Consensus of hybrid multi-agent systems with malicious nodes

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***Abstract*—This brief investigates resilient consensus problems of hybrid multi-agent systems containing both continuous-time dynamical agents and discrete-time dynamical agents. A hybrid censoring strategy is developed to reach resilient consensus for cooperative agents in the directed networks in which some Byzantine agents are present. The number, location, and dynamics of Byzantine agents are assumed to be unavailable to the cooperative agents. Sufficient conditions based on network robustness are established when the number of Byzantine agents is locally bounded. They are further extended to cope with resilient scaled hybrid consensus where dictated ratios instead of a common value can be reached. Numerical examples are presented to illustrate the theoretical results.**

***Index Terms*—Resilient; hybrid multi-agent system; continuous time; discrete time; scaled consensus.**

I. Introduction

Cooperative control of multi-agent systems has copious applications and has attracted considerable attention from many fields including system engineering, computer science, and sociology [1], [2]. A key research topic of multi-agent systems is reaching consensus, where the states of agents in the network reach an agreement based on distributed interaction in continuous-time or discrete-time. Advanced research themes such as convergence rate, control scheme, faulty tolerance, communication delay, and system uncertainty, have been intensively investigated in the recent years; see e.g. the comprehensive surveys [3]–[5] and references therein.

Until recently consensus problems have only been studied in networks composed of entirely discrete-time agents or entirely continuous-time agents. Nevertheless, complicated networked systems are oftentimes hybrid showing both discrete-time and continuous-time characteristics [6]–[8]. Robots with discrete-time dynamics, for example, are integrated into the collective behavior of a group of continuous-time cockroaches for modifying their shelter location selections [9]. Three consensus protocols are introduced in [10] to deal with consensus in first-order hybrid multi-agent systems. Sufficient and necessary conditions for consensus are characterized. The results are further extended in [11] to second-order consensus of hybrid multi-agent systems. In [12], consensus analysis is built upon a game-theoretic approach to regulating the interaction between discrete-time and continuous-time agents. Switched multi-agent systems [13], [14] containing continuous-time and discrete-time subsystems alternately have also been explored.

All results mentioned above concern consensus without fault tolerance ability and all agents in the network are assumed to be cooperative. However, in the real world, cyber physical attacks and malicious agents are not uncommon, which make the system vulnerable and undermine the consensus behavior [15], [16]. In this paper, we aim to study resilient consensus against malicious agents in hybrid multi-agent systems modeled by directed networks.

The main contribution of this brief is summarized as follows. First, we define the resilient consensus problem with multiple agents governed by both continuous-time and discrete-time control laws, and put forward the hybrid resilient consensus strategy. This strategy extends the Weighted-Mean-Subsequence-Reduced (W-MSR) algorithms by simultaneously accommodating both continuous-time and discrete-time dynamical systems. So far W-MSR has only been applicable for discrete-time [17], [18], continuous-time [19], [20], and switched [21] fault-tolerant consensus. Second, based upon the concept of graph robustness [17], sufficient conditions are established to enable cooperative agents to reach consensus in spite of the misbehavior of locally bounded Byzantine agents. The concurrency of continuous-time and discrete-time agents entails novel treatment of the convergence analysis (c.f. Remark 2). As a consequence, unlike [17], the positive lower bound for the weights of communication link is no longer required. Third, resilient scaled consensus is explored in hybrid systems as a further generalization. Scaled consensus has been proposed in [22] as a novel framework for controlling states such that any prescribed ratios between different agents can be achieved. We show that scaled consensus can be achieved not only between a pair of discrete- or continuous-time agents but also between a discrete-time agent and a continuous-time agent.

The rest of the brief is organized as follows. In Section 2, we present some preliminaries and formulate the hybrid system model with malicious nodes. The main results are provided in Section 3 with numerical simulations worked out in Section 4. Finally, the conclusion is drawn in Section 5.

II. Preliminaries

A. Graph theory

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of nonnegative integers and real numbers, respectively. Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a directed graph of order n, where $\mathcal{V}=\{v_1,\cdots,v_n\}$ is the node set and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the directed edge set. Interaction between nodes (or agents) is characterized by graph $\mathcal{G}$, which we split into two parts: $\mathcal{C}$ for cooperative nodes and $\mathcal{B}$ for Byzantine

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nodes such that $\mathcal{V} = \mathcal{C} \cup \mathcal{B}$. The number and identities of Byzantine nodes are in general not available to cooperative nodes, which aim to reach a common decision. An edge of $\mathcal{G}$ starting from v_i ending at v_j is denoted by $(v_i, v_j) \in \mathcal{E}$. The neighborhood of v_i consisting of all edges leading to v_i is signified by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. A directed path between a pair of distinct nodes v_i and v_j is a finite array of edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \cdots, (v_{i_l}, v_j)$. The graph $\mathcal{G}$ has a directed spanning tree with root $v_\ell \in \mathcal{V}$ if there is a directed path from v_ℓ to every other node in $\mathcal{V}$. For $S \subseteq \mathcal{V}$, if there is $v_i \in S$ satisfying $|\mathcal{N}_i \backslash S| \geq r$ for some $r \in \mathbb{N}$, then S is called r-reachable [17]. $\mathcal{G}$ is said to be r-robust if for any pair of nonempty and mutually exclusive subsets in $\mathcal{V}$, one of these two sets is r-reachable. The following results is shown in [17].

