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Geometrically exact extreme vibrations of cantilevers

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Abstract

This paper examines the extremely large nonlinear vibrations of a cantilever subject to base excitation in primary and secondary resonance regions for the first time. To predict extremely large vibration amplitudes accurately, a geometrically exact continuous model of the cantilever is developed for the centreline rotation of the cantilever; the proposed model's accuracy is verified for extremely large deformations through comparison to a nonlinear finite element model. The theory of Euler-Bernoulli, along with inextensibility assumption, and the Kelvin-Voigt material damping model are utilised to develop the geometrically exact model. The main feature of the geometrically exact model is that all nonlinear trigonometric terms in the model are kept intact before and after the discretisation process, which itself is performed utilising the Galerkin scheme. The numerical results show that the cantilever undergoes extremely large oscillations even at relatively small base excitation amplitudes. It is shown that for some cases the amplitude of the tip of the cantilever grows so large that it “bends backward”; a behaviour which can only be captured using the proposed geometrically exact model.

Keywords: Extreme vibrations; Cantilever; Primary resonance; Secondary resonance; Kelvin-Voigt; Geometrically exact model

1. Introduction

Beams with one end fixed and the other end free, also known as cantilevers, are present in different engineering applications and systems [1-7]. Such systems are usually subject to external base excitations in a wide range of frequencies. Examples of applications of cantilevers under excitations are: mass and bio sensors, scanning probe microscopy [8], micro/nano-electromechanical systems, and vibration-based energy harvesters [9-12]. Having one end free to oscillate, the cantilevers could undergo very large-amplitude oscillations even at relatively small excitation amplitudes or when subject to reverse axial flow. Additionally, as a result of one end of the cantilever being free, very small or zero strain is developed in the mid-plane of the beam; this feature, also known as the centreline inextensibility, relates the axial and transverse displacements and lead to inertial-type nonlinearities. Other nonlinearities such geometric and damping ones are also present in vibrating cantilevers. Hence, the nonlinear vibration analysis of a cantilever is a daunting task due to presence of all these nonlinearities as well as the capability of cantilevers to undergo extremely large oscillations. A summarised literature review is conducted in the following and then the main limitations of the models commonly used in the literature are highlighted.

The static and dynamic characteristics of cantilevers have been studied by many researchers over the last few decades. For instance, Crespo da Silva and Glynn [13, 14] derived the inextensional equations of motion of beams while accounting for nonlinearities of geometric and inertial type, and analysed the system response employing the multiple scales technique. They conducted further analysis [15] by examining the nonlinear resonance coupling in vibrating modes of a cantilever with asymmetries in the support. The

studies were continued by Nayfeh and Pai [16, 17], who examined the nonlinear out-of-plane oscillations of a cantilever subject to transverse base excitation; they derived the cubic nonlinear equation of motion using the inextensibility condition and solved it through use of the Galerkin technique along with multiple scales method. Hsieh et al. [18] derived the nonlinear normal vibration modes for finite amplitude response of a cantilever utilising an invariant manifold approach while accounting for nonlinearities of geometric and inertial types; they concluded that the linear cantilever mode shapes work very well for representing the nonlinear mode shapes. Further studies on this top were performed by Feng and Leal [19], who analysed the symmetries in inextensible cantilever beam equations, and utilised them to integrate the differential equations in two limiting cases. Anderson et al. [20] examined the vibration of a parametrically excited cantilever both experimentally and theoretically; they verified experimentally the significance of the nonlinear curvature and nonlinear inertia terms. Soon after, Oh and Nayfeh [21] conducted an experiment on the presence of combination resonances in composite cantilever plates subject to transverse excitation of harmonic type. Further investigations were conducted by Hamdan and Dado [22], who examined the large amplitude transverse free oscillations of a cantilever carrying a point mass; they neglected the effects of shear deformation and rotary inertia and utilised the inextensibility assumption taking into account the longitudinal inertia and nonlinear curvature. Nonlinear nonplanar dynamic response of a cantilever under parametric excitation was examined by Arafat et al. [23].

Alhazza et al. [24] conducted an investigation into the influence of feedback time delays on the nonlinear control of parametrically excited cantilevers; they employed the multiple scales technique to obtain the modulation equations governing the beam dynamics

of the beam and utilized them to study the time delay effects on the stability and vibration characteristics of the system. Further studies on nonlinear oscillations of cantilevers were performed by Herişanu and Marinca [25] who developed an explicit analytical approximation of nonlinear transverse free vibrations of a cantilever beam carrying a point mass utilising the inextensibility assumption. Phan et al. [26] studied the finite amplitude oscillations of a cantilever in a viscous fluid subject to base excitation of harmonic type; they modelled the interactions between fluid and structure using a hydrodynamic function that takes into account the damping effects as well as the added mass of the fluid in the presence of moderately large vibration amplitudes. Farokhi and Ghayesh [27] examined the effect of a motion constraint on large-amplitude oscillations of a cantilever; they showed that the resonance bandwidth is widened as a result of the presence of the motion constraint.

