A combinatorial necessary and sufficient condition for cluster consensus

Yilun Shang[1]

[1]*Department of Mathematics, Tongji University, Shanghai 200092, China*
Email: shyl@tongji.edu.cn

Abstract

In this letter, cluster consensus of discrete-time linear multi-agent systems is investigated. A set of stochastic matrices $\mathcal{P}$ is said to be a cluster consensus set if the system achieves cluster consensus for any initial state and any sequence of matrices taken from $\mathcal{P}$. By introducing a cluster ergodicity coefficient, we present an equivalence relation between a range of characterization of cluster consensus set under some mild conditions including the widely adopted inter-cluster common influence. We obtain a combinatorial necessary and sufficient condition for a compact set $\mathcal{P}$ to be a cluster consensus set. This combinatorial condition is an extension of the avoiding set condition for global consensus, and can be easily checked by an elementary routine. As a byproduct, our result unveils that the cluster-spanning trees condition is not only sufficient but necessary in some sense for cluster consensus problems.

Keywords:
Cluster consensus, multi-agent system, linear switched system, cooperative control.

1. Introduction

In the past two decades, consensus problems in multi-agent systems have gained increasing attention in various research communities, ranging from formation of unmanned air vehicles to data fusion of sensor networks, from swarming of animals to synchronization of distributed oscillators [1, 2, 3, 4]. The main objective of consensus problems is to design appropriate protocols and algorithms such that the states of a group of agents converge to a consistent value (see [5, 6] for a survey of this prolific field). In many distributed consensus algorithms, the agents update their values as linear combinations of the values of agents with which they can communicate:

$$x_i(t+1) = \sum_j p_{ij}(t+1)x_j(t), \tag{1}$$

where $x_i(t)$ is the value of agent i and $P(t) = (p_{ij}(t))$ for every discrete time instant $t \geq 0$ represents a *stochastic* matrix, i.e., $p_{ij}(t) \geq 0$ and $\sum_j p_{ij}(t) = 1$. The states of agents following such linear averaging algorithms tend to get closer over time. The problem of characterizing the complete sequence of matrices $P(t)$ for consensus is however known to be notoriously difficult [7]. A moderate goal would be to determine whether the system (1) converges to a state of consensus for all sequences of matrices $P(t)$ in a certain set $\mathcal{P}$. Remarkably, Blondel and Olshevsky in a recent work [8] presented an explicit combinatorial condition, which is both necessary and

sufficient for consensus of (1) in that sense. This condition, dubbed "avoiding sets condition", is easy to check with an algorithm and thus the consensus problem is decidable.

While most existing works are concerned with global consensus (namely, all the agents reach a common state), in varied real-world applications, there may be multiple consistent states as agents in a network often split into several groups to carry out different cooperative tasks. Typical situations include obstacle avoidance of animal herds, team hunting of predators, social learning under different environments, coordinated military operations, and task allocation over the network between groups. A possible solution is given by the cluster (or group) consensus algorithms [9, 10], where the agents in a network are divided into multiple subnetworks and different subnetworks can reach different consistent states asymptotically. Evidently, cluster consensus is an extension of (global) consensus. Various sufficient conditions and necessary conditions (although much fewer) for cluster consensus have been reported in the literature for discrete-time systems [10, 11, 12, 13, 14], simple first- or second-order continuous-time systems [9, 15, 16, 17, 18, 19, 20, 21], and high-order dynamics [22], to name a few. However, most of these conditions rely on either complicated linear matrix inequalities or algebraic conditions involving eigenvalues of the system matrices, which are in general difficult to check.

With the above inspiration, we aim to work on efficiently verifiable conditions for cluster consensus by extending the results in [8] for global consensus, which are highly non-trivial. The main contribution of this paper is to establish a combinatorial necessary and sufficient condition which guarantees the cluster consensus of system (1) under some common assumptions, i.e., self-loops, either undirected graph or doubly stochastic state-update matrices, and inter-cluster common influence. Some of the previous convergence criteria can be quickly reproduced from our results. It is noteworthy that the authors in [10] showed that, under some mild assumptions, the cluster consensus of (1) can be achieved provided the graphs associated with $P(t)$ contain cluster-spanning trees. Interestingly, our result implies that the cluster-spanning trees condition is essentially necessary.

2. Preliminaries

In this section, some definitions and lemmas on graph theory and matrix theory are given as the preliminaries. We refer the reader to the textbooks [6, 23] for more details.

Let $G = (V, E)$ be a directed graph of order n with the set of vertices $V = \{1, \cdots, n\}$ and the set of edges $E \subseteq V \times V$. For a stochastic matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$ (namely, $p_{ij} \geq 0$ and $\sum_{j=1}^{n} p_{ij} = 1$ for all i, j), a corresponding directed graph $G(P) = (V, E)$ can be constructed by taking $V = \{1, \cdots, n\}$, and $E = \{(i, j) : p_{ij} > 0\}$. $G(P)$ is assumed to be unweighted throughout the paper. Given a subset $S \subseteq V$, denote by $N_G(S)$ the set of out-neighbors of S in G, i.e., $N_G(S) = \{j \in V : i \in S, (i, j) \in E\}$. A directed path from vertex i to j of length l is a sequence of edges $(i, i_1), (i_1, i_2), \cdots, (i_l, j)$ with distinct vertices $i_1, \cdots, i_l \in V$. If $(i, i) \in E$, then there exists a self-loop at vertex i.

