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Finite-Time Weighted Average Consensus and Generalized Consensus Over a Subset

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ABSTRACT In this paper, the finite-time consensus for arbitrary undirected graphs is discussed. We develop a parametric distributed algorithm as a function of a linear operator defined on the underlying graph and provide a necessary and sufficient condition guaranteeing weighted average consensus in K steps, where K is the number of distinct eigenvalues of the underlying operator. Based on the novel framework of generalized consensus meaning that consensus is reached only by a subset of nodes, we show that the finite-time weighted average consensus can always be reached by a subset corresponding to the non-zero variables of the eigenvector associated with a simple eigenvalue of the operator. It is interesting that the final consensus state is shown to be freely adjustable if a smaller subset of consensus is admitted. Numerical examples, including synthetic and real-world networks, are presented to illustrate the theoretical results. Our approach is inspired by the recent method of successive nulling of eigenvalues by Safavi and Khan.

INDEX TERMS Weighted average consensus, generalized consensus, finite-time, discrete-time, distributed algorithm.

I. INTRODUCTION

The study of consensus algorithms for multi-agent systems, where distributed processors or agents seek agreement upon a certain quantity of interest via only local information exchange between neighbors, has received a great deal of attention in diverse scientific fields (see e.g. [1]–[4]). Without a central coordinator, the states of all agents may converge to a common quantity by implementing consensus algorithms. When the limit state is restricted to the (weighted) average value of the initial states of agents over the network, it becomes the (weighted) average consensus problem, which has been widely investigated in the literature [1], [5]–[7]. Admittedly, convergence rate is an important performance indicator for the proposed algorithms. A standard problem in system theory, for example, requires to develop controllers that drive a system to consensus as fast as possible [8].

Finite-time convergence is often more appealing than asymptotic convergence when stringent convergence time and high precise performance are needed. Moreover, finite-time controllers can lead to better system performance in the disturbance rejection and robustness against uncertainties [9]. Various sufficient conditions for finite-time convergence have been established in the literature. For example, the authors in [10] propose a continuous-time distributed protocol which realizes finite-time consensus for both undirected and directed communication graphs if the sum of time intervals, in which the underlying interaction network is connected, is sufficiently large. The result is extended to more general nonlinear protocols in [11]. Finite-time weighted average consensus is dealt with in [12] for time-varying topologies. Reference [13] shows that average consensus can always be achieved for connected undirected graphs by using stochastic matrices with positive diagonals. Results regarding average consensus reached in a minimal number of steps corresponding to the diameter in the case of distance regular graphs have been established in [14]–[16]. Besides, sufficient conditions for finite-time average consensus are derived in [17] and [18] using the formalism of minimal polynomials.

Recently, by introducing a novel approach termed as *successive nulling of eigenvalues*, the authors in [19] provide a necessary and sufficient condition for reaching average consensus in finite time with linear protocols. They design parametric distributed iterations which are a function of a linear operator W on an arbitrary undirected graph, and show that average consensus can be reached in K steps with K being the number of distinct eigenvalues of the operator W if and only if W has at least one simple eigenvalue having the eigenvector of all constants. The construction of their algorithm is also related to graph filters [20]; see Remark 1. Clearly, when W represents an irreducible doubly stochastic matrix, the above criterion along with the Perron-Frobenius theorem would guarantee the average consensus in finite time. However, for more general operators, finite-time average consensus may not be achievable.

In this letter, we first relax the condition in [19] by showing that weighted average consensus can be reached in K steps if and only if the operator W has at least one simple eigenvalue whose eigenvector has non-zero entries for all variables. When there are zero entries in the eigenvector associated to the simple eigenvalue, we further prove that finite-time weighted average consensus is reached only by a subset $\mathcal{S}_o$ of nodes corresponding to the non-zero entries. Moreover, for any proper subset $\mathcal{S}'$ of $\mathcal{S}_o$, the final consensus state on $\mathcal{S}'$ can even be arbitrarily adjusted by tuning parameters in our distributed algorithm. These results are formulated using the novel framework of *generalized consensus*, meaning that consensus is reached on the network except for some prescribed subset. This framework contrasts our results with other existing algorithms of distributed average consensus, such as [19], where disagreement is prohibited and the final consensus state does not extend straightforwardly to other values.