Furthermore, many investigations have been conducted on piezoelectrically actuated cantilevers and their application in mass sensing and vibration-based energy harvesting [9-12, 28-30]. Piezoelectrically actuated cantilevers were examined, for instance, by Mahmoodi and Jalili [9], who analysed the nonlinear vibration response of piezoelectrically actuated microcantilevers using an inextensible beam model; Ghazavi et al. [28], who conducted a stability analysis on a piezoelectrically actuated microcantilever under parametric excitation. Kumar et al. [10] proposed a saddle-node bifurcation-based mass sensing approach through use of piezoelectrically actuated microcantilevers. The applications of cantilevers in energy harvesting have been investigated, for instance, by Liu et al. [12], who proposed a piezoelectric energy harvesting cantilever system with a stopper in an effort to widen the operation frequency band; Leadenham and Erturk [30], who

examined the nonlinear dynamic characteristics of a piezoelectric-cantilever system in energy harvesting, sensing, and actuation modes; Staaf et al. [5], who demonstrated the importance of asymmetry in conjoined cantilevers system through use of a piezoelectric self-tuning energy harvester.

When examining the dynamical characteristics of cantilevers, all the valuable studies in the literature utilise the transverse and longitudinal equations of motion or just the former using an inextensibility assumption. The main limitation of the equation of motion for the transverse displacement of the beam, when discretised using a modal decomposition technique, is that it could capture the response of the beam only when the tip angle varies between $\pm 90^\circ$. To better explain this, the centreline rotation for the case of an extensible (θ_e) and inextensible (θ_i) beam can be formulated as:

$$\begin{aligned}\theta_e &= \sin^{-1} \left(\frac{\partial w / \partial x}{1 + e_0} \right), \\ \theta_i &= \sin^{-1} (\partial w / \partial x),\end{aligned}\tag{1}$$

where e_0 stands for the centreline strain for the case of extensible beam. Both of these equations are then expanded using Taylor's series expansion method while assuming centreline angle variations between $\pm 90^\circ$. If such a beam model is solved using modal decomposition techniques, then the model is incapable of predicting the system response when the tip angle grows larger than 90° . When subject to base excitations or when subject to reverse axial flow, the 90° tip angle is reached quite easily for cantilevers. Additionally, even when the tip angle approaches the 90° limit, the commonly used third-order nonlinear model becomes inaccurate. To overcome this limitation, the beam equation of motion must be derived for the centreline rotation while keeping all terms exact. The geometrically exact

model developed in this study can be used not only for extremely large oscillations, but also for moderately large oscillations to ensure very accurate results.

The present study examines the nonlinear primary and secondary resonance responses of a cantilever undergoing extremely large vibrations for the first time. Compared to Ref. [27], this study focuses on detailed nonlinear dynamic analysis of an unconstrained cantilever in both primary and secondary resonance regions considering the effect of an added tip mass. A model is proposed through obtaining the motion of the cantilever in terms of the rotation of the centreline. To ensure accuracy, all nonlinear terms in the continuous model are kept exact before and after the discretisation process. To prove the accuracy of the developed exact model, a finite element model is developed using Abaqus/CAE using a nonlinear three-dimensional analysis. A comparison is made between the two models for extremely large static deformation of a cantilever and excellent agreement between the two models is achieved. The nonlinear vibration of the cantilever is examined in primary and secondary resonance regions while examining the effect of an added tip mass.

2. Development of a continuous model exact with respect to the geometry

A continuous model of a vibrating cantilever is developed in this section while keeping all terms geometrically exact. A cantilever under transverse base excitation is considered as shown in Fig. 1, with an added mass at the tip. In particular, the cantilever is subject to a base excitation of $z_0 \sin(\omega t)$ in Z direction with ω and z_0 denoting the base excitation frequency and amplitude, respectively. Two coordinate systems are shown in Fig. 1; XZ is

the undeformed inertial coordinate while xz is the curvilinear coordinate system. In this study, the cantilever dimensions, i.e. thickness, width, and length, are shown by h , b , and L , respectively.

The exact continuous model is derived making use beam theory of Euler-Bernoulli in conjunction with the inextensibility condition. The length of an element the beam in the two coordinates is shown by dX and dx , respectively, as shown in Fig. 1. Under the assumption of the centreline inextensibility, the length of the element remains unchanged after deformation.