A *clustering* $C = \{C_1, \cdots, C_K\}$ of the directed graph G is defined by dividing its vertex set into disjoint clusters $\{C_k\}_{k=1}^{K}$. In other words, C satisfies $\cup_{k=1}^{K} C_k = V$ and $C_k \cap C_{k'} = \emptyset$ for $k \neq k'$. Letting $x(t) = (x_1(t), \cdots, x_n(t))^T$, we recast the system (1) as

$$x(t + 1) = P(t + 1)x(t). \tag{2}$$

Definition 1. For a given clustering $C = \{C_1, \cdots, C_K\}$, a set of $n \times n$ stochastic matrices $\mathcal{P}$ is said

to be a *cluster consensus set* if for any initial state $x(0)$ and all sequences $P(1), P(2), \cdots \in \mathcal{P}$,

$$\lim_{t\to\infty} x(t) = \sum_{k=1}^{K} \alpha_k \mathbf{1}_{C_k}, \tag{3}$$

where $\mathbf{1}_{C_k}$ is the sum of ith n-dimensional basis vector $e_i = (0, \cdots, 0, \overset{i\text{th}}{1}, 0, \cdots, 0)^T$ over all $i \in C_k$, and α_k is some scalar.

Remark 1. In most of the literatures, given a sequence of stochastic matrices (or switching signal) $P(1), P(2), \cdots$, the system (2) is said to achieve *cluster consensus* if (3) holds for any initial state $x(0)$. This is also referred to as *intra-cluster synchronization* in [10, 18, 24], where the cluster consensus requires additionally the separation of states of agents in different clusters. Nevertheless, the inter-cluster separation can only be realized by incorporating adapted external inputs.

Definition 2. [10] A stochastic matrix P is said to have *inter-cluster common influence* if for all $k \neq k'$, $\sum_{j\in C_{k'}} p_{ij}$ is identical with respect to all $i \in C_k$.

Remark 2. Since the entries on each row of P sum up to one, the above statement automatically holds for $k = k'$ if P has inter-cluster common influence. Therefore, $\sum_{j\in C_{k'}} p_{ij}$ depends only on k and k'. This (and some closely related variants) is a common assumption for cluster consensus problems; see e.g. [9, 10, 11, 12, 13, 15, 24, 25]. It is direct to check that if P_1 and P_2 have inter-cluster common influence with respect to the same clustering C, so does P_1P_2.

To analyze the cluster consensus of the multi-agent system (2), we will need to estimate some characteristics of infinite product of stochastic matrices. For a stochastic matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$, we define the cluster ergodicity coefficient with respect to a clustering C as

$$\tau_C(P) = \frac{1}{2} \max_{1\le k\le K} \max_{i,j\in C_k} \sum_{s=1}^{n} |p_{is} - p_{js}| = \frac{1}{2} \max_{1\le k\le K} \max_{i,j\in C_k} \|p_i - p_j\|_1, \tag{4}$$

where $p_i = (p_{i1}, \cdots, p_{in})$ is the ith row of P and $\|\cdot\|_1$ represents the 1-norm of vector.

It can be seen that $0 \le \tau_C(P) \le 1$ and that $\tau_C(P) = 0$ if and only if $P = \sum_{k=1}^{K} \mathbf{1}_{C_k} y_k^T$, where y_k is a stochastic vector, namely, P has identical rows for each cluster. Hence, τ_C can be viewed as an extension of the well-known Dobrushin ergodicity coefficient [23] for clustering.

Lemma 1. *If $P_1 = (p'_{ij})$ and $P_2 = (p''_{ij})$ are two $n \times n$ stochastic matrices having inter-cluster common influence with respect to the same clustering C, then*

$$\tau_C(P_1P_2) \le \tau_C(P_1)\tau_C(P_2) \le \min\{\tau_C(P_1), \tau_C(P_2)\}.$$

Proof. We only need to show the first inequality. Suppose that $C = \{C_1, \cdots, C_K\}$. We first recall a useful lemma (see [26, p. 126, Lem 1.1]): For any stochastic matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$ and $i, j \in V = \{1, \cdots, n\}$,

$$\frac{1}{2} \sum_{s=1}^{n} |p_{is} - p_{js}| = \max_{A\subseteq V} \sum_{s\in A} (p_{is} - p_{js}). \tag{5}$$

It follows immediately from (5) that

$$\tau_C(P_1P_2) = \max_{1\le k\le K} \max_{i,j\in C_k} \max_{A\subseteq V} \sum_{s\in A} \sum_{l=1}^{n} (p'_{il} - p'_{jl}) p''_{ls}.$$

Denote by $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$ for $f \in \mathbb{R}$. Hence, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Fix $1 \le k \le K$ and $i, j \in C_k$. For any $1 \le k' \le K$, we have $0 = \sum_{l \in C_{k'}} (p'_{il} - p'_{jl}) = \sum_{l \in C_{k'}} (p'_{il} - p'_{jl})^+ - \sum_{l \in C_{k'}} (p'_{il} - p'_{jl})^-$ since P_1 has inter-cluster common influence. Accordingly,

$$\sum_{l \in C_{k'}} (p'_{il} - p'_{jl})^+ = \sum_{l \in C_{k'}} (p'_{il} - p'_{jl})^- = \frac{1}{2} \sum_{l \in C_{k'}} |p'_{il} - p'_{jl}|.$$