For the rest of the letter, we present our main results for finite-time behaviors in Section III after fixing notation quickly in Section II. Section IV contains some synthetic and real-life based numerical examples. We draw conclusions in Section V.

II. PROBLEM SETUP

Let $\mathcal{G}$ denote an undirected graph with a node set $\mathcal{V} = \{1, 2, \cdots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Assume that node i maintains a state variable $x_i \in \mathbb{R}$ and set $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. Let us consider a symmetric linear operator $W : \mathbb{R}^n \to \mathbb{R}^n$ defined on the graph $\mathcal{G}$. The operator W is an $n \times n$ matrix whose (i, j)-entry $w_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. The value of w_{ij} captures the strength of the connection between i and j. The pattern of W describes the local structure of $\mathcal{G}$, which allows local (and hence, distributed) computation of Wx at the nodes of the graph.

As is known, there exist an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ that can be used to decompose W as $W = V\Lambda V^T$. For $1 \le K \le n$, suppose that W has K distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_K$ with algebraic multiplicities $m_1, m_2, \cdots, m_K$, respectively. Letting I_n be the n-dimensional identity matrix, we can write $V = (v_1^1, \cdots, v_{m_1}^1, \cdots, v_1^K, \cdots, v_{m_K}^K)$ and $\Lambda = \mathrm{diag}(\lambda_1 I_{m_1}, \cdots, \lambda_K I_{m_K})$, where $v_1^k, \cdots, v_{m_k}^k$ are m_k orthonormal eigenvectors corresponding to λ_k. We mention that our results below can be extended to the case of a diagonalizable W on a directed graph with a little more effort, but for better presentation we confine ourselves to the symmetric situation.

We study the following discrete-time multi-agent system:

$$x(t) = A_t x(t-1), \tag{1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T$ represents the state vector at time $t \ge 1$, and the operator A_t is of the form

$$A_t = a_t I_n - \Gamma^{-1} W \Gamma, \tag{2}$$

where $a_t \in \mathbb{R}$, and $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ is a diagonal matrix with $\gamma_j \neq 0$ for every $j = 1, \cdots, n$.

Remark 1: First, the application of A_t on x can be implemented distributedly through local interactions among neighbors since a_t and Γ (Γ^{-1}) reserve the pattern of W. Second, if we interpret x as a signal, the successive application of A_t on it constructs a graph filter. For more details on the connection with graph filters, we refer the reader to [20] and [21].

III. MAIN RESULTS

A. FINITE-TIME WEIGHTED AVERAGE CONSENSUS

The following theorem is based on the method of successive nulling of eigenvalues [19].

Theorem 1: Suppose that W is a linear operator on $\mathcal{G}$ with K distinct eigenvalues $\lambda_1, \cdots, \lambda_K$, and multiplicities $m_1, \cdots, m_K$. For any $x(0)$, invertible diagonal matrix Γ and $1 \le k \le K$, there exists a sequence of a_t's such that the system (1) converges in K steps to the m_k-dimensional subspace $\mathrm{span}\{\Gamma^{-1}v_1^k, \cdots, \Gamma^{-1}v_{m_k}^k\}$, which is the image by Γ^{-1} of the kth eigenspace.

Proof: Fix an initial condition $x(0)$, an invertible diagonal matrix Γ, and a number k. It follows from (1) and (2) that

$$\begin{aligned} x(K) &= A_K \cdot A_{K-1} \cdots A_1 x(0) \\ &= \Gamma^{-1} V \left[(a_K I_n - \Lambda) \cdots (a_1 I_n - \Lambda)\right] V^T \Gamma x(0), \end{aligned} \tag{3}$$

where the product within the brackets is tantamount to a diagonal matrix with diagonal entries $(a_K - \lambda_i)(a_{K-1} - \lambda_i) \cdots (a_1 - \lambda_i)$ appearing repeatedly for m_i times for each $1 \le i \le K$.

