The 3-Wave Resonant Interaction Model: Spectra and Instabilities of Plane Waves

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NEWCASTLE

March 2019
The 3-Wave Resonant Interaction Model:
Spectra and Instabilities of Plane Waves

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Submitted in accordance with the requirements of the University of Northumbria at
Newcastle for the degree of PhD in Mathematics

Research undertaken in the Department of Mathematics, Physics and Electrical
Engineering

Under the Supervision of Dr Matteo Sommacal.

Northumbria University
Department of Mathematics, Physics and Electrical Engineering

October 12, 2019

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Abstract

The aim of this thesis is the analysis of the spectral stability of plane wave solutions of the 3-wave resonant interaction (3WRI) model, when such solutions undergo localised perturbations.

For the first time, we provide a comprehensive topological classification of the spatial stability spectra with respect to the parameters space and the gain functions associated to any stability spectrum. We find that all the stability spectra of the coupled nonlinear Schrödinger (CNLS) system are enclosed in those of the 3WRI system. The topological features of the CNLS stability spectra are gaps on the real axis (solutions not bounded in space), and branches and loops off the real axis (solutions bounded in space which can be linearly unstable in time). New topological components exist in the stability spectra of the 3WRI model: we name such components twisted loops. They are associated with explosive instability (the corresponding solutions blow up in a finite time) and their gain function is non-zero in a whole neighbourhood of the origin. We observe that the gain function associated to the branches is non-zero at low wave numbers, symmetrically located with respect to the zero-value of the wave number, but it is zero at the origin of the plot (linear instability of baseband-type). The gain function associated to the loops is non-zero only away from the origin (linear instability of passband-type).

We show that the plane wave solutions of the 3WRI model are linearly unstable in time for any choice of the physical parameters, including those ones associated to the solutions that are explosive. Thus, there is linear instability of the plane wave for any choice of the physical parameters corresponding to a positive gain-function.

Finally, we conjecture that the existence of branches in the stability spectra is a necessary condition for the onset of rogue waves ascribable to rational or semi-rational solutions obtained by Darboux Dressing Transformation. Indeed, we observe numerically linear instability of plane waves with the subsequent generation of localised structures whose onset, as a result of the perturbation of plane waves, must be investigated further due to the dispersionless nature of the 3WRI system.
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Topological components of the stability spectrum:

- $B$: Branch
- $G$: Gap
- $L$: Loop
- $SG$: Split Gap
- $TL$: Twisted Loop

$nA$: $n \in \mathbb{N}$ number of a topological component $A \in \{B, G, L, SG, TL\}$
Acknowledgements

Firstly, I would like to express my sincere gratitude to my supervisor Dr Matteo Sommacal for his inspiring guidance, friendly advice during the project work, for his patience and for the huge time dedicated to the research together. He has been an extraordinary mentor and a wise guide. I am thankful to him for giving me the opportunity to participate and meet a wider scientific community during the workshops that he organised: LMS Norther Regional Society Meeting and Workshop and The Onset of Rogue Waves both at Northumbria University.

I would like to thank Professor Sara Lombardo, for giving me this wonderful opportunity to undertake my PhD at Northumbria University and for introducing me to the fascinating subject of Nonlinear and Integrable Systems. I would like to express my gratitude for helping me in all the time of the PhD project and during the writing of this thesis. I am grateful for her advices both as a friendly woman and as a professional guide.

I want to thank Dr Benoit Huard for the scientific support and proof reading of the thesis. I am also thankful to Professor Antonio Degasperis for his comments and feedback; my PhD project is based on his, Professor Sara Lombardo’s and Dr Matteo Sommacal’s research work, and without it the accomplishment of this thesis would be impossible.

I want to express my profound gratitude to my parents for the financial support during the extension period and for the encouragement especially during the maternity leave and the difficult period of the childcare while I was undertaking the research.

The biggest thank-you is to my husband, Matteo, who has always encouraged and believed in me; our discussions on physics and mathematics have powered the willing to work ever harder.

Finally, to my daughter Lucrezia: I hope this thesis will inspire you to be determined to follow your dreams.
Declaration

I declare that the work contained in this thesis has not been submitted for any other award and that it is all my own work. I also confirm that this work fully acknowledges opinions, ideas and contributions from the work of others. Any ethical clearance for the research presented in this thesis has been approved. Approval has been sought and granted by the Faculty Ethics Committee on 11 May 2015.

The Word Count of this Thesis is 47213.

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Chapter 1

Introduction

1.1 Background and Current Research

1.1.1 Modulation Instability for Scalar Dispersive Equations

Many phenomena in nature can be explained via instabilities. Hydrodynamic instabilities can occur if the initial physical features (velocity, pressure and density) of a fluid flow are exposed to small disturbances. Between all hydrodynamic instabilities Kelvin-Helmholtz Instability (KHI) \[1, 2\] is well-known. It takes place between the interface of two fluids flowing with different velocities. The most famous examples of the manifestation of KHI in nature are the red giant vortex in the Jupiter’s atmosphere and the generation of clouds that are ”ocean wave-like” in the Earth’s atmosphere. If the two fluids flow with different density Rayleigh-Taylor Instability (RTI) \[3\] can occur: it is the explanation of the generation of ”mushroom clouds” in the volcanic eruption or atomic explosion. It can be also seen as the limit of another fascinating phenomenon that is the Richtmyer-Meshkov Instability (RMI) \[4, 5, 6\]. In general, it is when a shock wave interacts with the perturbed interface of two fluids with different physical properties. At the beginning of the interaction just a row of vortices with different signs are created, but then the perturbation grows in time and ”mushroom” structures arise. Plasmas exhibit a huge number of magnetohydrodynamics instabilities, for example ”sausage instability” and ”kink instability” are observed in solar corona \[7, 8\].
Chapter 1. Introduction

Modulational Instability (MI) is ubiquitous in nature and it was observed in several nonlinear wave phenomena [9], such as in radio waves in transmission lines [10], in light waves in dielectric materials [11] and in plasma waves during the nonlinear coupling of plasma cyclotron waves and magneto hydrodynamic modes [12]. In the context of water waves, MI is referred to as Benjamin-Feir Instability, so named because Benjamin and Feir first observed it in nonlinear Stokes waves on deep water surface [13, 14].

MI may arise if a plane wave of a scalar nonlinear dispersive equation is perturbed with a long wave perturbation such that the original waveform is deformed and the wave amplitude is said to be modulated [15, 16, 17]. Perturbing the amplitude of the plane wave \( u(x, t) = ae^{i(kx - \omega t)} \) with real amplitude \( a \), wave number \( k \) and frequency \( \omega \), by means of the plane waves \( b(x, t) = b_{01}e^{i(Kx - \Omega t)} + b_{02}e^{-i(Kx - \Omega t)} \), one gets the perturbed solution

\[
\bar{u}(x, t) = (a + b(x, t))e^{i(kx - \omega t)} = ae^{i(kx - \omega t)} + b_{01}e^{i((k + K)x - (\omega + \Omega))t} + b_{02}e^{i((k - K)x - (\omega - \Omega))t},
\]

(1.1)

where \( x \) is space, \( t \) is time, \( b_{0j} \) are real amplitudes of the perturbations and \( b_{0j} \ll a \), the modulation frequency is \( \Omega \ll \omega \) and the wave number of the perturbation is \( K \ll k \).
After perturbing, the amplitude of the resulting plane wave is not anymore a constant, being instead a function of space and time. The perturbed solution reads $u(x, t) + \delta u_1(x, t) + \delta u_2(x, t)$, where $\delta u_1(x, t) = b_{01} e^{i((k+K)x-(\omega+\Omega)t)}$ and $\delta u_2(x, t) = b_{02} e^{i((k-K)x-(\omega-\Omega)t)}$ are the so-called sidebands and $u(x, t)$ is known as the carrier wave.

MI has two stages: a linear and a nonlinear stage. [9]. The linear stage is the mechanism responsible for the amplification of the perturbation when the approximation $\delta u_j \ll u$ remains valid in time. In this limit, the equation for the perturbation is linear, and nonlinear terms in the perturbation are meant as higher order corrections to the linear dispersive equation. This approximated linear dispersive equation is named the *linearised equation*. When this approximation is no longer valid because the perturbations $\delta u_j$ are such that $\delta u_j \approx u$, and the order of magnitude of the nonlinear terms becomes comparable to the order of magnitude of the linear dispersive equation, then the nonlinear stage comes into play. After the amplitude of the perturbed plane wave reaches its maximum value, it may vanish by causing the formation of localised energy solutions, and solitons may occur as a result of the balance between dispersion and nonlinearity [18].

Since the linear stage of MI is a linear instability, it can be studied via the linearisation of the nonlinear equation around the perturbation. By substituting the explicit expression of $u(x, t) + \delta u_1(x, t) + \delta u_2(x, t)$ into the nonlinear equation, and keeping only the terms at the first order in the perturbation, the linearised equation can be written as a matrix equation (see, for instance, [19])

$$A b_0 = 0_{2 \times 2},$$

(1.2)

where $A$ is a $2 \times 2$ matrix, $0_{2 \times 2}$ is the $2 \times 2$ zero matrix and the solution is the vector $b_0 = (b_{01} e^{i(Kx-\Omega t)} \ b_{02} e^{-i(Kx-\Omega t)})^T$. The condition such that $b_0$ is solution of (1.2) is the vanishing of the determinant of the matrix $A$, i.e. $\det(A) = 0$, which gives the dispersion relation for the perturbations.

In 1965 Lighthill [21] obtained the so called *Benjamin-Feir-Lighthill (BFL) criterion* to determine if a scalar system can be linearly unstable or not. Specifically, he considered a weakly nonlinear Stokes wave (i.e. weakly nonlinear periodic progressive wave, see [22]) on deep water and he found that, if the linearised PDE is a hyperbolic equation, then the Stokes wave is neutrally stable, instead, when the linearised PDE is an elliptic equation, then MI may occur [17]. However, this result is obtained by considering negligible dispersion dispersion

$^1$The explicit expression of the matrix $A$ depends on the equation in study.
effect and, for this reason, the BFL criterion is not a sufficient condition for the existence of MI. Indeed, when dispersion is negligible, the dispersion relation for the perturbation is independent of the wave number \( K \); on the contrary, when dispersion is present, we can define a domain for the wave number \( K \) of the perturbation such that MI can occur. Thus, also in the case in which we have elliptic PDEs, the argument of the square root can be negative only for certain values of \( K \). Moreover, for \( \frac{\partial |\text{Im}(\Omega(K))|}{\partial K} = 0 \), we obtain the critical value \( K_{\text{critic}} = K(a, P, Q) \) corresponding to the largest growth rate of \( |\text{Im}(\Omega(K))| \). The value \( K_{\text{critic}} \) depends on the amplitude \( a \) of the plane wave solution, but not on its frequency \( \omega \) and its wave number \( k \), whereas the module \( |\text{Im}(\Omega(K))| \) is known as the gain function. Thus, MI occurs for values of the physical parameters for which \( |\text{Im}(\Omega(K))| \) is not zero\(^2\). For instance, the NLS equation can be written\(^3\) \[ u_t - iPu_{xx} + iQ|u|^2u = 0, \quad (1.3) \]

when \( PQ > 0 \), by the BFL criterion, we have an hyperbolic equation and there is no possibility to have MI; whereas if \( PQ < 0 \), we have an elliptic equation and we can have MI. Thus, the necessary condition to observe MI is \( PQ < 0 \) and we need also to compute for which values \( K \) the gain function \( |\text{Im}(\Omega(K))| \) takes real values different from zero. In this particular case, the physical parameters are related one with each other by the formula

\[
(\Omega - 2PkK)^2 = P^2K^2 \left( K^2 + 2\frac{Q}{P}a^2 \right)^2, \quad (1.4)
\]

such that, if \( PQ > 0 \), the imaginary part of \( \Omega \), that is the gain function \( |\text{Im}(\Omega(K))| \), does not exist. Instead, if \( PQ < 0 \), the explicit expression of the gain function is

\[
|\text{Im}(\Omega(K))| = 2PkK \pm |PK|\sqrt{K^2 + 2\frac{Q}{P}a^2} \quad (1.5)
\]

and the domain of the wave number \( K \), at which the gain function is non-zero, reads

\[
K^2 \leq -2\frac{Q}{P}a^2. \quad (1.6)
\]

The maximum value of the function \( |\text{Im}(\Omega(K))| \) is reached at the wave number \( K_{\text{critic}} = \pm \sqrt{-\frac{Q}{P}a^2} \). \(^1\)

\(^2\)That is true also for multicomponent models.

\(^3\)In the Chapter 2, we will write the NLS equation in a different manner.
Looking at the expression of the perturbed solution \((1.1)\), the existence of complex modulation frequency \(\Omega\) entails that, as time goes on, a perturbation grows with the effect of amplifying the amplitude of \((1.1)\). Nevertheless, in general, the amplitude might not grow up indefinitely because, at a certain time, the neglected nonlinear terms come into play to possibly bound such a growth. From the physical point of view, the BFL criterion corresponds to a necessary condition for the occurrence of the localisation of the energy due to a balance between dispersion and nonlinearity. This fact is reflected in the frequency spectrum of the waves interacting during MI. Indeed, MI can be also explained as a four-wave resonant interaction between two infinitesimal sidebands at frequencies \(\omega_{1,2} = \omega \pm \Omega\) with a strong carrier wave at frequency \(\omega\) \([23]\). A wave at frequency \(\omega\) interacts twice with a sideband at frequency \(\omega_1\) to produce another sideband with a different frequency \(\omega_2 = 2\omega - \omega_1\). Similarly, this last sideband interacts with the carrier wave and reinforces the first sideband. In other words, they interact under resonance conditions for the frequencies \(\omega_1 + \omega_2 = 2\omega\) and also for the wave numbers \(k_1 + k_2 = 2k\), with \(k_{1,2} = k \pm K\). Indeed, if we consider the perturbed plane wave \((1.1)\) as solution of the NLS equation, the nonlinearity term becomes proportional to

\[
|\tilde{u}|^2 \tilde{u} = a^3 e^{i(kx-\omega t)} + a^2 b_{01} e^{i((2k-(k+K))x-(2\omega-(\omega+\Omega))t)} + \\
+ a^2 b_{02} e^{i((2k-(k-K))x-(2\omega-(\omega-\Omega))t)} + c.c. + o(a^2 b_{0j}), \quad j = 1, 2,
\]

(1.7)

where \(c.c.\) stands for complex conjugate of the correcting terms and we have neglected all the terms whose order of magnitude is smaller than the terms multiplied by \(a^2 b_{0j}\). Here, the correcting terms to the plane waves have as arguments of the exponentials \(2k - (k + K) = 2k - k_1, 2k - (k - K) = 2k - k_2\) and \(2\omega - (\omega + \Omega) = 2\omega - \omega_1, 2\omega - (\omega - \Omega) = 2\omega - \omega_2\). Thus, \(2k - k_2 = k_1, 2k - k_1 = k_2\) and \(2\omega - \omega_2 = \omega_1, 2\omega - \omega_1 = \omega_2\).