In view of this relation, we obtain

$$\begin{aligned}
\sum_{s \in A} \sum_{l=1}^{n} (p'_{il} - p'_{jl}) p''_{ls} &= \sum_{s \in A} \sum_{1 \le k' \le K} \sum_{l \in C_{k'}} (p'_{il} - p'_{jl}) p''_{ls} \\
&= \sum_{1 \le k' \le K} \sum_{l \in C_{k'}} (p'_{il} - p'_{jl})^+ \sum_{s \in A} p''_{ls} - \sum_{1 \le k' \le K} \sum_{l \in C_{k'}} (p'_{il} - p'_{jl})^- \sum_{s \in A} p''_{ls} \\
&\le \sum_{1 \le k' \le K} \Big(\frac{1}{2} \sum_{l \in C_{k'}} |p'_{il} - p'_{jl}| \Big) \max_{l \in C_{k'}} \sum_{s \in A} p''_{ls} \\
&\quad - \sum_{1 \le k' \le K} \Big(\frac{1}{2} \sum_{l \in C_{k'}} |p'_{il} - p'_{jl}| \Big) \min_{l \in C_{k'}} \sum_{s \in A} p''_{ls} \\
&\le \sum_{1 \le k' \le K} \Big(\frac{1}{2} \sum_{l \in C_{k'}} |p'_{il} - p'_{jl}| \Big) \max_{l, l' \in C_{k'}} \sum_{s \in A} (p''_{ls} - p''_{l's}) \\
&\le \Big(\frac{1}{2} \sum_{l=1}^{n} |p'_{il} - p'_{jl}| \Big) \cdot \max_{1 \le k' \le K} \max_{l, l' \in C_{k'}} \sum_{s \in A} (p''_{ls} - p''_{l's}).
\end{aligned}$$

Therefore,

$$\tau_C(P_1 P_2) \le \Big(\max_{1 \le k \le K} \max_{i, j \in C_k} \frac{1}{2} \sum_{l=1}^{n} |p'_{il} - p'_{jl}| \Big) \cdot \Big(\max_{A \subseteq V} \max_{1 \le k' \le K} \max_{l, l' \in C_{k'}} \sum_{s \in A} (p''_{ls} - p''_{l's}) \Big),$$

where the first term on the right-hand side is exactly $\tau_C(P_1)$, while the second term on the right-hand side equals $\frac{1}{2} \max_{1 \le k' \le K} \max_{l, l' \in C_{k'}} \sum_{s=1}^{n} |p''_{ls} - p''_{l's}| = \tau_C(P_2)$ by employing (5). This completes the proof. □

3. Cluster consensus analysis

For a compact set $\mathcal{P}$ of $n \times n$ stochastic matrices, we have the following assumptions:

Assumption 1. For all $P = (p_{ij}) \in \mathcal{P}$, $p_{ii} > 0$ for $i \in V$. This means that each vertex in the graph $G(P)$ has a self-loop.

Assumption 2. For each $P = (p_{ij}) \in \mathcal{P}$, if $p_{ij} > 0$ then $p_{ji} > 0$. Namely, $G(P)$ is an undirected graph.

Assumption 3. For each $P = (p_{ij}) \in \mathcal{P}$, P is a doubly stochastic matrix, namely, $\sum_i p_{ij} = \sum_j p_{ij} = 1$.

Remark 3. The positive diagonal condition in Assumption 1 is widely adopted in the existing consensus algorithms, see e.g. [1, 2, 6, 10]. It reflects the "self-confidence" that agents give positive weights to their own states when updating [27, 32], and is also naturally satisfied by any

algorithm producing the sampling of a continuous-time process. The undirected graph condition in Assumption 2 plays an important role in a range of consensus problems, where information exchange goes in both directions [28]. The doubly stochastic property in Assumption 3 is important for many cooperative control problems including distributed averaging, optimization, and gossiping [29]. A characterization of directed graphs with doubly stochastic adjacency matrix was provided in [30].

3.1. Equivalence lemma

A key step towards our main result is the following equivalence lemma, which characterizes the cluster consensus set under the above assumptions, and can be seen as a "clustering" version of Lemma 2.8 [8].

Lemma 2. *Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices having inter-cluster common influence with respect to the same clustering $C = \{C_1, \cdots, C_K\}$. Suppose that either Assumptions 1, 2 hold or Assumptions 1, 3 hold. The following are equivalent:*

(1) $\mathcal{P}$ is a cluster consensus set.

(2) For every infinite sequence $P(1), P(2), \cdots \in \mathcal{P}$, $\lim_{t\to\infty} P(t)P(t-1)\cdots P(1) = \sum_{k=1}^{K} \mathbf{1}_{C_k} y_k^T$, where y_k is some stochastic vector.

(3) For any $\varepsilon > 0$, there is an integer $t(\varepsilon)$ such that if Π is the product of $t(\varepsilon)$ matrices from $\mathcal{P}$, then $\tau_C(\Pi) < \varepsilon$.

(4) For all $1 \le k \le K$, $i, j \in C_k$, and infinite sequences $P(1), P(2), \cdots \in \mathcal{P}$, $\lim_{t\to\infty}(e_i^T - e_j^T)P(1)P(2)\cdots P(t) = 0$.