As in [19], we choose $a_j = \lambda_j$ for all $j \neq k$ and $j \in \{1, \cdots, K\}$. In addition, we take $a_k = \left(\prod_{j\neq k}(\lambda_j - \lambda_k)\right)^{-1} + \lambda_k$. In view of (3), we obtain

$$\begin{aligned} x(K) &= \Gamma^{-1} V \mathrm{diag}(0, \cdots, 0, \underbrace{1, \cdots, 1}_{m_k}, 0 \cdots, 0) V^T \Gamma x(0) \\ &= \Gamma^{-1} \left[\sum_{j=1}^{m_k} v_j^k (v_j^k)^T\right] \Gamma x(0), \end{aligned} \tag{4}$$

where the m_k 1's lie in the place corresponding to λ_k's in Λ. The theorem follows immediately. □

Clearly, if we take Γ as a scalar matrix $\Gamma = \gamma I_n$ for $\gamma \neq 0$, then the main result of [19] is reproduced. The following corollary is obvious.

Corollary 1: If a_t's are taken as in Theorem 1, then the fact that $x(K)$ lies in a one-dimensional subspace implies that W has at least one simple eigenvalue.

Let $\mathbf{1}$ be a vector of all components being one. Recall that $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ is an invertible diagonal matrix. The next theorem provides a necessary and sufficient condition for finite-time weighted average consensus of the system (1).

Theorem 2: For any $x(0)$, the system (1) reaches weighted average consensus in K steps, namely,

$$x(K) = \frac{\gamma_1^2 x_1(0) + \cdots + \gamma_n^2 x_n(0)}{\gamma_1^2 + \cdots + \gamma_n^2}\mathbf{1} \tag{5}$$

if and only if W has at least one simple eigenvalue whose eigenvector is $\frac{\Gamma \mathbf{1}}{\sqrt{\mathrm{trace}(\Gamma^2)}}$. Here, $\mathrm{trace}(\Gamma^2) := \gamma_1^2 + \cdots + \gamma_n^2$ means the trace of Γ^2.

Proof: Sufficiency: Suppose that λ_K is a simple eigenvalue with $m_K = 1$ and $v_1^K = \frac{\Gamma \mathbf{1}}{\sqrt{\mathrm{trace}(\Gamma^2)}}$. By the same choice of a_t's as in Theorem 1, the expression (3) readily yields $x(K) = \Gamma^{-1} v_1^K (v_1^K)^T \Gamma x(0)$, which concludes the proof of sufficiency.

Necessity: Suppose that (5) holds. Notice from (3) that

$$x(K) = \Gamma^{-1}\left[\sum_{k=1}^{K} \alpha_k \sum_{j=1}^{m_k} v_j^k (v_j^k)^T\right]\Gamma x(0), \tag{6}$$

where $\alpha_k = (a_K - \lambda_k)\cdots(a_1 - \lambda_k)$ for $k = 1, \cdots, K$. Since (5) and (6) are equivalent, we obtain

$$\Gamma^{-1}\left[\sum_{k=1}^{K} \alpha_k \sum_{j=1}^{m_k} v_j^k (v_j^k)^T\right]\Gamma = \frac{\mathbf{1}\cdot(\gamma_1^2, \cdots, \gamma_n^2)}{\mathrm{trace}(\Gamma^2)}. \tag{7}$$

Suppose on the contrary that W has no simple eigenvalue. Note that the right-hand side of (7) has rank one. If there is some $\alpha_k \neq 0$, then the left-hand side of (7) would have rank at least two. But if all $\alpha_k = 0$, the left-hand side of (7) becomes zero. These arguments imply that W must admit a simple eigenvalue.

