Nonlocal nonlinear mechanics of imperfect carbon nanotubes

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Abstract

In this article, for the first time, a coupled nonlinear model incorporating scale influences is presented to simultaneously investigate the influences of viscoelasticity and geometrical imperfections on the nonlocal coupled mechanics of carbon nanotubes; large deformations, stress nonlocality and strain gradients are captured in the model. The Kelvin-Voigt model is also applied in order to ascertain the viscoelasticity effects on the mechanics of the initially imperfect nanoscale system. The modified coupled equations of motion are then derived via the Hamilton principle. A solution approach for the derived coupled equations is finally developed applying a decomposition-based procedure in conjunction with a continuation-based scheme. The significance of many parameters such as size parameters, initial imperfections, excitation parameters and linear and nonlinear damping effects in the nonlinear mechanical response of the initially imperfect viscoelastic carbon nanotube is assessed. The present results can be useful for nanoscale devices using carbon nanotubes since the viscoelasticity and geometrical imperfection are simultaneously included in the proposed model.

Keywords: Carbon nanotubes; Initial imperfections; Viscoelasticity; Nonlinear response; Scale influences

1. Introduction

Micro and nano structures have been widely used in micro/nano devices [1, 2]. Among them, carbon nanostructures have been used in a wide range of applications in nanotechnology, biotechnology and nanoengineering since they display interesting electrical, mechanical and chemical properties. Some important carbon nanostructures are carbon nanotubes (CNTs), graphene sheets and buckyballs. To appropriately use these precious nanostructures in different applications, especially in nanoengineering, our level of understanding of their mechanical properties should be increased. This is due to the fact that the overall performance of a nanoelectromechanical system (NEMS) depends greatly on the mechanical behaviour of its building blocks such as CNTs.

Scale-dependent models have been utilised for the investigation of the mechanics of many small-scale structures such as microbeams [3-9], microplates [10-13], nanobeams [14-20] and nanoplates [21-25]. A particular attention has been paid to the mechanics of CNTs. Although CNTs display a viscoelastic response when they are subject to an applied load [26, 27], many size-dependent theoretical models in the literature have not considered the effects of viscoelasticity. As some examples, a few size-dependent models for the mechanical response of elastic CNTs are reviewed. Setoodeh et al. [28] obtained an exact solution for the buckling instability of elastic CNTs with large deformations by applying a classical nonlocal model. Aydogdu [29] presented a size-dependent nonlocal rod theory to ascertain the axial vibration characteristics of nanorods. In addition, Malekzadeh and Shojaee [30] proposed a non-classical continuum theory to explore the free vibration of non-uniform beams at the nanoscale level. The nonlocal oscillations of mass nanosensors employing elastic CNTs with small deformations were also examined by Aydogdu and Filiz [31]. In addition to these interesting papers, a few studies have been carried out on the viscoelastic response of CNTs under mechanical stresses. Chang and Lee [32] developed a nonlocal model to study the viscoelastic vibration characteristics of carbon nanotubes. In another analysis, a linear study was performed by Lei et al. [33] on the damping effect on the vibration response of CNTs using a combination of the Kelvin-Voigt model and the Eringen theory. The time-dependent deformation of fluid-conveying CNTs taking into account the internal energy loss was also explored by Bahaadini and Hosseini [34]. Furthermore, the effect of initial stresses on the vibration of viscoelastic beams at nanoscale levels was investigated by Zhang et al. [35]. Karlicic et al. [36] also proposed a non-classical model for the dynamic characteristics of a CNT-based composite viscoelastic system under the action of a magnetic field.

The use of the classical nonlocal theory of elasticity for nanoscale structures such as CNTs is limited to a particular range of lengths since nonlocal effects usually disappear after a certain length. To overcome this problem, Lim et al. [37] has recently introduced a modified nonlocal elasticity theory by incorporating the strain gradient influences. Using the molecular dynamics, it has been indicated that this modified theory is able to better estimate the size-dependent mechanics of CNTs compared to the classical nonlocal theory [38]. However, few research papers have been reported on the size-dependent deformation of CNTs with consideration of viscoelastic effects using this modified nonlocal theory. Some linear models have been merely developed for the wave propagation analysis of viscoelastic carbon nanotubes [39-41].

In addition to the influence of viscoelasticity, the influence of geometrical imperfections becomes more and more important when large deformations are taken into consideration since these imperfections can change the nonlinear mechanical characteristics of ultrasmall structures.

In the current investigation, *for the first time*, the effects of viscoelasticity as well as geometrical imperfections on the mechanics of CNTs with large deformations are analysed via a modified nonlocal elasticity model. The consideration of both viscoelasticity and geometrical imperfections leads to a more comprehensive scale-dependent model for CNTs. Furthermore, the proposed model can be used in a wide range of lengths since the stiffness hardening and softening are included. As a viscoelastic theory, the Kelvin-Voigt model is applied in the analysis. The coupled nonlinear equations of ultrasmall tubes are presented applying the Hamilton principle together with a beam model. A solution approach is developed with the application of a decomposition-based procedure in conjunction with a continuation-based method. The importance of many parameters such as the size parameter, the initial imperfection, the excitation loading as well as the linear and nonlinear damping effects in the size-dependent coupled mechanics of the initially imperfect viscoelastic carbon nanotube with large deflections is explained.

2. Formulation

Shown in Fig. 1 is a clamped-clamped single-walled carbon nanotube with an initial deformation as a geometric imperfection. The viscoelastic and elastic constants of the CNT are denoted by η and *E*, respectively. Moreover, Poisson's ratio, the length and the mass density are denoted by *v*, *L* and ρ, respectively. w_0 denotes the initial deflection of the viscoelastic CNT while the axial and transverse time-dependent displacements are described by *u* and *w*, respectively. A harmonic load in the form of $q(x,t)=\cos(\omega t)F(x)$ is applied on the imperfect nanoscale system along the transverse direction; ω and F are the forcing frequency and amplitude, respectively.

To model the mechanics of nanostructures such as CNTs, scale-dependent continuum mechanics can be used [42-47]. For a single-walled CNT with an initial deformation, the axial strain (ε_{xx}) is given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{dw_0}{dx} - z\frac{\partial^2 w}{\partial x^2}. \quad (1)$$

On the other hand, based on the modified nonlocal elasticity, the total axial stress of the imperfect viscoelastic CNT (t_{xx}) is expressed as [48-51]

$$\left[1-\left(e_0 a\right)^2 \nabla^2\right] t_{xx} = \left(1 - l_{sg}^2 \nabla^2\right)\left(t_{xx(el)}^{cl} + t_{xx(vis)}^{cl}\right), \quad (2)$$

where $t_{xx(el)}^{cl}$ and $t_{xx(vis)}^{cl}$ are respectively the elastic and viscoelastic parts of the classical (local) stress; e_0, a, l_{sg} and ∇^2 stand for the calibration parameter associated with the nonlocal stress [52], the internal characteristic length, the strain gradient parameter and the Laplace operator, respectively [53, 54]. Equation (2) is the differential scale-dependent constitutive relation of the modified elasticity theory. Recently, integral scale-dependent constitutive relations have also been used for nanostructures [55-58]. The elastic and viscoelastic parts of the classical stress are

$$t_{xx(el)}^{cl} = E\varepsilon_{xx}, \; t_{xx(vis)}^{cl} = \eta\frac{\partial \varepsilon_{xx}}{\partial t}. \quad (3)$$

In view of Eqs. (1)-(3), the non-classical stress resultants of the imperfect viscoelastic CNT can be formulated as

$$\left[1-\left(e_0a\right)^2\nabla^2\right]N_{xx}=EA\left(1-l_{sg}^2\nabla^2\right)\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{dw_0}{dx}\right]$$
$$+\eta A\left(1-l_{sg}^2\nabla^2\right)\left(\frac{\partial^2 u}{\partial t\partial x}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial^2 w}{\partial t\partial x}\frac{dw_0}{dx}\right), \tag{4}$$

$$\left[1-\left(e_0a\right)^2\nabla^2\right]M_{xx}=-EI\left(1-l_{sg}^2\nabla^2\right)\frac{\partial^2 w}{\partial x^2}-\eta I\left(1-l_{sg}^2\nabla^2\right)\frac{\partial^3 w}{\partial t\partial x^2}, \tag{5}$$

where

$$\begin{Bmatrix} N_{xx} \\ M_{xx} \end{Bmatrix}=\int_A t_{xx}\begin{Bmatrix} 1 \\ z \end{Bmatrix}dA,\quad \begin{Bmatrix} A \\ I \end{Bmatrix}=\int_A \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix}dA. \tag{6}$$

The relations between different non-classical stresses are described by

$$\begin{Bmatrix} t_{xx} \\ t_{xx(el)} \\ t_{xx(vis)} \end{Bmatrix}=\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xx(el)} \\ \sigma_{xx(vis)} \end{Bmatrix}-\nabla\begin{Bmatrix} \sigma_{xx}^{(1)} \\ \sigma_{xx(el)}^{(1)} \\ \sigma_{xx(vis)}^{(1)} \end{Bmatrix},$$
$$\begin{Bmatrix} t_{xx} \\ \sigma_{xx} \\ \sigma_{xx}^{(1)} \end{Bmatrix}=\begin{Bmatrix} t_{xx(el)} \\ \sigma_{xx(el)} \\ \sigma_{xx(el)}^{(1)} \end{Bmatrix}+\begin{Bmatrix} t_{xx(vis)} \\ \sigma_{xx(vis)} \\ \sigma_{xx(vis)}^{(1)} \end{Bmatrix}, \tag{7}$$

where ∇, $\sigma_{ij(\alpha)}$ and $\sigma_{ij(\alpha)}^{(1)}$ represent the gradient operator, the axial classical nonlocal stress and the axial higher-order nonlocal stress, respectively. The energy variation due to the total elastic stress (δU_{el}) of the imperfect CNT and the work variation due to its total viscoelastic stress (δW_{vis}) are as follows

$$\delta U_{el}=\int_0^L\int_A t_{xx(el)}\delta\varepsilon_{xx}dAdx+\left[\int_A \sigma_{xx(el)}^{(1)}\delta\varepsilon_{xx}dA\right]_0^L, \tag{8}$$

$$\delta W_{vis}=-\int_0^L\int_A t_{xx(vis)}\delta\varepsilon_{xx}dAdx-\left[\int_A \sigma_{xx(vis)}^{(1)}\delta\varepsilon_{xx}dA\right]_0^L. \tag{9}$$