(5) There do not exist $1 \le k \le K$, $i, j \in C_k$, and an infinite sequence $P(1), P(2), \cdots \in \mathcal{P}$ such that $e_i^T P(1)P(2)\cdots P(t)$ and $e_j^T P(1)P(2)\cdots P(t)$ have disjoint supports for all $t \ge 1$.

(6) For all infinite sequences $P(1), P(2), \cdots \in \mathcal{P}$, $1 \le k \le K$, and two stochastic vectors y_1 and y_2 whose supports are within C_k, $\lim_{t\to\infty}(y_1^T - y_2^T)P(1)P(2)\cdots P(t) = 0$

Proof. The relations (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and (6) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (3): This can be proved by contradiction. Suppose that (3) fails. In the light of Lemma 1, there must be some $\varepsilon > 0$ such that for every $i = 1, 2, \cdots$, there exists a product $\Pi_i := P_{i,i}P_{i,i-1}\cdots P_{i,1}$ of i matrices with $\tau_C(\Pi_i) > \varepsilon$, where $P_{i,j} \in \mathcal{P}$ for all $1 \le j \le i$. Since $\mathcal{P}$ is compact, we choose a subsequence from $\{P_{i,1}\}_{i\ge1}$, with the index set signified by I_1, so that it converges to some accumulation point $P_1 \in \mathcal{P}$. Next, we choose a new subsequence from $\{P_{i,2}\}_{i\in I_1}$, with the index set signified by I_2, so that it converges to some $P_2 \in \mathcal{P}$. By repeating this procedure, we obtain a sequence of stochastic matrices $P_1, P_2, \cdots$, so that, for every integer ℓ, we have matrices $\Delta_1, \Delta_2, \cdots, \Delta_\ell$ sufficiently close to zero satisfying $\tau_C(Q(t')\cdots Q(\ell+1)(P_\ell + \Delta_\ell)\cdots(P_2+\Delta_2)(P_1+\Delta_1)) > \varepsilon$, where $t' \ge \ell+1$ and $Q(t'), \cdots, Q(\ell+1)$ are some stochastic matrices. It follows from Lemma 1 and the continuity of $\tau_C(\cdot)$ that $\tau_C(P_\ell \cdots P_2P_1) > \varepsilon$. Clearly, the sequence $P_1, P_2, \cdots$ does not meet the statement of (2), which is a contradiction. This establishes the relation (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2): Given any infinite sequence $P(1), P(2), \cdots \in \mathcal{P}$. If Assumptions 1 and 2 hold, the limit $\lim_{t\to\infty} P(t)P(t-1)\cdots P(1)$ exists [27, Thm. 2]. Therefore, the statement (3) implies that the limit must have identical rows for each cluster. Hence, (2) is true. On the other hand, suppose that Assumptions 1 and 3 hold. Notice that a doubly stochastic matrix is cut-balanced

(see Remark 5). Thus, Theorem 3 in [32] implies that the product $P(t)P(t-1)\cdots P(1)$ has a limit. An application of (3) again yields (2).

(4) $\Rightarrow$ (3): Suppose that (3) does not hold. In view of Lemma 1, there exists $\varepsilon > 0$ such that for every $i = 1, 2, \cdots$, there exists a product $\Pi_i := P_{i,i}P_{i,i-1}\cdots P_{i,1}$ of i matrices with $\tau_C(\Pi_i) > \varepsilon$, where $P_{i,j} \in \mathcal{P}$ for all $1 \le j \le i$. Since $\mathcal{P}$ is compact, we choose a subsequence from $\{P_{i,i}\}_{i\ge1}$, with the index set signified by I_1, so that it converges to some $P_1 \in \mathcal{P}$. We then choose a new subsequence from $\{P_{i,i-1}\}_{i\in I_1}$, with the index set signified by I_2, so that it converges to some $P_2 \in \mathcal{P}$. By repeating this procedure, we have a sequence of stochastic matrices $P_1, P_2, \cdots$, so that, for every integer ℓ, we can choose matrices $\Delta_1, \Delta_2, \cdots, \Delta_\ell$ sufficiently close to zero satisfying $\tau_C((P_1+\Delta_1)\cdots(P_\ell+\Delta_\ell)Q(\ell+1)\cdots Q(t')) > \varepsilon$, where $t' \ge \ell+1$ and $Q(t'), \cdots, Q(\ell+1)$ are some stochastic matrices. Thus, $\tau_C(P_1P_2\cdots P_\ell) > \varepsilon$ by employing Lemma 1 and the continuity of $\tau_C(\cdot)$. By (4), this means that there are $1 \le k \le K$, and $i(\ell), j(\ell) \in C_k$ such that $\|(e^T_{i(\ell)} - e^T_{j(\ell)})P_1\cdots P_\ell\|_1 > 2\varepsilon$. Consequently, we have $\|(e^T_i - e^T_j)P_1\cdots P_\ell\|_1 > 2\varepsilon$, where (i, j) appears infinitely often in the set $\{(i(\ell), j(\ell)) : \ell = 1, 2, \cdots\}$. This contradicts item (4).