Without loss of generality, we assume that λ_1 is a simple eigenvalue and $\alpha_1 \neq 0$. Since V is an orthogonal matrix, similarly as in [19] we know that $\Gamma^{-1}\left[\alpha_1 v_1^1 (v_1^1)^T\right]\Gamma = \frac{\mathbf{1}\cdot(\gamma_1^2, \cdots, \gamma_n^2)}{\mathrm{trace}(\Gamma^2)}$. Hence, we are led to the conclusion $v_1^1 = \frac{\Gamma \mathbf{1}}{\sqrt{\mathrm{trace}(\Gamma^2)}}$. The proof is complete. □

Remark 2: The consensus time in Theorem 2, i.e., K, is bounded for any initial values. Such finite-time behavior is also called fixed-time consensus or stability, which is highly desirable especially when the knowledge of initial conditions is not available in advance [22].

Note that the necessary and sufficient condition for finite-time weighted average consensus in Theorem 2 has a prerequisite that Γ is non-degenerate, namely, the eigenvector corresponding to some simple eigenvalue cannot have zero components. In the following we will tackle the finite-time consensus behavior when this condition is violated capitalizing the framework of generalized consensus.

B. FINITE-TIME GENERALIZED CONSENSUS

Given a subset $\mathcal{S} \subseteq \mathcal{V}$, we say that the multi-agent system (1) reaches (generalized) consensus with respect to $\mathcal{S}$ if the states of all agents in $\mathcal{S}$ converge to a common value for any initial $x(0)$. In other words, we do not impose any requirement on the behavior of the states of agents in $\mathcal{V}\backslash\mathcal{S}$. The generalized consensus ideally provides more flexibility than the usual consensus or even group consensus [23], [24], where consensus in each subgroup is required.

For $\mathcal{S} \subseteq \mathcal{V}$, we define a real diagonal matrix $\hat{\Gamma}_{\mathcal{S}} = \mathrm{diag}(\hat{\gamma}_1, \cdots, \hat{\gamma}_n)$ such that $\hat{\gamma}_j = \gamma_j$ for $j \in \mathcal{S}$.

Theorem 3: If W has at least one simple eigenvalue whose eigenvector is $\frac{\hat{\Gamma}_{\mathcal{S}} \mathbf{1}}{\sqrt{\mathrm{trace}(\hat{\Gamma}_{\mathcal{S}}^2)}}$, then for any $x(0)$, the system (1) reaches consensus with respect to $\mathcal{S}$ in K steps.

Proof: Without loss of generality, assume that λ_K is a simple eigenvalue with $m_K = 1$ and $v_1^K = \frac{\hat{\Gamma}_{\mathcal{S}} \mathbf{1}}{\sqrt{\mathrm{trace}(\hat{\Gamma}_{\mathcal{S}}^2)}}$. Employing the same choice of a_t's as in Theorem 1, it follows from (3) that $x(K) = \Gamma^{-1} v_1^K (v_1^K)^T \Gamma x(0)$. Hence, we have for any $i \in \mathcal{S}$ that

$$x_i(K) = \frac{1}{\mathrm{trace}(\hat{\Gamma}_{\mathcal{S}}^2)}\left[\sum_{j\in\mathcal{S}} \hat{\gamma}_j^2 x_j(0) + \sum_{j\notin\mathcal{S}} \gamma_j \hat{\gamma}_j x_j(0)\right]. \tag{8}$$

The desired result is obtained. □

A major new implication of Theorem 3 is the following fact, which indicates that the system (1) can always reach (generalized) weighted average consensus if W admits a simple eigenvalue. This is achieved by designing appropriate matrix Γ (as well as a_t's) in (2).

Corollary 2: Let $\hat{\Gamma} = \mathrm{diag}(\hat{\gamma}_1, \cdots, \hat{\gamma}_n)$ be a real diagonal matrix. Suppose that W has at least one simple eigenvalue whose eigenvector is $\frac{\hat{\Gamma} \mathbf{1}}{\sqrt{\mathrm{trace}(\hat{\Gamma}^2)}}$.

