta_1}\bigg|_{\theta=0}\theta_1 + \frac{\partial\theta_3}{\partial\theta_2}\bigg|_{\theta=0}\theta_2 = c_1\theta_1 + c_2\theta_2. \tag{10}$$

Thus, to first order, the expansion reads: $Q_1 = Q_1(\theta_1, \theta_2) \approx Ph((c_1 - 1)\theta_1 + c_2\theta_2)$ and $Q_2 = Q_2(\theta_1, \theta_2) \approx Ph((c_1\theta_1 + (c_2 - 1)\theta_2)$. The condition of conservativity of the loading then reads $c_1 = c_2 = c$. The kinematic relation is $\theta_3 = c(\theta_1 + \theta_2)$ and the quadratic potential is

$$U(\theta) \approx -Ph\Big(\frac{c-1}{2}(\theta_1^2 + \theta_2^2) + c\theta_1\theta_2\Big). \tag{11}$$

For $c = 0$, we find the first generic kinematic constraint $\theta_3 = 0$, and for $c \neq 0$ it is, as expected, a linear combination of both generic constraints.

Suppose now that the problem is not resolvable with respect to the variable θ_3, meaning that $\partial\phi/\partial\theta_3 = 0$. We then deduce that, linearly, the relation only concerns θ_1 and θ_2 and reads linearly as

$$\frac{\partial\phi}{\partial\theta_1}\bigg|_{\theta=0}\theta_1 + \frac{\partial\phi}{\partial\theta_2}\bigg|_{\theta=0}\theta_2 = a_1\theta_1 + a_2\theta_2 \approx 0, \tag{12}$$

and that $a_1\theta_1^* + a_2\theta_2^* = 0$. Resolving these relations, for example, with respect to the variable θ_1 ($\theta_1 = -b\theta_2 = -(a_2/a_1)\theta_2$ and reporting this relation in (8) shows that $\mathcal{P}^*(P) = Q_2\theta_2^*$ with $Q_2 = Q_2(\theta_2, \theta_3) \approx Ph(-b(\theta_3 - b\theta_2) + (\theta_3 - \theta_2)) \approx Ph((b^2 - 1)\theta_2 + (1 - b)\theta_3)$. The condition of integrability then reads $b = 1$, the kinematic relation is $\theta_1 + \theta_2 = 0$, and the potential is nil up to order two. Then we will come back precisely to the second generic kinematic constraint. To sum up, the direct calculations lead to both generic constraints obtained from our algebraic method.

3.1.4. *Three-DOF Ziegler column with partial follower force.* The geometric stiffness matrix is

$$K_{\text{ext}} = \begin{pmatrix} 1 & 0 & -\gamma \\ 0 & 1 & -\gamma \\ 0 & 0 & 1-\gamma \end{pmatrix}.$$

Its skew-symmetric part is

$$K_{a,\text{ext}} = \tfrac{1}{2}\gamma \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Obviously $\operatorname{rank}(K_{a,\text{ext}}) = 2$. The square of $K_{a,\text{ext}}$ is

$$K^2_{a,\text{ext}} = -\frac{1}{4}\gamma^2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Calculations give $-\mu_1^2 = -\gamma^2/2$ and

$$E_{-\gamma^2/2}(K^2_{a,\text{ext}}) = \operatorname{Vec}\left\{ \alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, K_{a,\text{ext}}\alpha = \frac{\gamma}{2} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

leading to the same two generic constraints as previously which convert the system into a conservative one: $\theta_3 = 0$ and $\theta_1 + \theta_2 = 0$.