Lemma 1.*Given $s, r \in \mathbb{N}$ and $s < r$. Suppose that $\mathcal{H}$ is obtained by deleting up to s incoming edges of every node in an r-robust directed graph $\mathcal{G}$. Then $\mathcal{H}$ is $(r-s)$-robust. In addition, $\mathcal{G}$ being a 1-robust directed graph is equivalent to $\mathcal{G}$ having a directed spanning tree.*

B. System model

Recall that hybrid multi-agent system contains both continuous-time and discrete-time agents. Let $\mathcal{V}^C = \{v_1, \cdots, v_m\}$ and $\mathcal{V}^D = \{v_{m+1}, \cdots, v_n\}$ represent the groups of continuous-time and discrete-time agents, respectively. The state of the agent v_i at time $t \geq 0$ if it has continuous-time dynamics (or at time $k \in \mathbb{N}$ if it has discrete-time dynamics) is denoted by $x_i(t) \in \mathbb{R}$ (or $x_i(k) \in \mathbb{R}$).

Definition 1. (resilient consensus for hybrid systems) We say that the cooperative nodes achieve resilient consensus if for any initial conditions $\{x_i(0)\}_{v_i \in \mathcal{V}}$, we have $\lim_{k\to\infty} x_i(k) - x_j(k) = 0$ for $v_i, v_j \in \mathcal{C}$ and $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$ for $v_i, v_j \in \mathcal{C} \cap \mathcal{V}^C$.

For $k \in \mathbb{N}$, the dynamics of a continuous-time cooperative node $v_i \in \mathcal{C} \cap \mathcal{V}^C$ is described by

$$\begin{aligned}\dot{x}_i(t) =& \varphi_i^C\big(\{x_j^i(t) : v_j \in (\mathcal{N}_i \cup \{v_i\}) \cap \mathcal{V}^C\} \\ &\cup \{x_j^i(k) : v_j \in \mathcal{N}_i \cap \mathcal{V}^D\}\big), t \in [k, k+1) \end{aligned} \quad (1)$$

and the dynamics of a discrete-time cooperate node $v_i \in \mathcal{C} \cap \mathcal{V}^D$ is described by

$$x_i(k+1) = \varphi_i^D\left(\{x_j^i(k) : v_j \in \mathcal{N}_i \cup \{v_i\}\}\right), \quad (2)$$

where $\varphi^C(\cdot)$ and $\varphi^D(\cdot)$ define the state evolution of cooperative node v_i, $x_j^i(t)$ is the state value transmitted from node v_j to node v_i at time t, and $x_j^i(t) = x_j(t)$ for $v_j \in \mathcal{C}$. For ease of notation, for $v_i \in \mathcal{V}^D$ we will conveniently set $x_i(t) := x_i(k)$ for $t \in [k, k+1)$ throughout the paper. Namely, we assume that the information passed from a cooperative $v_i \in \mathcal{V}^D$ to its neighbor during $[k, k+1)$ is invariably equal to $x_i(k)$.

Malicious nodes can exert different individual control laws that are unavailable to the cooperative ones. In particular, the Byzantine nodes are defined as follows.

Definition 2. (Byzantine node) $v_i \in \mathcal{B} \cap \mathcal{V}^C$ (or $v_i \in \mathcal{B} \cap \mathcal{V}^D$, respectively) is called Byzantine if it exerts a different control law $\tilde{\varphi}_i^C$ (or $\tilde{\varphi}_i^D$, respectively), or at some time $t > 0$ not all of its neighbors receive the same value from it.

Byzantine nodes are oftentimes thought to be the worst attackers [18], [20], [21], who possess a perfect knowledge of the entire network and are capable of transmitting faulty information within their neighborhoods via point-to-point communication or broadcasting. We naturally assume that the number of Byzantine nodes in $\mathcal{G}$ is constrained in some way. In particular, given $R \in \mathbb{N}$, for each $v_i \in \mathcal{C}$ we assume $|\mathcal{N}_i \cap \mathcal{B}| \leq R$, which is referred to as *R-locally bounded model* [19], [20] in the literature.

C. Hybrid R-censoring strategy

In order for cooperative nodes to reach consensus, we design the following purely distributed censoring strategy generalizing the well-known W-MSR algorithm [17], [18].

Fix $R \in \mathbb{N}$. For any $k \in \mathbb{N}$, cooperative node $v_i \in \mathcal{C} \cap \mathcal{V}^C$ at $t \in [k, k+1)$ receives the information $\{x_j^i(t)\}$ of its neighbors, and arranges $\{x_j^i(t)\}_{v_j \in \mathcal{N}_i}$ in an decreasing order (recall that $x_j^i(t) = x_j^i(k)$ for $v_j \in \mathcal{V}^D$). The highest R values which are higher than $x_i(t)$ in this ordered list are discarded (if there exist fewer than R such values, all of them are discarded). The analogous censoring procedure is adopted to the lowest R values. The values (or equivalently, their corresponding nodes) that are deleted by v_i at time t is signified by a set $\mathcal{R}_i(t)$. $v_i \in \mathcal{C} \cap \mathcal{V}^C$ changes its state using the following $\varphi_i^C(\cdot)$ in (1):