In this section, the equation of motion of the cantilever is derived for the centreline rotation. As will be shown later, the centreline rotation equation of motion allows for examining extremely large oscillations while the transverse motion equation is capable of capturing oscillations only when centreline angle magnitude is much smaller than 90° . As a first step in constructing the centreline rotation motion equation, the strain developed in the cantilever in the axial direction based on the inextensibility assumption is formulated as

$$\varepsilon(t, x, z) = -z \left[\frac{\partial}{\partial x} \theta(t, x) \right], \quad (2)$$

in which t represents time and θ denotes the centreline rotation angle.

Using Eq. (2) and the Kelvin-Voigt internal dissipation model, the stress developed in the axial direction of the cantilever is obtained as

$$\sigma(t, x, z) = -Ez \frac{\partial}{\partial x} \theta(t, x) - \alpha z \frac{\partial^2}{\partial t \partial x} \theta(t, x), \quad (3)$$

in which E stands for the material Young's modulus while α denotes its viscosity. Using the elastic part of the axial stress, the variation of potential strain energy developed in the cantilever is formulated as

$$\delta II = \int_0^L EI \left(\frac{\partial \theta(t, x)}{\partial x} \right) \frac{\partial}{\partial x} (\delta \theta(t, x)) dx, \quad (4)$$

in which δ stands for the variational operator and I denotes the second moment of area of the cross-section.

The viscous part of the axial stress is the source of internal energy dissipation in the system whose dissipative virtual work is obtained as

$$\delta D_{kv} = - \int_0^L \alpha I \left(\frac{\partial^2 \theta(t, x)}{\partial x \partial t} \right) \frac{\partial}{\partial x} (\delta \theta(t, x)) dx. \quad (5)$$

Utilising the inextensibility assumption, the axial displacement, $u(t, x)$, and the transverse displacement relative to the base, $w(t, x)$, are expressed as functions of the centreline rotation angle $\theta(t, x)$ as

$$u(t, x) = - \int_0^x (1 - \cos[\theta(t, \zeta)]) d\zeta, \quad (6)$$

$$w(t, x) = \int_0^x \sin[\theta(t, \zeta)] d\zeta. \quad (7)$$

The kinetic energy of the cantilever of density ρ with a tip mass M_t , taking into account the base motion, can be obtained as

$$\begin{aligned}
KE = & \frac{1}{2} \int_0^L \left((M_t \delta_{dr}(x-L) + \rho A) \left[\int_0^x \frac{\partial \theta(t, \zeta)}{\partial t} \sin(\theta(t, \zeta)) d\zeta \right]^2 \right) dx + \frac{1}{2} \int_0^L \rho I \left(\frac{\partial \theta(t, x)}{\partial t} \right)^2 dx \\
& + \frac{1}{2} \int_0^L \left((M_t \delta_{dr}(x-L) + \rho A) \left[\int_0^x \frac{\partial \theta(t, \zeta)}{\partial t} \cos(\theta(t, \zeta)) d\zeta + z_0 \omega \cos(\omega t) \right]^2 \right) dx,
\end{aligned} \tag{8}$$

where A denotes the cross-sectional area and δ_{dr} represents the Dirac delta function.

Through use of the generalised Hamilton's principle, the exact rotational motion equation can be derived as

$$\begin{aligned}
& -EI \frac{\partial^2 \theta}{\partial x^2} - \alpha I \frac{\partial^3 \theta}{\partial x^2 \partial t} + \rho I \frac{\partial^2 \theta}{\partial t^2} \\
& + \cos \theta \int_L^x \left((M_t \delta_{dr}(x-L) + \rho A) \left\{ \int_0^x \left[\left(\frac{\partial \theta}{\partial t} \right)^2 \sin \theta - \frac{\partial^2 \theta}{\partial t^2} \cos \theta \right] dx + z_0 \omega^2 \sin(\omega t) \right\} \right) dx \\
& - \sin \theta \int_L^x \left((M_t \delta_{dr}(x-L) + \rho A) \int_0^x \left[\left(\frac{\partial \theta}{\partial t} \right)^2 \cos \theta + \frac{\partial^2 \theta}{\partial t^2} \sin \theta \right] dx \right) dx = 0,
\end{aligned} \tag{9}$$