If only dispersion was present, the resonance conditions would not be satisfied because, in the linear dispersion relation, frequency would depend only on the wave number. Therefore, even if the waves would interact with different frequencies, in general, they do not match the resonant conditions. The interacting waves would propagate with different phase velocities and, as a result, the dispersion effect pulls apart the resulting wave. On the other hand, if also the nonlinearity is present, then the perturbation satisfies a linearised equation whose dispersion relation involves both the wave number and the amplitude of the carrier.
wave. Although with different phase velocities, they propagate on the background of the carrier wave and the wave numbers, as well as the frequencies, match the resonant conditions. Indeed, because of the self-interaction term, the sideband, say, with frequency $\omega_1$ satisfies a linearised equation plus a forcing correction term proportional to the sideband at frequency $\omega_2$ and to the squared carrier wave, (see, for instance, the second corrective term in the expression (1.7)). If this forcing term oscillates with the same frequency of the other sideband, then they resonate. If nonlinearity and dispersion compensate, the sidebands grow linearly as time goes on and drive the carrier wave to oscillate around the resonant frequency $\omega$ with greater amplitude. In this case, we say that nonlinearity contrasts dispersion by compressing the plane wave, namely, MI causes the localisation of the energy. As already mentioned above, since when nonlinearity and dispersion balance one each other, we say that the nonlinear stage of MI comes into play and the occurrence of solitons was observed in this stage \[24, \, 25\]. From here it is clear the importance of studying the linear stability of physical systems undergoing small perturbations and searching for the existence conditions of localised solutions. In this respect, the interest in mathematical methods to treat MI has witnessed an explosion during the '70s, after the powerful Inverse Scattering Transform (IST) method to find solutions of nonlinear PDEs was developed \[26, \, 27, \, 28\]. It was applied first to the Nonlinear Schrödinger (NLS) equation \[29\], and then, after the development of the Ablowitz, Kaup, Newell and Segur (AKNS) scheme \[30\], it was applied to the Kortweg de Vries (KdV) equation \[31\] and to other physical models. The key idea is that IST is suitable for the investigation of the asymptotic behaviour of solutions of nonlinear equations. A pioneering research in this sense was conducted by Kuznetsov and Mikhailov \[32\]. Using the Shabat-scheme \[33\], they studied the stability of periodic stationary waves of the KdV equation as time goes on. The asymptotic stability of the system is explained by the vanishing of the continuous spectrum, while the discrete spectrum survives in time, that is the existence of a set of stable solitons as asymptotic state. Later, several other works were conducted using the Shabat-scheme. For instance, the NLS equation, whose solution goes to the amplitude of the Langmuir wave as $|x| \to \infty$ was used for the investigation of the parametric instability of solitons in a homogeneous plasma \[34\] and the KP equation, whose solution goes to the cnoidal wave as $|x| \to \infty$ was used

\[34\] In the case of the NLS equation, we have chosen the frequency and the wave number of the perturbations so that they resonate with the carrier wave.
for the analysis of the stability of periodic waves in a weakly dispersive medium [35].
In the framework of the IST method, particular boundary conditions are required, for example by imposing that the solution and its first derivative are in the class of potentials vanishing sufficiently fast asymptotically. Indeed, it is difficult to deal technically with the IST when solutions have more complicated asymptotic behaviour. Nevertheless, some research works have been conducted to reformulate the IST such that one can work with more sophisticated asymptotic solutions, for instance, solitons with nonzero boundary conditions as solutions of the focusing Nonlinear Schrödinger equation [36] [37], and it has been shown that combinations of the growing exponential solutions of the inverse problem saturate the MI leading, then, to the formation of solitons in the nonlinear stage. In recent times, this subsequent nonlinear stage of MI for the scalar NLS equation has been the subject of intensive investigation and, using numerical and analytical techniques, the stability of plane wave solutions with respect to localised and random perturbations has been studied [38, 39]. Similarly, the stability of plane wave solutions of the scalar NLS equation with respect to periodic perturbations has attracted much attention, and has been investigated in [40, 41, 42, 43] using the theory of finite-gaps and matched asymptotics.
In 1974, Ablowitz developed an alternative and powerful spectral method to lead stability analysis [44]. The method is based on the fact that the solutions of the linearised equation can be written in terms of the so called squared eigenfunctions (SE) (see for example [44], [45, 46] for an introduction to squared eigenfunctions as solutions of the linearised NLS equation), which in turn can be written in terms of the Lax operators [47]. Once a solution of a nonlinear equation is perturbed, the problem to investigate the linear stability of the system is equivalent to investigate the behaviour in time of the SE solutions of the linearised equation (see, for example, [48]). Indeed, via the construction of the SE, one is able to compute the corresponding eigenfrequency whose imaginary part, i.e. the gain function, provides information about the linear stability.

1.1.2 Modulational Instability and Other Linear Instabilities for Multi-Component Systems

As mentioned above, MI for scalar equations is the deformation of the form of the amplitude of a plane wave as a result of the balancing of dispersion and nonlinearity. When these
effects are balanced, solitons may appear in the nonlinear stage. In this case, MI arises when two sidebands are added on the background of the carrier wave with which they interact. MI is the result of the self-interaction of the perturbed solution, such that there exists a nonlinear interaction between plane waves linearly superimposed. When dispersive nonlinear multi-component systems are taken into account, the single component of a system may exhibit MI, but the entire system can be linearly unstable or not. In this regard, the MI of two-component system of counter-propagating waves has been analysed in the research work [49]. The authors considered two-component solutions of two coupled sine-Gordon equations, which are travelling with different group velocities and each component is composed by two counter-propagating waves linearly superimposed. Then, they used a multiple scale approach to obtain three asymptotic models at different length scales and different timescales, which are systems of four evolution equations, one for each wave. By considering the models so obtained, first, they took into account the two components composed by only one plane wave. At the super-long length scales, the leading order in the dispersion relations is represented by a dispersion term, such that dispersion is considered responsible for the MI. As soon as one considers the long-length scales, a term coming from the self-interaction is added to the linear dispersion relation, so, dispersion and nonlinearity compete with one another as described in the previous subsection. However, without any restriction on the wave number of the perturbation, it has been observed that, besides the linear instability of the single plane wave arising from the competition between the terms of dispersion and self-interaction, any wave can be affected by the coupling between the components. Indeed, such coupling can suppress or enhance this linear instability as a result of the addition of a term coming from the self-interaction. In this way, the effect of nonlinearity becomes stronger or weaker compared to when coupling is absent. The plane wave can gain or lose energy due to the sharing and the exchange of energy occurring when the components overlap, and so, finally, leading to the linear stability or instability of the system [49] [50]. For these two-component systems, dispersion plays a key-role in the occurrence of MI, and it has also been shown that coupled dispersionless nonlinear systems are, indeed, stable with respect to the MI [51]. The linear instability can occur if the dispersive term is taken into account also if it is a higher order correction to the transport and nonlinear terms [49] [50]. Furthermore, the presence of an additional counter-propagating mode in any component can completely inhibit or enable the linear
Chapter 1. Introduction

instability because of the exchange of energy between the counter-propagating plane waves in each component and between two coupled components as well. Indeed, because of these couplings, nonlinearity competes with dispersion in a stronger or weaker manner \cite{49, 50}. The classification of the linear instabilities in the parameters space was carried out in the research works for two coupled focusing or for two coupled defocusing NLS equations \cite{71} and for mixed coupled NLS equations, that is a focusing and a defocusing scalar NLS equations are coupled \cite{52}. Taking into account the coupling between two plane waves considered as the two modes of a wave packet and whose perturbations are two Fourier modes, the classification of the linear instabilities is performed on the basis of the energy exchange between plane waves undergoing linear perturbations. These works confirm the existence of instabilities also in defocusing regime (besides the long-wave instability) when there is a coupling of two propagating modes. Several scenarios can occur. For instance, when two focusing NLS equations are coupled, there can be cross-phase instability if one unstable mode, or both unstable modes, of the perturbation excite each plane wave and the energy is shared by the two co-propagating plane waves; or there can be self-phase instability if one mode of the perturbation excites a plane wave more than the other one, there is less shared energy between the two plane waves, and, so, there is linear instability of just one plane wave.

Besides to dispersive nonlinear multi-component systems mentioned above, there is a non-dispersive nonlinear multi-component system which can exhibit linear instability, that is the 3-wave resonant interaction (3WRI) system \cite{53}. The solitons of the 3WRI system interact in a different manner from the solitons of dispersive nonlinear systems, thus one can expect that the mechanism leading to the linear instability (and then to the possible formation of solitons) is different for the two kinds of systems. For this reason, we refer to this phenomenon as linear instability rather than as MI. Indeed, differently from the solitons of dispersive multi-component systems with quadratic coupling, originating from the balance between dispersion and nonlinearity \cite{54}, the solitons of the 3WRI system originate from the mismatch of the group velocities of the interacting wave packets, once provided the resonant conditions (see formulas (3.1) in Chapter 3) \cite{53}. In order to highlight how special the 3WRI system is, we remind the reader about the research work \cite{55} where the generation of dispersive shock waves was observed in absence of dispersion and whose behaviour

\footnote{These parameters are the wave numbers of the two component plane waves and of the disturbance.}
resembles the generation of shock waves in dispersive equations, such as the NLS and the KdV equation. Moreover, the role of the dispersion in the system was investigated in [56]. The authors studied the linear stability in space of a triplet of dark solitons and showed that the introduction of a quasi-negligible second order dispersion reduces the instability allowing the three dark solitons propagation. It is worth pointing out that there is no galilean transformation for which all the three waves have zero group velocity and the velocities mismatch allows the envelopes to overlap when the nonlinearity becomes important. Moreover, due to the fact that the 3WRI system is non-dispersive, in the asymptotic limits $t \to \pm \infty$, the envelopes are well separated and do not overlap; however, during the interaction, the envelopes can exchange both solitons and radiation. Thus, "the solitons and radiation (continuos spectrum) are on an equal footing" [57, 58]. This feature of the 3WRI system makes its soliton solutions remarkably different from those ones of other dispersive systems for which radiation decays as time goes on [44, 59]. In fact, solitons and radiation interact nonlinearly and radiation never decays asymptotically [57, 58]. Kaup and collaborators carried out the linear stability analysis of the 3WRI system when its solutions have a vanishing background, and gave necessary and sufficient conditions for linear instabilities to occur [57]. The carrier wave and the two sidebands have finite and infinitesimal amplitudes, respectively, and they interact under the three-wave resonant conditions, $\omega_1 + \omega_2 + \omega_3 = 0$ and $k_1 + k_2 + k_3 = 0$, where the frequencies $\omega_i$ and the wave numbers $k_i$ can take any value, not necessarily infinitesimally close with one another. These resonant conditions do not originate as a result of selfinteraction, but they must be written together with the 3WRI system [57, 58]. Under particular conditions, the carrier wave can exhibit explosive and the decay instability [60, 61], but only if the carrier wave travels with intermediate group velocity with respect to the group velocities of the two sidebands and if it possesses solitons. We stress that the solitons in question move on a vanishing background and can be obtained, in principle, via IST. In this case, we say that the carrier wave is linearly unstable [57]. Nevertheless, thanks to the lack of the dispersion, all we need to know is the linear behaviour at time $t \to -\infty$, and then "we can completely determine how this system evolves, even in nonlinear regime" [57], looking only at the time evolution of the scattering data. This allows to turn around the matter of solving the inverse problem in order to see if an envelope possesses solitons in the nonlinear 

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6We will clarify this concept in Chapter 3.
regime. Some works on the scalar NLS \cite{36, 37, 29} were carried out via this procedure. However, obtaining \( N \)-soliton solutions, especially for multi-component systems, can be rather difficult task to achieve. Hence, the necessity to develop mathematical tools to investigate the linear stability of multi-component systems. The application of perturbation theory to the IST can be extended, in principle, to investigate the linear stability of any integrable system, also those ones which are multi-component \cite{62, 63}. The starting point is still the Lax Pair. Once the scattering data are given by the \textit{direct problem}, one can construct the associated potential by the \textit{inverse problem}. Perturbing the direct and inverse problems, the linear perturbations of the potential are written in terms of the variations of the scattering data (and vice versa) by squared combinations of components of eigenfunctions and its adjoints: the so called \textit{squared eigenfunctions} (SEs) and the \textit{adjoint squared eigenfunctions} (ASEs) \cite{62, 63}. Although, the research works by Kaup and collaborators were carried out both on scalar and multicomponent systems, one needs to use the IST machinery to get a representation of the perturbation in terms of the SEs, and thus their expression in terms of the eigenmodes. It is clear that in the framework of the IST method, the stability analysis is cumbersome to apply to multicomponent systems, in particular for soliton solutions with a non vanishing background. However, the property of the SEs to be solutions of the linearised equation is local and it follows directly from the Lax pair without the need to apply the IST machinery (see Chapters 2 and 3). In this respect, recently a new spectral approach has been developed in \cite{64}. In this research work, the authors investigate the linear stability of continuous waves in all regimes in the framework of the integrability and their method can be applied to other more complicated solutions such as dark-dark, bright-dark, and higher-order solitons travelling on a continuos wave background.

\section{Motivations and Purposes}

In the recent years MI has been proposed as a possible mechanism for the generation of rogue waves. \"In oceanography, rogue waves are defined as waves whose height is more than twice the significant wave height, that is the average height of the highest one-third of the waves in a wave record\" \cite{66, 23}.\footnote{In oceanography, rogue waves are defined as waves whose height is more than twice the significant wave height, that is the average height of the highest one-third of the waves in a wave record\cite{66, 23}.}
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From an experimental point of view, researchers are interested on their reproducibility and observability in wave tank experiments. From a mathematical point of view, this is equivalent to study the linear stability of solutions which can describe the dynamics of rogue waves.

The dynamics and the physical features of rogue waves can be described by solutions of the NLS equation, such as the homoclinic orbits \[67\] of an unstable Stokes wave. Calini and collaborators \[69\] discovered that, beside the mechanism of MI, rogue waves are created as a result of phase modulation. Using the gauge form of the Bäcklund transformation \[68\], they constructed the associated solution of the linearised NLS equation, i.e. a squared eigenfunction and so they explored the stability of Stokes waves. Finally, they stated the following selection criterion for rogue waves: “among the homoclinic orbits of a Stokes wave with M unstable modes, the only ‘good’ candidate for rogue wave is the maximally iterated homoclinic orbit, with all its spatial modes coalesced through phase modulation” \[69\].

A similar investigation was performed on the spatially periodic breathers on an unstable plane wave background, obtaining a similar conclusion \[70\].

If one considers two coupled NLS equations (CNLS), besides the already mentioned non-focusing instabilities between two unstable and two stable CNLS fields \[71\] or between a stable and an unstable CNLS field \[52\], baseband MI can exists. It is triggered by zero-frequency disturbances, and it is believed to be responsible for the formation of rogue waves \[72, 73, 74\].