3.1.5. *An n-DOF Ziegler column with complete follower force* ($\gamma = 1$). The stiffness matrix is

$$K(p) = \begin{pmatrix} 2-p & -1 & 0 & 0 & \cdots & 0 & p \\ -1 & 2-p & -1 & 0 & \cdots & 0 & p \\ 0 & -1 & 2-p & -1 & \cdots & 0 & p \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2-p & -1+p \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

Its skew-symmetric part is

$$K_a(p) = \frac{p}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & 0 \end{pmatrix}.$$

Obviously $\operatorname{rank}(K_a(p)) = 2$. The square of $K_a(p)$ is

$$K_a^2(p) = -\frac{p^2}{4} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n-1 \end{pmatrix} = p^2 \widetilde{K}_a^2.$$

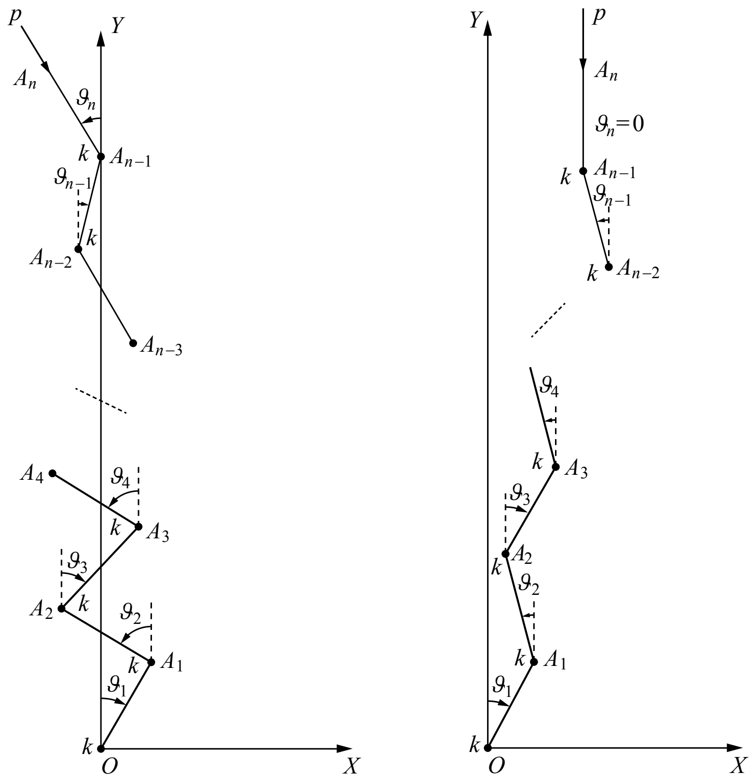

Figure 1. An n-DOF Ziegler column with complete follower force: the case $\theta_1 + \cdots + \theta_{n-1} = 0$ (left) and the case $\theta_n = 0$ (right).

Calculations give $-\mu_1^2 = -(n-1)/4$ and

$$E_{-(n-1)/4}(\widetilde{K}_a^2) = \mathrm{Vec}\left\{ \alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \widetilde{K}_a^2 \alpha = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ 0 \end{pmatrix} \right\}$$

leading to two generic constraints converting the system into a conservative one: $\theta_1 + \cdots + \theta_{n-1} = 0$, meaning that the motion of A_{n-1} is constrained to remain on the axis OY (Figure 1, left), and $\theta_n = 0$ (Figure 1, right).

3.2. ***The Bigoni system or discretized Leipholz column.*** We now turn to the n-DOF Bigoni system [Bigoni and Noselli 2011], which can also be regarded as an n-DOF Leipholz column [1987]. The system Σ consists of n bars OA_1, A_1A_2,

$\dots, A_{n-1}A_n$, with $OA_1 = A_1A_2 = \cdots = A_{n-1}A_n = h$ linked with n elastic springs with the same stiffness k. Adopting the same device at the end of each bar of Σ leads to a family of follower forces $\vec{P}_1, \dots, \vec{P}_n$ (see Figure 2, left). The pure follower forces $\vec{P}_1, \vec{P}_2, \dots, \vec{P}_n$ are applied at the ends of $OA_1, A_1A_2, \dots, A_{n-1}A_n$, respectively. Adopting a dimensionless format, we use $p_i = \|\vec{P}_i\|h/k$, for $i = 1, \dots, n$, as loading parameters. The stiffness matrix is $K(p) = K(p_1, p_2, \dots, p_n)$:

$$K(p)=\begin{pmatrix} 2-\sum_{i=2}^{n} p_i & -1+p_2 & p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ -1 & 2-\sum_{i=3}^{n} p_i & -1+p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ 0 & -1 & 2-\sum_{i=4}^{n} p_i & -1+p_4 & p_5 & \cdots & p_{n-1} & p_n \\ 0 & 0 & -1 & 2-\sum_{i=5}^{n} p_i & -1+p_5 & \cdots & p_{n-1} & p_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1+p_{n-1} & p_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2-p_n & -1+p_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