(5) $\Rightarrow$ (4): From item (5) we see that for any $1 \le k \le K$, $i, j \in C_k$, and infinite sequences $P(1), P(2), \cdots \in \mathcal{P}$ there exists an integer ℓ such that the supports of the ith and jth rows of $P(1)P(2)\cdots P(\ell)$ have nonempty intersection. We claim that there further exists a uniform ℓ' such that for all sequences $P(1), P(2), \cdots \in \mathcal{P}$ of length ℓ' the following statement is true: For any $1 \le k \le K$, $i, j \in C_k$, the supports of the ith and jth rows of $P(1)P(2)\cdots P(\ell')$ have nonempty intersection. Indeed, if there is no such ℓ', then for any $s = 1, 2, \cdots$ we can find a product of s matrices from $\mathcal{P}$, which has two rows in some $C_{k'}$ whose supports do not intersect. By similar argument used in the proof of "(4) $\Rightarrow$ (3)", we have $\tau_C(P_1P_2\cdots P_s) = 1$ involving Lemma 1 and the continuity. Hence, there must exist $1 \le k'' \le K$ and $i, j \in C_{k''}$ such that the ith and jth rows of $P_1P_2\cdots P_s$ have disjoint supports for infinitely many s. Notice that the initial statement says that there is an integer ℓ satisfying $\tau_C(P_1\cdots P_\ell) < 1$. But now we can pick $\bar{\ell} \ge \ell+1$ such that $\tau_C(P_1\cdots P_\ell\cdots P_{\bar{\ell}}) = 1$. We reach a situation that is at odds with Lemma 1. This proves the claim.

Fix the above obtained ℓ'. For an integer s, denote by $\Pi_s := P(1)\cdots P(s)$. Define $\beta := \sup\{\tau_C(\Pi_{\ell'}) : P(1), \cdots, P(\ell') \in \mathcal{P}\}$. It is clear that $0 \le \beta \le 1$. If $\beta = 1$, then there must exist some product $P(1)\cdots P(\ell')$ which has two rows within the same cluster having disjoint supports since any continuous function on a compact set attains its supremum. This contradicts the above claim. Hence, $\tau_C(\Pi_{\ell'}) \le \beta < 1$ for any $\Pi_{\ell'}$ from $\mathcal{P}$. For any $\varepsilon > 0$, by taking $m = \lceil \ln_\beta \varepsilon \rceil + 1$ and $t = m\ell'$ we have $\tau_C(\Pi_t) = \tau_C(\Pi_{m\ell'}) \le \beta^m < \varepsilon$. Thus, item (3) holds and (4) follows.

(4) $\Rightarrow$ (6): Given any $1 \le k \le K$, and stochastic vectors y_1 and y_2 whose supports are within C_k. Without loss of generality, we assume $k = 1$ and $C_1 = \{1, 2, \cdots, n_1\}$ $(1 \le n_1 \le n)$. Since $\{e_i - e_{i+1} : i = 1, \cdots, n_1 - 1\}$ is a base of the subspace of $\mathbb{R}^{n_1}$ that is orthogonal to $\mathbf{1}_{C_1}$, we have $y_1 - y_2 = \sum_{i=1}^{n_1-1} a_i(e_i - e_{i+1}) = \sum_{i<j,\ i,j\in C_1} a_{ij}(e_i - e_j)$ for some numbers a_i and a_{ij}. Therefore, for any infinite sequence $P(1), P(2), \cdots \in \mathcal{P}$, $\lim_{t\to\infty}(y_1^T - y_2^T)P(1)\cdots P(t) = \sum_{i<j,\ i,j\in C_1} a_{ij} \lim_{t\to\infty}(e_i^T - e_j^T)P(1)\cdots P(t) = 0$. □

Remark 4. In Lemma 2, Assumptions 1, 2, and 3 are only used in "(3) $\Rightarrow$ (2)". The equivalence of (1) and (5) will be critical in our following combinatorial characterization of cluster consensus set. As such, even without the three assumptions, we still have "(1) $\Rightarrow$ (5)".

Remark 5. We mention that the doubly stochasticity in Assumption 3 can be replaced with a weaker (but more sophisticated) condition, which is called *cut-balance* [28, 31]. $P = (p_{ij})$ is cut-balanced if there exists $C \ge 1$ such that for every $S \subseteq V$, $\sum_{i\in S}\sum_{j\in V\setminus S} p_{ij} \le C \sum_{i\in V\setminus S}\sum_{j\in S} p_{ij}$.

3.2. Avoiding sets condition

Given $P, P(1), P(2), \cdots \in \mathcal{P}$ and $S \subseteq V$, we will write $N_P(S) := N_{G(P)}(S)$, $N_i(S) := N_{G(P(i))}(S)$, $N^1(S) := N_1(S)$, $N^2(S) := N_2(N_1(S))$, etc. following [8] for ease of notation. The "clustering" version of the avoiding sets condition is as follows.

Theorem 1. *Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices having inter-cluster common influence with respect to the same clustering $C = \{C_1, \cdots, C_K\}$. Suppose that either Assumptions 1, 2 hold or Assumptions 1, 3 hold. $\mathcal{P}$ is not a cluster consensus set if and only if there exist two sequences of nonempty subsets of V,*

$$S_1, S_2, \cdots, S_\ell \quad \textit{and} \quad S'_1, S'_2, \cdots, S'_\ell,$$

of length $\ell \leq 3^n - 2^{n+1} + 1$ and a sequence of matrices $P(1), P(2), \cdots, P(\ell) \in \mathcal{P}$ satisfying (i) $S_l \cap S'_l = \emptyset$, $l = 1, \cdots, \ell$; (ii) For any integer $s \geq 0$, $N_l(S_l) \subseteq S_{l+1}$, $l \equiv s(\mod \ell) + 1$; and (iii) There exist $i \in S_1$ and $j \in S'_1$ such that $i, j \in C_k$ for some $1 \leq k \leq K$.