To transform Eq. (9) into nondimensional form, the following nondimensional quantities are defined

$$\begin{aligned}
x^* &= \frac{x}{L}, & m_R &= \frac{M_t}{\rho A L}, & z_b &= \frac{z_0}{L}, & \Omega_b &= \omega T, \\
t^* &= \frac{t}{T}, & \xi &= \frac{A L^2}{I}, & \alpha_d &= \frac{\alpha}{E T},
\end{aligned} \tag{10}$$

in which $T = L^2 \sqrt{\rho A / (EI)}$. Substitution of these quantities into Eq. (9) and disregarding the asterisk notation for convenience gives

$$\begin{aligned}
& -\frac{\partial^2 \theta}{\partial x^2} - \alpha_d \frac{\partial^3 \theta}{\partial x^2 \partial t} + \frac{1}{\xi} \frac{\partial^2 \theta}{\partial t^2} \\
& + \cos \theta \int_1^x \left(m_R \delta_{dr} (x-1) + 1 \right) \left\{ \int_0^x \left[\left(\frac{\partial \theta}{\partial t} \right)^2 \sin \theta - \frac{\partial^2 \theta}{\partial t^2} \cos \theta \right] dx + z_b \Omega_b^2 \sin(\Omega_b t) \right\} dx \\
& - \sin \theta \int_1^x \left(m_R \delta_{dr} (x-1) + 1 \right) \int_0^x \left[\left(\frac{\partial \theta}{\partial t} \right)^2 \cos \theta + \frac{\partial^2 \theta}{\partial t^2} \sin \theta \right] dx dx = 0.
\end{aligned} \tag{11}$$

Equation (11) represents the geometrically exact model of the cantilever capable of predicting extremely large vibration responses. To numerically solve this equation, it should first be discretised to remove the dependency on spatial domain. To this end, the Galerkin modal decomposition scheme is employed in which the rotational motion is written as a finite series expansion of time-dependent coordinates $r_k(t)$ multiplied by spatial trial

functions $\Gamma_k(x)$, i.e. $\theta = \sum_{k=1}^M r_k(t) \Gamma_k(x)$. In this study, the following hyperbolic trial function is

used

$$\Gamma_k(x) = \left(\sin(\lambda_k x) + \beta_k \cos(\lambda_k x) \right) + \left(\sinh(\lambda_k x) - \beta_k \cosh(\lambda_k x) \right), \tag{12}$$

$$\beta_k = \left(\sinh(\lambda_k) + \sin(\lambda_k) \right)^{-1} \left(\cosh(\lambda_k) + \cos(\lambda_k) \right),$$

where λ_k is the k th root of the equation $\cos(\lambda) \cosh(\lambda) + 1 = 0$. Substituting the assumed series expansion for θ into Eq. (11), multiplying the resultant equation by the corresponding trial function, and integrating over the spatial domain from 0 to 1 yields

$$\begin{aligned}
& -\sum_{k=1}^M \left(\int_0^1 \Gamma_j \Gamma_k'' dx \right) r_k - \alpha_d \sum_{k=1}^M \left(\int_0^1 \Gamma_j \Gamma_k'' dx \right) \dot{r}_k + \frac{1}{\xi} \sum_{k=1}^M \left(\int_0^1 \Gamma_j \Gamma_k dx \right) \ddot{r}_k \\
& + \int_0^1 \Gamma_j \left\{ \cos \left(\sum_{k=1}^M r_k(t) \Gamma_k(x) \right) \int_1^x \left(m_R \delta_{dr}(x-1) + 1 \right) \left\{ \int_0^x \left(\sum_{k=1}^M \dot{r}_k(t) \Gamma_k(x) \right)^2 \sin \left(\sum_{k=1}^M r_k(t) \Gamma_k(x) \right) \right. \right. \\
& \quad \left. \left. - \left(\sum_{k=1}^M \ddot{r}_k(t) \Gamma_k(x) \right) \cos \left(\sum_{k=1}^M r_k(t) \Gamma_k(x) \right) \right\} dx + z_b \Omega_b^2 \sin(\Omega_b t) \right\} dx \right\} dx \quad (13) \\
& - \int_0^1 \Gamma_j \left\{ \sin \left(\sum_{k=1}^M r_k(t) \Gamma_k(x) \right) \int_1^x \left(m_R \delta_{dr}(x-1) + 1 \right) \int_0^x \left(\sum_{k=1}^M \dot{r}_k(t) \Gamma_k(x) \right)^2 \cos \left(\sum_{k=1}^M r_k(t) \Gamma_k(x) \right) \right. \\
& \quad \left. + \left(\sum_{k=1}^M \ddot{r}_k(t) \Gamma_k(x) \right) \sin \left(\sum_{k=1}^M r_k(t) \Gamma_k(x) \right) \right\} dx \right\} dx = 0, \quad j = 1, 2, \dots, M.
\end{aligned}$$

It should be noted that in order to keep the developed discretised model geometrically exact, it is essential to keep the trigonometric terms intact; the reason for the significance of keeping the model geometrically exact is that for the cases examined in this study, the cantilever undergoes very large-amplitude motions with the centreline tip angle growing as large as π . To predict such large angles accurately, the model must be geometrically exact and all the terms must be kept intact. The main challenge which is faced in discretisation of the geometrically exact model is that none of the integrations involving trigonometric functions in Eq. (13) can be conducted in closed form. Consequently, these terms are numerically integrated with sufficient terms being retained to ensure accurate results; this will result in significantly large discretised equations requiring high computational costs but ensures accurate predictions.