Its skew-symmetric part is

$$K_a(p) = \frac{1}{2}\begin{pmatrix} 0 & p_2 & p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ -p_2 & 0 & p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ -p_3 & -p_3 & 0 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ -p_4 & -p_4 & -p_4 & 0 & p_5 & \cdots & p_{n-1} & p_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -p_{n-1} & -p_{n-1} & -p_{n-1} & -p_{n-1} & -p_{n-1} & \cdots & 0 & p_n \\ -p_n & -p_n & -p_n & -p_n & -p_n & \cdots & -p_n & 0 \end{pmatrix}$$

and

$$\operatorname{rank}(K_a(p)) = \begin{cases} n & \text{if } n \text{ even,} \\ n-1 & \text{if } n \text{ odd,} \end{cases}$$

thus

$$\ell = \begin{cases} n/2 & \text{if } n \text{ even,} \\ (n-1)/2 & \text{if } n \text{ odd.} \end{cases}$$

For $n = 2$, calculations give

$${K_a}^2 = -\frac{1}{4}p_2^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -\mu_1^2 = -\frac{1}{4}p_2^2.$$

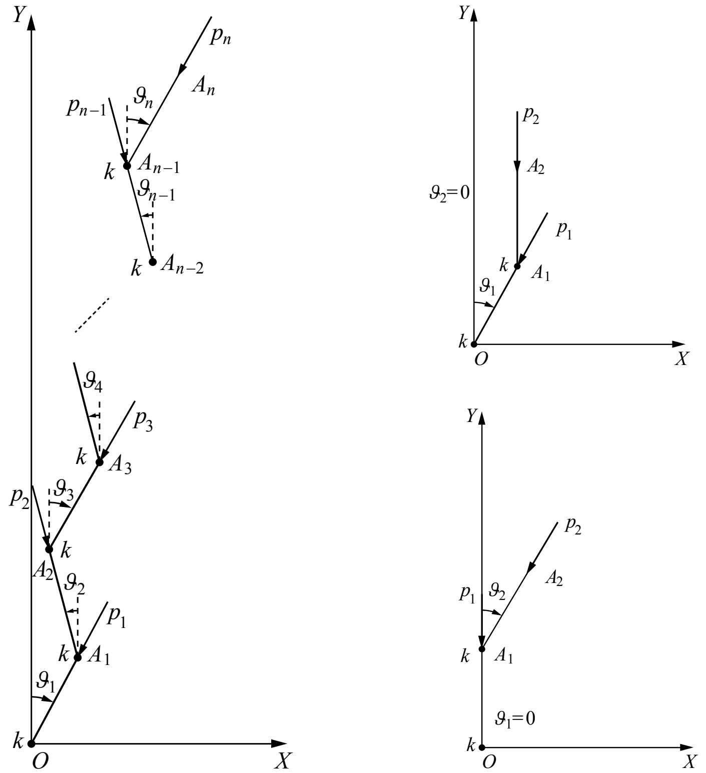

Figure 2. Bigoni systems with n DOF (left) and two DOF (right).

K_a^2 is spherical, $E_{-\mu_1^2}(K_a^2) = \mathbb{R}^2$, and α is then any vector in $\mathbb{R}^2$. The geometric degree of nonconservativity is equal to 1 and the constraint is a linear combination of the two generic constraints $\theta_1 = 0$ and $\theta_2 = 0$: this is any linear constraint! (See Figure 2, right.)