The kinetic energy variation (δK_e) of the imperfect CNT and the work variation (δW_q) due to $q(x,t)$ are also formulated as [59]

$$\delta K_e = m\int_0^L \left(\frac{\partial u}{\partial t}\delta\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t}\delta\frac{\partial w}{\partial t} \right) dx, \tag{10}$$

$$\delta W_q = \int_0^L q(x,t)\delta w \, \mathrm{d}x. \tag{11}$$

In Eq. (10), m denotes the mass per unit length of the imperfect CNT. The Hamilton principle is now used for the derivation of the motion equations of the imperfect viscoelastic tube. This principle is generally written as follows

$$\int_{t_1}^{t_2} \left(\delta K_e + \delta W_q + \delta W_{vis} - \delta U_{el} \right) \mathrm{d}t = 0. \tag{12}$$

Using Eqs. (8)-(11), one obtains the following motion equations

$$\frac{\partial N_{xx}}{\partial x} = m\frac{\partial^2 u}{\partial t^2}, \tag{13}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial}{\partial x}\left[N_{xx}\left(\frac{\partial w}{\partial x} + \frac{dw_0}{dx} \right) \right] + q = m\frac{\partial^2 w}{\partial t^2}. \tag{14}$$

Application of the above equations to Eqs. (4) and (5) gives the following expressions for the non-classical stress resultants

$$\begin{aligned} N_{xx} &= EA\left(1 - l_{sg}^2\nabla^2\right)\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x}\frac{dw_0}{dx} \right] \\ &+ \eta A\left(1 - l_{sg}^2\nabla^2\right)\left(\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t \partial x} + \frac{\partial^2 w}{\partial t \partial x}\frac{dw_0}{dx} \right) + m\left(e_0 a\right)^2 \frac{\partial^3 u}{\partial x \partial t^2}, \end{aligned} \tag{15}$$

$$\begin{aligned} M_{xx} &= -EI\left(1 - l_{sg}^2\nabla^2\right)\frac{\partial^2 w}{\partial x^2} - \eta I\left(1 - l_{sg}^2\nabla^2\right)\frac{\partial^3 w}{\partial t \partial x^2} \\ &+ \left(e_0 a\right)^2 \left\{ m\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[N_{xx}\left(\frac{\partial w}{\partial x} + \frac{dw_0}{dx} \right) \right] - q \right\}, \end{aligned} \tag{16}$$

Substituting the obtained stress resultants into Eqs. (13) and (14) and assuming the harmonic load as $q = F_1 \cos(\omega t)$, one can obtain

$$
\begin{aligned}
& EA\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{dw_0}{dx}+\frac{\partial w}{\partial x}\frac{d^2 w_0}{dx^2}\right)-EAl_{sg}^2\left(\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\right. \\
& \left.+\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^4}\frac{dw_0}{dx}+3\frac{\partial^3 w}{\partial x^3}\frac{d^2 w_0}{dx^2}+3\frac{\partial^2 w}{\partial x^2}\frac{d^3 w_0}{dx^3}+\frac{\partial w}{\partial x}\frac{d^4 w_0}{dx^4}\right) \\
& +\eta A\left(\frac{\partial^3 u}{\partial t\partial x^2}+\frac{\partial^2 w}{\partial t\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^3 w}{\partial t\partial x^2}\frac{dw_0}{dx}+\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^2 w_0}{dx^2}\right) \\
& -\eta Al_{sg}^2\left(3\frac{\partial^3 w}{\partial x^3}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^5 w}{\partial t\partial x^4}+\frac{\partial^5 u}{\partial t\partial x^4}+\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial t\partial x}+3\frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial x^3\partial t}\right. \\
& \left.+\frac{\partial^5 w}{\partial t\partial x^4}\frac{dw_0}{dx}+3\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^2 w_0}{dx^2}+3\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^3 w_0}{dx^3}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^4 w_0}{dx^4}\right)=m\left[\frac{\partial^2 u}{\partial t^2}-(e_0 a)^2\frac{\partial^4 u}{\partial x^2\partial t^2}\right],
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
& -EI\left(\frac{\partial^4 w}{\partial x^4}-l_{sg}^2\frac{\partial^6 w}{\partial x^6}\right)-\eta I\left(\frac{\partial^5 w}{\partial t\partial x^4}-l_{sg}^2\frac{\partial^7 w}{\partial t\partial x^6}\right) \\
& +EA\left[\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}-(e_0 a)^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\left[\frac{dw_0}{dx}\frac{\partial w}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial u}{\partial x}\right] \\
& +EA\left[\frac{\partial w}{\partial x}+\frac{dw_0}{dx}-3(e_0 a)^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{dw_0}{dx}+\frac{\partial w}{\partial x}\frac{d^2 w_0}{dx^2}\right) \\
& -EA\left\{\left[l_{sg}^2+3(e_0 a)^2\right]\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)-(e_0 a)^2 l_{sg}^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right\}\times \\
& \left[\frac{\partial^3 u}{\partial x^3}+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x^3}\frac{dw_0}{dx}+2\frac{\partial^2 w}{\partial x^2}\frac{d^2 w_0}{dx^2}+\frac{\partial w}{\partial x}\frac{d^3 w_0}{dx^3}\right] \\
& -EA\left\{\left[l_{sg}^2+(e_0 a)^2\right]\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)-3(e_0 a)^2 l_{sg}^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right\}\times \\
& \left(\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\frac{\partial^4 w}{\partial x^4}+3\frac{d^2 w_0}{dx^2}\frac{\partial^3 w}{\partial x^3}+\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^2 w}{\partial x^2}\frac{d^3 w_0}{dx^3}+3\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\right) \\
& +3EAl_{sg}^2(e_0 a)^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)\left[3\left(\frac{\partial^3 w}{\partial x^3}\right)^2+\frac{\partial w}{\partial x}\frac{\partial^5 w}{\partial x^5}+\frac{\partial^5 u}{\partial x^5}+4\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial x^2}\right.
\end{aligned}
$$

$$
+\frac{\partial^5 w}{\partial x^5}\frac{dw_0}{dx}+4\frac{\partial^4 w}{\partial x^4}\frac{d^2 w_0}{dx^2}+6\frac{\partial^3 w}{\partial x^3}\frac{d^3 w_0}{dx^3}+4\frac{\partial^2 w}{\partial x^2}\frac{d^4 w_0}{dx^4}+\frac{\partial w}{\partial x}\frac{d^5 w_0}{dx^5}\Bigg]
$$

$$
+EA\left(e_0 a\right)^2 l_{sg}^2\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)\Bigg[\frac{\partial^6 u}{\partial x^6}+10\frac{\partial^3 w}{\partial x^3}\frac{\partial^4 w}{\partial x^4}+5\frac{\partial^2 w}{\partial x^2}\frac{\partial^5 w}{\partial x^5}+\frac{\partial w}{\partial x}\frac{\partial^6 w}{\partial x^6}
$$

$$
+\frac{\partial^6 w}{\partial x^6}\frac{dw_0}{dx}+5\frac{\partial^5 w}{\partial x^5}\frac{d^2 w_0}{dx^2}+10\frac{\partial^4 w}{\partial x^4}\frac{d^3 w_0}{dx^3}+10\frac{\partial^3 w}{\partial x^3}\frac{d^4 w_0}{dx^4}+5\frac{\partial^2 w}{\partial x^2}\frac{d^5 w_0}{dx^5}+\frac{\partial w}{\partial x}\frac{d^6 w_0}{dx^6}\Bigg]
$$

$$
+\eta A\left[\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}-\left(e_0 a\right)^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\left(\frac{dw_0}{dx}\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial^2 u}{\partial t\partial x}+\frac{\partial^2 w}{\partial t\partial x}\frac{\partial w}{\partial x}\right)
$$

$$
+\eta A\left[\frac{dw_0}{dx}+\frac{\partial w}{\partial x}-3\left(e_0 a\right)^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\Bigg(\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial t}
$$

$$
+\frac{\partial^3 w}{\partial t\partial x^2}\frac{dw_0}{dx}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^2 w_0}{dx^2}\Bigg)-\eta A\left\{\left[l_{sg}^2+3\left(e_0 a\right)^2\right]\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)-\left(e_0 a\right)^2 l_{sg}^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right\}
$$

$$
\times\left(2\frac{d^2 w_0}{dx^2}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial^4 u}{\partial t\partial x^3}+\frac{\partial^3 w}{\partial x^3}\frac{\partial^2 w}{\partial t\partial x}+2\frac{\partial^3 w}{\partial t\partial x^2}\frac{\partial^2 w}{\partial x^2}+\frac{dw_0}{dx}\frac{\partial^4 w}{\partial t\partial x^3}+\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial t\partial x^3}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^3 w_0}{dx^3}\right)
$$

$$
-\eta A\left\{\left[l_{sg}^2+\left(e_0 a\right)^2\right]\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)-3\left(e_0 a\right)^2 l_{sg}^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right\}\Bigg(\frac{\partial^5 u}{\partial t\partial x^4}+\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial t\partial x}+3\frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial t\partial x^3}
$$