Six generalised coordinates are considered in this study (i.e. setting $M=6$ in Eq. (13)) which results in a 6-degree-of-freedom (DOF) system; the 6-DOF model ensures converged results. The resultant discretised model is solved through use of a continuation method.

Noting that solving Eq. (13) gives the centreline rotation, the axial and transverse displacements are then evaluated using Eqs. (6) and (7), respectively.

3. Verification of the proposed geometrically exact model

This section examines the extremely large static deflection of a cantilever under tip load using the proposed exact continuous model and a nonlinear finite element (FE) model in three-dimensional (3D) space. The tip load is assumed to remain perpendicular to the tip of the cantilever; such a force will cause extremely large deflections in the cantilever. The geometrically exact model for such a cantilever can be obtained as

$$f_d \left[\sin \theta \int_x^1 (\sin \theta \delta_{dr} (x-1)) dx + \cos \theta \int_x^1 (\cos \theta \delta_{dr} (x-1)) dx \right] + \frac{\partial^2 \theta}{\partial x^2} = 0, \quad (14)$$

in which f_d is the nondimensional tip load related to its dimensional counterpart f through $f_d = fL^2/(EI)$. A similar discretisation procedure as the one proposed in the previous section is followed here to obtain a 6-DOF discretised model.

The nonlinear 3D static FE analysis is conducted using Abaqus/CAE which is commonly used for benchmark structural analysis. For enhanced accuracy, the cantilever is modelled as a 3D solid and meshed using a continuum shell element; in particular, the 8-node quadrilateral continuum shell element with reduced integration is used with a size of 0.5 mm to ensure converged results. A cantilever of $h=1$ mm, $L=150$ mm, and $b=6$ mm is selected for comparison of the two models. The nonlinear FE results are made nondimensional as well.

Figure 2 shows the comparison of the two models in predicting the deflected cantilever shape at different tip load values. The results of the proposed exact continuous model are shown by a solid line; the 3D nonlinear FE results are demonstrated by solid circles. X and Z axes denote the nondimensional (with respect to L) undeformed coordinates. The FE results are obtained by extracting the deformed configuration of the cantilever midplane at different tip loads. It is seen clearly that the results obtained by the proposed geometrically exact model are in excellent agreement with those obtained via 3D nonlinear FE model. Hence, the proposed model is verified for accuracy even when the deflections are extremely large. To ensure the accuracy of the FE results, a mesh convergence study is conducted and the deflected beam configuration corresponding to $f_d=9.31$ is obtained using three mesh sizes, i.e. 0.5 mm, 1.0 mm, and 2.0 mm and the results of which are plotted in Fig. 3. As seen, the results are almost identical indicating that the mesh size selected in this study, i.e. 0.5 mm, ensures converged results. Figure 4 illustrates the transverse displacement contour plots of all the cases examined in Fig. 2, obtained via the 3D nonlinear FE model.

4. Extreme resonance response

The nonlinear vibration response of the cantilever in primary and secondary resonance regions at very large amplitudes is studied in detail in this section. It should be noted that in all the results presented in this section, w and u represent the nondimensional (with respect to length) transverse and axial displacements, respectively. Additionally, ω_1 and ω_2 denote the first two nondimensional natural frequencies of the transverse motion. It is important to note that the only parameter in the equation of motion whose value

requires defining specific dimensions is ζ ; however, the goal of the present study is to perform a nondimensional analysis with the results being applicable to any cantilever. The value of ζ depends on the length-to-thickness ratio of the cantilever. A parametric analysis is performed and it is found that for cantilevers with $L/h > 50$, ζ will be sufficiently large so that the term $1/\zeta$ can be safely neglected. This finding ensures that the nondimensional results of the present study are applicable to any cantilever with $L/h > 50$. The nondimensional Kelvin-Voigt nondimensional viscosity coefficient, α_d , is set to 0.0025 for all the cases examined in this section. Additionally, for all the cases presented, the solid line style implies the solution is stable while a dotted line style indicates the solution is unstable.

4.1 Primary resonance

The primary resonance behaviour of the cantilever subject to transverse base excitation is studied in this section. To this end, the base excitation frequency is selected as the control parameter and changed in a range close to the first nondimensional transverse natural frequency (ω_1). Next, all six rotational generalised coordinates are calculated via solving the discretised set of equations. Then the rotational, transverse, and axial displacements of the tip of the cantilever are calculated for various cases.