For $n = 4$, calculations give

$$-\mu_1^2 = -\tfrac{3}{8}p_4^2 - \tfrac{1}{4}p_3^2 - \tfrac{1}{8}p_2^2 + \tfrac{1}{8}a, \qquad -\mu_2^2 = -\tfrac{3}{8}p_4^2 - \tfrac{1}{4}p_3^2 - \tfrac{1}{8}p_2^2 - \tfrac{1}{8}a,$$

where $a = \sqrt{9p_4^4 + 12p_3^2p_4^2 + 2p_4^2p_2^2 + 4p_3^4 + 4p_3^2p_2^2 + p_2^4}$, and

$$\begin{aligned}
E_{-\mu_1^2}(K_a^2) = \mathrm{Vec}\big\{\alpha_1 = \big[&[2(-p_3^2p_4^2a + 2p_3p_2p_4^2a - p_4^2p_2^2a + p_3p_2^3a \\
&+ p_3^3p_2a - p_4^2p_2^4 + 3p_3p_2^3p_4^2 + 5p_3^3p_2p_4^2 + 2p_3p_2p_4^4 + p_3^2p_2^2p_4^2 + p_4^4p_2^2 \\
&+ 7p_4^4p_3^2 + 6p_4^6 + 2p_4^2p_3^4 - 2p_4^4a + 2p_3^5p_2 + 3p_3^3p_2^3 + p_3p_2^5)], \\
&[-(2p_3^3p_2 + 5p_3p_2p_4^2 + p_3p_2^3 + 3p_4^2p_2^2 + p_3p_2a + 3p_4^4 \\
&+ 2p_3^2p_4^2 - p_4^2a)(-p_4^2 + p_2^2 + a)], [(-3p_4^4 - 2p_3^2p_4^2 + 2p_4^2p_2^2 \\
&+ p_4^2a + 2p_3^2p_2^2 + p_2^4 + p_2^2a)(-{p_4}^2 + {p_2}^2 + a)], [0]\big]\big\}, \qquad (13)
\end{aligned}$$

$$\begin{aligned}E_{-\mu_2^2}(K_a^2) = \mathrm{Vec}\big\{\alpha_2 = \big[&[2p_4(p_3^3a + p_2^3a + p_3^2p_2a + p_3p_2^2a \\ &+ 2p_4^2p_2a + 2p_4^2p_3a + p_3p_2^4 + 2p_4^4p_2 + 3p_4^2p_2^3 + 3p_3^2p_2^3 - 7p_3^3p_4^2 \\ &- 6p_3p_4^4 - 2p_3^5 - p_4^2p_2^2p_3 + 5p_4^2p_3^2p_2 - p_2^2p_3^3 + 2p_3^4p_2 + p_2^5)], \\ &[-p_4(2p_3^2p_2 + 5p_4^2p_2 + p_2^3 - 3p_3p_2^2 + p_2a - 3p_4^2p_3 - 2p_3^3 \\ &+ p_3a)(-p_4^2 + p_2^2 + a)], [0], [(-3p_4^4 - 2p_3^2p_4^2 \\ &+ 2p_4^2p_2^2 + p_4^2a + 2p_3^2p_2^2 + p_2^4 + p_2^2a)(-p_4^2 + p_2^2 + a)]\big]\big\}. \quad (14)\end{aligned}$$

For $p_i = \dfrac{c}{ih}$, we have

$$-\mu_1^2 = \frac{c^2}{1152h^2}\left(-95 + \sqrt{7729}\right),$$
$$-\mu_2^2 = \frac{c^2}{1152h^2}\left(-95 - \sqrt{7729}\right),$$
$$a = \frac{1}{144}\,\frac{\sqrt{7729}c^2}{h^2},$$

so

$$E_{-\mu_1^2}(K_a^2) = \mathrm{Vec}\left\{\alpha_1 = \frac{c^6}{h^6}\begin{pmatrix} \frac{35}{6}\,\frac{\sqrt{7729}+137}{27+\sqrt{7729}} \\ -\frac{1}{24}\left(281 - \sqrt{7729}\right) \\ \frac{1}{8}\left(57 + \sqrt{7729}\right) \\ 0 \end{pmatrix}, K_a\alpha_1\right\},$$