Between all the possible kinds of solutions suitable to model rogue waves, we focus on rational solutions: they are solutions with a rational, or semi-rational\(^7\) dependence on the variables \(x\) and \(t\), in contrast with the standard solitons whose expression is given in terms of exponentials only. They are also solutions of multicomponent wave equations such as the CNLS equation and the 3WRI system \[75, 76, 77\]. In this thesis we take into account the system of a resonant triad, not only because it encloses this kind of solitons, but even because the nonlinearity term is the simplest that can occur between three interacting waves \[78, 79\], and the interaction of waves under resonance conditions is of great interest in many fields of research (see, for instance, \[80, 81\] and the literature therein). We investigate the linear stability of the 3WRI model by considering solutions on finite background.

\(^7\)Here, semi-rational solutions are meant to have an expression that is both rational and exponential in the variables \(x\) and \(t\).
Due to the technical mathematical issues already stated in the previous section, the linear stability of this system and the mechanism leading to its linear instability are less studied than those of other systems, such as the CNLS equations. However, further outcomes can be obtained thanks to the approach developed in [64], and we have applied such an approach to the plane wave solutions of the 3WRI system in order to investigate their linear stability and to confirm if linear instability can be considered a necessary condition to explain the formation of rogue waves mathematically represented by rational solutions. In order to do that, we take advantage of the integrability of this system [82, 83]. This fact is not obvious at all, since the stability of a solution is a local phenomenon, while the integrability is a global characteristic of the equation. As it has already been mentioned (see [62, 63]), solutions of a linearised equation can be written by squared eigenfunctions, and, in turn, by means of the associated Lax pair. If an equation is Lax-integrable [84], the stability of its solutions can be analysed (see, for instance, [71] and [52]). Once the squared eigenfunctions are obtained, we look at their temporal behaviour: if they have an exponential growth in time (i.e. the gain function is different from zero), then the provided solution of the nonlinear equation is (neutrally) linearly unstable. We underline that this approach allows us to generalise the formalism to \(N \times N\) matrices. As a consequence, the solutions of the Lax pair equations are written in matrix form and the squared eigenfunctions are more general than those ones used so far.

In addition, we use the spectral method developed in [64] for a technical reason: IST is algebraically cumbersome and it becomes more and more complicated to apply if one wants to find solutions which are rational, semi-rational or, more generally, with a finite background. Indeed, it is necessary to solve the inverse problem to see if localised structures exist asymptotically (after the interaction). For the 3WRI system, the linear stability analysis which can lead to the formation of solitons on vanishing background has led to the conclusion that the envelope with intermediate velocity can decay surrounding all its \(N\) solitons and, after the interaction, the fast and the slow envelopes own additional \(N\) solitons each one. However, it has been impossible to find the final \(N\) soliton solutions which, instead, were written only in implicit form and the findings were supported by numerical analysis [58]. Instead, this alternative approach [64] allows us to use Darboux Dressing Transformation (DDT) [85, 86] for envelopes whose background are plane waves interacting resonantly, and in this respect, it turns out that DDT is potentially more useful and algebraically simpler.
for finding rational solutions [76]. Another reason for choosing this method is that this approach does not depend on the functional class of the potential, contrary to the IST, which requires the imposition of boundary conditions. Therefore, a wider class of solutions can be taken into account, such as rational or semi-rational solitons, whose dependence on space $x$ and time $t$ usually is asymptotically polynomial, but also other solutions such as breathers. We have numerically observed the generation of potential rogue waves and breathers generated via the linear instability of such a system. This remarkable observation would not have been possible without the application of the approach [64], which is tailor-made for multi-component systems with solutions on finite background.

In this thesis, first, we carry out a comprehensive topological classification of the stability spectra of the 3WRI system in the parameters space, where the parameters in question are combinations of the physical parameters involved in the plane waves and in the system. This classification is topological and we associate a gain function to any topology. We observe that the stability spectra of the CNLS system are included in this classification and, indeed, the gain function shows the presence of MI-baseband-like and MI-passband-like for such spectra. However, the stability analysis for the 3WRI system is richer, because it presents additional topological features. Indeed, we observe a new gain function associated to such topological features which can be considered neither a MI-passband type nor a MI-baseband type, and are, instead, associated to a stronger linear instability around the zero wave number. In particular, we have observed, via numerical simulations, that this kind of topology and its gain is associated to explosive instability, i.e. the three interacting waves blow up in a finite time [60, 61].

We show that the plane wave solutions of the 3WRI system are linearly unstable in time for any choice of the physical parameters, including those ones associated to the solutions that are explosive. The linear instability of the plane wave correspond to the observation of a positive gain-function. Although the onset of the linear instability of the 3WRI system has not been clarified, this thesis is meant to be a prelude for its understanding. Indeed, we aim to investigate the mechanism leading to this phenomenon in future research works. It is worth highlighting that, in the research work [87], the authors have analysed the stability of two coupled ordinary differential equations (instead of three PDEs) whose forcing term is the interaction with a third wave, with large amplitude, which does not obey the same dynamical equa-
tions of the other two interacting waves with finite amplitudes [88]. They speculate that the third wave is a cosine as a function of the time. The authors claimed that a sufficient and necessary condition for the onset of instability is that the two interacting waves have modes of opposite sign.

In this thesis, we perform the linear stability analysis on three coupled PDEs, and after linearising every equation, the wave solution interacts with the perturbations of the other two waves. Moreover, every solution propagates with different velocity and we cannot find a reference frame in which they have the same velocity. Instead, we can obtain a system of three ordinary differential equations only in the particular case in which the three waves propagate at the same velocity [57]. For the 3WRI system with wave solutions propagating at the same velocity, the Hasselmann’s criterion [89, 90] states that instability occurs if the two sidebands modes sum together with the same signs, and in the case they interact resonantly with the same signs, there is neutral stability.

Hence, we underline the relation between the presence of baseband MI type and the possible existence of rational solitons which can model rogue waves, and in this regard, we provide a necessary condition for the existence of rational solutions on a finite background constructed by means of the DDT method. For sake of simplicity, we will refer sometimes to these rational and semi-rational solutions as rational solutions of Darboux type or DT-FB rational solutions, FB standing for “finite background”. We do that by the stability analysis of the plane waves which are solutions of the 3WRI system. These plane waves are meant as the possible background for solitons, namely as the seed solutions for the algebraic construction of rational solutions of Darboux type. When needed, we will specify if we deal with semi-rational solutions, instead of purely rational solutions.

The motivation to conduct this research is because, so far, the mechanism which causes rogue waves formation is unknown, although several hypothesis have been formulated and MI (like passband and baseband) is one of them [91], we assume that their onset can be due to the more general phenomenon of the linear instability, that is a phenomenon that can occur in the 3WRI system. Moreover, rogue waves are ubiquitous in nature and are observed in several physical settings, such as in water tank [92], in fibre optics [24] and in plasma [93]. They are also predicted in the atmosphere [94], in superfluids [95], in Bose-Einstein condensates [96] and in capillary waves [97].
1.3 Overview of the thesis

In Chapter 2, we describe the spectral method developed in [64] and we use the NLS equation as a case study.

In Chapter 3, we give the 'set up' of the formalism for the 3WRI system: Lax pair and Lax operators, construction of the squared eigenfunctions, characteristic polynomials and useful rescalings of the physical parameters of the problem.

In Chapter 4, we conduct the stability linear analysis through the topological classification of the stability-spectra in the parameters space and the associated gain functions. Thus, we state a necessary criterion for the existence of rational and semi-rational solitons on a finite background of Darboux type. However, the computations of the expressions of these solutions is not the aim of this thesis.

In Chapter 5, we show some numerical observations of breathers and potential rogue waves.
Chapter 2

Linear Stability of Plane Waves of the NLS and CNLS Systems

In this chapter we present the formalism developed in [64] by providing the NLS equation as an example. Because NLS is a scalar equation, the space stability spectrum coincides with the spectrum of the spatial part of the Lax pair [64]. This coincidence is not met in the general case of the multi-component systems. Thus, in order to understand where the spatial stability spectrum comes from, we provide the CNLS system as a further example [64]. In this way we are able to write the general definition of stability spectra for multi-component systems.

2.1 Universal Nature of the Nonlinear Schrödinger Equation and Modulational Instability

The origin of the Nonlinear Schrödinger equation is rooted in the theory of self-focusing waves in electrical fields. In 1964, R. Y. Chiao et al. wrote a NLS-type equation [98]. They considered the wave equation for an electric field plus a nonlinear term and concluded that an optical beam of a single frequency, whose dynamics is described by such equation, cannot spread in a nonlinear media, namely the beam is self-focused. Since then, more investigations were carried out on the self-focusing phenomenon (see, for instance, [99, 100])
and, in this context, the NLS equation was derived in [99] as we know it today,

\[ u_t - iu_{xx} + 2is|u|^2u = 0, \quad s^2 = 1, \]  

(2.1)

where the subscripts denote the derivatives with respect to the space \( x \) and the time \( t \), \( |\cdot| \) is the module of the classical field \( u(x,t) \), and \( s \) is a sign.

The NLS equation with the sign \( s = -1 \) is known as focusing NLS equation and was written for the first time in the research paper on the self-focusing optical beams in dispersive and nonlinear media [99]. Using the IST method, Zakharov and Shabat derived its solutions classified as bright solitons which decay at zero at the spatial infinity, and as breathers which decay at a constant background at the spatial infinity [29].

For \( s = 1 \), the equation is named defocusing NLS. Its solutions are known as dark solitons due to their feature to have a nontrivial background intensity and a spatial local dip. They were obtained by IST in [101].

The universal nature of the NLS equation lies in the fact that it describes the dynamics of many systems in nature whose behaviour is that one of an envelope of a monochromatic wave packet in a dispersive and nonlinear media when the dissipation can be neglected [19].

Zakharov was the first to derive the NLS equation in the context of water waves [103]. Nevertheless, it can be derived in the limit of weak nonlinearity from several equations via the multiple scale method, for instance, by the Sine-Gordon equation [15]. The defocusing NLS equation can also be derived from the KdV equation and the focusing NLS equation from the modified-KdV (see for example [102]).

After Stokes wrote the approximate solutions of the Laplace problem [22], i.e. the Stokes waves, Benjamin and Feir discovered that nonlinear Stokes waves are modulationally unstable [13]. It is well known the pivotal role of the NLS equation in the study of MI and the NLS equation is one of the most used equation for modelling modulation of waves. In this regard, we refer the reader to the book [15] for the linear stability analysis of plane waves, solutions of the NLS equation.

Because the NLS equation had a key role in the understanding of the MI and because it is a scalar equation, in the next section, we shall describe the spectral method developed in [64], by using the NLS equation as a case study.
2.2 Matrix Form of the Nonlinear Schrödinger Equation

Before we proceed with the stability analysis, we make a ’set up’ for a new formalism in which the NLS is meant as a matrix equation, where the matrices involved are $2 \times 2$. In order to do that, we take advantage of mathematical techniques provided by the integrability. Indeed, since the NLS equation is integrable [29], it admits a representation via two differential equations, which are called Lax equations,

$$
\psi_x = X\psi, \quad \psi_t = T\psi, \quad (2.2)
$$

where the subscripts are the derivatives with respect to the spatial variable $x$ and the time variable $t$, $X$ and $T$ are $2 \times 2$ matrices named Lax operators, being $\psi = \psi(x, t, \kappa)$ a common solution of the two linear differential matrix equations (2.2), while $X = X(x, t, \kappa)$ and $T = T(x, t, \kappa)$ depend on the variables $x$, $t$ and on a complex quantity $\kappa$ called spectral parameter, according to the definitions

$$
X = i\kappa \sigma_3 + Q, \quad (2.3a)
$$

$$
T = 2i\kappa^2 \sigma_3 + 2\kappa Q + i\sigma_3(Q^2 - Q_x), \quad (2.3b)
$$

where

$$
\sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad (2.4)
$$

is a Pauli matrix. The matrix $Q = Q(x, t)$ depends on the variables $x$ and $t$, contains the complex dynamical variable $u = u(x, t)$ and introduces the sign $s^2 = 1$

$$
Q = \begin{pmatrix}
0 & su^* \\
u & 0
\end{pmatrix}, \quad (2.5)
$$

where $u^*$ is the complex conjugate of the dynamical variable $u$. We are interested in nontrivial common solutions $\psi$ of the Lax equations (2.2), which are given by the compatibility condition

$$
\psi_{xt} = \psi_{tx}. \quad (2.6)
$$
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The NLS equation can be obtained by (2.6), because it is integrable. By using both (2.2) and (2.6), we get an equation for the Lax operators $X$ and $T$

$$X_t - T_x + [X, T] = 0,$$

(2.7)

where $[X, T] = XT - TX$ is the commutator of the two matrices $X$ and $T$. Finally, by the definitions of the Lax operators (2.3) and using the fact that

$$Q^2 = s||Q||^2 I_{2\times2},$$

(2.9)

where $I_{2\times2}$ is the $2 \times 2$ identity matrix and $|| \cdot ||$ denotes the spectral norm of $Q$, we obtain the scalar NLS equation in matrix form

$$Q_t + i\sigma_3 Q_{xx} - 2i\sigma_3 ||Q||^2 Q = 0.$$

(2.10)

In computing the equation (2.10), we have used the property of the Pauli matrix $\sigma_3$ to be involutory, that is $\sigma_3^2 = I_{2\times2}$. Moreover, the anti-commutators $\{\sigma_3, Q\} = 0$ and $\{\sigma_3, Q_x\} = 0$. The anti-commutator between two matrices $A$ and $B$ is defined as $\{A, B\} = AB + BA$. One can check that (2.10) includes the NLS equation in scalar form. By writing (2.10) as follows

$$\begin{pmatrix}
0 & s(u^*_t + i u^*_x - 2isu^2) \\
u_t - iu_{xx} + 2isu^*u^2 & 0
\end{pmatrix} = 0_{2\times2},$$

(2.11)

where $0_{2\times2}$ denotes the null matrix $2 \times 2$, hence,

$$u^*_t + i u^*_x - 2isu^2 = 0.$$

(2.12a)

\footnote{By (2.2), the left-hand side and the right-hand side of the condition (2.6) are rewritten in the following way

$$\psi_{xt} = (X\psi)_t = X_t\psi + X\psi_t = X_t\psi + XT\psi,$$

(2.8a)

and

$$\psi_{tx} = (T\psi)_x = T_x\psi + T\psi_x = T_x\psi + TX\psi.$$}

The spectral norm of $Q$ is defined as

$$||Q|| = \sqrt{\tau_{\max}}, \quad Q^\dagger = \begin{pmatrix} 0 & u^* \\ su & 0 \end{pmatrix},$$

where $\tau_{\max}$ is the biggest eigenvalue of the matrix $QQ^\dagger$, where $Q^\dagger$ is the conjugate transpose of the matrix $Q$.\footnote{The spectral norm of $Q$ is defined as

$$||Q|| = \sqrt{\tau_{\max}}, \quad Q^\dagger = \begin{pmatrix} 0 & u^* \\ su & 0 \end{pmatrix},$$

where $\tau_{\max}$ is the biggest eigenvalue of the matrix $QQ^\dagger$, where $Q^\dagger$ is the conjugate transpose of the matrix $Q$.}
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\[ u_t - iu_{xx} + 2isu^* u^2 = 0, \quad (2.12b) \]

where the equation (2.12a) is the complex conjugate of the equation (2.12b). In particular, (2.12b) is the NLS equation for \( u \) and (2.12a) is the NLS equation for \( u^* \).