Figure 5 illustrates the frequency-amplitude results when $m_R=0$ and $z_b=0.008$ and 0.024. Sub-figure (a-d) shows the maximum tip rotation, maximum tip transverse displacement, tip transverse displacement corresponding to maximum tip rotation, and maximum axial displacement, respectively. The reason for plotting the frequency-amplitude for w twice, i.e. sub-figure (b) and (c), is to better illustrate the extreme vibration behaviour

as will be explained in more detail in the following. It is seen in Fig. 5 that for the case with $z_b=0.008$, the cantilever displays a weekly nonlinear response of hardening type. For this case, two bifurcations of saddle-node type occur in the primary resonance region, rendering the motion in between unstable (as shown by dotted line). It is seen that even for base excitation amplitudes as small as $z_b=0.008$, the cantilever undergoes very large oscillations, with a transverse displacement more than 80% of the beam length. Additionally, it is seen that for the case with $z_b=0.008$, the maximum tip w curve, i.e. sub-figure (b), is the same as the curve for w corresponding to maximum tip θ , i.e. sub-figure (c), indicating that the maximum tip transverse displacement occurs the same time as when the maximum tip rotation occurs.

The other case examined in Fig. 5, i.e. when $z_b=0.024$, better highlights the extreme vibration of the cantilever. As seen, the general behaviour is similar to the case with $z_b=0.008$. However, in the case, the cantilever undergoes extremely large oscillations with the tip angles reaching 2.63 radians, which is equivalent to 150° . As a result, the maximum transverse displacement in each period of oscillation does not necessarily occur when the tip rotation reaches its maximum, as can be observed by comparing Fig. 5(b) and (c). In other words, as the tip rotation increases, the tip transverse displacement increases until reaching a maximum and then decreases. To better explain this phenomenon, the time histories of the tip rotation and tip transverse displacement at two different excitation frequencies, i.e. $\Omega_b/\omega_1=1.0111$ and 1.0352 are plotted in Fig. 6. As seen in Fig. 6(a, b), at $\Omega_b/\omega_1=1.0111$, the maximum tip transverse displacement and tip rotation occur at the same time in a period of oscillation. However, at $\Omega_b/\omega_1=1.0352$, as shown in Fig. 6(c, d), the maximum tip transverse displacement occurs before the tip angle reaching its maximum

value. In fact, the transverse displacement reaches a local minimum when the tip angle reaches its maximum value in a period of oscillation. Now going back to Fig. 5, it is seen that as the excitation frequency is increased from $\Omega_b/\omega_1=1.02$, the tip transverse displacement corresponding to maximum tip rotation (shown in sub-figure (c)) starts to decrease while the maximum tip transverse displacement (shown in sub-figure(b)) increases. The decrease in the amplitude indicates that the cantilever tip rotates so much that it “bends backward” causing the transverse displacement to decrease. This is clearly demonstrated in Fig. 7 in which the cantilever oscillation in one period is plotted at $\Omega_b/\omega_1=1.0111$ and 1.0352 in sub-figures (a) and (b), respectively. As seen, for oscillations as large as the magnitude shown in sub-figure (a), the maximum tip transverse displacement occurs at maximum tip rotation. However, for extremely large oscillation amplitudes, such as the case shown in sub-figure (b), the tip of the cantilever reaches its maximum transverse displacement before reaching maximum rotation.

It is worth mentioning that the extremely large vibration responses examined in Fig. 5 can only be captured using a geometrically exact model in which all the nonlinearities are accounted for and all the trigonometric terms are kept intact. To show the importance of keeping a large number of generalised coordinates in Galerkin discretisation, and that a 6-DOF model ensures converged results, Fig. 8 is constructed showing the frequency-amplitude of the first four rotational generalised coordinates of the cantilever of Fig. 5. As seen, the amplitude of fourth generalised coordinate is much smaller than that of the first generalised coordinate indicating convergence.

Figure 9 illustrates the tip mass ratio (m_R) effect on the first transverse natural frequency of the cantilever. As seen, the first natural frequency decreases as m_R is

increased. It is interesting to note that the reduction in ω_1 is faster for smaller values of m_R . Having examined the effect of the tip mass ratio on the linear natural frequency, the next figure, i.e. Fig. 10, shows the impact of m_R on nonlinear primary resonance behaviour of the cantilever. A similar trend can be observed here that at smaller values of m_R , a larger shift in the resonant region is observed; more specifically, a larger shift in the resonant region is observed as m_R is increased from 0 to 0.05 compared to the case when it is increased from 0.15 to 0.20. Additionally, due to increased tip mass ratio, there is a slight increase in the maximum tip rotational motion while the maximum tip transverse displacement remains almost unchanged. Furthermore, the tip axial displacement magnitude increases slightly as a result of increased tip mass ratio.