- If $\hat{\Gamma}$ is invertible, then for any $x(0)$, the system (1) reaches weighted average consensus in K steps. In particular,

$$x(K) = \frac{\hat{\gamma}_1^2 x_1(0) + \cdots + \hat{\gamma}_n^2 x_n(0)}{\mathrm{trace}(\hat{\Gamma}^2)}\mathbf{1}; \tag{9}$$

- If $\hat{\Gamma}$ is non-invertible, then for any $x(0)$, the system (1) reaches weighted average consensus with respect to $\mathcal{S}_o$ in K steps, where $\mathcal{S}_o := \{j \in \mathcal{V} : \hat{\gamma}_j \neq 0\}$. In particular, for any $i \in \mathcal{S}_o$,

$$x_i(K) = \frac{1}{\mathrm{trace}(\hat{\Gamma}^2)} \sum_{j\in\mathcal{S}_o} \hat{\gamma}_j^2 x_j(0). \tag{10}$$

Proof: The first statement holds from Theorem 2 by taking $\Gamma = \hat{\Gamma}$. The second statement holds from Theorem 3 by taking $\hat{\Gamma}_{\mathcal{S}_o} = \hat{\Gamma}$ and $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ with $\gamma_j = \hat{\gamma}_j$ for $j \in \mathcal{S}_o$, while $\gamma_j \neq 0$ for $j \notin \mathcal{S}_o$. □

Theorem 3 also implies the following practically appealing result, which says that generalized consensus with respect to any subset $\mathcal{S}' \subseteq \mathcal{S}_o$ can be reached in finite time.

Furthermore, if $\mathcal{S}'$ is a proper subset of $\mathcal{S}_o$, then the final (generalized) consensus state can be freely chosen by designing appropriate Γ.

Corollary 3: Let $\hat{\Gamma} = \mathrm{diag}(\hat{\gamma}_1, \cdots, \hat{\gamma}_n)$ be a real diagonal matrix. Suppose that W has at least one simple eigenvalue whose eigenvector is $\frac{\hat{\Gamma}1}{\sqrt{\mathrm{trace}(\hat{\Gamma}^2)}}$, and $\mathcal{S}_o$ is defined in Corollary 2. For any $\mathcal{S}' \subseteq \mathcal{S}_o$, define $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ such that $\gamma_j = \hat{\gamma}_j$ for $j \in \mathcal{S}'$. Then we have

$$x_i(K) = \frac{1}{\mathrm{trace}(\hat{\Gamma}^2)} \left[\sum_{j \in \mathcal{S}'} \hat{\gamma}_j^2 x_j(0) + \sum_{j \notin \mathcal{S}'} \gamma_j \hat{\gamma}_j x_j(0) \right]. \quad (11)$$

for all $i \in \mathcal{S}'$.

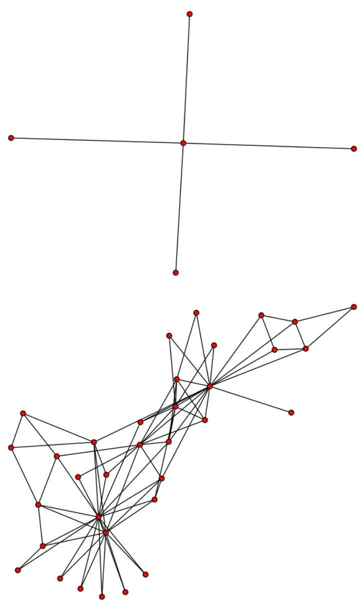

FIGURE 1. Upper panel: star graph; Lower panel: Zachary's Karate Club graph.