2.3 Integrability and Linear Stability

In addition to the matrix NLS equation, we need to introduce Lax operators which are independent of the spatial variable \( x \) and of the time variable \( t \). In this way, the resulting Lax equations are integrable by the separation of variables method and the obtained solution is mapped into the solution of the original Lax problem [31].

2.3.1 Lax Problem Revisited

Let us consider the simplest solution of the NLS equation (2.12b), namely the plane wave

\[ u_0(x,t) = ae^{i(qx - \eta t)}, \quad (2.13) \]

depending on \( x \) and \( t \), where \( a \) is the amplitude, \( q \) is the wave number and \( \eta \) is the frequency of the wave solution. Let \( a \in \mathbb{R} \) be independent of the variables \( x \) and \( t \). The frequency depends on \( a \) and \( q \) by the relation

\[ \eta = q^2 + 2sa^2, \quad (2.14) \]

that is obtained by substituting (2.13) in the NLS equation (2.12b).

In order to simplify the calculations, we introduce the transformation on the matrix solution \( Q_0 = Q_0(x,t) \),

\[ Q_0 = \begin{pmatrix} 0 & su_0^* \\ u_0 & 0 \end{pmatrix} = G \begin{pmatrix} 0 & sa \\ a & 0 \end{pmatrix} G^{-1}, \quad (2.15) \]

where \( u_0 = u_0(x,t) \) is the plane wave given in (2.13), \( u_0^* \) is its complex conjugate, \( G = G(x,t) \) is a diagonal matrix which contains all the dependence on the variables \( x \) and \( t \),
and $G^{-1} = G^{-1}(x, t)$ is its inverse. More in detail, the matrix $G$ can be written as follows (see Appendix A):

$$G = e^{-i|q x - (q^2 + 2sa^2)t|} \sigma_3. \quad (2.16)$$

The transformation (2.15) induces also a transformation on the solution $\psi$ of the Lax pair (2.2),

$$\psi = G \phi, \quad (2.17)$$

such that it introduces the function $\phi = \phi(x, t)$.

By putting the transformation (2.17) into the the Lax pair (2.2), we obtain the PDEs (see Appendix B):

$$\phi_x = iW \phi, \quad \phi_t = -iZ \phi, \quad (2.18)$$

where the operators $W = W(\kappa)$ and $Z = Z(\kappa)$ are defined as follows

$$iW = G^{-1}XG - G^{-1}G_x, \quad (2.19a)$$

$$-iZ = G^{-1}TG - G^{-1}G_t. \quad (2.19b)$$

The operators $W$ and $Z$ in (2.19) are $2 \times 2$ matrices independent of $x$ and $t$ and depending only on the spectral parameter $\kappa$, whose expressions, in terms of their entries, are

$$W = \begin{pmatrix} \kappa + \frac{q}{2} & -isa \\ -ia & -\kappa - \frac{q}{2} \end{pmatrix}, \quad Z = \begin{pmatrix} -2\kappa^2 + \frac{q^2}{2} & isa(2\kappa - q) \\ ia(2\kappa - q) & 2\kappa^2 - \frac{q^2}{2} \end{pmatrix}, \quad (2.20)$$

and, in addition, they are related with one another is

$$Z = -2 \left( \kappa - \frac{q}{2} \right) W. \quad (2.21)$$

---

We have used the formulas

$$G^{-1}G_x = -\frac{i}{2} q \sigma_3, \quad G^{-1}G_t = \frac{i}{2} (q^2 + 2sa^2) \sigma_3,$$

and the anti-commutators

$$\{\sigma_3, Q\} = 0.$$
The advantage of the revisited Lax Pair (2.18) is that we can integrate it by separation of variables and a solution is

$$\phi = e^{i(Wx-Zt)}, \quad (2.22)$$

where we have used the fact that $[W, Z] = 0$. In turn, by using the transformation (2.17), the solution of the Lax Pair (2.2) is

$$\psi = e^{-\frac{i}{2}(q(x-(q^2+2\alpha^2)t))\sigma_3} e^{i(Wx-Zt)}.$$  

(2.23)

2.4 Investigating Stability via Lax Pair

The perturbations of the NLS equation can be written as combinations of the SEs which are solutions of the linearised equation [44]. Since the SEs are written starting from the Lax operators, it turns out that one can characterise the solutions of the linearised equation by using such operators.

2.4.1 Squared Eigenfunctions

The starting point of our investigation is the Lax problem (2.2) for the NLS equation. Using the solutions $\psi$ and $\psi^{-1}$ of the Lax problem (2.2), we define the SE $\Psi = \Psi(x, t)$

$$\Psi = \psi M \psi^{-1}, \quad (2.24)$$

where $M = M(\kappa)$ is a $2 \times 2$ matrix dependent only on the spectral parameter $\kappa$. By (2.2), $\Psi$ satisfies the PDEs [64]

$$\Psi_x = [X, \Psi], \quad \Psi_t = [T, \Psi], \quad (2.25)$$

which are compatible with one another because of (2.7).

The transformation (2.17) induces the similarity transformation

$$\Psi = G \Phi G^{-1}, \quad (2.26)$$
where the matrix $G$ is given in (2.16), and $\Phi = \Phi(x, t)$, defined as

$$\Phi := \phi M \phi^{-1},$$  

(2.27)

satisfies the PDEs (Appendix C)

$$\Phi_x = i[W, \Phi], \quad \Phi_t = -i[Z, \Phi],$$  

(2.28)

whose solution is

$$\Phi = e^{i(Wx-Zt)} \Phi(0, 0) e^{-i(Wx-Zt)}.$$  

(2.29)

where $\Phi(0, 0)$ is the initial condition at $x = 0$ and $t = 0$. Because of (2.27) and (2.22), $\Phi(0, 0) \equiv M(\kappa)$. Finally, because of the transformations (2.26) and (2.17), the SE, solution of the PDEs (2.25), reads

$$\Psi = G \phi \Phi(0, 0) \phi^{-1} G^{-1} = \psi \Phi(0, 0) \psi^{-1},$$  

(2.30)

where $\psi$ is provided by (2.23).

### 2.4.2 Solution of the Linearised Equation and its Connection with Integrability

In this section we introduce the linearised equation (LE) obtained perturbing a generic solution of the NLS.

Let us suppose $\Phi(x, t) = \alpha(x) \beta(t)$, by integrating in $x$ the first of the equations (2.28)

$$\alpha(x) = \alpha(0) + i \int_0^x [W, \alpha(x_1)] dx_1,$$

and, via iteration,

$$\alpha(x) = \alpha(0) + i \int_0^x [W, \alpha(0)] dx_1 - \int_0^x W \left[ \int_0^{x_1} [W, \alpha(x_2)] dx_2 \right] dx_1 = \alpha(0) + i \int_0^x [W, \alpha(0)] dx_1 - \int_0^x \left[ W, \int_0^{x_1} [W, \alpha(x_2)] dx_2 \right] dx_1.$$

Similarly, by integrating in $t$ the second of the equations (2.25), one gets

$$\beta(t) = \beta(0) - i[Z, \beta(0)] t - \int_0^t \left[ Z, \int_0^{t_1} [Z, \alpha(t_2)] dt_2 \right] dt_1,$$

thus,

$$\Phi(x, t) = \alpha(x) \beta(t) = \alpha(0) \beta(0) + i \int_0^x [W, \alpha(0)] \beta(0) dx - i[Z, \alpha(0)] \beta(0) t + \ldots,$$

and, by setting $\alpha(0) \beta(0) = \Phi(0, 0)$, the solution is (2.29).
Let us suppose to have a solution of a nonlinear equation and to add to it a small perturbation, so that the perturbed solution is \( u + \delta u \). As a consequence, the matrix solution is perturbed

\[
Q \rightarrow Q + \delta Q, \tag{2.31}
\]

and the perturbed Lax operators are

\[
X \rightarrow X + \delta X, \quad T \rightarrow T + \delta T. \tag{2.32}
\]

By substituting (2.32) in the equations (2.7), at the first order in the perturbation, we get the LE

\[
(\delta X)_t - (\delta T)_x + [\delta X, T] + [X, \delta T] = 0, \tag{2.33}
\]

which is an evolution equation for the perturbation \( \delta Q \). Moreover, we stress that the expression of the linearised equation, as it is written in (2.33), is independent of the model until one chooses the Lax pair, namely the matrix \( Q \) and, for this reason, the LE (2.33) is written more generally as follows

\[
A_t - B_x + [A, T] + [X, B] = 0. \tag{2.34}
\]

We are interested in searching for solutions \( A = A(x, t, \kappa) \) and \( B = B(x, t, \kappa) \) related to the fundamental matrix solution \( \psi \) of the Lax pair. In this respect, we provide the following propositions, which are also stated in [64].

**Proposition 2.4.1.** If the pair \( A, B \) solves the linearised equation (2.34), then also the pair \( F = F(x, t), H = H(x, t) \) defined as

\[
F = [A, \Psi], \quad H = [B, \Psi], \tag{2.35}
\]

satisfies the linearised equation (2.34), namely

\[
F_t - H_x + [F, T] + [X, H] = 0. \tag{2.36}
\]

This is a consequence of the Jacobi identity and of the fact that \( \Psi \) is a solution of (2.25) [64].
Proposition 2.4.2. The following expressions

\[ F = \left[ \frac{\partial X}{\partial \kappa}, \Psi \right], \quad H = \left[ \frac{\partial T}{\partial \kappa}, \Psi \right], \quad (2.37) \]

are solutions of the linearised equation (2.36).

The validity of this statement follows from the fact that the matrices

\[ A = \frac{\partial X}{\partial \kappa}, \quad B = \frac{\partial T}{\partial \kappa}, \quad (2.38) \]

are solutions of the equation (2.34) and from the Proposition 2.4.1 [64].

A consequence of the Proposition 2.4.2 is that any sum or integral of \( F \) over the spectral variable \( \kappa \) is a solution \( \delta Q \) of the LE (2.33). As in the paper [64], we assume that the perturbation \( \delta Q \) has the integral representation

\[ \delta Q = \int d\kappa F(x, t, \kappa), \quad (2.39) \]

which provides a solution \( \delta Q \) bounded and localised in \( x \) at any fixed time \( t \). We require that the perturbation is localised, so that the absolute value of the perturbed solution goes to a constant as \( |x| \to \infty \).

The matrix \( F(x, t, \kappa) \) and the perturbation \( \delta Q \) satisfy the same linearised equation (2.33) provided that only local terms are involved in their expressions [64]. The solution \( F \) plays the same role as the exponential solution of any linearised equation with constant coefficients, namely, by varying \( \kappa \) over the spectrum, it provides the set of "Fourier-like" modes of the linear PDE (2.33) and it takes the general expression [64]

\[ F(x, t, \kappa) = G(x, t) \sum_{j, m=1}^{N} \mu_{jm}(\kappa) e^{i\left[(x(w_j - w_m) - t(z_j - z_m))\right]} F^{(jm)}(\kappa) G^{-1}(x, t), \quad (2.40) \]

where \( w_j \) and \( z_j \) are the eigenvalues of the matrices \( W \) and \( Z \), respectively and \( N \) is the dimension of the matrices \( W \) and \( Z \). The coefficients \( \mu_{jk}(\kappa) \) are arbitrary functions of the spectral parameter \( \kappa \), whereas \( F^{(jm)}(\kappa) \) constitute a basis (we will obtain explicitly the formula (2.77) in the section 2.5).

If we want \( \delta Q \) to be bounded, then the solution \( F \) must be bounded in \( x \) for any fixed \( t \) and \( \kappa \), and the subset of the complex \( \kappa \)-plane, over which the integral (2.39) runs, constitutes the so-called stability \( x \)-spectrum of the solution \( Q \), denoted by \( S_x \) [64]. The spatial spectrum \( S_x \) can be geometrically defined as follows:
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Definition 2.4.3. The x-spectrum \( S_x \), namely the spectral curve on the complex \( \kappa \)-plane, is the set of values of the spectral variable \( \kappa \) such that at least one of these complex numbers

\[ k_j = w_{j+1} - w_{j+2}, \quad j = 1, 2, ..., N \text{ (mod } N) \]

is real.

Similarly, one can define the stability t-spectrum, denoted by \( S_t \), for which the perturbation \( \delta Q \) and the solution \( F \) is bounded and localised in time for any fixed \( x \) and \( \kappa \). Here, we are interested in finding the values of the spectral parameter \( \kappa \) for which the perturbation \( \delta Q \) is bounded in space but it can grow up in time, such that the solution \( Q \) can even be linearly unstable in time [64]. In the case of the NLS equation, we will do that by finding both the \( S_x \) and \( S_t \) spectrum. The values of \( \kappa \) belonging to the \( S_x \) spectrum, but not to the \( S_t \) spectrum are those ones for which linear instability occurs. We highlight that, for the scalar NLS equation, the spectrum of the operator \( \frac{d}{dx} - X \) and the spectrum of the operator \( \frac{d}{dt} - T \) coincide with the stability spectrum \( S_x \) and with the stability spectrum \( S_t \), respectively (see [64] for more details about this point).

In the following, we find the general expression of \( F \) for the NLS equation. Although \( H \) is also a solution of the LE, we choose to work with \( F \) only for the sake of simplicity. Because we are in the case of \( 2 \times 2 \) matrices, we use the algebraic basis generated by the Pauli matrices.