The extremely large resonance response of the cantilever when the base excitation amplitude is changed as the bifurcation parameter is studied in Fig. 11. As seen, the cantilever shows hysteresis type nonlinear response with two bifurcations of saddle-node type at points S_1 and S_2 . More specifically, as the amplitude of the base excitation is increased from 0, a saddle-node bifurcation occurs at $z_b=0.0262$ (i.e. point S_1) causing the cantilever to undergo a snap-through motion to the large-amplitude stable branch. The cantilever undergoes a different solution path as z_b is decreased from 0.1 where the occurrence of another saddle-node bifurcation at $z_b=0.0136$ (i.e. point S_2) causes the cantilever to jump to the small-amplitude stable branch. To better illustrate the snap-through motion of the cantilever, Fig. 12 is constructed which shows the oscillation of the cantilever in one period at $z_b=0.0262$ (i.e. point S_1) before and after the snap-through motion in sub-figures (a) and (b), respectively.

4.2 Secondary resonance

The secondary resonance vibration of the cantilever under transverse base excitation is studied in this section. To excite the cantilever in secondary resonance region, the base excitation frequency is changed in a range close to the second transverse natural frequency (ω_2).

Figure 13 illustrates the frequency-amplitude diagrams in the secondary resonance region when $m_R=0$ and $z_b=0.03$. As opposed to the case of primary resonance where the cantilever displays a hardening response, in the secondary resonance region, as shown in Fig. 13, the cantilever shows a softening-type behaviour. However, the number of saddle-node bifurcations remains the same as the case of the primary resonance. In particular two bifurcations of saddle-node type occur at points S_1 and S_2 , i.e. at $\Omega_b/\omega_2=0.8460$ and 0.6955 , respectively. The detailed dynamics of the cantilever at $\Omega_b/\omega_2=0.6955$ are shown in Fig. 14 through time traces and phase-plane plots of the tip rotation and tip transverse displacement; additionally, to better illustrate the vibrational behaviour in the secondary resonance region, the oscillation of the cantilever in one period is depicted in Fig. 15. The secondary resonance response of the first four rotational generalised coordinates of the cantilever of Fig. 13 are depicted in Fig. 16, clearly showing the second generalised coordinate being the dominant one.

The tip mass ratio (m_R) effect on second linear transverse natural frequency is shown in Fig. 17. As seen, similar to the case of the first natural frequency, the second natural frequency decreases as well with increasing tip mass ratio. Furthermore, the reduction in ω_2 is much steeper at small values of m_R . The next figure, i.e. Fig. 18, shows the effect of m_R on the frequency-amplitude diagrams of the cantilever in the secondary resonance region. As

seen, the added tip mass shifts the secondary resonance region to smaller excitation frequencies and reduces the maximum transverse displacement significantly. The axial displacement is decreased as well with increasing m_R , while the tip angle is not affected much.

5. Conclusions

This study examined the extremely large oscillations of a cantilever subject to base excitation in transverse direction in primary and secondary resonance regions. To examine such large oscillation amplitudes accurately, a geometrically exact model was developed for centreline rotation of the cantilever while ensuring all trigonometric terms are kept intact before and after the discretisation. The proposed geometrically exact model was verified for accuracy at extremely large deformations via comparison to a 3D nonlinear FE model.

Examining the primary resonance response characteristics of the cantilever revealed that the cantilever could reach extremely large oscillation amplitudes even at very small base excitation amplitudes. This highlights the necessity of developing a geometrically exact model capable of capturing such extreme responses accurately. It was shown that it is possible for the cantilever to undergo extreme oscillations such that the tip of the cantilever “bends backward” and the tip angle grows larger than 150° . Additionally, it was shown that even in the secondary resonance region, the cantilever could undergo very large-amplitude motions with tip angle reaching almost 90° . The nonsymmetric behaviour of the cantilever in the secondary resonance region was highlighted as well. It was shown that a cantilever undergoing extreme vibrations shows a hardening-type nonlinear behaviour in the primary resonance region and a softening one in the secondary resonance region.

The added tip mass effect on both linear and nonlinear vibration characteristics of the cantilever was examined. It was shown that the added tip mass reduces the first and second natural frequencies of the cantilever and hence causes a shift in the resonance region to smaller excitation frequencies. Additionally, it was shown that in the primary resonance, the added tip mass does not affect the transverse and axial motion amplitudes

much. In the secondary resonance region, on the other hand, it was shown that the added tip mass significantly reduces the transverse and axial motion amplitudes.