Proof. *Sufficiency.* Suppose the sequences of sets $S_1, \cdots, S_\ell, S'_1, \cdots, S'_\ell$, and $P(1), P(2), \cdots, P(\ell) \in \mathcal{P}$ satisfying (i), (ii), and (iii) exist. Consider the infinite sequence of matrices made up of $P(1), P(2), \cdots, P(\ell)$ occurring periodically in this order. In the light of (iii), we pick $i \in S_1$ and $j \in S'_1$ such that $i, j \in C_k$ for some $1 \leq k \leq K$. Now we claim that the two vectors $e_i^T P(1) \cdots P(t)$ and $e_j^T P(1) \cdots P(t)$ have disjoint supports for all $t \geq 1$. Therefore, $\mathcal{P}$ is a cluster consensus set by Lemma 2 "(1) $\Rightarrow$ (5)".

It remains to show the claim. Indeed, for any $t \geq 1$, the support of $e_i^T P(1) \cdots P(t)$ is just $N^t(\{i\})$, which is contained in $S_{t(\mod \ell)+1}$. Likewise, the support of $e_j^T P(1) \cdots P(t)$ is $N^t(\{j\})$, which is contained in $S'_{t(\mod \ell)+1}$. But by assumption, $S_{t(\mod \ell)+1} \cap S'_{t(\mod \ell)+1} = \emptyset$. The proves the claim.

Necessity. We will show the necessity by contradiction. Suppose that there are no such sets $S_1, \cdots, S_\ell, S'_1, \cdots, S'_\ell$, and $P(1), P(2), \cdots, P(\ell) \in \mathcal{P}$ exist. Take any sequence $Q(1), Q(2), \cdots, \in \mathcal{P}$. Pick $i, j \in C_k$ for some $1 \leq k \leq K$, and define $U_t := N^t(\{i\})$ and $U'_t := N^t(\{j\})$ for $t \geq 1$. We claim that there must exist some $t_0 \geq 1$ such that $U_{t_0} \cap U'_{t_0} \neq \emptyset$. Since U_t is the support of $e_i^T Q(1) \cdots Q(t)$ and U'_t is the support of $e_j^T Q(1) \cdots Q(t)$, the claim would imply that $\mathcal{P}$ is a cluster consensus set by using Lemma 2 "(5) $\Rightarrow$ (1)". This contradicts our assumption and completes the proof of necessity.

To show the claim, we again assume the opposite. Suppose that, for any $t \geq 1$, $U_t \cap U'_t = \emptyset$. There exist two integers $a < b$ such that $(U_a, U'_a) = (U_b, U'_b)$. Define $S_1 := U_a$, $S'_1 := U'_a$, $P(1) := Q(a)$, $P(2) := Q(a+1)$, $\cdots$, $P(\ell) := Q(\ell + a - 1)$, where $\ell := b - a$. Let $S_{l+1} := N_l(S_l)$ and $S'_{l+1} := N_l(S'_l)$ for $1 \leq l \leq \ell$. Thereby we get two avoiding set cycles satisfying (i) and (ii) of Theorem 1. Since $P(1), \cdots, P(\ell)$ all have positive diagonals, we see that $i \in S_1$, $j \in S'_1$ and hence item (iii) is also satisfied. Finally, by a basic combinatorial outcome that the number of ordered partitions (A, B, C) of V with nonempty A, B and empty intersection of any two of them is $3^n - 2^{n+1} + 1$ (see e.g. [33, p. 90]), we have $\ell \leq 3^n - 2^{n+1} + 1$ by deleting some possible repetitions. This is at odds with our initial assumption, which in turn proves the claim. □

Remark 6. The condition (iii) is an essential difference as compared to Theorem 2.2 [8]. Since the two sequences are set cycles, S_1 and S'_1 in (iii) can well be substituted by any pair of S_l and S'_l. Roughly speaking, Theorem 1 says that $\mathcal{P}$ is a cluster consensus set if and only if there do not exist two set cycles which are disjoint at every step and contain vertices from the same cluster at some step.

Remark 7. It is clear that, under the assumptions of Theorem 1, deciding whether a finite set $\mathcal{P}$ of stochastic matrices is a cluster consensus set is algorithmically decidable (c.f. [8, Prop. 2.6]).

In the following, we illustrate how Theorem 1 can be used to establish some concrete results with several examples.