$$\begin{aligned}\dot{x}_i(t) =& \sum_{v_j \in [(\mathcal{N}_i \cup \{v_i\}) \backslash \mathcal{R}_i(t)] \cap \mathcal{V}^C} \varphi_{ij}(x_j^i(t), x_i(t)) \\ &+ \sum_{v_j \in [\mathcal{N}_i \backslash \mathcal{R}_i(t)] \cap \mathcal{V}^D} \varphi_{ij}(x_j^i(k), x_i(t)), \quad t \in [k, k+1), \end{aligned} \quad (3)$$

where the function $\varphi_{ij} : \mathbb{R}^2 \to \mathbb{R}$ satisfies (C1) φ_{ij} is locally Lipschitz continuous, (C2) $\varphi_{ij}(x, y) = 0 \Leftrightarrow x = y$, and (C3) $\varphi_{ij}(x, y)(x - y) > 0$ for $x \neq y$. In a similar way, cooperative node $v_i \in \mathcal{C} \cap \mathcal{V}^D$ at time k receives the information $\{x_j^i(k)\}$ of its neighbors, and arranges $\{x_j^i(k)\}_{v_j \in \mathcal{N}_i}$ in an decreasing order. The highest R values that are higher than $x_i(t)$ in the above ordered list are discarded (if there are fewer than R such values, all these values are discarded). The analogous censoring procedure is adopted to the lowest R values. The nodes that are deleted by v_i at time k is signified by $\mathcal{R}_i(k)$. $v_i \in \mathcal{C} \cap \mathcal{V}^D$ updates its state using the following $\varphi_i^D(\cdot)$ in (2):

$$x_i(k+1) = \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \backslash \mathcal{R}_i(k)} w_{ij}(k) x_j^i(k), \quad (4)$$

where $w_{ij}(k)$ represents non-negative weight on edge $(v_j, v_i) \in \mathcal{E}$ satisfying (D1) $w_{ij}(t) = 0$ if $v_j \notin \mathcal{N}_i \cup \{v_i\}$, and (D2) $\sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \backslash \mathcal{R}_i(k)} w_{ij}(k) = 1$.

Remark 1. For discrete-time dynamics (4), a positive lower bound for $w_{ij}(k)$ is no longer needed as we will apply different techniques from those in [17], [21]. A uniform choice for $w_{ij}(k)$ is $w_{ij}(k) = (|\mathcal{N}_i| + 1 - |\mathcal{R}_i(k)|)^{-1}$ for all $k \in \mathbb{N}$. Regarding continuous-time dynamics (3), $\varphi_{ij}(x, y) = a_{ij}(x - y)$ with $a_{ij} > 0$ being the adjacency weights of the network is canonical in the literature of consensus problems [3], [4]. We call the above algorithm as *hybrid R-censoring strategy*.

III. Main results

we in this section investigate the consensus of hybrid system (1)-(4) with malicious nodes characterized by R-locally bounded model. The highest and lowest states of all cooperative nodes are defined, respectively, as $M(t) := \max_{v_i \in \mathcal{C}} x_i(t)$ and $m(t) := \min_{v_i \in \mathcal{C}} x_i(t)$ for $t \geq 0$. Note that these definitions are valid for both continuous- and discrete-time agents as per our notation.

Theorem 1. *Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which cooperative nodes adopt the hybrid R-censoring strategy. In R-locally bounded model, for $v_i \in \mathcal{C} \cap \mathcal{V}^C$, we have $x_i(t) \in [m(0), M(0)]$ for $t \geq 0$; for $v_i \in \mathcal{C} \cap \mathcal{V}^D$, we have $x_i(k+1) \in [m(k), M(k)]$ for $k \in \mathbb{N}$.*

Proof. For $v_i \in \mathcal{C} \cap \mathcal{V}^D$, it is clear from (4) that $x_i(k+1)$ is in the form of a convex combination of $\{x^i_j(k)\}_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \backslash \mathcal{R}_i(k)}$, each of which sits in the range $[m(k), M(k)]$ in R-locally bounded model when hybrid R-censoring strategy is invoked. Hence, $x_i(k+1) \in [m(k), M(k)]$ for $k \in \mathbb{N}$.

For $v_i \in \mathcal{C} \cap \mathcal{V}^C$, we only prove the upper bound $x_i(t) \leq M(0)$ and the lower bound follows with similar arguments. If the upper bound does not hold, there exists $t^* \in [k^*, k^*+1)$ and $t^* < t$ for some $k^* \in \mathbb{N}$ such that (i) $x_j(t') \leq M(0)$ for all $t' \leq t^*$ and $v_j \in \mathcal{C}$ and (ii) $x_i(t^*) = M(0)$ and $\dot{x}_i(t^*) > 0$. It follows from (3) that

$$\begin{aligned} 0 < \dot{x}_i(t^*) = &\sum_{v_j \in [(\mathcal{N}_i \cup \{v_i\}) \backslash \mathcal{R}_i(t^*)] \cap \mathcal{V}^C} \varphi_{ij}(x^i_j(t^*), x_i(t^*)) \\ &+ \sum_{v_j \in [\mathcal{N}_i \backslash \mathcal{R}_i(t^*)] \cap \mathcal{V}^D} \varphi_{ij}(x^i_j(k^*), x_i(t^*)). \end{aligned} \tag{5}$$