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Fig. 1. Undeformed and deformed schematic configurations of a cantilever with a tip mass under transverse base excitation.

Fig.2. Comparison of the large-amplitude static deformation of the cantilever under different tip load magnitudes (indicated on the curves) obtained via the 3D nonlinear FE model (shown by solid circles) and the present study (shown by solid lines). X and Z are nondimensional (relative to L) undeformed coordinates.

Fig. 3. Large-amplitude static deformation of the cantilever at $f_d=9.31$ obtained via FE analysis using different mesh sizes: 0.5 mm shown by black solid line, 1.0 mm shown by red dashed line, and 2.0 mm shown by green dots.

Fig. 4. Transverse displacement FE contour plots (in mm) of four deformed configurations of the cantilever of Fig. 2.

(a)

(b)

(c)

(d)

Fig.5. Frequency-amplitude diagrams of the base excited cantilever in primary resonance region for two base excitation amplitudes; (a) maximum tip rotation; (b) maximum tip transverse displacement; (c) tip transverse displacement corresponding to maximum tip rotation; (d) maximum tip axial displacement. $m_R=0$ and $\omega_1=3.5160$.

Fig.6. Dynamics of the cantilever of Fig. 5 for the case with $z_b=0.024$; (a, b) time histories of θ and w at tip, respectively, when $\Omega_b/\omega_1=1.0111$; (c, d) time histories of θ and w at tip, respectively, when $\Omega_b/\omega_1=1.0352$. t_n : time normalised relative to the oscillation period.

Fig.7. Extremely large vibrations of the cantilever of Fig. 5 for the case with $z_b=0.024$ at (a) $\Omega_b/\omega_1=1.0111$ and (b) $\Omega_b/\omega_1=1.0352$. X and Z are nondimensional (relative to L) undeformed coordinates.

Fig.8. (a-d) Frequency-amplitude plots of the rotational generalised coordinates r_1 , r_2 , r_3 , and r_4 , respectively, in primary resonance region, corresponding to the cantilever of Fig. 5 for the case with $z_b=0.024$.

Fig.9. Effect of m_R on the first transverse natural frequency of the cantilever.

(a)

(b)

(c)

(d)

Fig.10. Effect of tip mass ratio (m_R) on frequency-amplitude diagrams in the primary resonance region; (a) maximum tip rotation; (b) maximum tip transverse displacement; (c) tip transverse displacement corresponding to maximum tip rotation; (d) maximum tip axial displacement. $z_b=0.015$.

(a)

(b)

(c)

(d)

Fig.11. Force-amplitude plots of the base excited cantilever in the primary resonance region; (a) maximum tip rotation; (b) maximum tip transverse displacement; (c) tip transverse displacement corresponding to maximum tip rotation; (d) maximum tip axial displacement. $m_R=0$, $\Omega_b=1.03$, and $\omega_1=3.5160$.

Fig.12. Vibrations of the cantilever of Fig. 11 at $z_b=0.0262$ (point S_1); (a) before snap-through; (b) after snap-through. X and Z are nondimensional (relative to L) undeformed coordinates.

(a)

(b)

(c)

Fig.13. Frequency-amplitude diagrams of the base excited cantilever in secondary resonance region; (a) maximum tip rotation; (b, c) maximum tip transverse and axial displacements, respectively. $m_R=0$, $z_b=0.03$, and $\omega_2=22.0345$.

Fig.14. Dynamics of the cantilever of Fig. 13 at $\Omega_b/\omega_2=0.6955$ (i.e. point S_2). (a, b) time histories of θ and w at tip, respectively. (c, d) phase-planes of θ and w at tip, respectively. t_n : time normalised relative to the oscillation period.

(a)

(b)

Fig.15. Vibrations of the cantilever of Fig. 13 in one period at $\Omega_b/\omega_2=0.6955$ (i.e. point S_2); (a) vibration response as tip angle varies from maximum to minimum value; (a) vibration response as tip angle varies from minimum to maximum value. X and Z are nondimensional (relative to L) undeformed coordinates.

Fig.16. (a-d) Frequency-amplitude plots of the rotational generalised coordinates r_1 , r_2 , r_3 , and r_4 , respectively, in the secondary resonance region, corresponding to the cantilever of Fig. 13.

Fig.17. Effect of m_R on the second transverse natural frequency of the cantilever.

(a)

(b)

(c)

Fig.18. Effect of m_R on frequency-amplitude plots of the cantilever under transverse base excitation in the secondary resonance region; (a) maximum tip rotation; (b, c) maximum tip transverse and axial displacements, respectively. $z_b=0.025$.