From (2.37), the solution \( F \) can be written as

\[ F = 2i\sigma_3 \Psi^{(o)}, \quad (2.41) \]

where we have used the commutation rules between the Pauli matrices. Therefore, we are interested only on the computation of the off-diagonal part of the SE \( \Psi \); in particular, because \( G \) is a diagonal matrix \(^5\) we consider only the following SE:

\[ \Psi^{(o)} = G\Phi^{(o)}G^{-1}, \quad (2.42) \]

where \( \Phi^{(o)} = e^{i(W_0x-Z_0t)}\Phi^{(o)}(0,0)e^{-i(W_0x-Z_0t)} \) (see [64] for more details about the procedure for obtaining this solution after the diagonalisation of the matrices \( W \) and \( Z \)), where \( W_0 \) and \( Z_0 \) are the diagonalised matrices of \( W \) and \( Z \), and the latter are simultaneously

\(^5\)By the definition of \( \Psi \) (2.26), we have

\[ \Psi = G\Phi G^{-1} = G(\Phi^{(o)} + \Phi^{(d)})G^{-1} = G\Phi^{(o)}G^{-1} + \Phi^{(d)}, \]

where \( \Phi^{(o)} \) and \( \Phi^{(d)} \) are the off-diagonal part and the diagonal part of the matrix \( \Phi \), respectively.
diagonalised because they commute with one another. We cannot have the term proportional to $\Phi^{(d)}(0, 0)$, because, when $x = 0$ and $t = 0$, the solution must be still off-diagonal. Thus, we impose $\Phi^{(o)}(0, 0) \equiv \Phi(0, 0)$ and $\Phi^{(d)}(0, 0) \equiv 0$. Moreover, if the matrices $W$ and $Z$ were not diagonalised, then $\Phi(x, t)$ would have also had entries on its diagonal part, but we want to take only its off-diagonal entries. Indeed, once put $\Phi^{(o)}(0, 0) \equiv \Phi(0, 0)$, one can check this by assuming that the exponential $e^{i((W^{(d)}+W^{(o)})x - (Z^{(d)}+Z^{(o)})t)}$ can be approximated by the Taylor series $I_2 \times I_2 + i((W^{(d)}+W^{(o)})x - (Z^{(d)}+Z^{(o)})t)$ + .... and, then, by substituting it into the expression of $\Phi(x, t)$.

Let us suppose that the expression of the initial condition is

$$\Phi^{(o)}(0, 0) = \alpha \sigma_1 + \beta \sigma_2,$$

where $\alpha$ and $\beta$ are arbitrary constants, and $\sigma_1$ and $\sigma_2$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. $$

Let $W_0$ and $Z_0$ be proportional to $\sigma_3$. Indeed, because of the property of the trace to be invariant under cyclic permutations, it turns out that the trace of the similarity transformations for diagonalising the matrices $W$ and $Z$ in \eqref{2.19} is zero as well as $W_0$ and $Z_0$ are $2 \times 2$ traceless matrices.

Since the off-diagonal Pauli matrices, $\sigma_1$ and $\sigma_2$, anti-commute with the diagonal Pauli matrix $\sigma_3$, we get

$$\Psi^{(o)} = \alpha \sigma_1 e^{-2i(W_0 x - Z_0 t)} e^{i(q x - (q^2 + 2a^2) t) \sigma_3}, \quad \text{if} \quad s = +1,$$

$$\Psi^{(o)} = \beta \sigma_2 e^{-2i(W_0 x - Z_0 t)} e^{i(q x - (q^2 - 2a^2) t) \sigma_3}, \quad \text{if} \quad s = -1. $$

We have supposed $\beta = 0$ for the defocusing case $s = +1$ in \eqref{2.44}, and $\alpha = 0$ for the focusing case $s = -1$ in \eqref{2.45}. This is because $Q$ is an off-diagonal matrix equals to $Q = a a_1 e^{i(q x - (q^2 + 2a^2) t) \sigma_3}$ if $s = +1$, and equals to $Q = -i a_2 e^{i(q x - (q^2 - 2a^2) t) \sigma_3}$ if $s = -1$, and we choose to perturb with a matrix $\delta Q$ with amplitude proportional to $\sigma_1$ if $s = +1$ or proportional to $\sigma_2$ if $s = -1$. Moreover, since $F$ satisfies the same evolution equation of $\delta Q$ and they are related with one another in \eqref{2.39}, then the perturbation $\delta Q$ and $F$ must be proportional to the same Pauli matrix. In turn, $F$ is given in \eqref{2.41}, so that the squared
eigenfunctions must be those in (2.44) and (2.45). Finally, using the commutation and anti-commutation rules of the Pauli matrices applied to the expression (2.41), the solution of the LE is

\[ F = -2\alpha\sigma_2 e^{-2i(W_0x - Z_0t)} e^{i(qx - (q^2 + 2a^2)t)}\sigma_2, \quad \text{if } s = +1, \tag{2.46} \]

\[ F = 2\beta\sigma_1 e^{-2i(W_0x - Z_0t)} e^{i(qx - (q^2 - 2a^2)t)}\sigma_3, \quad \text{if } s = -1. \tag{2.47} \]

We observe that, in the case of the NLS equation written via \(2 \times 2\) matrices, the differences of the eigenvalues \(w_j\) and \(z_j\) are \(w_j - w_{j+1} = 2w_j\) and \(z_j - z_{j+1} = 2z_j\) for \(j = 1, 2 \mod(2)\), because of the commutation rules of the Pauli matrices. This does not hold for matrices \(W\) and \(Z\) whose dimension is \(N > 2\).

\[ \text{2.4.3 Spectral Stability Analysis} \]

The matrices \(W\) and \(Z\) are simultaneously diagonalised and are related one with the other by means of formula (2.21) and thus their eigenvalues are related as well \(^6\)

\[ w = \pm \sqrt{\kappa^2 - sa^2}, \quad z = -2\kappa w, \tag{2.48} \]

where we are considering the plane wave with \(q = 0\).

In both the focusing and defocusing (\(s = -1\) and \(s = +1\)) cases, the eigenvalues \(w\) and \(z\) are reals if and only if \(\kappa \in S_x\) and \(S_t\), respectively. In the defocusing case \(s = +1\) the solution \(u = ae^{-2ia^2t}\) is linearly stable because the \(x\)-spectrum

\[ S_x = \{ -\infty < \kappa \leq -a \} \oplus \{ a \leq \kappa < +\infty \}, \tag{2.49} \]

is included in the spectrum of the operator \(T\)

\[ S_t = S_x \oplus \{ \kappa = i\gamma : -\infty < \gamma < +\infty \}. \tag{2.50} \]

Both the spectra are shown in the figure 2.1

In the focusing case \(s = -1\), the solution \(u = ae^{2ia^2t}\) is linearly unstable. Indeed, the

\[^6\text{In order to obtain the eigenvalues } w \text{ and } z, \text{ we have solved the equations} \]

\[ \text{Det}(W - w I_{2\times2}) = 0, \quad \text{Det}(Z - z I_{2\times2}) = 0, \]

with unknowns \(w\) and \(z\), and \(\text{Det}(\cdot)\) is the determinant.
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Figure 2.1: Defocusing case, $a=1$.

$x$-spectrum is

\[ S_x = \{ -\infty < \kappa < +\infty \} \oplus \{ \kappa = -i\gamma : -a \leq \gamma \leq +a \}, \]  \hspace{1cm} (2.51)

while $t$-spectrum is

\[ S_t = \{ -\infty < \kappa < +\infty \} \oplus \{ \kappa = i\gamma : \gamma < -a \text{ or } \gamma > +a \}. \]  \hspace{1cm} (2.52)

In this case the branch $-a < \gamma < +a$ belongs to $S_x$ but not to $S_t$, as shown in the figure 2.2. This is related to the MI phenomenon, indeed, in such a case, the squared eigenfunctions in (2.46) and (2.47) grow exponentially as time goes on.

Figure 2.2: Focusing case, $a=1$. 
2.5 Wave Coupling and Solution of the Linearised Equation for the CNLS System

In the following we give an introduction to the linear stability problem of the plane wave solutions of the CNLS system within the integrability framework to prove that the main objects to be computed are the eigenmodes’ wave numbers and frequencies defined on the stability spectrum. In the presentation, we will follow [64].

For the CNLS system, we start by choosing the Lax operators $X(x, t, \kappa)$ and $T(x, t, \kappa)$ to be

$$
X(\kappa) = i\kappa \Sigma + Q, \quad T(\kappa) = 2i\kappa^2 \Sigma + 2\kappa Q + i\Sigma(Q^2 - Q_x),
$$

(2.53)

where $\Sigma, Q$ are matrix-valued functions of $x$ and $t$, and $\kappa$ is the spectral parameter, and

$$
\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$

(2.54a)

$$
Q = \begin{pmatrix} 0 & v_1^* & v_2^* \\ u_1 & 0 & 0 \\ u_2 & 0 & 0 \end{pmatrix}.
$$

(2.54b)

Here and below the asterisk denotes complex conjugation and the four field variables $u_1, u_2, v_1, v_2$ are considered as independent functions of $x$ and $t$, and are conveniently arranged as two two-dimensional vectors, that is

$$
u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
$$

(2.55)

Then the matrix PDE (2.7) becomes, in this case,

$$
Q_t = -i\Sigma(Q_{xx} - 2Q_x),
$$

(2.56)

which is equivalent to the two vector PDEs

$$
\begin{align*}
u_t &= i[u_{xx} - 2(v^\dagger u)u] \\
v_t &= i[v_{xx} - 2(u^\dagger v)v].
\end{align*}
$$

(2.57)
equivalent to the CNLS system upon setting

\[ v_1 = s_1 u_1, \quad (2.58a) \]

\[ v_2 = s_2 u_2. \quad (2.58b) \]

Here the dagger notation denotes the Hermitian conjugation (which takes column-vectors into row-vectors). In this simpler setting, if \( Q(x,t) \) is a given solution of the equation \((2.56)\), the linearised equation \((2.33)\) for a small change \( \delta Q(x,t) \) reads

\[ \delta Q_t = -i \Sigma [ \delta Q_{xx} - 2(\delta QQ^2 + Q\delta QQ + Q^2 \delta Q)]. \quad (2.59) \]

Moreover, the Propositions 2.4.1 and 2.4.2 are still satisfied for the CNLS system (and any multi-component system, satisfying certain hypothesis, see [64]), and the fact that the perturbation \( \delta Q \) and the solution \( F \) are linked by the integral \((2.39)\) guarantees that the matrix \( F(x,t,\kappa) \) satisfies this same linear PDE, namely

\[ F_t = -i \Sigma [F_{xx} - 2(FQ^2 + QFQ + Q^2 F)], \quad (2.60) \]

and, for \( \kappa \in S_x \), these solutions should be considered as eigenmodes of the linearised equation.

The spectral analysis is based on the following

**Proposition 2.5.1.** The matrix

\[ F = i[\Sigma, \Psi], \quad (2.61) \]

defined in the Proposition 2.4.2 along with \((2.53)\), in the case of the CNLS system, satisfies the same linear equation satisfied by \( \delta Q \).

The Proposition 2.5.1 has been specialised to the case of the CNLS equation, but it is stated in a general form in [64] (see Proposition 4 of the paper), in which it is formulated for any multi-component system provided that each term in the expression of the Lax operators has a local character.

We can compute analytically the matrix \( F \) if the fundamental matrix solution \( \psi(x,t,\kappa) \) of
the Lax pair corresponding to the solution $Q(x,t)$ is explicitly known. Here we devote our attention to the stability of the plane wave solution of (2.56), or of the equivalent vector system (2.57).

$$u(x,t) = e^{i(qx-\nu t)}a, \quad v(x,t) = e^{i(qx-\nu t)}b, \quad \nu = q^2 + 2b^1a.$$  \tag{2.62}

In these expressions $a$ and $b$ are arbitrary, constant and, with no loss of generality, real 2-dim vectors:

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.  \tag{2.63}$$

The plane wave solution of the CNLS system is obtained by setting

$$b_1 = s_1 a_1, \tag{2.64a}$$

$$b_2 = s_2 a_2. \tag{2.64b}$$

The reduced version of this system is the NLS one-component version, for $u_2 = v_2 = 0, v_1 = -u_1$, that turns out to be a good model of the Benjamin-Feir (or modulational) instability which is of great physical relevance (see Chapter 2).

The main focus of this section is to understand how the spectrum $S_x$ changes by varying the parameters $a_1, a_2, b_1, b_2$ and $q$. In matrix notation, see (2.54b), this plane wave solution (2.62) reads

$$Q = G\Xi G^{-1}, \quad \Xi = \begin{pmatrix} 0 & b_1 & b_2 \\ a_1 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}, \quad G(x,t) = e^{i(qx-\nu t^2 + pt\Sigma)},  \tag{2.65}$$

where the matrix $\Sigma$ has the expression (2.54a) while the matrix $\sigma$ is

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.66}$$

and we conveniently introduce the real parameters

$$p = b_1a_1 + b_2a_2. \tag{2.67a}$$
Chapter 2. Linear Stability of Plane Waves of the NLS and CNLS Systems

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which will be handy in the following. Next we observe that a fundamental matrix solution

\[ \psi(x, t, \kappa) \]

of the Lax equations has the expression

\[ \psi(x, t, \kappa) = G(x, t) e^{i(x W(\kappa) - t Z(\kappa))}, \] (2.68)

where the \( x, t \)-independent matrices \( W \) and \( Z \) are found to be

\[
W(\kappa) = \begin{pmatrix}
\kappa & -ib_1 & -ib_2 \\
-ia_1 & -\kappa - q & 0 \\
-ia_2 & 0 & -\kappa + q
\end{pmatrix} = \kappa \Sigma - q \sigma - i \Xi,
\] (2.69)

\[
Z(\kappa) = \begin{pmatrix}
-2\kappa^2 & i(2\kappa - q)b_1 & i(2\kappa + q)b_2 \\
2\kappa^2 - q^2 - a_2b_2 & a_1b_2 \\
2\kappa^2 - q^2 - a_1b_1
\end{pmatrix} = \kappa^2 - 2\kappa W(\kappa) - W^2(\kappa) - p,
\] (2.70)

with the property that they commute, \([W, Z] = 0\), consistently with the compatibility condition \( \psi_{xt} = \psi_{tx} \). We consider here the eigenvalues \( w_j(\kappa) \) and \( z_j(\kappa), j = 1, 2, 3, \) of \( W(\kappa) \) and, respectively, of \( Z(\kappa) \) as simple, as indeed they are for generic values of \( \kappa \). In this case both \( W(\kappa) \) and \( Z(\kappa) \) are diagonalized by the same matrix \( U(\kappa) \), namely

\[
W(\kappa) = U(\kappa) W_D(\kappa) U^{-1}(\kappa), \quad \text{where} \quad W_D(\kappa) = \text{diag}\{w_1, w_2, w_3\}
\]

\[
Z(\kappa) = U(\kappa) Z_D(\kappa) U^{-1}(\kappa), \quad \text{where} \quad Z_D(\kappa) = \text{diag}\{z_1, z_2, z_3\}.
\] (2.71)

Next we construct the matrix \( F(x, t, \kappa) \) via its definition, see (2.61), (2.26) and (2.27),

\[ F(x, t, \kappa) = [\Sigma, \phi(x, t, \kappa) M(\kappa) \phi^{-1}(x, t, \kappa)], \] (2.72)

which, because of the explicit expression (2.68), reads

\[ F(x, t, \kappa) = G(x, t) \left[ \Sigma, e^{i(x W(\kappa) - t Z(\kappa))} M(\kappa) e^{-i(x W(\kappa) - t Z(\kappa))} \right] G^{-1}(x, t). \] (2.73)

As for the matrix \( M(\kappa) \), it lies in a nine-dimensional linear space whose standard basis is given by the matrices \( B^{(jm)} \), whose entries are

\[ B_{kn}^{(jm)} = \delta_{jk} \delta_{mn}, \] (2.74)
where $\delta_{jk}$ is the Kronecker symbol ($\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise). However the alternative basis $V^{(jm)}$, which is obtained via the similarity transformation

$$V^{(jm)}(\kappa) = U(\kappa)B^{(jm)}U^{-1}(\kappa), \quad (2.75)$$

where $U(\kappa)$ diagonalizes $W$ and $Z$ (see (2.71)), is more convenient to our purpose. Indeed, expanding the generic matrix $M(\kappa)$ in this basis as

$$M(\kappa) = \sum_{j,m=1}^{3} \mu_{jm}(\kappa)V^{(jm)}(\kappa), \quad (2.76)$$

the scalar functions $\mu_{jm}$ being its components, and inserting this decomposition into the expression (2.73), leads to the following representation of $F$

$$F(x, t, \kappa) = G(x, t) \sum_{j,m=1}^{3} \mu_{jm}(\kappa)e^{i[(x(w_j - w_m) - t(z_j - z_m)]}F^{(jm)}(\kappa)G^{-1}(x, t), \quad (2.77)$$

where we have introduced the $x, t$-independent matrices

$$F^{(jm)}(\kappa) = \begin{bmatrix} \Sigma, & V^{(jm)}(\kappa) \end{bmatrix}. \quad (2.78)$$

The advantage of expression (2.77) is to explicitly show the dependence of the matrix $F$ on the six exponentials $e^{i[(x(w_j - w_m) - t(z_j - z_m)]}$. The elements $\mu_{jk}(\kappa)$ are arbitrary because they are the coefficients on the basis $V^{jk}(\kappa)$ used to write the matrix $M(\kappa)$ in (2.76), that is, in turn, arbitrary.