Recall that there cannot be $R+1$ Byzantine nodes in any cooperative node's neighborhood. Under the hybrid R-censoring strategy, $x_i(t^*) = M(0) \geq x^i_j(t^*)$ for $v_j \in [(\mathcal{N}_i \cup \{v_i\}) \backslash \mathcal{R}_i(t^*)] \cap \mathcal{V}^C$. On the other hand, $x^i_j(k^*) = x^i_j(t^*) \leq M(0) = x_i(t^*)$ for $v_j \in [\mathcal{N}_i \backslash \mathcal{R}_i(t^*)] \cap \mathcal{V}^D$. In the light of (C2) and (C3), we arrive at that the right-hand side of (5) is at most zero. This contradicts with $\dot{x}_i(t^*) > 0$, and hence concludes the proof. □

From Theorem 1, it can be seen that both sequences $\{m(k)\}_{k \in \mathbb{N}}$ and $\{M(k)\}_{k \in \mathbb{N}}$ are monotonic and bounded. The network topology $\mathcal{G}$ is essentially dynamic due to the censoring strategy implemented. We make the following assumption on the rate of change, i.e., the dwell time.

Assumption 1. Signify by $\{\tau_l\}_{l \in \mathbb{N}}$ the array of time instances where $\mathcal{R}_i(t)$ varies for some i. There exists $\tau > 0$ satisfying $|\tau_{l+1} - \tau_l| \geq \tau$.

Theorem 2. *Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which cooperative nodes adopt the hybrid R-censoring strategy. In R-locally bounded model, resilient consensus is reached if $\mathcal{G}$ is $(2R+1)$-robust and Assumption 1 holds.*

Proof. For any $t > 0$, without loss of generality we assume $t \in [k, k+1)$ for some $k \in \mathbb{N}$. Denote by $\Phi(t) = M(t) - m(t)$ the difference between the two extreme states of cooperative nodes and $\Phi(t)$ is non-negative. Given a function $\varphi(t) : \mathbb{R} \to \mathbb{R}$, its Dini derivative is defined as $D^+\varphi(t) = \limsup_{h \to 0+} (\varphi(t+h) - \varphi(t))/h$. Furthermore, define $\mathcal{V}_M(t) := \{v_i \in \mathcal{C} : x_i(t) = M(t)\}$ and $\mathcal{V}_m(t) := \{v_i \in \mathcal{C} : x_i(t) = m(t)\}$.

If $\mathcal{V}_M(t) \cap \mathcal{V}^C \neq \emptyset$, we define the index i_M satisfying $\dot{x}_{i_M}(t) = \max_{v_i \in \mathcal{V}_M(t) \cap \mathcal{V}^C} \dot{x}_i(t)$. By the basic property of Dini derivative [23], the Dini derivative of $M(t)$ taken with respect to the trajectory of (3) follows

$$\begin{aligned} D^+M(t) = \dot{x}_{i_M}(t) = &\sum_{v_j \in [(\mathcal{N}_{i_M} \cup \{v_{i_M}\}) \backslash \mathcal{R}_{i_M}(t)] \cap \mathcal{V}^C} \varphi_{i_M j}(x^{i_M}_j(t), x_{i_M}(t)) \\ &+ \sum_{v_j \in [\mathcal{N}_{i_M} \backslash \mathcal{R}_{i_M}(t)] \cap \mathcal{V}^D} \varphi_{i_M j}(x^{i_M}_j(k), x_{i_M}(t)). \end{aligned} \tag{6}$$

We obtain $x_{i_M}(t) \geq x^{i_M}_j(t)$ when $v_j \in \mathcal{V}^C$ in the first term on the right-hand side of (6); $x_{i_M}(t) \geq x^{i_M}_j(t) = x^{i_M}_j(k)$ when $v_j \in \mathcal{V}^D$ in the second term on the right-hand side of (6). By the assumption (C3), we know that $D^+M(t) \leq 0$. Similarly, if $\mathcal{V}_m(t) \cap \mathcal{V}^C \neq \emptyset$, we define the index i_m satisfying $\dot{x}_{i_m}(t) = \max_{v_i \in \mathcal{V}_m(t) \cap \mathcal{V}^C} \dot{x}_i(t)$. Therefore, the Dini derivative of $m(t)$ taken with respect to the trajectory of (3) follows

$$\begin{aligned} D^+m(t) = \dot{x}_{i_m}(t) = &\sum_{v_j \in [(\mathcal{N}_{i_m} \cup \{v_{i_m}\}) \backslash \mathcal{R}_{i_m}(t)] \cap \mathcal{V}^C} \varphi_{i_m j}(x^{i_m}_j(t), x_{i_m}(t)) \\ &+ \sum_{v_j \in [\mathcal{N}_{i_m} \backslash \mathcal{R}_{i_m}(t)] \cap \mathcal{V}^D} \varphi_{i_m j}(x^{i_m}_j(k), x_{i_m}(t)). \end{aligned} \tag{7}$$

A similar analysis as above leads to $D^+m(t) \geq 0$. If $\mathcal{V}_M(t) \cap \mathcal{V}^C = \emptyset$, we define the node v_{i_M} to be any one in the set $\mathcal{V}_M(t)$ and $D^+M(t) = \dot{x}_{i_M}(t) = 0$ when $t \neq k$ by the property of Dini derivative [23]. If $\mathcal{V}_m(t) \cap \mathcal{V}^C = \emptyset$, we likewise define the node v_{i_m} to be any one in the set $\mathcal{V}_m(t)$ and $D^+m(t) = \dot{x}_{i_m}(t) = 0$ when $t \neq k$. Combining the above discussion, we have $D^+\Phi(t) = D^+M(t) - D^+m(t) \leq 0$ for $t \in (k, k+1)$.