$$
+3\frac{\partial^3 w}{\partial x^3}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^5 w}{\partial t\partial x^4}+\frac{\partial^5 w}{\partial t\partial x^4}\frac{dw_0}{dx}+3\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^2 w_0}{dx^2}+3\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^3 w_0}{dx^3}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^4 w_0}{dx^4}\Bigg)
$$

$$
+3\eta A\left(e_0 a\right)^2 l_{sg}^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)\Bigg(\frac{\partial^6 u}{\partial t\partial x^5}+\frac{\partial^5 w}{\partial x^5}\frac{\partial^2 w}{\partial t\partial x}+4\frac{\partial^4 w}{\partial x^4}\frac{\partial^3 w}{\partial t\partial x^2}+6\frac{\partial^3 w}{\partial x^3}\frac{\partial^4 w}{\partial t\partial x^3}+4\frac{\partial^2 w}{\partial x^2}\frac{\partial^5 w}{\partial t\partial x^4}
$$

$$
+\frac{\partial w}{\partial x}\frac{\partial^6 w}{\partial t\partial x^5}+\frac{\partial^6 w}{\partial t\partial x^5}\frac{dw_0}{dx}+4\frac{\partial^5 w}{\partial t\partial x^4}\frac{d^2 w_0}{dx^2}+6\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^3 w_0}{dx^3}+4\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^4 w_0}{dx^4}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^5 w_0}{dx^5}\Bigg)
$$

$$
+\eta A\left(e_0 a\right)^2 l_{sg}^2\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)\Bigg(\frac{\partial^7 u}{\partial t\partial x^6}+\frac{\partial^6 w}{\partial x^6}\frac{\partial^2 w}{\partial t\partial x}+5\frac{\partial^5 w}{\partial x^5}\frac{\partial^3 w}{\partial t\partial x^2}+10\frac{\partial^4 w}{\partial x^4}\frac{\partial^4 w}{\partial t\partial x^3}
$$

$$
+10\frac{\partial^3 w}{\partial x^3}\frac{\partial^5 w}{\partial t\partial x^4}+\frac{\partial^7 w}{\partial x^6\partial t}\frac{dw_0}{dx}+5\frac{\partial^6 w}{\partial t\partial x^5}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^7 w}{\partial t\partial x^6}\frac{\partial w}{\partial x}+5\frac{\partial^6 w}{\partial t\partial x^5}\frac{d^2 w_0}{dx^2}+10\frac{\partial^5 w}{\partial t\partial x^4}\frac{d^3 w_0}{dx^3}
$$

$$
+10\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^4 w_0}{dx^4}+5\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^5 w_0}{dx^5}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^6 w_0}{dx^6}\Bigg)
$$

$$+m\left(e_0a\right)^2\left[\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}-\left(e_0a\right)^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\frac{\partial^3 u}{\partial x\partial t^2}$$

$$+m\left(e_0a\right)^2\left[\frac{\partial w}{\partial x}+\frac{dw_0}{dx}-3\left(e_0a\right)^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\frac{\partial^4 u}{\partial x^2\partial t^2}$$

$$-3m\left(e_0a\right)^4\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)\frac{\partial^5 u}{\partial x^3\partial t^2}-m\left(e_0a\right)^4\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)\frac{\partial^6 u}{\partial x^4\partial t^2}$$

$$=m\frac{\partial^2 w}{\partial t^2}-m\left(e_0a\right)^2\frac{\partial^4 w}{\partial x^2\partial t^2}-F_1\cos\left(\omega t\right). \tag{18}$$

3. Solution method

In this section, a numerical solution procedure is presented for the derived coupled equations of motion given by Eqs. (17) and (18). First of all, it is better to rewrite these differential equations in the non-dimensional form via the following set of parameters

$$\begin{gathered}x^*=\frac{x}{L},\ \left\langle u^*,w^*,w_0^*\right\rangle=\frac{1}{R_g}\left\langle u,w,w_0\right\rangle,\ R_g=\sqrt{\frac{I}{A}},\ \left\langle\chi_{nl},\chi_{sg}\right\rangle=\frac{1}{L}\left\langle e_0a,l_{sg}\right\rangle,\\ \eta^*=\eta\sqrt{\frac{I}{EmL^4}},\ \ \beta=\frac{L}{R_g},\ \ F_1^*=\frac{F_1L^3}{EI},\ \ t^*=t\sqrt{\frac{EI}{mL^4}},\ \Omega=L^2\sqrt{\frac{m}{EI}}\omega,\end{gathered} \tag{19}$$

in which β represents the ratio of the CNT length (L) to its gyration radius (R_g). In view of these non-dimensional parameters, Eqs. (17) and (18) can be expressed as

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\chi_{nl}^2\frac{\partial^4 u}{\partial t^2\partial x^2}-\beta\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\frac{d^2 w_0}{dx^2}+\frac{dw_0}{dx}\frac{\partial^2 w}{\partial x^2}+\beta\frac{\partial^2 u}{\partial x^2}\right)\\
&+\beta\chi_{sg}^2\left(\beta\frac{\partial^4 u}{\partial x^4}+\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}+3\frac{\partial^3 w}{\partial x^3}\frac{\partial^2 w}{\partial x^2}\right.\\
&\left.+3\frac{d^3 w_0}{dx^3}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^4 w}{\partial x^4}\frac{dw_0}{dx}+3\frac{d^2 w_0}{dx^2}\frac{\partial^3 w}{\partial x^3}+\frac{\partial w}{\partial x}\frac{d^4 w_0}{dx^4}\right)\\
&-\beta\eta\left(\frac{dw_0}{dx}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial^2 w}{\partial t\partial x}\frac{\partial^2 w}{\partial x^2}+\beta\frac{\partial^3 u}{\partial t\partial x^2}+\frac{d^2 w_0}{dx^2}\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial t\partial x^2}\right)\\
&+\beta\eta\chi_{sg}^2\left(\beta\frac{\partial^5 u}{\partial t\partial x^4}+3\frac{\partial^3 w}{\partial x^3}\frac{\partial^3 w}{\partial x^2\partial t}+3\frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial t\partial x^3}+\frac{\partial^2 w}{\partial t\partial x}\frac{\partial^4 w}{\partial x^4}+\frac{\partial w}{\partial x}\frac{\partial^5 w}{\partial t\partial x^4}\right.\\
&\left.+3\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^2 w_0}{dx^2}+\frac{\partial^5 w}{\partial t\partial x^4}\frac{dw_0}{dx}+3\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^3 w_0}{dx^3}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^4 w_0}{dx^4}\right)=0,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}-\chi_{nl}^2\frac{\partial^4 w}{\partial x^2\partial t^2}-F_1\beta\cos(\Omega t)+\frac{\partial^4 w}{\partial x^4}-\chi_{sg}^2\frac{\partial^6 w}{\partial x^6}+\eta\left(\frac{\partial^5 w}{\partial t\partial x^4}-\chi_{sg}^2\frac{\partial^7 w}{\partial t\partial x^6}\right)\\
&-\left[\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}-\chi_{nl}^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{dw_0}{dx}\frac{\partial w}{\partial x}+\beta\frac{\partial u}{\partial x}\right]\\
&-\left[\frac{dw_0}{dx}+\frac{\partial w}{\partial x}-3\chi_{nl}^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\left(\beta\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{dw_0}{dx}+\frac{\partial w}{\partial x}\frac{d^2 w_0}{dx^2}\right)\\
&+\left[\left(\chi_{sg}^2+3\chi_{nl}^2\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)-\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\times\\
&\left[\beta\frac{\partial^3 u}{\partial x^3}+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x^3}\frac{dw_0}{dx}+2\frac{\partial^2 w}{\partial x^2}\frac{d^2 w_0}{dx^2}+\frac{\partial w}{\partial x}\frac{d^3 w_0}{dx^3}\right]\\
&+\left[\left(\chi_{sg}^2+\chi_{nl}^2\right)\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)-3\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\times\\
&\left(\beta\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^4}\frac{dw_0}{dx}+3\frac{\partial^3 w}{\partial x^3}\frac{d^2 w_0}{dx^2}+3\frac{\partial^2 w}{\partial x^2}\frac{d^3 w_0}{dx^3}+\frac{\partial w}{\partial x}\frac{d^4 w_0}{dx^4}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-3\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)\left[\beta\frac{\partial^5 u}{\partial x^5}+3\left(\frac{\partial^3 w}{\partial x^3}\right)^2+4\frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial x^4}+\frac{\partial w}{\partial x}\frac{\partial^5 w}{\partial x^5}\right.\\
&\left.+\frac{\partial^5 w}{\partial x^5}\frac{dw_0}{dx}+4\frac{\partial^4 w}{\partial x^4}\frac{d^2 w_0}{dx^2}+6\frac{\partial^3 w}{\partial x^3}\frac{d^3 w_0}{dx^3}+4\frac{\partial^2 w}{\partial x^2}\frac{d^4 w_0}{dx^4}+\frac{\partial w}{\partial x}\frac{d^5 w_0}{dx^5}\right]\\
&-\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)\left[\beta\frac{\partial^6 u}{\partial x^6}+10\frac{\partial^3 w}{\partial x^3}\frac{\partial^4 w}{\partial x^4}+5\frac{\partial^2 w}{\partial x^2}\frac{\partial^5 w}{\partial x^5}+\frac{\partial w}{\partial x}\frac{\partial^6 w}{\partial x^6}\right.\\
&\left.+\frac{\partial^6 w}{\partial x^6}\frac{dw_0}{dx}+5\frac{\partial^5 w}{\partial x^5}\frac{d^2 w_0}{dx^2}+10\frac{\partial^4 w}{\partial x^4}\frac{d^3 w_0}{dx^3}+10\frac{\partial^3 w}{\partial x^3}\frac{d^4 w_0}{dx^4}+5\frac{\partial^2 w}{\partial x^2}\frac{d^5 w_0}{dx^5}+\frac{\partial w}{\partial x}\frac{d^6 w_0}{dx^6}\right]\\
&-\eta\left[\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}-\chi_{nl}^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}+\frac{\partial^2 w}{\partial t\partial x}\frac{dw_0}{dx}+\beta\frac{\partial^2 u}{\partial t\partial x}\right)\\
&-\eta\left[\frac{dw_0}{dx}+\frac{\partial w}{\partial x}-3\chi_{nl}^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\left(\beta\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t\partial x}\right.\\
&\left.+\frac{\partial^3 w}{\partial t\partial x^2}\frac{dw_0}{dx}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^2 w_0}{dx^2}\right)+\eta\left[\left(3\chi_{nl}^2+\chi_{sg}^2\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)-\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\times\\
&\left(\frac{\partial^2 w}{\partial t\partial x}\frac{\partial^3 w}{\partial x^3}+\beta\frac{\partial^4 u}{\partial t\partial x^3}+\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial t\partial x^3}+2\frac{\partial^3 w}{\partial t\partial x^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^4 w}{\partial t\partial x^3}\frac{dw_0}{dx}+2\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^2 w_0}{dx^2}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^3 w_0}{dx^3}\right)\\
&+\eta\left[\left(\chi_{sg}^2+\chi_{nl}^2\right)\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)-3\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\left(\beta\frac{\partial^5 u}{\partial x^4\partial t}+3\frac{\partial^4 w}{\partial x^3\partial t}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial t\partial x}\right.\\
&\left.+3\frac{\partial^3 w}{\partial x^3}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^5 w}{\partial t\partial x^4}+\frac{\partial^5 w}{\partial t\partial x^4}\frac{dw_0}{dx}+3\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^2 w_0}{dx^2}+3\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^3 w_0}{dx^3}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^4 w_0}{dx^4}\right)\\
&-3\eta\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)\left(\frac{\partial^2 w}{\partial t\partial x}\frac{\partial^5 w}{\partial x^5}+4\frac{\partial^2 w}{\partial x^2}\frac{\partial^5 w}{\partial t\partial x^4}+\beta\frac{\partial^6 u}{\partial t\partial x^5}+4\frac{\partial^3 w}{\partial t\partial x^2}\frac{\partial^4 w}{\partial x^4}+6\frac{\partial^4 w}{\partial t\partial x^3}\frac{\partial^3 w}{\partial x^3}\right.\\
&\left.+\frac{\partial w}{\partial x}\frac{\partial^6 w}{\partial t\partial x^5}+\frac{\partial^6 w}{\partial t\partial x^5}\frac{dw_0}{dx}+4\frac{\partial^5 w}{\partial t\partial x^4}\frac{d^2 w_0}{dx^2}+6\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^3 w_0}{dx^3}+4\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^4 w_0}{dx^4}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^5 w_0}{dx^5}\right)\\
&-\eta\chi_{nl}^2\chi_{sg}^2\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)\left(\beta\frac{\partial^7 u}{\partial t\partial x^6}+\frac{\partial^6 w}{\partial x^6}\frac{\partial^2 w}{\partial t\partial x}+5\frac{\partial^5 w}{\partial x^5}\frac{\partial^3 w}{\partial t\partial x^2}+10\frac{\partial^4 w}{\partial x^4}\frac{\partial^4 w}{\partial t\partial x^3}\right.\\
&+10\frac{\partial^3 w}{\partial x^3}\frac{\partial^5 w}{\partial t\partial x^4}+5\frac{\partial^2 w}{\partial x^2}\frac{\partial^6 w}{\partial t\partial x^5}+\frac{\partial w}{\partial x}\frac{\partial^7 w}{\partial t\partial x^6}+\frac{\partial^7 w}{\partial t\partial x^6}\frac{dw_0}{dx}+5\frac{\partial^6 w}{\partial t\partial x^5}\frac{d^2 w_0}{dx^2}\\
&\left.+10\frac{\partial^5 w}{\partial t\partial x^4}\frac{d^3 w_0}{dx^3}+10\frac{\partial^4 w}{\partial t\partial x^3}\frac{d^4 w_0}{dx^4}+5\frac{\partial^3 w}{\partial t\partial x^2}\frac{d^5 w_0}{dx^5}+\frac{\partial^2 w}{\partial t\partial x}\frac{d^6 w_0}{dx^6}\right)
\end{aligned}
$$