$$E_{-\mu_2^2}(K_a^2) = \mathrm{Vec}\left\{\alpha_2 = \frac{c^6}{h^6}\begin{pmatrix} \frac{35}{3}\,\frac{\sqrt{7729}+37}{27+\sqrt{7729}} \\ -\frac{1}{12}\left(1 - \sqrt{7729}\right) \\ 0 \\ \frac{1}{8}\left(57 + \sqrt{7729}\right) \end{pmatrix}, K_a\alpha_2\right\}.$$

In this example, the geometric degree of nonconservativity is equal to 2: two additional kinematic constraints $\phi_1(\theta_1, \ldots, \theta_4) = 0$ and $\phi_2(\theta_1, \ldots, \theta_4) = 0$ are then necessary to convert the system into a conservative one, each constraint ϕ_i being chosen in $E_{-\mu_i^2}(K_a^2)$ for $i = 1, 2$. For example,

$$\phi_1(\theta_1, \ldots, \theta_4) = \tfrac{35}{6}\,\tfrac{\sqrt{7729}+137}{27+\sqrt{7729}}\theta_1 - \tfrac{1}{24}\left(281 - \sqrt{7729}\right)\theta_2 + \tfrac{1}{8}\left(57 + \sqrt{7729}\right)\theta_3,$$

$$\phi_2(\theta_1, \ldots, \theta_4) = \tfrac{35}{6}\,\tfrac{\sqrt{7729}+137}{27+\sqrt{7729}}\theta_1 - \tfrac{1}{12}\left(1 - \sqrt{7729}\right)\theta_2 + \tfrac{1}{8}\left(57 + \sqrt{7729}\right)\theta_4.$$

Conclusion

In this paper, we investigate nonconservative systems, meaning here elastic systems with a nonsymmetric stiffness matrix. We associate with each mechanical system a minimal number ℓ of additional kinematic constraints allowing this system to be converted into a conservative one. As this integer measure of the nonconservativity of the mechanical system is linked with the dimension of a vector space, it is called the geometric degree of nonconservativity of the system. Computations of this integer and of the corresponding additional kinematic constraints are constructive and several examples illustrate the results. The extension to the nonlinear case will be developed in a forthcoming paper.

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JEAN LERBET: jean.lerbet@ibisc.univ-evry.fr
Informatique Biologie Intégrative Systèmes Complexes, Université d'Evry-Val-d'Essonne, UFR Sciences & Technologies, 40, Rue du Pelvoux, CE 1455 Courcouronnes, 91020 Evry cedex, France

MARWA ALDOWAJI: marwadoh@yahoo.com
Informatique Biologie Intégrative Systèmes Complexes, Université d'Evry-Val-d'Essonne, UFR Sciences & Technologies, 40, Rue du Pelvoux, CE 1455 Courcouronnes, 91020 Evry cedex, France

NOËL CHALLAMEL: noel.challamel@univ-ubs.fr
Laboratoire d'Ingénierie des Matériaux de Bretagne, Université de Bretagne Sud, Rue de saint Maudé, BP 92116, 56321 Lorient cedex, France

OLEG N. KIRILLOV: o.kirillov@hzdr.de
Department of Magnetohydrodynamics, Institute of Fluid Dynamics, Helmholtz-Zentrum Dresden-Rossendorf, Bautzner Landstraße 400, P.O. Box 510119, 01314 Dresden, Germany

FRANÇOIS NICOT: Francois.Nicot@irstea.fr
Geomechanics Group, Irstea, Domaine Universitaire BP 76, 38402 Saint Martin d'Hères cedex, France

FÉLIX DARVE: felix.darve@3sr-grenoble.fr
Laboratoire Sols Solides Structures, UJF-INPG-CNRS, BP 53, 38041 Grenoble cedex 9, France

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