### 2.6 Eigenmodes’ Wave Numbers and Frequencies for Multi-Components Systems

The Proposition 2.5.1 stated in the previous section guarantees that, for any choice $\kappa$ of the functions $\mu_{jm}(\kappa)$, the expression (2.77) be a solution of the linearized equation (2.59), see (2.60). The requirement to have $\delta Q$ bounded is equivalent to require $F$ bounded. Looking at the formula (2.77), since $\mu_{jk}$ are arbitrary, it is sufficient to impose that only one difference, say $k_1 - k_2$, is real and the corresponding $\mu_{12} \neq 0$, and to impose that the other coefficients are $\mu_{23} = \mu_{31} = 0$. The further condition that the solution $\delta Q(x, t)$ be bounded in $x$ at any fixed time $t$ results in integrating expression (2.77) with respect to the
variable $\kappa$ over the spectral curve $S_x$ of the complex $\kappa$-plane:

$$\delta Q(x,t) = \int_{S_x} d\kappa F(x,t,\kappa). \tag{2.79}$$

The spatial spectrum $S_x$ can be geometrically defined as follows:

**Definition 2.6.1.** The $x$-spectrum $S_x$, namely the spectral curve on the complex $\kappa$-plane, is the set of values of the spectral variable $\kappa$ such that at least one of the three complex numbers $k_j = w_{j+1} - w_{j+2}$, $j = 1, 2, 3$ (mod 3), or explicitly

$$k_1(\kappa) = w_2(\kappa) - w_3(\kappa), \quad k_2(\kappa) = w_3(\kappa) - w_1(\kappa), \quad k_3(\kappa) = w_1(\kappa) - w_2(\kappa), \tag{2.80}$$

is real.

Observe that the $k_j$’s play the role of eigenmode wave-numbers (see (2.77)).

The requirement to have $\delta Q$ bounded is equivalent to require $F$ bounded. Looking at the formula (2.77), since $\mu_{jk}$ are arbitrary, it is sufficient to impose that only one difference, say $k_1 - k_2$, to be real and the corresponding $\mu_{12} \neq 0$ and to impose that the other coefficients are $\mu_{23} = \mu_{31} = 0$.

To the purpose of establishing the stability properties of the continuous wave solution (2.62) we do not need to compute the integral representation (2.79) of the solution $\delta Q$ of (2.59). Indeed, it is sufficient to compute the eigenfrequencies

$$\omega_1(\kappa) = z_2(\kappa) - z_3(\kappa), \quad \omega_2(\kappa) = z_3(\kappa) - z_1(\kappa), \quad \omega_3(\kappa) = z_1(\kappa) - z_2(\kappa), \tag{2.81}$$

as suggested by the exponentials which appear in (2.77). Their expression follows from the matrix relation (2.70)

$$z_j = \kappa^2 - 2\kappa w_j - w_j^2 - p, \tag{2.82}$$

and read

$$\omega_j = -k_j(2\kappa + w_{j+1} + w_{j+2}), \quad j = 1, 2, 3 \ (\text{mod} \ 3). \tag{2.83}$$

This expression looks even simpler by using the relation $w_1 + w_2 + w_3 = -\kappa$ implied by the trace of the matrix $W(\kappa)$ (see (2.69)) and finally reads

$$\omega_j = k_j(w_j - \kappa), \quad j = 1, 2, 3. \tag{2.84}$$

The consequence of this expression (2.84), which is relevant to our stability analysis, is given by the following
Proposition 2.6.2. The continuous wave solution $Q_0$ is stable against perturbations $\delta Q$ whose representation (2.39) is given by an integral which runs only over those values of $\kappa \in S_x$ which are strictly real.

The proof of this Proposition is provided in [64].
Chapter 3

The 3-Wave Resonant Interaction Model

In this Chapter we introduce the 3WRI system and we reformulate it in order to include all the possible velocity orderings and all the possible choices of signs in just one Lax pair. This reformulation simplifies the computations in view of a complete classification of the spatial stability spectra in the parameters space (see Chapter 4). Then, we apply the formalism developed in [64] as introduced in the Chapter 2. Finally, we write the differences of the eigenvalues of the matrices $W$ in terms of the differences of the eigenvalues of the matrices $Z$.

3.1 Linear Stability Analysis of the 3WRI Equations: Historical Overview and State of the Research

The 3-wave resonant interaction (3WRI) model describes the dynamics of three waves interacting by a quadratic nonlinearity and without dispersion and dissipation. The nonlinearity term can be considered like a perturbation at the first order of the linear dynamics [55, 78]. The weak, quadratic nonlinearity and the neglected dispersion make the system the simplest model to be analysed in case of resonant interaction. The interaction is called ‘resonant’ because it takes place only when the frequencies $\eta_j$ and the wave numbers $\nu_j$ of each wave
Chapter 3. The 3-Wave Resonant Interaction Model

with \( j = 1, 2, 3 \) satisfy particular relations, named resonant conditions:

\[
\eta_1 + \eta_2 + \eta_3 = 0, \quad \nu_1 + \nu_2 + \nu_3 = 0. \tag{3.1}
\]

Moreover, the dispertonless dynamics entails that for time going to \( \pm \infty \) the solutions are well separated and do not decay, and the model is integrable \[58\]. The dynamics with dispersion can be also described by adding a dispersive term that, in general, makes the model non-integrable, unless it is a second order correction to the linear dynamics that acts in a timescale longer than the nonlinearity. However, this correction affects substantially the stability of the system. For instance, in the research work \[56\], a triplet of dark solitons with locked velocity has been found to be always unstable. Nevertheless, a quasi-negligible second order dispersion balances the nonlinearity effect so that a stable triplet of dark solitons can propagate.

In this thesis we are interested in the linear stability analysis of the simplest solutions of the 3WRI system, i.e. the plane waves. The stability of a resonant triad was studied first in 1967 by Hasselmann for spatially uniform plane waves and he formulated a stability criterion (named Hasselmann’s criterion)\[89]\:

“the nonlinear coupling between two infinitesimal components 1 and 2 and a finite component 3 whose wave-numbers and frequencies satisfy the resonant-interaction conditions

\[
\eta_1 \pm \eta_2 = \eta_3, \quad \nu_1 \pm \nu_2 = \nu_3,
\]

is unstable for the sum interaction and neutrally stable for the difference interaction.”

In other words, the criterion above states that the wave with highest frequency exhibits instability \[44, 90\]. The dynamics of a conservative system of coupled plane waves with amplitudes modulated in time is described by the ordinary differential equations

\[
\begin{align*}
\frac{dA_1}{dt} &= s_1 A_2^* A_3^*, \\
\frac{dA_2}{dt} &= s_2 A_1^* A_3^*, \\
\frac{dA_3}{dt} &= s_3 A_1^* A_2^*,
\end{align*}
\tag{3.2}
\]

where \( A_j = A_j(t) \), with \( j = 1, 2, 3 \), are complex slowly varying amplitudes, \( s_j \) are signs such that \( s_j^2 = 1 \), and the asterisk denotes the complex conjugation. Let us suppose that the wave \( A_3 \) is the ‘pump’, whose amplitude is initially finite and it is approximately constant in time, while the amplitudes of the other two interacting waves are infinitesimal.
and dependent on time. These assumptions lead the system (3.2) to reduce to \[44, 57\]

\[
\begin{align*}
\frac{dA_1}{dt} &= s_1 A_2^* A_3^* , \\
\frac{dA_2}{dt} &= s_2 A_1^* A_3^* , \\
\frac{dA_3}{dt} &= 0 .
\end{align*}
\] (3.3)

and because \(A_3\) is constant in time, we obtain

\[
\frac{d^2 A_j}{dt^2} \approx s_1 s_2 A_j |A_3|^2 , \quad j = 1, 2 .
\] (3.4)

If \(s_1 = s_2 = 1\) and \(s_3 = -1\), then the infinitesimal amplitudes grow exponentially until the linear approximation is not longer valid, the amplitudes \(A_1\) and \(A_2\) become comparable to the amplitude \(A_3\) which, in turn, depletes. However, this process is periodic, in the sense that, after an exact period \(2t_0^1\) \(A_1\) and \(A_2\) deplete and \(A_3\) grows \[57\]. Thus, \(A_1\) and \(A_2\) are periodic, and \(A_3\) is nonlinearly unstable: the linear approximation is not valid anymore when the infinitesimal amplitudes reach the value of the finite one. If \(s_1 = s_2 = s_3\), all the three amplitudes grow indefinitely and the system exhibits explosive instability \[57, 60, 61, 104\].

Beside research works on the system of ordinary differential equations (3.2), further research has been carried out on the system (3.2) and on the stability of its solutions. In particular, if the amplitudes \(A_j\) depend on both time and space, such that the system (3.2) involves partial derivatives with respect to both time \(t\) and space \(x\), general solutions are wave packets \[105, 106\]. The starting point of the investigation in this thesis is the 3WRI model, that is written in general form as an integrable system of three PDEs in \(1 + 1\) dimensions \[55\]

\[
\begin{align*}
q_{1t} + c_1 q_{1x} &= s_1 g_2^* g_3^* , \\
q_{2t} + c_2 q_{2x} &= s_2 g_1^* g_3^* , \\
q_{3t} + c_3 q_{3x} &= s_3 g_1^* g_2^* ,
\end{align*}
\] (3.5)

where \(q_j = q_j(x, t)\) are complex amplitudes, \(c_j\) is the group velocity of the \(j\)th-packet, \(s_j\) such that \(s_j^2 = 1\). The subscripts \(x\) and \(t\) denote the partial derivatives with respect to space \(x\) and time \(t\), respectively, while the asterisk stands for the complex conjugation.

By assuming that the resonant conditions (3.1) are satisfied, the system (3.5) describes
several processes that can be classified mainly by the signs $s_j$ and by the ordering of the velocities $c_j$. Indeed, the system (3.5) comes from perturbing the differential equation $u_{TT} + \Omega(-i\partial_X)^2u = N(u)$, where $\Omega(\eta)$ is a real nonnegative polynomial, with $\Omega^2(0) > 0$, and $N$ is twice differentiable function, with $N(0) = N'(0) = 0$, enclosing nonlinear terms. After using the multi-scale method, one gets an equation for the lower order error term and, in order to avoid that this term grows linearly, we need to set equal to zero some terms such that, after rescaling, we obtain the system (3.5) with signs $s_j = \text{sgn}(N''(0)\nu_j)$, $j = 1, 2, 3$. Therefore, the system (3.5), with the resonant conditions (3.1) and with a sign $s_j$ different from the other two signs, is associated to decay instability. On the other hand, the system (3.5) can also be obtained, with the same procedure described above, by replacing the term $N(u)$ with the term $2N(u_x)$, such that the signs are defined as $s_j = -\text{sgn}(N''(0)\nu_j\eta_m\eta_l)$, for $j, l, m = 1, 2, 3$ and all distinct. As a result, the conditions (3.1) are satisfied for $s_1 = s_2 = s_3$, provided that the ratios $\eta_j/\nu_j$ are all positive. This is the case known as explosive instability. Indeed, by considering wave packets as solutions, we can individuate the solutions with explosive behaviour via the analysis of the Manley-Rowe relations:

\begin{align}
\int_R |q_1|^2 \, dx - s_1s_2 \int_R |q_2|^2 \, dx &= I_{12}, \quad (3.6a) \\
\int_R |q_2|^2 \, dx - s_2s_3 \int_R |q_3|^2 \, dx &= I_{23}, \quad (3.6b) \\
\int_R |q_3|^2 \, dx - s_1s_3 \int_R |q_1|^2 \, dx &= I_{31}, \quad (3.6c)
\end{align}

with $q_j$ being a smooth function satisfying the condition $|q_j|^2 \to 0$ for $|x| \to \infty$, where $|\cdot|$ is the modulus and $I_{12}$, $I_{23}$ and $I_{13}$ being constants. The equations (3.6) are conservation laws that denote the derivative with respect to $x$ of the solution $u$. If we multiply each equation for the envelope $q_j$ by its complex conjugate $q_j^*$, and if we add to this equation the equation for $q_j^* \nu_j$ multiplied by $q_j$, we obtain an equation for the action of every envelope.

\[ \frac{d}{dt} \int_R |q_j|^2 \, dx = 2s_j \int_R \text{Re} \{ q_1q_2q_3 \} \, dx, \quad j = 1, 2, 3, \]

and $\text{Re} \{ \cdot \}$ is the real part of the product $q_1q_2q_3$. Finally, by summing two by two these equations and by integrating them with respect to the time, we get the Manley-Rowe relations.

This is a consequence of the fact that the solutions $q_j$ are wave packets and so they are also square-integrable $\int_R |q_j|^2 \, dx < \infty$. 

laws of the action exchanged between the interacting envelopes \[108, 109\].