$$-\frac{\chi_{nl}^2}{\beta}\left[\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}-\chi_{nl}^2\left(\frac{\partial^4 w}{\partial x^4}+\frac{d^4 w_0}{dx^4}\right)\right]\frac{\partial^3 u}{\partial x\partial t^2}$$

$$-\frac{\chi_{nl}^2}{\beta}\left[\frac{\partial w}{\partial x}+\frac{dw_0}{dx}-3\chi_{nl}^2\left(\frac{\partial^3 w}{\partial x^3}+\frac{d^3 w_0}{dx^3}\right)\right]\frac{\partial^4 u}{\partial x^2\partial t^2} \quad (21)$$

$$+3\frac{\chi_{nl}^4}{\beta}\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2 w_0}{dx^2}\right)\frac{\partial^5 u}{\partial x^3\partial t^2}+\frac{\chi_{nl}^4}{\beta}\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)\frac{\partial^6 u}{\partial x^4\partial t^2}=0.$$

In Eqs. (20) and (21), asterisk superscripts are neglected for the sake of simplification. As the second step, the non-dimensional nonlinear coupled equations are discretised employing the following expressions

$$u(x,t)=\sum_{i=1}^{N_x} r_i[\text{as a function of } t]\Phi_i[\text{as a function of } x],$$

$$w(x,t)=\sum_{i=1}^{N_z} q_i[\text{as a function of } t]\Psi_i[\text{as a function of } x]. \quad (22)$$

Here (r_i,q_i) and (Φ_i,Ψ_i) indicate the generalised coordinates and the shape functions of the imperfect viscoelastic CNT, respectively. Assuming the initial deflection as $w_0=A_0\Psi_1(x)$ and applying Eq. (22), a set of coupled discretised equations are obtained, where, a continuation-based approach is applied so as to the frequency response of the imperfect viscoelastic CNT is obtained.

4. Numerical results

A nonlinear investigation is performed in the following to examine the effect of initial deflections on the nonlinear coupled response of viscoelastic CNTs. All results are plotted for the case of a zigzag (10,0) single-walled CNT. The scale and geometrical parameters of the imperfect viscoelastic nanosystem are as (χ_{nl} =0.1, χ_{sg}=0.05) and (L=20, h=0.34, d=0.7829) nm, respectively.

Here the thickness and the average diameter are, respectively, shown by h and d. For the described geometry, the slenderness ratio is as β= 66.2751. The material features of the imperfect viscoelastic zigzag CNT are considered as E=1.0 TPa, ν=0.19, η=0.00045 and ρ=2300 kg/m^3 for all the cases.

Plotted in Fig. 2 is the size-dependent frequency-amplitude responses of the initially imperfect viscoelastic CNT for χ_{nl} =0.1, χ_{sg}=0.05, F_1=0.35, A_0=0.7, and η=0.00045. The coupled resonance behaviour of this nanoscale system is of hardening nonlinearity; two saddle nodes at Ω/ω_1=1.1554 and Ω/ω_1=1.0292 are found. The natural frequency of the initially imperfect viscoelastic CNT is ω_1= 23.4998. It is worth pointing out that between the two saddle nodes, the nonlinear response is unstable while it is stable in other regions.

The frequency-amplitude responses of the initially imperfect viscoelastic CNT obtained via the nonlocal strain gradient and classical continuum theories are indicated in Fig. 3. The dimensional parameters of the imperfect viscoelastic nanotube are set to F_1=0.35, A_0=0.7 and η=0.00045. Using the classical continuum theory causes overestimated results for the motion amplitudes in both directions (i.e. the axial and transverse ones). In addition, the resonance frequency of the modified nonlocal theory is slightly lower than the frequency estimated by the classical continuum theory.

Shown in Fig.4 is the force-amplitude responses of the initially imperfect viscoelastic nanotube obtained via the nonlocal strain gradient and classical continuum theories for Ω=25.0, A_0=0.7, and η=0.00045. The size parameters for the nonlocal strain gradient and classical continuum theories are taken as (χ_{nl} =0.1, χ_{sg}=0.05) and (χ_{nl} =0, χ_{sg}=0), respectively. Applying the classical continuum theory generally yields higher values of q_1 and r_2. Moreover, ignoring the influence of size parameters causes significantly underestimated results for the value of F_1 related to the saddle node.

Figure 5 represents the variation of the resonance forcing amplitude versus the resonance frequency for initially imperfect viscoelastic CNTs for two damping mechanisms. For the linear damping, it is assumed that ζ=0.006 where ζ denotes the modal damping ratio. Moreover, a value of η=0.00045 is assumed for the nonlinear damping in this figure. For relatively small values of F_1, there is not an important difference between the results of the two mechanisms. By contrast, for high values of F_1, ignoring nonlinear damping effects causes overestimated results for the resonance frequencies.