Example 1. Assume that $P = (p_{ij})$ is an $n \times n$ stochastic matrix having positive diagonal and inter-cluster common influence with respect to a clustering $C = \{C_1, \cdots, C_K\}$. Suppose further that either $G(P)$ is undirected or P is doubly stochastic. If there is some $1 \leq k \leq K$ such that C_k can be partitioned into two disjoint nonempty sets U_1 and U_2 with $p_{ij} = p_{ji} = 0$ for all $i \in U_1, j \in U_2$ and $\sum_{j \in U_1} p_{i_1,j} = \sum_{j \in U_2} p_{i_2,j} = 1$ for some $i_1 \in U_1$ and $i_2 \in U_2$, then the singleton $\mathcal{P} = \{P\}$ obviously is not a cluster consensus set. Indeed, we can take $S_1 = U_1$, $S'_1 = U_2$, and $\ell = 1$. The inter-cluster common influence condition implies that $N_P(S_1) \subseteq S_1$ and $N_P(S'_1) \subseteq S'_1$.

Example 2. Suppose $\mathcal{P}$ is a compact set of $n \times n$ stochastic matrices having inter-cluster common influence with respect to the same clustering $C = \{C_1, \cdots, C_K\}$. In addition, either Assumptions 1, 2 hold or Assumptions 1, 3 hold. If for all $P \in \mathcal{P}$ the induced subgraphs of $G(P)$ on C_k for all $1 \leq k \leq K$ are strongly connected, then $\mathcal{P}$ is a cluster consensus set. This can be justified by Theorem 1 as follows. For any sequence of matrices $P(1), P(2), \cdots \in \mathcal{P}$ and any pair of subsets S_1 and S'_1 such that there are $i \in S_1$, $j \in S'_1$, and $i, j \in C_k$ for some $1 \leq k \leq K$, we must have $C_k \subseteq N^{n-1}(S_1)$ and $C_k \subseteq N^{n-1}(S'_1)$. Thus, two avoiding set cycles cannot exist.

Example 3. Suppose that P is an $n \times n$ stochastic matrix having positive diagonal and inter-cluster common influence with respect to a clustering $C = \{C_1, \cdots, C_K\}$. Suppose further that either $G(P)$ is undirected or P is doubly stochastic. If $G(P)$ has *cluster-spanning trees* with respect to C (i.e., for each cluster C_k, $1 \leq k \leq K$, there is a vertex $i_k \in V$ such that there exist paths in $G(P)$ from all vertices in C_k to i_k), then $\{P\}$ is a cluster consensus set [10, Thm 1]. This can be deduced quickly from Theorem 1. Indeed, for any pair of subsets S_1 and S'_1 such that there are $i \in S_1$, $j \in S'_1$, and $i, j \in C_{k'}$ for some $1 \leq k' \leq K$, we obtain $i_k \in N^{n-1}(S_1) \cap N^{n-1}(S'_1)$. Clearly, two avoiding set cycles cannot occur in this case.

Example 4. Example 3 can be generalized to tackle switching topologies. Consider the infinite products of stochastic matrices $\cdots Q(2)Q(1)$ such that $\{Q(t) : t \geq 1\}$ have inter-cluster common influence with respect to the same clustering C. Assume that (i) either Assumptions 1, 2 hold or Assumptions 1, 3 hold for $\{Q(t) : t \geq 1\}$; and (ii) the graph obtained by joining the edge sets of the graphs $G(Q(iL+1)), \cdots, G(Q((i+1)L))$ contains cluster-spanning trees with respect to C for every integer $i \geq 0$. Then, the dynamic system $x(t+1) = Q(t+1)x(t)$ achieves cluster consensus (c.f. [10, Thm. 3] and [11, Thm. 2]). We can derive this from Theorem 1 by first noting that the product $P(i) := Q((i+1)L) \cdots Q(iL+1)$ has inter-cluster common influence with respect to C, and it still satisfies Assumptions 1, 2 or Assumptions 1, 3 (according to whether the former or the latter holds in (i)). Define $\mathcal{P} := \{P(i) : i \geq 0\}$. The same reasoning in Example 3 implies that two avoiding set cycles cannot occur. Hence, the system $x(t+1) = Q(t+1)x(t)$ reaches cluster consensus.

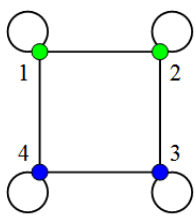

Figure 1: Schematic illustration of a 4-cycle with one self-loop at each node. $C = \{C_1, C_2\}$ with $C_1 = \{1, 2\}$ and $C_2 = \{3, 4\}$. The stochastic transition matrix corresponding to this graph forms a cluster consensus set. The similar result holds true if the edge $(1, 2)$ is removed, but not if $(3, 4)$ is further removed.

Example 5. Fig. 1 depicts the possible structure of a wireless sensor network G consisting of 4 nodes, where each node communicates with neighboring nodes within a fixed physical distance [2]. To such a graph we associate a stochastic matrix P by defining $p_{ij} = 1/d_i$ if (i, j) is an edge and $p_{ij} = 0$ otherwise, where d_i is the number of neighbors of node i. Here, for example, $d_i = 3$ for $i = 1, 2, 3, 4$ in G. We consider the following clustering $C = \{C_1 = \{1, 2\}, C_2 = \{3, 4\}\}$. It is straightforward to see that Theorem 1 is applicable, and there do not exist two avoiding set cycles in view of Example 3. Hence, P itself is a cluster consensus set. Next, suppose that G' is the graph obtained by removing the edge $(1, 2)$ possibly due to terrain obstacle or communication malfunction. The associated stochastic matrix is denoted by P'. By using Example 3 again, we see that P' still forms a cluster consensus set. Now suppose that the graph disintegrates further by removing the edge $(3, 4)$. We write P'' for the corresponding stochastic matrix. P'' is no longer a cluster consensus set since we can take $S_1 = \{1, 4\}$, $S'_1 = \{2, 3\}$ and $\ell = 1$ (note that $N_{P''}(S_1) = S_1$ and $N_{P''}(S'_1) = S'_1$). This example epitomizes a realistic situation where fragmentation of network structure leads to the collapse of system consensus.