By looking at the relations (3.6), we deduce that when all the signs \(s_j\) are equal one to another, the system (3.5) can exhibit solutions with spikes at a finite time because the conserved quantity \(I_{kj}\) may not bound the action \(\int_{\mathbb{R}} |q_j|^2 \, dx\) of any packet in the left-hand side of the equations (3.5). Indeed, the envelope can grow up indefinitely and blow up at a finite time although their mutual exchange of action allows \(I_{kj}\) to remain constant \[60, 55, 110\]. If instead one sign is different from the others, the system (3.5) describes two kind of interactions: decay instability or stimulated backscatter \[57\]. For example, let us consider \(s_3 = -1\) and \(s_1 = s_2 = 1\), then the Manley-Rowe relations become\(^5\)

\[
\begin{align*}
\int_{\mathbb{R}} |q_1|^2 \, dx - \int_{\mathbb{R}} |q_2|^2 \, dx &= I_{12}, \\
\int_{\mathbb{R}} |q_3|^2 \, dx + \int_{\mathbb{R}} |q_1|^2 \, dx &= I_{13}, \\
\int_{\mathbb{R}} |q_2|^2 \, dx + \int_{\mathbb{R}} |q_3|^2 \, dx &= I_{23}.
\end{align*}
\]  

(3.7a) 

(3.7b) 

(3.7c)

With this choice of signs, the constants \(I_{kj}\) are nonnegative and bound any norm of the envelopes at any time, such that the solutions do not grow up indefinitely and remain bounded (see \[55\] and \[110\] for more details). This process is interesting because, during the interaction between the envelopes, linear instability may occur. In this respect, in \[111\], by using the IST formalism and numerical techniques, the authors studied the stability of a finite amplitude wave interacting with two initially infinitesimal amplitude waves. In this way, the system (3.5) reduces to a system of two linear equations for the initial infinitesimal waves. Because the small amplitudes can grow up during the interaction, after a certain time, nonlinearity comes into play and the linear approximation is not valid. However, there is a connection between the linearisation of the 3WRI model and the Zakharov-Shabat problem describing its nonlinear evolution. In particular, after some transformations, the 3WRI model describing the interaction between a 'pump' and two infinitesimal sidebands, reduces to three Zakharov-Shabat problems \[57\]. This is because the system is dispersionless, the envelopes are well separated in the initial and the final states, and hence the

\(^5\)Only two conserved quantities \(I_{kj}\) are linearly independent. Indeed, for \(I_{12}\) and \(I_{13}\) linearly independent, then \(I_{23} = I_{12} - I_{13}\).
Zhakarov-Shabat scattering data contain all the information about the final state. A conclusion about the linear stability analysis of the 3WRI model is that, if the highest-frequency pump has the middle velocity and contains solitons, then linear instability occurs and in the final state the pump can show a spike at a finite time (explosive instability) or can deplete (decay instability) [57]. We highlight that this analysis was conducted by supposing that the potential of the eigenvalues problem goes to zero as $x \to \pm \infty$ (see also [58]). Although, the Manley-Rowe relations are useful to predict if square-integrable solutions blow up at a finite time, we can not use them when plane waves solutions are considered. For this reason, even if we do not exclude the possibility of the existence of solutions with explosive instability, in our analysis, we need to consider any choice of signs, also that one in general associated to explosive instability, i.e. $s_1 = s_2 = s_3$.

Moreover, most of the research works lead so far dealt with wave packets, not with plane waves as, instead, we shall do. In addition, the linear stability analysis was carried out by considering the two side-bands and the pump as solutions of the three 3WRI system. The framework of our research is more general, in the sense that we deal with a system of three plane waves, we perturb every solution and then every solution interacts with the perturbations of the other two interacting waves. In other words, we are considering a system of three pumps interacting in resonance one with each other and every pump interacts with two side-bands.

In the literature, the system (3.5) has been written in different forms, obtained by redefining the variables $q_j$ and the velocities $c_j$. This is because, from the computational point of view and depending on the applications, a form may be more convenient than others. For instance, let us consider the case in which one sign is different from the others. If one puts $q_1 = \bar{q}_1$, $q_2 = \bar{q}_2^*$ and $q_3 = \bar{q}_3$ and $s_1 = -p_1$, $s_2 = p_2$ and $s_3 = -p_3$, the system (3.5) can be written in a more general form by introducing a complex coupling constant $K$ [112,56]

\[
\begin{align*}
\bar{q}_1 \frac{t}{t} + c_1 \bar{q}_1 x &= -p_1 K^* \bar{q}_2 \bar{q}_3^*, \\
\bar{q}_2 t + c_2 \bar{q}_2 x &= p_2 K \bar{q}_1 \bar{q}_3 , \\
\bar{q}_3 t + c_3 \bar{q}_3 x &= -p_3 K^* \bar{q}_1 \bar{q}_2.
\end{align*}
\] (3.8)

Although the equations (3.8) describe the interactions between three waves in a homogeneous medium [58,111], they can be mapped into a system with a phase factor describing the interactions in a medium with weak inhomogeneity and an IST problem is formulated in order to understand the effect of inhomogeneities on the three waves interaction [112].
In this thesis we apply the spectral analysis method developed in [64] to plane waves only. On the other hand, several other researches were carried out on more complicated solutions of the 3WRI system. In fact, it is well known that the 3WRI model has solutions that are triplets of solitons travelling together with a common velocity, the so called *simultons*. Interesting research works were conducted on simultons in the last years [82, 113, 114, 115, 116]. Indeed, the system (3.12) has also been used in a co-variant form [56] obtained transforming the fields \( \tilde{q}_j \) as

\[
\begin{align*}
    u_1(z, y) &= K \sqrt{\frac{c_2}{c_1}} \tilde{q}_1(x, t), \\
    u_2(z, y) &= K \sqrt{\frac{c_1}{c_2}} \tilde{q}_2(x, t), \\
    u_3(z, y) &= K \sqrt{\frac{c_1}{c_2}} \tilde{q}_3(x, t)
\end{align*}
\]

and with ‘velocities’ of the solitons \( V_1 = \frac{1}{c_1 - c_3}, \ V_2 = \frac{1}{c_2 - c_3} \) satisfying the condition \( 0 < V_1 < V_2 \). After that transformations, \( z \) and \( y \) are the temporal and the spatial variable, respectively. The equations so obtained have a simulton solution constituted by three dark solitons which are always unstable unless the 3WRI system is perturbed via a weak dispersion. It was observed that this weak dispersion reduces the MI [56]. Even if the perturbed system should not have solitons, the dispersion parameters are chosen such that only the shape of the soliton are slightly modified and at the same time the instability is reduced [56]. In the same work, the analysis and the classification of the stabilities according to the signs is carried out.

The stability of a simulton composed by two bright and a dark solitons (BBD) was studied in the paper [114]. Let \( c_1, c_2 \) and \( c_3 \) be the velocities of the triad solutions of the 3WRI system and \( c \) be the velocity of the simulton. It was found that when \( c \) is brought below a critical value, that is

\[
c_{uns} = \frac{2c_1c_2}{c_1 + c_2 - Q(c_1 - c_2)},
\]

the simulton becomes unstable. \( Q \) depends on the parameters involved on the expression of the simulton, and when \( -1 < Q < 1 \), we have \( c_1 < c_{uns} < c_2 \). Furthermore, an unstable simulton decays in a stable one emitting a pulse when \( c \) is brought above the value \( c_{uns} \) and the simulton becomes a ‘boomeron’ in the sense that its final velocity is different from the initial one. The stability was also analysed under collision between two stable simultons, and it was found they can pass through each other maintaining their shape if their velocities are different, and repulse or attract each other if their velocities are the same. A similar analysis was carried out on the interaction between a stable or an unstable simulton with a linear wave [115]. Bearing in mind the outcomes described above, an interesting future direction of research is the stability analysis of rational triads, whose analytical expressions were found by Darboux Dressing Transformation in [72, 83].
Chapter 3. The 3-Wave Resonant Interaction Model

3.2 Lax Pair

As explained in Chapter 1, the main aim of this thesis is to individuate values of the physical parameters for which linear instability occurs. In general, we observe that a change of the ordering of the velocities is reflected into a change of the Lax pair associated to the system. This would be quite impractital in view of a complete classification of the instabilities (and of the stability spectra that we will introduce later in this work) with respect to the parameters. For this reason, in the following discussion, first we observe some symmetries in the 3WRI system, and then we use such symmetries to write a general expression for the Lax operators including all possible orderings of the velocities. This general expression allows us to compute all the necessary analytic tools and, only in the end, deduce what happens if we change the velocities ordering without further complicated computations.

3.2.1 Symmetries

The Lax operators associated with the system (3.5) [53, 58] make our computations hard to carry out because of the square roots in their expressions. In order to write the Lax operators in the easy form, one can rescale the fields $q_j$ as follows [55]

$$q_j = \sqrt{\frac{\Delta_1 \Delta_2 \Delta_3}{\Delta_j}} s_j u_j, \quad j = 1, 2, 3,$$

(3.10)

where $s_j^2 = 1$ and

$$\Delta_1 = c_2 - c_3, \quad \Delta_2 = c_1 - c_3, \quad \Delta_3 = c_1 - c_2,$$

(3.11)

and the system (3.5) becomes

$$
\begin{align*}
&u_{1t} + c_1 u_{1x} = s_2 s_3 |\Delta_1| u_2^* u_3^*, \\
&u_{2t} + c_2 u_{2x} = s_1 s_3 |\Delta_2| u_1^* u_3^*, \\
&u_{3t} + c_3 u_{3x} = s_1 s_2 |\Delta_3| u_1^* u_2^*.
\end{align*}
$$

(3.12)

The symbol $|\cdot|$ denotes the absolute value of the differences of the velocities which is defined as

$$|c_i - c_j| = \begin{cases} 
  c_i - c_j & \text{if } c_i > c_j, \\
  c_j - c_i & \text{if } c_j > c_i,
\end{cases} \quad i, j = 1, 2, 3.$$

(3.13)
Thus, once fixed, say, the velocity \( c_3 \), the system (3.12) takes two different expressions corresponding to the orderings \( c_1 > c_2 \) and \( c_2 > c_1 \). The same argument holds when \( c_1 \), or \( c_2 \), is fixed. Therefore, the system (3.12) encloses six systems associated to every velocities ordering and they correspond to six Lax Pairs. We can not write all the six Lax pairs in one, because the differences of the velocities appear only after computing the compatibility condition for the Lax operators. Every Lax operator contains terms proportional to the single velocities \( c_j \), but not terms proportional to their differences, that, instead, appear after the computation of the commutator between the two operators. Precisely, the six 3WRI systems are:

\[ C_1 \) \( c_1 > c_2 > c_3 \]
\[
\begin{align*}
  u_{1t} + c_1 u_{1x} &= s_2 s_3 (c_2 - c_3) u_2^* u_3^* , \\
  u_{2t} + c_2 u_{2x} &= s_1 s_3 (c_1 - c_3) u_1^* u_3^* , \\
  u_{3t} + c_3 u_{3x} &= s_1 s_2 (c_1 - c_2) u_1^* u_2^* ,
\end{align*}
\]

\[ C_2 \) \( c_1 > c_3 > c_2 \]
\[
\begin{align*}
  u_{1t} + c_1 u_{1x} &= -s_2 s_3 (c_2 - c_3) u_2^* u_3^* , \\
  u_{2t} + c_2 u_{2x} &= s_1 s_3 (c_1 - c_3) u_1^* u_3^* , \\
  u_{3t} + c_3 u_{3x} &= s_1 s_2 (c_1 - c_2) u_1^* u_2^* ,
\end{align*}
\]

\[ C_3 \) \( c_3 > c_1 > c_2 \]
\[
\begin{align*}
  u_{1t} + c_1 u_{1x} &= -s_2 s_3 (c_2 - c_3) u_2^* u_3^* , \\
  u_{2t} + c_2 u_{2x} &= -s_1 s_3 (c_1 - c_3) u_1^* u_3^* , \\
  u_{3t} + c_3 u_{3x} &= s_1 s_2 (c_1 - c_2) u_1^* u_2^* ,
\end{align*}
\]

\[ C_4 \) \( c_3 > c_2 > c_1 \]
\[
\begin{align*}
  u_{1t} + c_1 u_{1x} &= -s_2 s_3 (c_2 - c_3) u_2^* u_3^* , \\
  u_{2t} + c_2 u_{2x} &= -s_1 s_3 (c_1 - c_3) u_1^* u_3^* , \\
  u_{3t} + c_3 u_{3x} &= -s_1 s_2 (c_1 - c_2) u_1^* u_2^* ,
\end{align*}
\]

\[ ^6 \text{In this discussion, we are not considering the choices of the signs.} \]
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\[ C_5 \] \( c_2 > c_3 > c_1 \)

\[
\begin{align*}
    u_{1t} + c_1 u_{1x} &= s_2 s_3 (c_2 - c_3) u_2^* u_3^*, \\
    u_{2t} + c_2 u_{2x} &= -s_1 s_3 (c_1 - c_3) u_1^* u_3^*, \\
    u_{3t} + c_3 u_{3x} &= -s_1 s_2 (c_1 - c_2) u_1^* u_2^*,
\end{align*}
\]  
\( (3.18) \)

\[ C_6 \] \( c_2 > c_1 > c_3 \)

\[
\begin{align*}
    u_{1t} + c_1 u_{1x} &= s_2 s_3 (c_2 - c_3) u_2^* u_3^*, \\
    u_{2t} + c_2 u_{2x} &= s_1 s_3 (c_1 - c_3) u_1^* u_3^*, \\
    u_{3t} + c_3 u_{3x} &= -s_1 s_2 (c_1 - c_2) u_1^* u_2^*,
\end{align*}
\]  
\( (3.19) \)

Since our aim is to classify the spectra of the Lax operators with regard to the parameters involved in the 3WRI model, we should examine every Lax Pair \( (3.14)-(3.19) \) to get a complete classification. Nevertheless, our analysis can be further simplified because in the system of equations \( (3.12) \) only the product of signs appear. This implies there are only four possible products of signs, each one corresponding to two different combinations of the signs.

\[ S_1 \] for \( s_1 = s_2 = -s_3 \)

\[
\begin{align*}
    s_2 s_3 &= -1, \\
    s_1 s_3 &= -1, \\
    s_1 s_2 &= 1,
\end{align*}
\]  
\( (3.20) \)

\[ S_2 \] for \( s_1 = s_3 = -s_2 \)

\[
\begin{align*}
    s_2 s_3 &= -1, \\
    s_1 s_3 &= 1, \\
    s_1 s_2 &= -1,
\end{align*}
\]  
\( (3.21) \)

\(^7\)For example, the case \( s_1 = s_2 = -s_3 \) encloses two combinations of signs that are \( s_1 = s_2 = -1 \) and \( s_3 = 1 \) or \( s_1 = s_2 = 1 \) and \( s_3 = -1 \).
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\( S_3 \) for \( s_2 = s_1 = -s_1 \)

\[
\begin{align*}
    s_2 s_3 &= 1, \\
    s_1 s_3 &= -1, \\
    s_1 s_2 &= -1, \\
\end{align*}
\]  

(3.22)

\( S_4 \) for \( s_1 = s_2 = s_3 \)

\[
\begin{align*}
    s_2 s_3 &= 1, \\
    s_1 s_3 &= 1, \\
    s_1 s_2 &= 1. \\
\end{align*}
\]  

(3.23)

At this point, it is worth reminding that we are including also the "explosive case" (i.e. the case (3.23)). In fact, as already mentioned, we do not have reason to exclude this case. A priori, we cannot know if the system with the combination of signs in (3.23) is actually explosive for the plane wave solutions because we cannot predict the possibility of explosive behaviour via the Manley-Rowe relations, for the plane wave solutions are not square-integrable.