Plotted in Fig. 6 is the size-dependent frequency-amplitude responses of the initially imperfect viscoelastic CNT for a higher imperfection amplitude (A_0=1.4). Other CNT parameters are set to χ_{nl} =0.1, χ_{sg}=0.05, F_1=0.80, and η=0.00045. This time the coupled resonance behaviour of the imperfect viscoelastic zigzag CNT is significantly changed. Four saddle nodes at Ω/ω_1=0.9419, 0.9142, 1.0566 and 0.9413 are found for the softening-hardening behaviour. In this case, the natural frequency of the initially imperfect viscoelastic zigzag CNT is as ω_1=28.7136. Figure 7 also represents the frequency-amplitude responses of the initially imperfect viscoelastic nanosystem obtained via the nonlocal strain gradient (χ_{nl} =0.1, χ_{sg}=0.05) and classical continuum (χ_{nl} =0, χ_{sg}=0) theories for F_1=0.80, A_0=1.4, and η=0.00045. It is found that the classical continuum theory leads to overestimated results for the motion amplitudes of imperfect viscoelastic zigzag CNTs in both directions.

Figure 8 indicates the force-amplitude responses of the initially imperfect viscoelastic nanotube obtained via the nonlocal strain gradient and classical continuum theories; this time a larger imperfection amplitude is chosen A_0=1.4. The excitation frequency and the viscoelastic coefficient are, respectively, set to Ω=27.5, and η=0.00045. Ignoring the size effect generally yields higher values of q_1 and r_2. Plotted in Fig. 5 is the variation of the resonance forcing amplitude versus the resonance frequency for initially imperfect viscoelastic CNTs for two damping mechanisms. For the linear damping, it is assumed that ζ=0.0072 while a value of η=0.00045 is assumed for the nonlinear damping. For small values of F_1, no important difference between the results of the two mechanisms is found. Nonetheless, for relatively high values of F_1, nonlinear damping effects become important. Ignoring them causes highly overestimated results for the resonance frequency.

5. Concluding remarks

A nonlocal coupled nonlinear beam model was proposed in this paper in order to extract the mechanical response of initially imperfect viscoelastic CNTs. The effect of viscoelasticity was modelled using a viscoelastic model. Moreover, the influence of being geometrically imperfect was captured by considering an initial deflection along the transverse direction. The coupled nonlinear equations of the initially imperfect viscoelastic CNT were derived and solved by applying a work/energy law and a Galerkin procedure.

It was found that the coupled resonance behaviour of viscoelastic CNTs is of hardening nonlinearity with two saddle nodes when a relatively small imperfection is imposed. In addition, using the classical continuum theory causes overestimated amplitudes of motion along both directions. The resonance frequency of the coupled nonlocal model is lower than the frequency estimated by the classical model. For relatively small forcing amplitudes, there is not an important difference between the results of the linear and nonlinear damping mechanisms. By contrast, for high values of this parameter, ignoring nonlinear damping causes overestimated resonance frequencies. It was also seen that a change in the initial deflection can alter the number of the saddle nodes. Four saddle nodes are found for CNTs when a large enough initial deflection is imposed.

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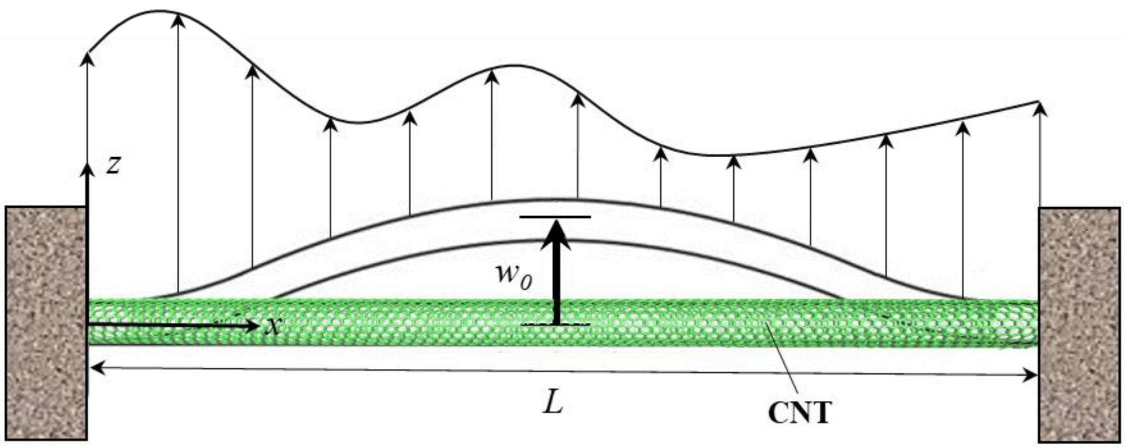
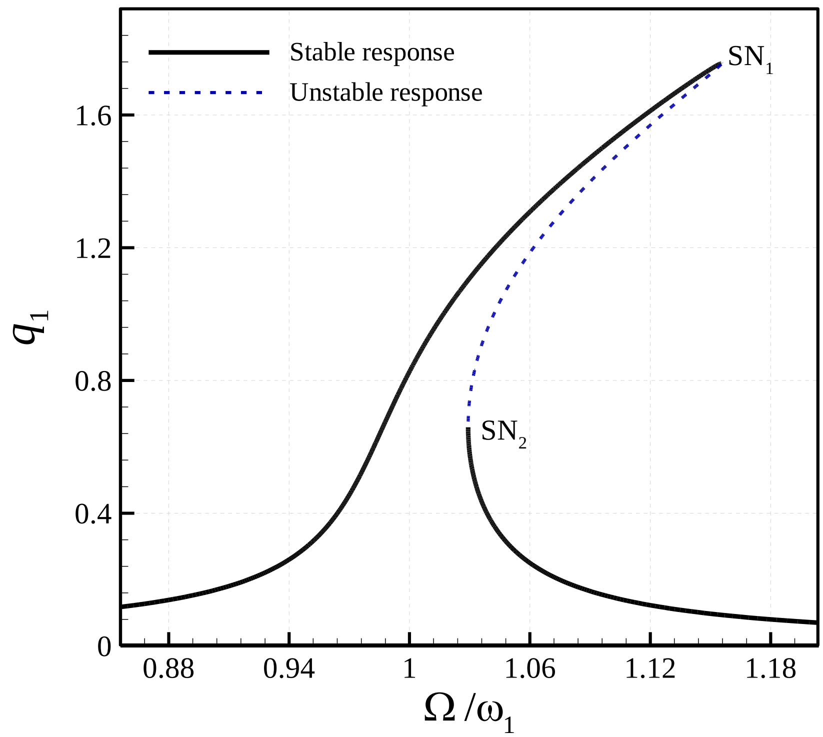


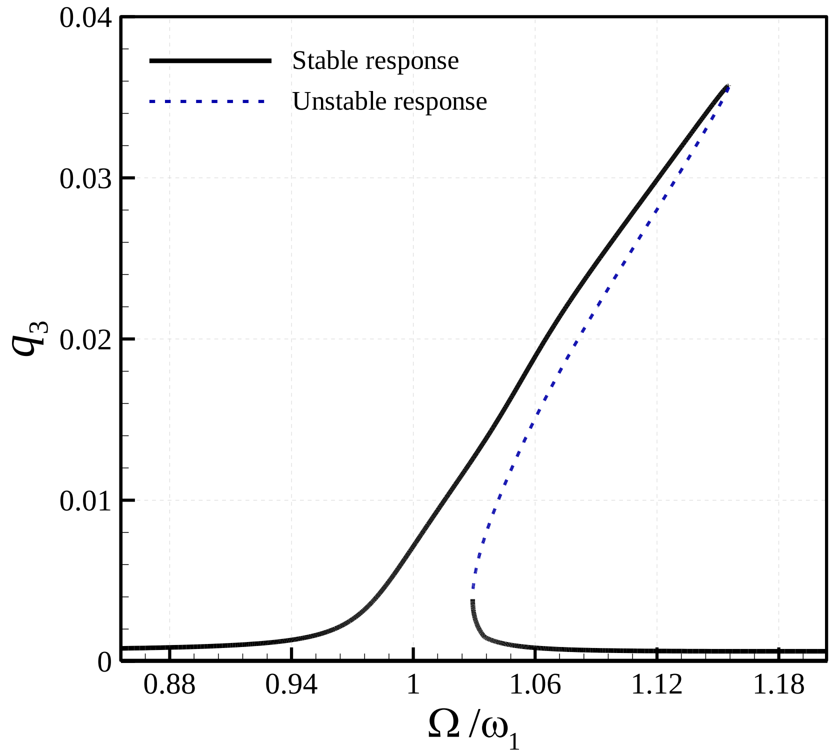
Fig. 1. An initially imperfect viscoelastic CNT with clamped-clamped edges.

(a)



Stable response
Unstable response
SN_1
SN_2
1.6
1.2
0.8
0.4
0
q_1
0.88
0.94
1
1.06
1.12
1.18
Ω / ω_1

(b)



Stable response
Unstable response
0.04
0.03
0.02
0.01
0
q_3
0.88
0.94
1
1.06
1.12
1.18
Ω / ω_1

(c)