From the comments below Theorem 1, we know that $m(k)$ and $M(k)$ change monotonically and both are bounded. We define

$$\rho_m := \lim_{k \to \infty} m(k) \leq \rho_M := \lim_{k \to \infty} M(k). \tag{8}$$

Therefore, $\lim_{k \to \infty} D^+\Phi(k) = 0$. Next, we will show $\lim_{t \to \infty} D^+\Phi(t) = 0$. Assume the contrary, that this does not hold. There exist $\varepsilon_0 > 0$, $\delta_0 > 0$, and a sequence of $\{s_p\}_{p \in \mathbb{N}}$ such that (i) $s_p \to \infty$ as $p \to \infty$ and (ii) $D^+\Phi(s_p) \leq -\varepsilon_0$ and $|s_{p+1} - s_p| > \delta_0$ for $p \in \mathbb{N}$.

Consider any range I satisfying $I \cap \mathbb{N} = \emptyset$ and $I \cap \{\tau_l\}_{l \in \mathbb{N}} = \emptyset$, where $\{\tau_l\}_{l \in \mathbb{N}}$ are given in Assumption 1. Note that $D^+\Phi(t)$ is continuous in I and $\dot{x}_i(t)$ is bounded for any $v_i \in \mathcal{C}$ (by condition (C1) if $v_i \in \mathcal{C} \cap \mathcal{V}^C$, and $\dot{x}_i(t) = 0$ for $t \in I$ if $v_i \in \mathcal{C} \cap \mathcal{V}^D$). Therefore, $D^+\Phi(t)$ is uniformly continuous within I. There exists $\delta_1 > 0$ such that for any pair of time points $t^1, t^2 \in I$ and $|t^1 - t^2| < \delta_1$, $|D^+\Phi(t^1) - D^+\Phi(t^2)| < \varepsilon_0/2$ holds. It follows from Assumption 1 that there is $\delta_2 \in (0, \delta_1)$ satisfying for any $p \in \mathbb{N}$, the interval $[s_p - \delta_2, s_p + \delta_2]$ is a subset of a range I delineated above. For $t \in [s_p - \delta_2, s_p + \delta_2]$, we estimate that $D^+\Phi(t) = -|D^+\Phi(s_p) - (D^+\Phi(s_p) - D^+\Phi(t))| \leq -(|D^+\Phi(s_p)| - |D^+\Phi(s_p) - D^+\Phi(t)|) \leq -\frac{\varepsilon_0}{2}$. We choose $0 < \delta < \delta_2$ satisfying that the intervals $\{[s_p -$

$\delta, s_p+\delta]\}_{p\in\mathbb{N}}$ are mutually exclusive. Drawing on the fact that $D^+\Phi(t)\le 0$, we have

$$\int_0^\infty D^+\Phi(t)dt \le -\lim_{N\to\infty}\sum_{p=1}^{N}\int_{s_p-\delta}^{s_p+\delta}\frac{\varepsilon_0}{2}dt=-\infty. \quad (9)$$

This contradicts with the non-negativity of $\Phi(t)$. Hence, we proved $\lim_{t\to\infty} D^+\Phi(t)=0$ by the method of contradiction.

From the discussion in the beginning of the proof, we know for any $t\ge 0$, $D^+M(t)$ is non-positive and $D^+m(t)$ is non-negative. In view of (8), $\lim_{t\to\infty}M(t)=\lim_{t\to\infty}x_{i_M}(t)=\rho_M$ and $\lim_{t\to\infty}m(t)=\lim_{t\to\infty}x_{i_m}(t)=\rho_m$. Assume that $\rho_M>\rho_m$, and we will prove this is not the case. Recall that $\mathcal{G}$ is $(2R+1)$-robust, and the interaction topology in the hybrid multi-agent system contains a directed spanning tree at any time under the hybrid R-censoring strategy by Lemma 1. There exist some time $T>0$ and $\varepsilon>0$ satisfying $x_{i_M}(t)>\rho_M-\varepsilon>\rho_m+\varepsilon>x_{i_m}(t)$ for $t\ge T$. Under our assumptions and the hybrid R-censoring strategy, we have the following observation.