IV. NUMERICAL EXAMPLES

Example 1: We first consider a star network of $n = 5$ nodes (see Fig. 1 left panel) with the center node, say, being node 5. The linear operator W is defined as its adjacency matrix, namely, $W = \begin{bmatrix} 0 & 1 \\ 1^T & 0 \end{bmatrix} \in \mathbb{R}^{5\times5}$. By straightforward computation, we obtain the eigenvalues $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 2$ with multiplicities $m_1 = 3, m_2 = m_3 = 1$. In particular, we have two eigenvectors $v_1^2 = \frac{\sqrt{2}}{4}(1, 1, 1, 1, -2)^T$ and $v_1^3 = \frac{\sqrt{2}}{4}(1, 1, 1, 1, 2)^T$. It is clear that the system (1) can not achieve finite-time average consensus on this network, where the special case of $\Gamma = I_5$ is observed in [19]. However, in the current framework, by taking $a_1 = \lambda_1 = 0$, $a_2 = \lambda_2 = -2, a_3 = \frac{1}{(0-2)(-2-2)} + 2 = 17/8$, and $\Gamma = \mathrm{diag}(1, 1, 1, 1, 2)$, we conclude from Theorem 2 that for any initial condition $x(0)$,

$$x(3) = \frac{x_1(0) + x_2(0) + x_3(0) + x_4(0) + 4x_5(0)}{8} 1. \quad (12)$$

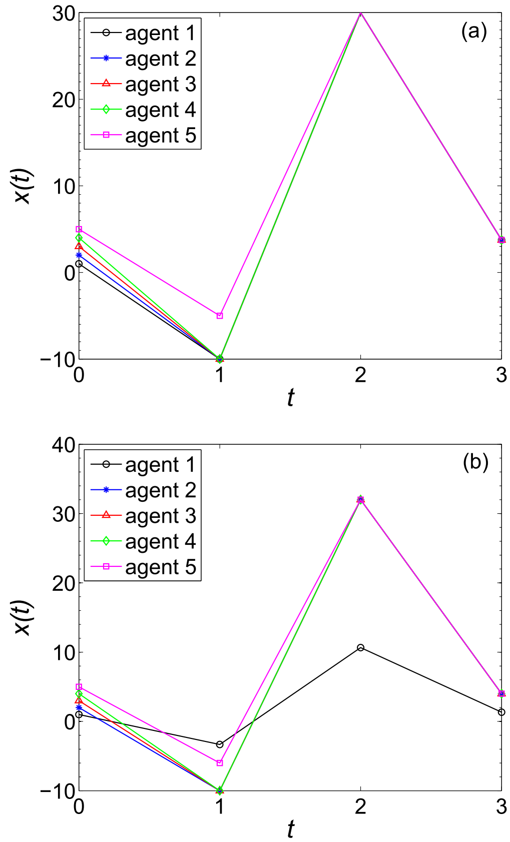

FIGURE 2. State trajectories $x(t)$ as a function of t for Example 1. Initial conditions are $x_i(0) = i$ ($i = 1, \cdots, 5$). (a): weighted average consensus in 3 steps; (b) generalized weighted average consensus with respect to $\mathcal{S}'$ in 3 steps.

In other words, the system (1) reaches weighted average consensus in $K = 3$ steps.

Take $x_i(0) = i$ for $i = 1, \cdots, 5$. The evolution of the states of the agents is shown in Fig. 2(a). We observe that all states converge to 15/4 in 3 steps confirming our theoretical prediction.

Note that $\mathcal{S}_o = \mathcal{V}$ in this example. We take $\mathcal{S}' = \{2, 3, 4, 5\}$, which is a proper subset of $\mathcal{S}_o$. Define a new $\Gamma = \mathrm{diag}(3, 1, 1, 1, 2)$. Hence, Corollary 3 implies that the system (1) reaches generalized consensus with respect to $\mathcal{S}'$ in $K = 3$ steps, and that the final consensus state is

$$\begin{aligned} x_i(3) &= \frac{3x_1(0) + x_2(0) + x_3(0) + x_4(0) + 4x_5(0)}{8} \\ &= 4 \end{aligned} \quad (13)$$

for any $i \in \mathcal{S}'$. This is shown in Fig. 2(b).

Here, we highlight that with the considered operator W average consensus cannot be reached. By designing an operator like $\Gamma^{-1}W\Gamma$, the weighted average consensus can be achieved in finite time (c.f. (12)) and the final consensus state reached by a subset of nodes can even be at our disposal (c.f. (13)).