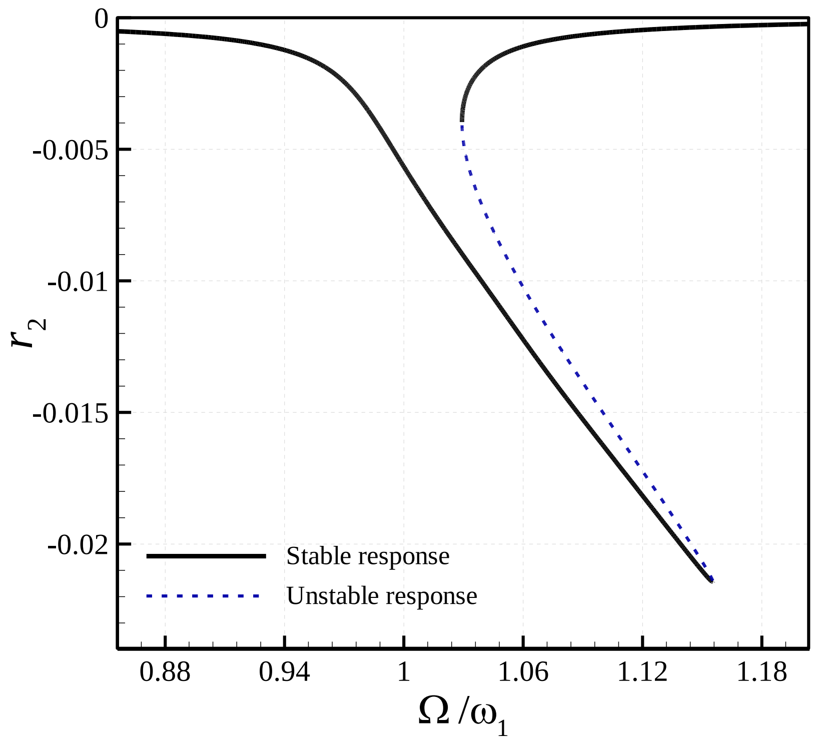
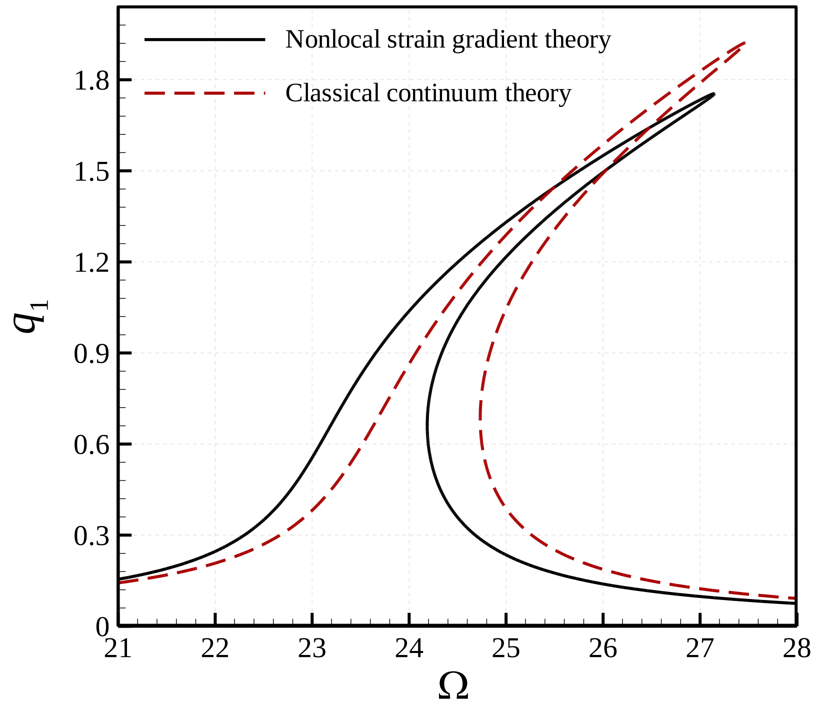


Fig. 2. Frequency-amplitude response of the initially imperfect viscoelastic CNT; (a, b) the maximum of q_1 and q_3, respectively; (c) the minimum of r_2; A_0=0.7.

(a)



(b)

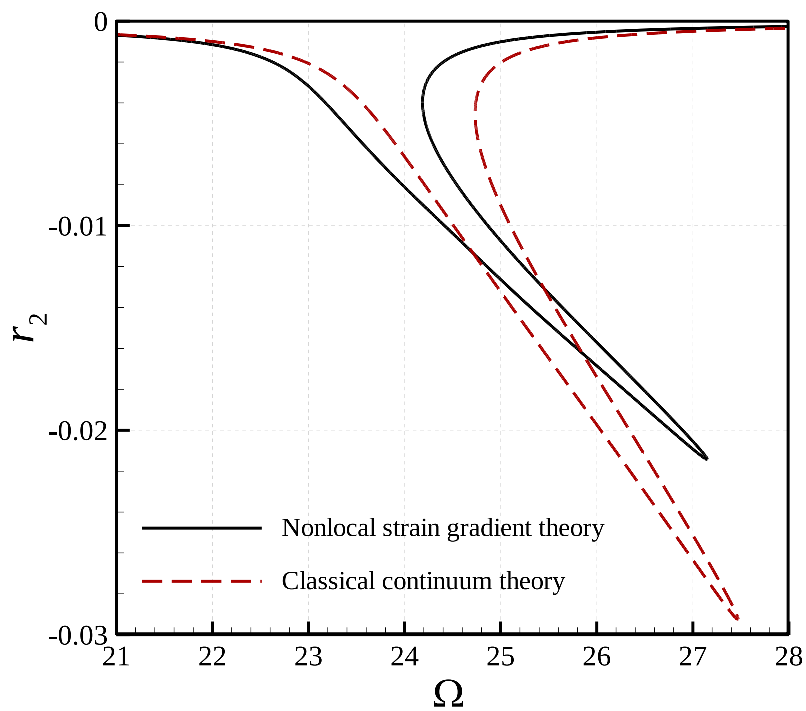
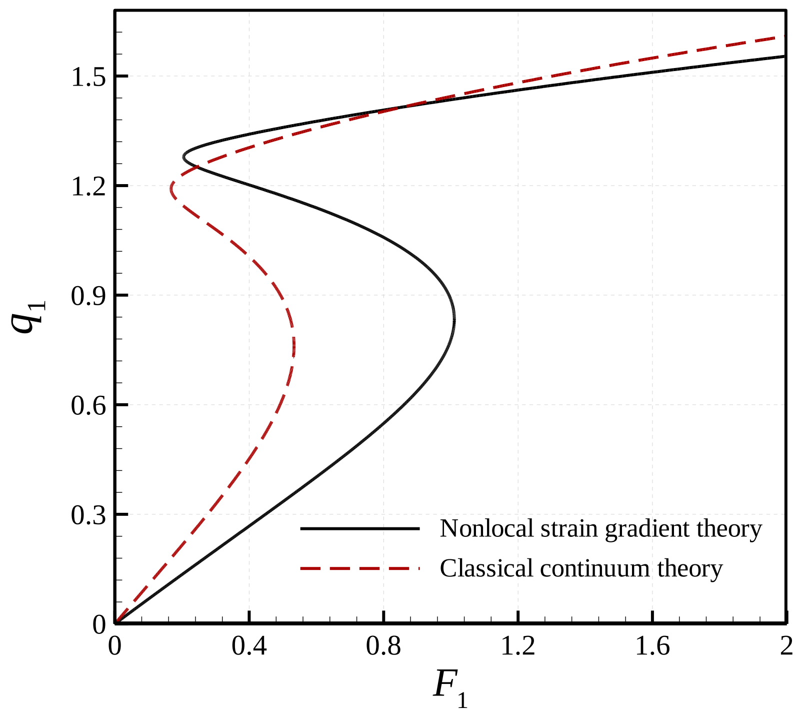


Fig. 3. Comparison of frequency-amplitude responses of the initially imperfect viscoelastic CNT obtained via the nonlocal strain gradient (χ_{nl} =0.1, χ_{sg}=0.05) and classical continuum (χ_{nl} =0, χ_{sg}=0) theories; (a) the maximum of q_1; (b) the minimum of r_2; A_0=0.7.

(a)



(b)

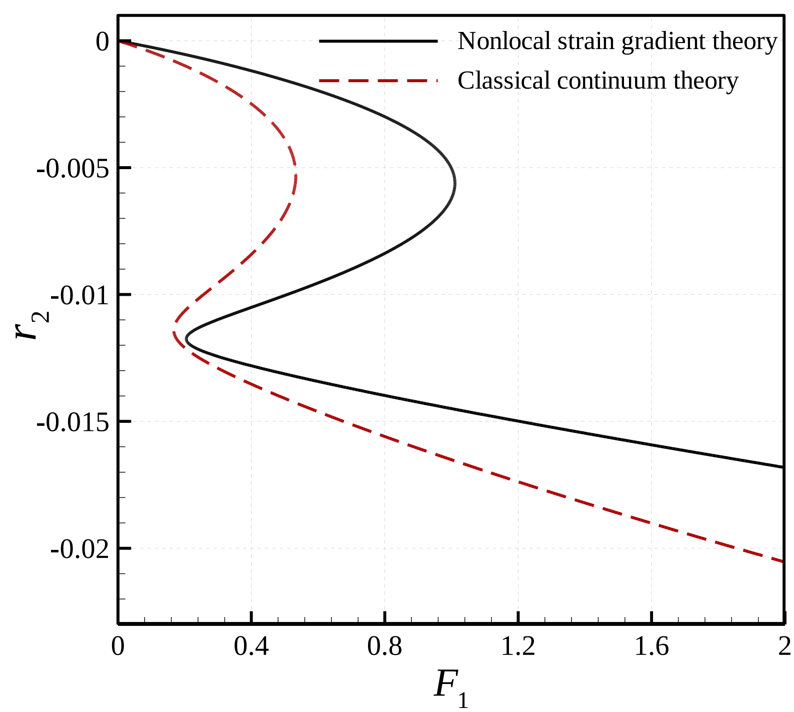


Fig. 4. Comparison of force-amplitude responses of the initially imperfect viscoelastic CNT obtained via the nonlocal strain gradient (χ_{nl} =0.1, χ_{sg}=0.05) and classical continuum (χ_{nl} =0, χ_{sg}=0) theories; (a) the maximum of q_1; (b) the minimum of r_2; A_0=0.7.