(i) If $v_{i_M}\in\mathcal{V}^C$, then $\lim_{t\to\infty}\dot{x}_{i_M}(t)=0$, which means $\lim_{t\to\infty}x_j^{i_M}(t)-x_{i_M}(t)=0$ for $v_j\in[(\mathcal{N}_{i_M}\cup\{v_{i_M}\})\backslash\mathcal{R}_{i_M}(t)]\cap\mathcal{V}^C$ and $\lim_{t\to\infty,t\in[k,k+1)}x_j^{i_M}(k)-x_{i_M}(t)=0$ for $v_j\in[\mathcal{N}_{i_M}\backslash\mathcal{R}_{i_M}(t)]\cap\mathcal{V}^D$;

(ii) If $v_{i_M}\in\mathcal{V}^D$, then $\lim_{k\to\infty}x_{i_M}(k)=\rho_M$, which means $\lim_{k\to\infty}x_j^{i_M}(k)=\rho_M$ for $v_j\in(\mathcal{N}_{i_M}\cup\{v_{i_M}\})\backslash\mathcal{R}_{i_M}(k)$;

(iii) If $v_{i_m}\in\mathcal{V}^C$, then $\lim_{t\to\infty}\dot{x}_{i_m}(t)=0$, which means $\lim_{t\to\infty}x_j^{i_m}(t)-x_{i_m}(t)=0$ for $v_j\in[(\mathcal{N}_{i_m}\cup\{v_{i_m}\})\backslash\mathcal{R}_{i_m}(t)]\cap\mathcal{V}^C$ and $\lim_{t\to\infty,t\in[k,k+1)}x_j^{i_m}(k)-x_{i_m}(t)=0$ for $v_j\in[\mathcal{N}_{i_m}\backslash\mathcal{R}_{i_m}(t)]\cap\mathcal{V}^D$;

(iv) If $v_{i_m}\in\mathcal{V}^D$, then $\lim_{k\to\infty}x_{i_m}(k)=\rho_m$, which means $\lim_{k\to\infty}x_j^{i_m}(k)=\rho_m$ for $v_j\in(\mathcal{N}_{i_m}\cup\{v_{i_m}\})\backslash\mathcal{R}_{i_m}(k)$.

Since $\mathcal{G}$ is finite, there is time $T'\ge T$ such that there are two direction paths in the interaction network at T' one starting from the root vertex v_ℓ ending at v_{i_M} the other starting from v_ℓ ending at v_{i_m}. Moreover, $x_\ell(T')>\rho_M-\varepsilon$ and $x_\ell(T')<\rho_m+\varepsilon$. This is impossible. Hence, we derive that $\rho_M=\rho_m$, which means the consensus has been achieved. □

Remark 2. Due to the concurrency of both continuous-time and discrete-time dynamical agents, a discrete-time agent may have both continuous-time and discrete-time neighbors, which can be cooperative and/or Byzantine. A continuous-time Byzantine node can not be well kept in check when only discrete time steps are examined as in [17], while the information flow of a discrete-time Byzantine node is not differentiable. These obstacles have been worked around by our proposed algorithm and mathematical analysis.

Given real number $\gamma_i\neq 0$ for every agent $v_i\in\mathcal{G}$, we define hybrid resilient scaled consensus as below.

Definition 3. (resilient scaled consensus for hybrid systems) Given $(\gamma_1,\cdots,\gamma_n)$, the cooperative nodes are said to achieve resilient scaled consensus if for any initial conditions $\{x_i(0)\}_{v_i\in\mathcal{V}}$, we have $\lim_{k\to\infty}\gamma_i x_i(k)-\gamma_j x_j(k)=0$ for $v_i,v_j\in\mathcal{C}$ and $\lim_{t\to\infty}\gamma_i x_i(t)-\gamma_j x_j(t)=0$ for $v_i,v_j\in\mathcal{C}\cap\mathcal{V}^C$.

Scaled consensus has numerous applications in real life ranging from water distribution systems to compartmental mass-action systems [22]. Based upon the hybrid R-censoring strategy described in Section II.C, we consider a modified *hybrid scaled R-censoring strategy*, which sorts $\{\gamma_j x_j^i(t)\}_{v_j\in\mathcal{N}_i}$ for each cooperative node v_i, compares with $\gamma_j x_i(t)$, and replaces the control laws (3) and (4), respectively, with

$$\begin{aligned}\dot{x}_i(t)=&\operatorname{sgn}(\gamma_i)\sum_{v_j\in[(\mathcal{N}_i\cup\{v_i\})\backslash\mathcal{R}_i(t)]\cap\mathcal{V}^C}\varphi_{ij}(\gamma_j x_j^i(t),\gamma_i x_i(t))\\&+\gamma_{\rm i}\sum_{v_j\in[\mathcal{N}_i\backslash\mathcal{R}_i(t)]\cap\mathcal{V}^D}\varphi_{ij}(\gamma_j x_j^i(k),\gamma_i x_i(t)),\end{aligned} \quad (10)$$

and

$$x_i(k+1)=\operatorname{sgn}(\gamma_i)\sum_{v_j\in(\mathcal{N}_i\cup\{v_i\})\backslash\mathcal{R}_i(k)}w_{ij}(k)\gamma_j x_j^i(k), \quad (11)$$

with $\operatorname{sgn}(\cdot)$ indicating the signum function, φ_{ij} satisfying the same conditions (C1)-(C3), $w_{ij}(k)$ satisfying again (D1) and (D2') $\sum_{v_j\in(\mathcal{N}_i\cup\{v_i\})\backslash\mathcal{R}_i(k)}|\gamma_i|w_{ij}(k)=1$.