Example 2: Next, we consider a well-known social network of friendships between $n = 34$ members of a karate club at a US university in the 1970s [25]. In this graph, each node represents a member of the club, and each edge represents a tie between two members of the club (see Fig. 1 right panel). We take W again to be the adjacency matrix, which is described (with respect to a specific labeling) in [25, p. 457]. The operator W has $K = 25$ distinct eigenvalues $\lambda_1 = 0, \lambda_2, \cdots, \lambda_{24}, \lambda_{25} = -2$ with multiplicities $m_1 = 10$, $m_2 = \cdots = m_{25} = 1$. The eigenvector corresponding to λ_{25} is $v_1^{25} = \frac{1}{2}(0, 0, 0, 0, -1, -1, 1, 0, 0, 0, 1, 0, \cdots, 0)^T$.

Let $\hat{\Gamma} = \text{diag}(v_1^{25})$ and $\mathcal{S}_o = \{5, 6, 7, 11\}$. Clearly, $\hat{\Gamma}$ is non-invertible. In view of the proof of Corollary 2, by taking $a_1, \cdots, a_{25}$ in the above standard manner, and $\Gamma = \text{diag}(*, *, *, *, -1, -1, 1, *, *, *, 1, *, \cdots, *)^T$, with all the wildcards $*$ being non-zero, we have for any $x(0)$,

$$x_i(25) = \frac{x_5(0) + x_6(0) + x_7(0) + x_{11}(0)}{4} \tag{14}$$

for all $i \in \mathcal{S}_o$.

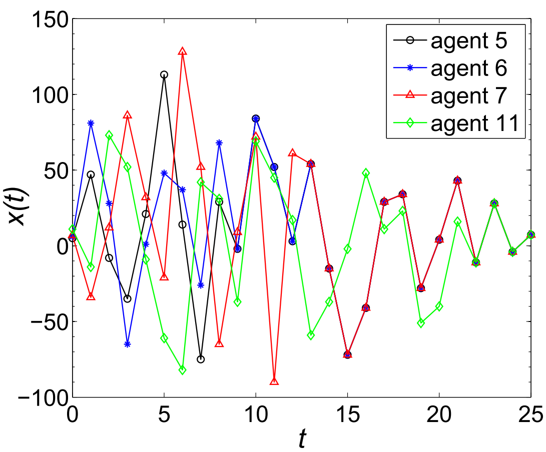

FIGURE 3. State trajectories $x(t)$ as a function of t for Example 2. Initial conditions are $x_i(0) = i$ ($i = 1, \cdots, 25$). Generalized weighted average consensus with respect to $\mathcal{S}_o$ in 25 steps is shown for the 4 agents in $\mathcal{S}_o$.

Take $x_i(0) = i$ for $i = 1, \cdots, 25$. We choose independently each $*$ following a uniform distribution on [0, 1]. We show in Fig. 3 a sample of the evolution of $x(t)$ (for clarity, only the values of x_i for $i \in \mathcal{S}_0$ are displayed). We observe that the final consensus state becomes $(5 + 6 + 7 + 11)/4 = 29/4$, which agrees with our theory. This example further illustrates that not only the desired finite-time consensus can be reached by using $\Gamma^{-1} W \Gamma$ but the choice of Γ is by no means unique.

V. CONCLUSIONS

In this letter, we study the finite-time consensus behavior for arbitrary undirected graphs, on which a linear operator, W, is defined. Based on the method of successive nulling of eigenvalues, we present a necessary and sufficient condition for finite-time weighted average consensus. Moreover, by introducing the novel framework of generalized consensus, we show that finite-time consensus can be reached only by a subset of nodes, where the final consensus state may even be freely adjusted. In the analysis of the letter, it has been assumed that W is symmetric (or diagonalizable), and that the information exchange among agents is noise-free. Therefore, it would be desirable to extend our results to general directed networks, and to address finite-time generalized consensus against system uncertainties and stochastic perturbations.

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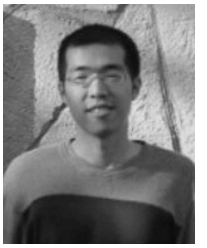

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