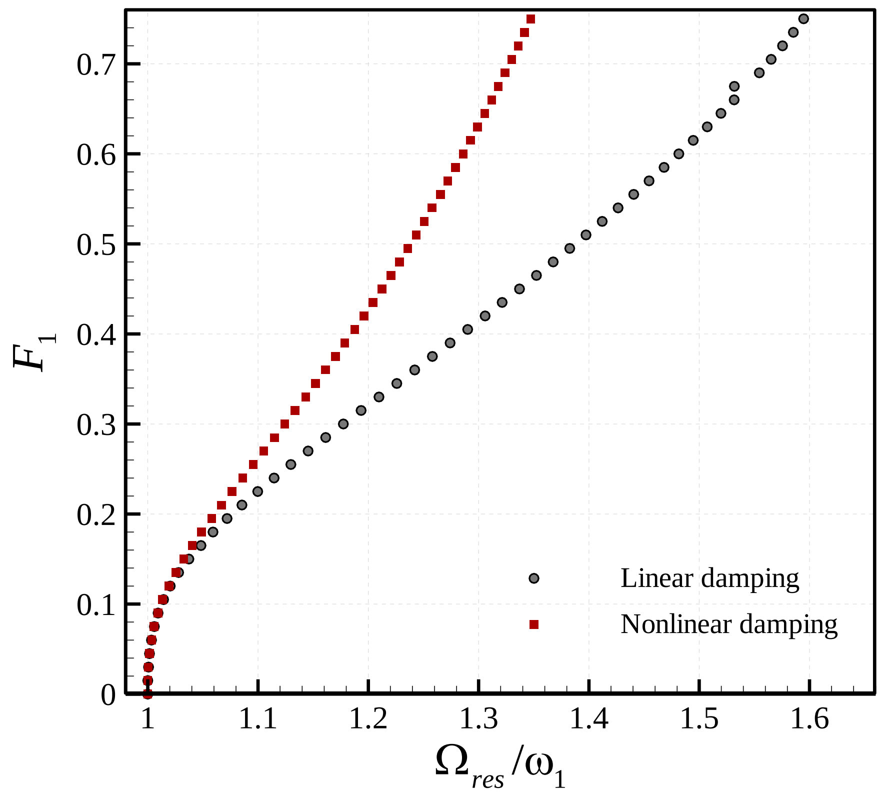
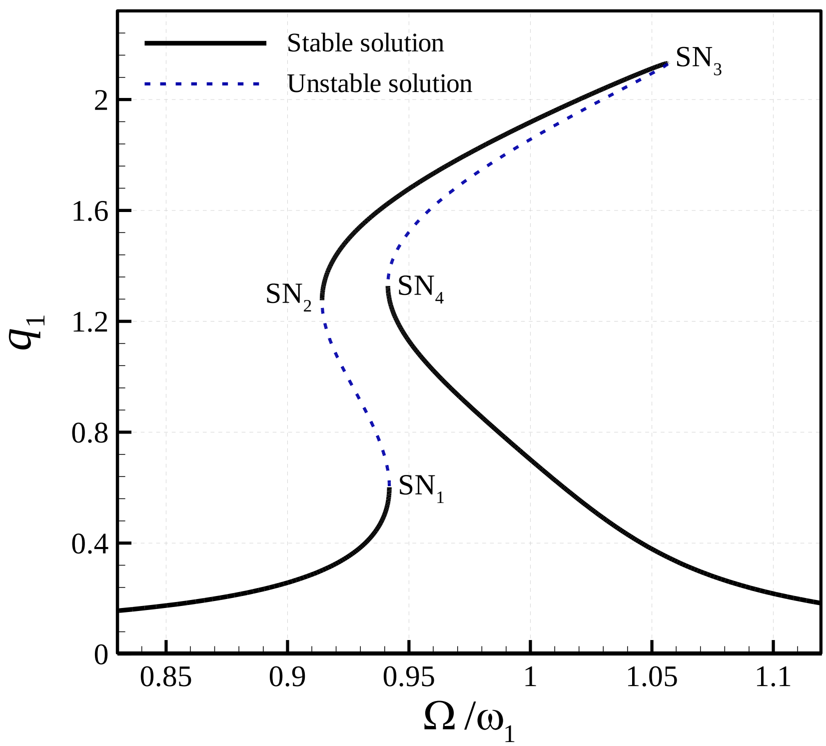
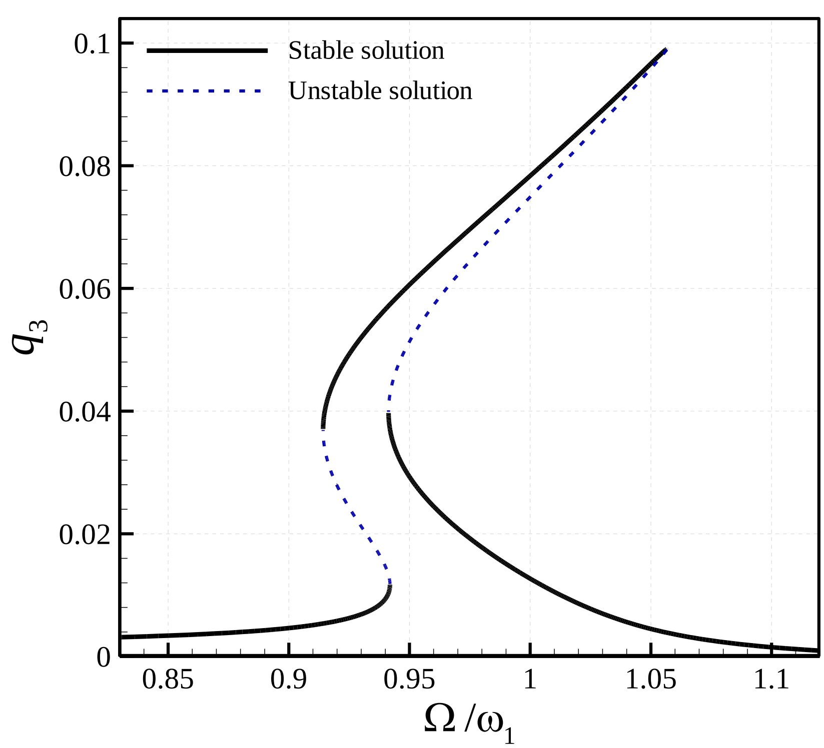


Fig. 5. Resonance forcing amplitude versus the resonance frequency for two damping mechanisms; a circle denotes the linear damping mechanism (ζ=0.006) while a square denotes the nonlinear one (η=0.00045); A_0=0.7.

(a)


Stable solution
Unstable solution
SN$_3$
SN$_2$
SN$_4$
SN$_1$
q_1
0
0.4
0.8
1.2
1.6
2
0.85
0.9
0.95
1
1.05
1.1
Ω/ω_1

(b)


Stable solution
Unstable solution
q_3
0
0.02
0.04
0.06
0.08
0.1
0.85
0.9
0.95
1
1.05
1.1
Ω/ω_1

(c)

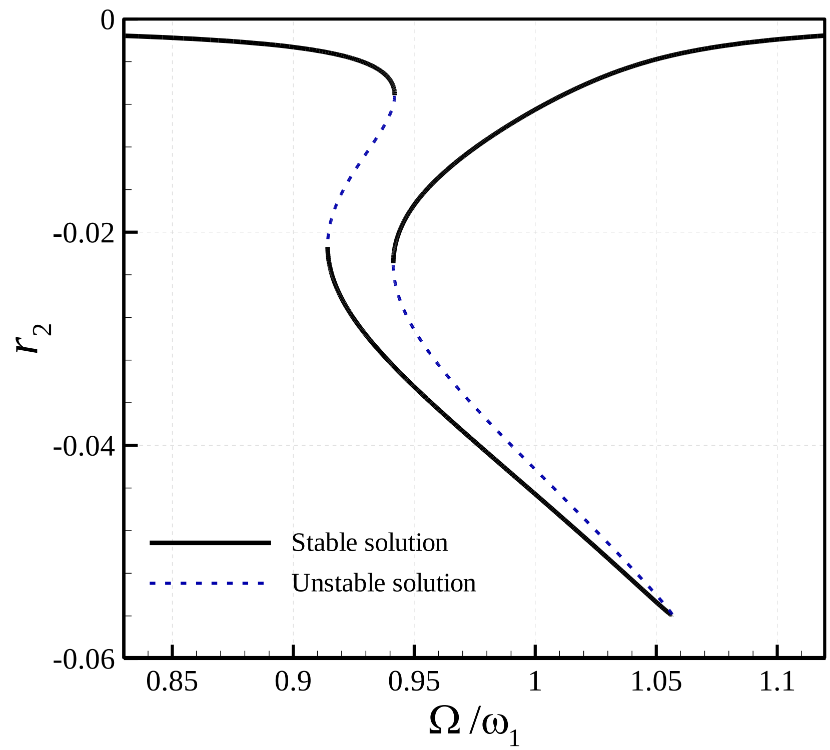
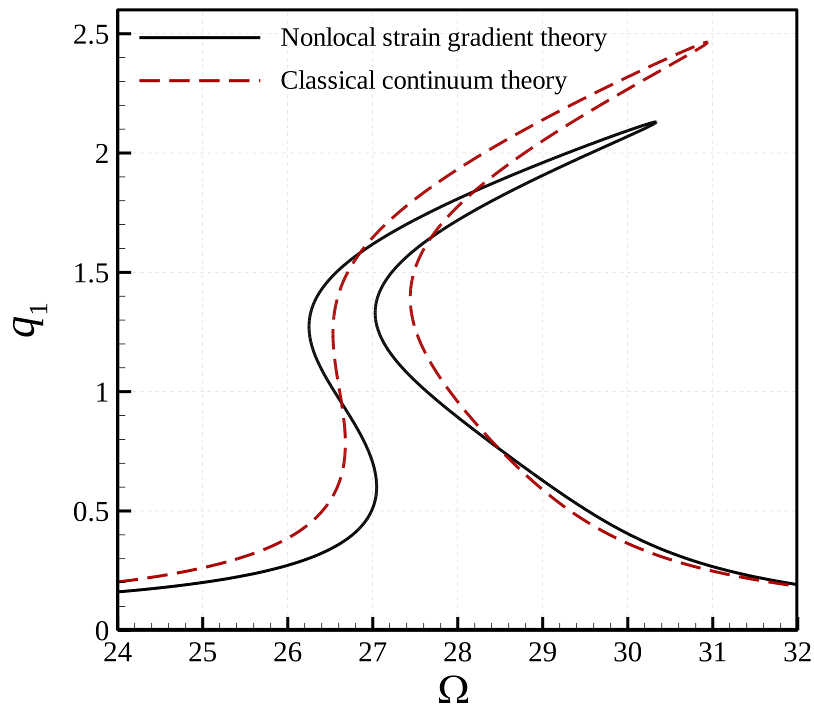


Fig. 6. Frequency-amplitude diagrams of the initially imperfect viscoelastic nanotube; (a, b) the maximum of q_1 and q_3, respectively; (c) the minimum of r_2; A_0=1.4.