The following theorem can be shown as Theorem 2 following the similar arguments (hence omitted here) by resetting $M(t):=\max_{v_i\in\mathcal{C}}\gamma_i x_i(t)$ and $m(t):=\min_{v_i\in\mathcal{C}}\gamma_i x_i(t)$.

Theorem 3. *Consider a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, in which cooperative nodes adopt the hybrid scaled R-censoring strategy. Given $(\gamma_1,\cdots,\gamma_n)$. In R-locally bounded model, resilient scaled consensus is reached if $\mathcal{G}$ is $(2R+1)$-robust and Assumption 1 holds.*

Remark 3. Theorem 3 generalizes Theorem 2 in the sense that the restriction $\gamma_1=\gamma_2=\cdots=\gamma_n=1$ is lifted. This provides desired flexibility especially in hybrid systems since autonomous robots, for example, are allowed to have different or even opposite tasks from natural critters in a group [9].

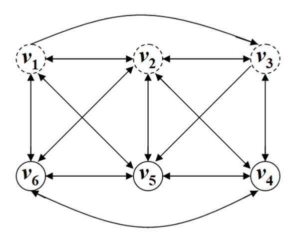

Fig. 1. 3-robust graph $\mathcal{G}$ having $\mathcal{V}^C=\{v_1,v_2,v_3\}$ and $\mathcal{V}^D=\{v_4,v_5,v_6\}$.

IV. Simulations

In this section, we consider a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with $\mathcal{V}^C=\{v_1,v_2,v_3\}$ and $\mathcal{V}^D=\{v_4,v_5,v_6\}$; see Fig. 1. It is direct to check that $\mathcal{G}$ is 3-robust. We choose the initial configuration of the agents as $x_1(0)=3$, $x_2(0)=2$, $x_3(0)=-1$, $x_4(0)=0$, $x_5(0)=1$, $x_6(0)=-2$.

Example 1. We choose $\mathcal{C}=\{v_1,\cdots,v_5\}$ and $\mathcal{B}=\{v_6\}$. v_6 is the only Byzantine node follows the discrete-time dynamics $x_6(k+1)=x_6(k)+\ln(k)/20$. Every continuous-time cooperative node $v_i\in\mathcal{C}$ adopts $\varphi_{ij}(x,y)=0.1\cdot(x-y)$ in (3) and discrete-time cooperative nodes take $w_{ij}(k)=(|\mathcal{N}_i|+1-|\mathcal{R}_i(k)|)^{-1}$ for $v_j\in(\mathcal{N}_i\cup\{v_i\})\backslash\mathcal{R}_i(k)$ in (4). By using the hybrid 1-censoring strategy, the evolution of agents' states are shown in Fig. 2(a). We observe that all cooperative agents converge despite the malicious behavior of v_6 as one would expect according to Theorem 2. In the inset of Fig. 2(a), the malicious node v_6 succeeds when the protocol without censoring is performed.

Example 2. In this example, we choose $\mathcal{C} = \{v_2, \cdots, v_6\}$ and $\mathcal{B} = \{v_1\}$. v_1 is the only Byzantine node follows the continuous-time dynamics $\dot{x}_1(t) = x_1(t)/5$. The cooperative nodes follow the same protocols as in Example 1. We observe from Fig. 2(b) that resilient hybrid consensus has been achieved asymptotically in line with Theorem 2. Similarly, in the inset of Fig. 2(b), we see that the states of other nodes are led by the malicious node v_1 when the censoring mechanism is absent.

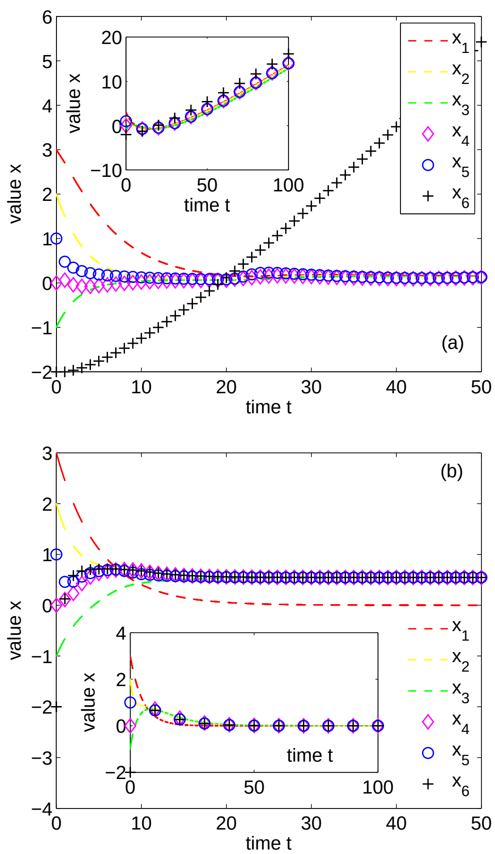

Fig. 2. States of the agents over network $\mathcal{G}$, where (a) v_6 is the Byzantine node and (b) v_1 is the Byzantine node, with censoring (main panels) and without censoring (insets).