An interesting implication of Theorem 1 is the following result, which says that having cluster-spanning trees is also necessary for cluster consensus.

Corollary 1. *Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices having inter-cluster common influence with respect to the same clustering $C = \{C_1, \cdots, C_K\}$. Suppose that either Assumptions 1, 2 hold or Assumptions 1, 3 hold. If $\mathcal{P}$ is a cluster consensus set, then, for any $P \in \mathcal{P}$, $G(P)$ has cluster-spanning trees with respect to C.*

Proof. If $\mathcal{P}$ is a cluster consensus set, then so it is with $\{P\}$ for any $P \in \mathcal{P}$. The condensation of $G(P)$, denoted by $CG(P)$, is a directed acyclic graph formed by contracting the strongly connected components of $G(P)$. It is well known that the condensation of a graph has at least one sink, i.e., a vertex with no out-neighbors. We claim that if $CG(P)$ has at least two sinks, then we cannot find two sinks both containing vertices in the same cluster.

Indeed, if this is not true, then we obtain two nodes $Sink_1$ and $Sink_2$ from $CG(P)$ such that $i \in Sink_1$ and $j \in Sink_2$ for some $i, j \in C_k$ and $1 \le k \le K$. Take $S_1 = Sink_1$ and $S'_1 = Sink_2$. Then $S_1 \cap S'_1 = \emptyset$, $N_P(S_1) \subseteq S_1$, and $N_P(S'_1) \subseteq S'_1$. It follows from Theorem 1 that $\{P\}$ is not a cluster consensus set, which contradicts our assumption. This establishes the claim.

Therefore, if $CG(P)$ has precisely one sink, then each vertex in this sink represents the root of a spanning tree of $G(P)$. Of course, $G(P)$ has cluster-spanning trees with respect to C. If $CG(P)$ has at least two sinks, by our above claim, for any cluster C_k, the vertices of C_k lie in at most one sink (some vertices of C_k may lie in non-sink nodes of $CG(P)$). It is easy to see that $G(P)$ has cluster-spanning trees with respect to C in this case. □

We observe that, without Assumptions 1, 2, 3, the proof for sufficiency of Theorem 1 above still holds by recalling Remark 4. Therefore, we obtain the following necessary condition for cluster consensus without requiring any of the three assumptions.

Corollary 2. *Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices having inter-cluster common influence with respect to the same clustering $C = \{C_1, \cdots, C_K\}$. If $\mathcal{P}$ is a cluster consensus set, then there do not exist two sequences of nonempty subsets of V,*

$$S_1, S_2, \cdots, S_\ell \quad \text{and} \quad S'_1, S'_2, \cdots, S'_\ell,$$

of length $\ell \le 3^n - 2^{n+1} + 1$ and a sequence of matrices $P(1), P(2), \cdots, P(\ell) \in \mathcal{P}$ satisfying (i) $S_l \cap S'_l = \emptyset$, $l = 1, \cdots, \ell$; (ii) For any integer $s \ge 0$, $N_l(S_l) \subseteq S_{l+1}$, $l \equiv s(\mod \ell) + 1$; and (iii) There exist $i \in S_1$ and $j \in S'_1$ such that $i, j \in C_k$ for some $1 \le k \le K$.

4. Concluding remarks

In this technical note, we have presented a combinatorial necessary and sufficient condition for cluster consensus of discrete-time linear systems. This combinatorial condition can be thought of as an extension of the original avoiding sets condition (i.e., $K = 1$) that was shown to be responsible for global consensus [8]. The result can be used to show that the cluster-spanning trees condition proposed in [10] is not only sufficient but necessary in some sense for achieving cluster consensus.

Note that the concept of cluster consensus in our paper is built on an underlying fixed clustering of the networks. In a general context, the multi-agent system may have changing clusterings over time. How to extend the presented method to deal with dynamical clustering is very appealing. Another direction worth investigation is the complexity of decidability of cluster consensus. It is revealed in [8] that checking consensus of a finite set $\mathcal{P}$ of matrices is NP-hard. This is the case when there are two or more matrices in the set, but, for undirected matrices, it is true when there are three or more matrices in the set. Here, we emphasize that the present technical note generalizes the consensus set to cluster consensus set, but does not address the complexity questions.

Finally, we mention that a variant notion of cluster consensus, referred to as scaled consensus [34, 35, 36], is recently introduced, where the final consensus states of nodes converge to prescribed ratios. Cluster consensus can be achieved in this framework by appropriately choosing the ratios. Scaled consensus allows for non-stochastic, non-positive state-update matrices, and remarkably, does not require inter-cluster common influence. Therefore, exploring an analogous notion of scaled consensus set would be an interesting future work.

Acknowledgement

Work funded by the National Natural Science Foundation of China (11505127), the Shanghai Pujiang Program (15PJ1408300), and a grant from the Fundamental Research Funds for the Central Universities. The author would like to thank the three anonymous referees for careful reading and constructive comments.

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