(a)



(b)

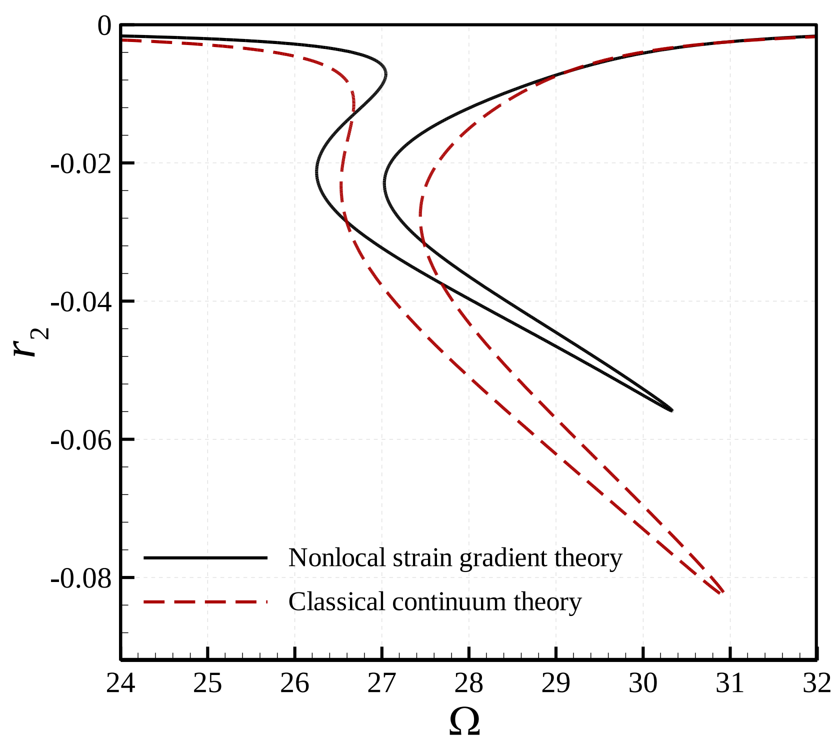
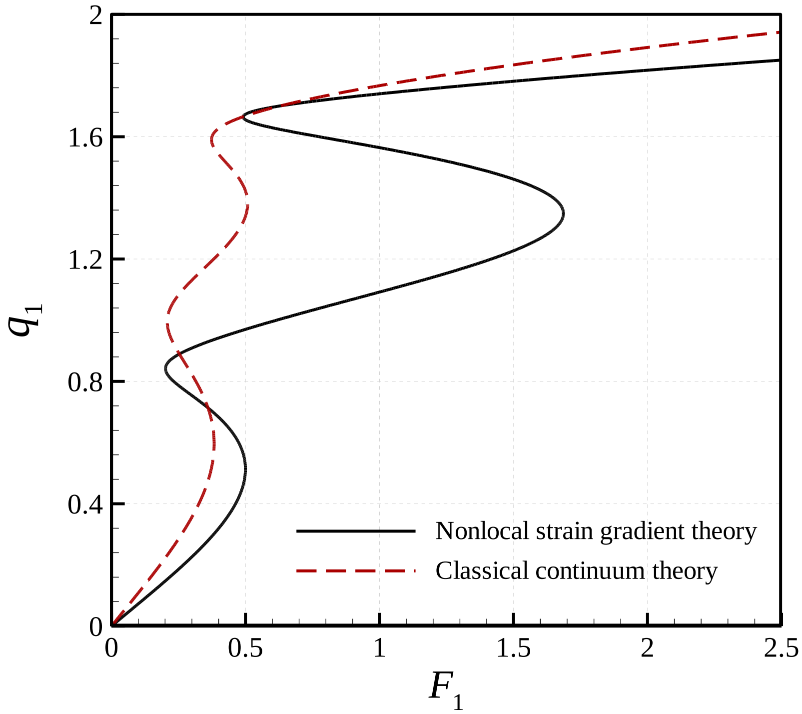


Fig. 7. Comparison of frequency-amplitude responses of the initially imperfect viscoelastic CNT obtained via the nonlocal strain gradient and classical continuum theories; (a) the maximum of q_1; (b) the minimum of r_2; A_0=1.4.

(a)



(b)

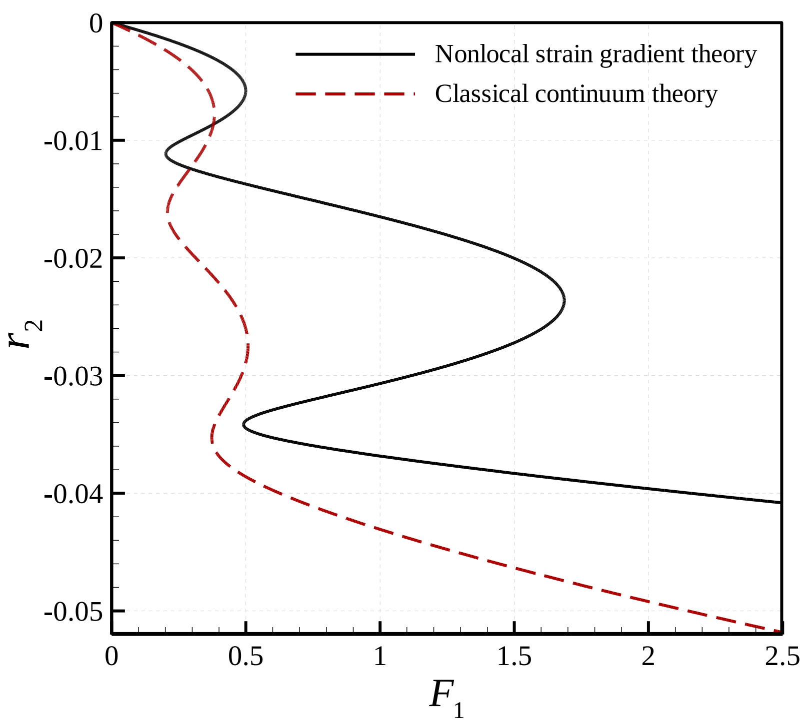


Fig. 8. Comparison of force-amplitude responses of the initially imperfect viscoelastic CNT obtained via the nonlocal strain gradient (χ_{nl} =0.1, χ_{sg}=0.05) and classical continuum (χ_{nl} =0, χ_{sg}=0) theories; (a) the maximum of q_1; (b) the minimum of r_2; A_0=1.4.

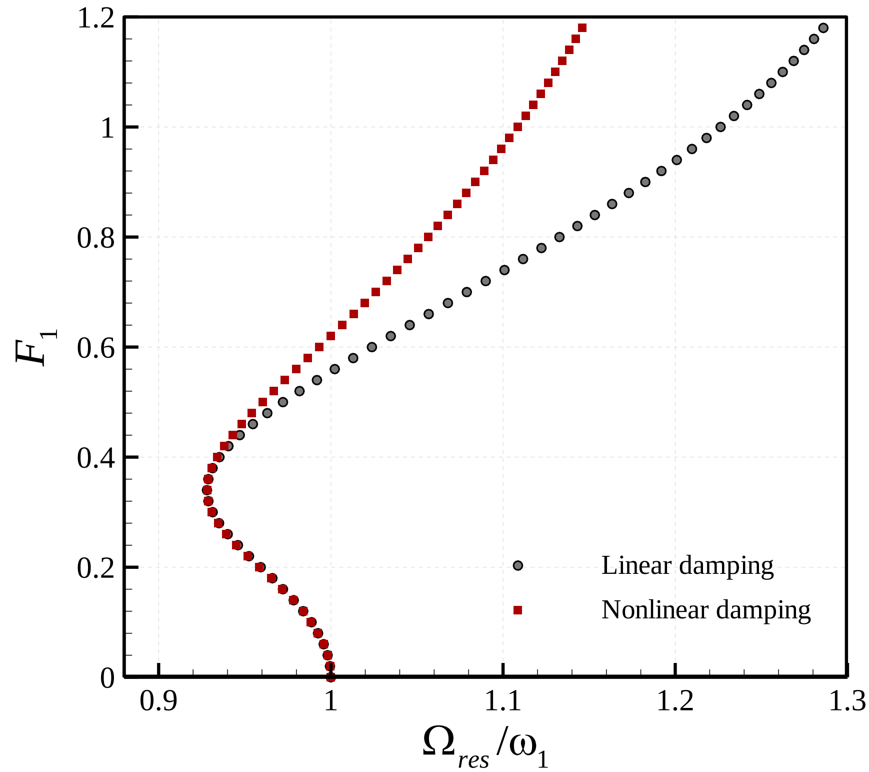


Fig. 9. Resonance forcing amplitude versus the resonance frequency for two damping mechanisms; a circle denotes the linear damping mechanism (ζ=0.006) while a square denotes the nonlinear one (η=0.00045); A_0=1.4.