V. Conclusion

In this brief, resilience consensus problems are studied for systems composing of multiple dynamical agents governed by both continuous-time and discrete-time control laws. We frame hybrid R-censoring strategies to withstand possible Byzantine nodes enabling cooperative nodes to reach consensus when malicious behaviors are bounded in neighborhoods of cooperative nodes. The designed strategy is purely distributed and has low complexity. It is further generalized to deal with resilient scaled hybrid consensus where dictated ratios instead of a common state can be achieved. Sufficient conditions are established to solves resilient scaled hybrid consensus problems. For future work, it would be desirable to consider the effect of time delay and event-triggered consensus [5] in hybrid multi-agent systems.

References

[1] Y. Cao, W. Yu, W. Ren, G. Chen, An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Trans. Ind. Inf.*, 9(1)(2013) 427–438

[2] X. Ge, F. Yang, Q.-L. Han, Distributed networked control systems: a brief overview. *Inf. Sci.*, 380(2017) 117–131

[3] R. Olfati-Saber, J. A. Fax, R. M. Murray, Consensus and cooperation in networked multi-agent systems. *Proc. IEEE*, 95(2007) 215–233

[4] J. Qin, Q. Ma, Y. Shi, L. Wang, Recent advances in consensus of multi-agent systems: a brief survey. *IEEE Trans. Ind. Inf.*, 64(6)(2017) 4972–4983

[5] L. Ding, Q.-L. Han, X. Ge, X.-M. Zhang, An overview of recent advances in event-triggered consensus of multiagent systems. *IEEE Trans. Cybern.*, 48(4)(2018) 1110–1123

[6] R. Goebel, R. G. Sanfelice, A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, New Jersey, 2012

[7] Y. Susuki, T. J. Koo, H. Ebina, T. Yamazaki, T. Ochi, T. Uemura, T. Hikihara, A hybrid system approach to the analysis and and design of power grid dynamic performance. *Proc. IEEE*, 100(1)(2012) 225–239

[8] Y. Shang, Hybrid consensus for averager-copier-voter networks with non-rational agents. *Chaos, Solitons & Fractals*, 110(2018) 244–251

[9] J. Halloy, G. Sempo, G. Caprari, C. Rivault, M. Asadpour, F. Tâche, I Saïd, V. Durier, S. Canonge, J. M. Amé, C. Detrain, N. Correll, A. Martinoli, F. Mondada, R. Siegwart, J. L. Deneubourg, Social integration of robots into groups of cockroaches to control self-organized choices. *Science*, 318(2007) 1155–1158

[10] Y. Zheng, J. Ma, L. Wang, Consensus of hybrid multi-agent systems, Consensus of hybrid multi-agent systems. *IEEE Trans. Neural Netw. Learn. Syst.*, 29(4)(2018) 1359–1365

[11] Y. Zheng, Q. Zhao, J. Ma, L. Wang, Second-order consensus of hybrid multi-agent systems. *Syst. Cont. Lett.*, 125(2019) 51–58

[12] J. Ma, M. Ye, Y. Zheng, Y. Zhu, Consensus analysis of hybrid multiagent systems: a game-theoretic approach. *Int. J. Robust Nonlinear Control*, doi:10.1002/rnc.4462, 2019

[13] Y. Zheng, L. Wang, Consensus of switched multiagent systems. *IEEE Trans. Circuits Syst. Express Briefs*, 63(2016) 314–318

[14] Y. Shang, scaled consensus of switched multi-agent systems. *IMA J. Math. contr. Inform.*, doi:10.1093/imamci/dnx062, 2018

[15] R. Moghadam, H. Modares, Resilient autonomous control of distributed multiagent systems in contested environments. *IEEE Trans. Cybern.*, doi:10.1109/TCYB.2018.2856089, 2018

[16] P. Wang, G. Wen, X. Yu, W. Yu, Y. Wan, Synchronization of resilient complex networks under attacks. *IEEE Trans. Syst. Man, Cybern., Syst.*, doi:10.1109/TSMC.2019.2895027, 2019

[17] H. J. LeBlanc, H. Zhang, X. Koutsoukos, S. Sundaram, Resilient asymptotic consensus in robust networks. *IEEE J. Select. Areas Commun.*, 31(4)(2013) 766–781

[18] A. Mitra, J. A. Richards, S. Bagchi, S. Sundaram, Resilient distributed state estimation with mobile agents: overcoming Byzantine adversaries, communication losses, and intermittent measurements. *Auton. Robots*, 43(2019) 743–768

[19] Y. Wu, X. He, Secure consensus control for multi-agent systems with attacks and communication delays. *IEEE/CAA J. Autom. Sinica*, 4(1)(2017) 136–142

[20] H. J. LeBlanc, X. Koutsoukos, Resilient first-order consensus and weakly stable, higher order synchronization of continuous-time networked multiagent systems. *IEEE Trans. Contr. Netw. Syst.*, 5(3)(2018) 1219–1231

[21] Y. Shang, Resilient consensus of switched multi-agent systems. *Syst. Contr. Lett.*, 122(2018) 12–18

[22] S. Roy, Scaled consensus. *Automatica*, 51(2015) 259–262

[23] J. M. Danskin, The theory of max-min, with applications. *SIAM J. Appl. Math.*, 14(4)(1966) 641–664