By combining the systems (3.14)-(3.19) with every possible choice of signs (3.20)-(3.23), we get in total twenty-four systems of equations, everyone denoted by the letter \( C_j \), associated to a Lax pair, and by the letter \( S_j \), associated to the choice of signs. Therefore, for instance, the system of equations \( C_1 S_1 \) is the one corresponding to the Lax pair \( C_1 \) with a choice of signs \( S_1 \). However, only twelve cases are relevant because the others can be obtained by the former via symmetry, as will be shown below.

Proposition 3.2.1. For every fixed choice of signs, if the velocities ordering is reversed, the resulting system is symmetric to the former one by relabelling of the indices, that is equivalent to exchange the bigger velocity with the smaller velocity.

Proof. Let us consider, say, the velocities orderings \( C_1 \), (3.14), and \( C_4 \), (3.17), and let us suppose that the signs \( s_1 s_2 \), \( s_1 s_3 \) and \( s_2 s_3 \) are fixed and are the same for both systems. In the system (3.14) the differences of the velocities are all positive, while in the system (3.17) they are all negative. However, there is a minus in front of the negative differences in (3.17), such that the sign in front of the interaction term is the same of the system
In particular, the systems (3.14) and (3.17) are equivalent, in the sense that one can be obtained from the other by a relabelling. Indeed, after the substitutions $1 \rightarrow 3$, $3 \rightarrow 1$ and $2 \rightarrow 2$ in the system (3.17) we get the system (3.14), and vice versa. Thus, if we consider first the system (3.17) in which the solution $u_1$ has the smaller velocity and $u_3$ has the bigger one, after relabelling, $u_1$ has the bigger velocity and $u_3$ has the smaller one. This reasoning can be extended to all orderings of velocities. □

As a consequence of the Proposition 3.2.1, we may consider only three orderings instead of six. In particular, the equivalent orderings are: $C_1$ with $C_4$, $C_2$ with $C_5$ and $C_3$ with $C_6$. We note that once the signs are fixed, for example after choosing the signs $S_4$, the signs of the interactions are positive in all the systems (3.14)-(3.19), and this is a consequence of the presence of the modulus $|\Delta_j|$ in the system (3.5). Since the choice $s_1s_2 = s_1s_3 = s_2s_3 = -1$ is not allowed, it looks like the negative interactions are not allowed. Nevertheless, there is another symmetry in the model that makes the interaction of any sign possible.

**Proposition 3.2.2.** For any choice of signs and any velocities ordering, the 3WRI system admits interaction with both positive and negative signs.

**Proof.** Let us consider the system (3.14) with fixed signs. If we change every solution $u_j \rightarrow -u_j$ and then we define $-u_j = \bar{u}_j$, then we get that the solutions $\bar{u}_j$ satisfy the same system of $u_j$ with the same interaction, namely, $\bar{u}_j$ are also solutions of the model. On the other hand, if we come back to $u_j$, we get a 3WRI system for $u_j$, but with the sign of the interaction opposite to the former, i.e. negative. □

Proposition 3.2.1 and Proposition 3.2.2 suggest that we can change the interaction sign by changing simultaneously the signs of all the amplitudes of the solutions $u_j$. Below, we shall show that, once the ordering of the velocities is fixed, the systems for $\bar{u}_j$ and for $u_j$ describe the same processes.
3.2.2 A General Expression for the 3WRI System

In order to fulfil the stability analysis, we need to deal with six Lax pairs (or with three Lax pairs if we want to rediscover the other three orderings by symmetry) and, once fixed the velocities ordering, we will vary the signs to reproduce the four cases $S_1$, $S_2$, $S_3$ and $S_4$. However, our analysis can be simplified further because the cases analysed in the previous section can be rediscovered if the 3WRI model is interpreted and written in a particular fashion (as explained below).

Let us consider the 3WRI model

\[
\begin{align*}
    u_{1t} + c_1 u_{1x} &= s_1 (c_2 - c_3) u_2^* u_3^*, \\
    u_{2t} + c_2 u_{2x} &= s_2 (c_1 - c_3) u_1^* u_3^*, \\
    u_{3t} + c_3 u_{3x} &= s_3 (c_1 - c_2) u_1^* u_2^*,
\end{align*}
\]

(3.24)

where $s_j$ are signs such that $s_j^2 = 1$, $j = 1, 2, 3$, and the velocities and their orderings can be whatever\(^8\). Indeed, in some of the systems (3.14)-(3.19), the differences $c_j - c_{j+1}$ appear with a minus in front of them. In the system (3.24), the sign minus is included in the definition of $s_j$ and, for this reason, there are eight possible choices of signs:

- $S_1^+$: $s_1 = s_2 = 1$, $s_3 = -1$;
- $S_1^-$: $s_1 = s_2 = -1$, $s_3 = 1$;
- $S_2^+$: $s_1 = s_3 = 1$, $s_2 = -1$;
- $S_2^-$: $s_1 = s_3 = -1$, $s_2 = 1$;
- $S_3^+$: $s_2 = s_3 = 1$, $s_1 = -1$;
- $S_3^-$: $s_2 = s_3 = -1$, $s_1 = 1$;
- $S_4^+$: $s_1 = s_2 = s_3 = -1$;
- $S_4^-$: $s_1 = s_2 = s_3 = 1$.

\(^8\)We are taking into account also the combinations of signs not included in the classification of the choices $S_j$. 
The combination \( S^+_3 \) is obtained from the combination \( S^-_1 \) by a symmetry. To prove that, let us consider the 3WRI system for the solution \( u = u(x, t) \)

\[
\begin{align*}
 u_{1t} + c_1 u_{1x} &= s_1(c_2 - c_3) u_2^* u_3^*, \\
 u_{2t} + c_2 u_{2x} &= s_2(c_1 - c_3) u_1^* u_3^*, \\
 u_{3t} + c_3 u_{3x} &= s_3(c_1 - c_2) u_1^* u_2^*.
\end{align*}
\tag{3.25}
\]

**Proposition 3.2.3.** In the system (3.25), the interaction term is left invariant by changing all the signs \( s_j \), (see also [57]).

**Proof.** The system (3.25) admits solutions like \( \bar{u}_j = e^{i \theta_j} u_j \), where \( \theta_j \) are arbitrary phases. We also write \( \bar{s}_j = |\bar{s}_j| e^{i \arg(s_j)} \), with \( |\bar{s}_j| = 1 \) and

\[
\arg(\bar{s}_j) = \begin{cases} 
0 & \text{if } \bar{s}_j = 1, \\
\pi & \text{if } \bar{s}_j = -1,
\end{cases} \quad j = 1, 2, 3, \tag{3.26}
\]

the signs \( \bar{s}_j \) are mapped into the signs \( s_j \) via the following transformation

\[
\bar{s}_j = e^{i(\arg(s_j) + \phi_j)}, \quad j = 1, 2, 3 \tag{3.27}
\]

and when we come back to the system for \( u_j \), we have

\[
\begin{align*}
 u_{1t} + c_1 u_{1x} &= (c_2 - c_3) u_2^* u_3^* e^{-i(\theta_1 + \theta_2 + \theta_3 - \arg(s_1) - \phi_1)} \\
 u_{2t} + c_2 u_{2x} &= (c_1 - c_3) u_1^* u_3^* e^{-i(\theta_1 + \theta_2 + \theta_3 - \arg(s_2) - \phi_2)} \\
 u_{3t} + c_3 u_{3x} &= (c_1 - c_2) u_1^* u_2^* e^{-i(\theta_1 + \theta_2 + \theta_3 - \arg(s_3) - \phi_3)}.
\end{align*}
\tag{3.28}
\]

The system (3.28) is equivalent to the system

\[
\begin{align*}
 \bar{u}_{1t} + c_1 \bar{u}_{1x} &= (c_2 - c_3) \bar{u}_2^* \bar{u}_3^* e^{i \arg(\bar{s}_1)} \\
 \bar{u}_{2t} + c_2 \bar{u}_{2x} &= (c_1 - c_3) \bar{u}_1^* \bar{u}_3^* e^{i \arg(\bar{s}_2)} \\
 \bar{u}_{3t} + c_3 \bar{u}_{3x} &= (c_1 - c_2) \bar{u}_1^* \bar{u}_2^* e^{i \arg(\bar{s}_3)},
\end{align*}
\tag{3.29}
\]

only if \( \phi_1 = \phi_2 = \phi_3 \equiv \phi \) and \( \phi = \theta_1 + \theta_2 + \theta_3 \). Because \( s_j \) and \( \bar{s}_j \) are just signs, \( \phi \) has value \( \pi \) or 0. Moreover, looking at the system for the solution \( \bar{u}_j \), one has also \( e^{-i(\theta_1 + \theta_2 + \theta_3 - \arg(\bar{s}_j))} = e^{-i(\theta_1 + \theta_2 + \theta_3 - \arg(s_j) - \phi)} \), with \( j = 1, 2, 3 \), and so \( e^{i(\arg(s_j))} = e^{i(\arg(\bar{s}_j))} \), such that \( \arg(\bar{s}_j) = \arg(s_j) + \phi \). Let us suppose \( \bar{s}_1 = 1, \bar{s}_2 = -1 \) and \( \bar{s}_3 = -1 \), there are two possibilities: \( \phi = 0 \), and \( \arg(s_1) = 0, \arg(s_2) = \arg(s_3) = \pi \), or \( \phi = \pi \), and
signs $s_j$ & Velocities ordering  \\ 
$S_1^+$ & $-S_1C_1$ & $S_2C_2$ & $-S_4C_3$ & $S_1C_4$ & $-S_2C_5$ & $S_4C_6$  
$S_1^-$ & $S_1C_1$ & $-S_2C_2$ & $S_4C_3$ & $-S_1C_4$ & $S_2C_5$ & $-S_4C_6$  
$S_2^+$ & $-S_2C_1$ & $S_1C_2$ & $-S_3C_3$ & $S_2C_4$ & $-S_1C_5$ & $S_3C_6$  
$S_2^-$ & $S_2C_1$ & $-S_1C_2$ & $S_3C_3$ & $-S_2C_4$ & $S_1C_5$ & $-S_3C_6$  
$S_3^+$ & $-S_3C_1$ & $S_4C_2$ & $-S_2C_3$ & $S_3C_4$ & $-S_4C_5$ & $S_2C_6$  
$S_3^-$ & $S_3C_1$ & $-S_4C_2$ & $S_2C_3$ & $-S_3C_4$ & $S_4C_5$ & $-S_2C_6$  
$S_4^+$ & $-S_4C_1$ & $S_3C_2$ & $-S_1C_3$ & $S_4C_3$ & $-S_3C_5$ & $S_1C_6$  
$S_4^-$ & $S_4C_1$ & $-S_3C_2$ & $S_1C_3$ & $-S_4C_4$ & $S_3C_5$ & $-S_1C_6$ 

Table 3.1: 3WRI cases.

$\arg(s_1) = \pi$, $\arg(s_2) = \arg(s_3) = 0$.

This means that if we want to leave unchanged the interaction, we need to change both the signs $s_j$ and the signs of all the amplitudes, i.e. $\bar{u}_j = -u_j$. In other words, changing all the signs $s_j$ is equivalent to changing all the signs in front of the amplitudes of the solutions $u_j$.

As a consequence of the Proposition 3.2.3, we take into account only four possible choices of signs (instead of eight) that, combined with the six possible velocities orderings, give twenty-four systems in total, although we expect twelve relevant cases only. This means there is another symmetry, in fact the Proposition 3.2.1 holds for the system (3.24) as well. We can associate every case of the 3WRI model (3.12) to every case of the model described by the system (3.24), as shown in the Table 3.1. Let us focus on the choices $S_1^+$ and $S_1^-$. Once the signs $S_1^+$ are fixed, the system 3WRI (3.24) gives us the right cases $S_2C_2$, $S_1C_4$ and $S_4C_6$ for the solution $u_j$, while the other remaining cases are reversed, $-S_1C_1$, $-S_4C_3$, $-S_2C_5$ and the sign minus denotes this reversion. Nevertheless, these cases are the right ones for $\bar{u}_j = -u_j$, and can be also obtained with the right signs by changing the signs $s_j \to -s_j$. Moreover, by the Proposition 3.2.1 the case $S_1C_1$ is equivalent to the case $S_1C_4$. The same argument holds for the other orderings.

Finally, we observe that with the model (3.12) one would have to deal with three Lax pairs and then, to obtain the other orderings, we have to swap the bigger velocity with the smaller
one. As we show in the following, with (3.24) we deal with only one Lax pair and, if we wish, we can cover all cases by a change of sign or by changing the values of the velocities without changing Lax pair.

3.3 Plane Wave Solutions

The system (3.24) admits as solutions the plane waves

\[ u_1 = a_1 e^{i(\eta_1 t - \nu_1 x)}, \quad u_2 = a_2 e^{i(\eta_2 t - \nu_2 x)}, \quad u_3 = a_3 e^{i(\eta_3 t - \nu_3 x)}, \]

(3.30)

where \( a_j \) are the amplitudes, \( \eta_j \) are the frequencies and \( \nu_j \) are the wave numbers.

By choosing the reference frame moving with the wave \( u_3 \), such that the velocity \( c_3 = 0 \) and by substituting the solution \( u_3 \) in the last equation of the system (3.24), we get the resonant conditions

\[ \eta_1 + \eta_2 + \eta_3 = 0, \quad \nu_1 + \nu_2 + \nu_3 = 0. \]

(3.31)

The amplitude \( a_3 \) takes the expression

\[ a_3 = is_3a_1a_2 \frac{c_1 - c_2}{\eta_1 + \eta_2}, \quad a_1, a_2 \in \mathbb{R}, \]

(3.32)

thus, the solution \( u_3 \) is

\[ u_3 = is_3a_1a_2 \frac{c_1 - c_2}{\eta_1 + \eta_2} e^{-i(\eta_1 + \eta_2)t - (\nu_1 + \nu_2)x}. \]

(3.33)

Moreover, by setting the expression (3.33) and the other two plane waves \( u_1 \) and \( u_2 \) in (3.30) into the first two equations of the system (3.24), the nonlinear dispersion relations are obtained

\[ \nu_1 = \frac{\eta_1}{c_1} + s_1 s_3 a_2 \frac{c_2 (c_1 - c_2)}{c_1 (\eta_1 + \eta_2)}, \]

\[ \nu_2 = \frac{\eta_2}{c_2} + s_2 s_3 a_1 \frac{c_1 (c_1 - c_2)}{c_2 (\eta_1 + \eta_2)}. \]

(3.34)

3.3.1 Galilean Invariance

The system (3.24) is invariant under the substitutions

\[ u_j(x, t) = \bar{u}_j(\bar{x}, \bar{t}), \]

(3.35)
where space $x$ and time $t$ transform as the Galilei transformations
\[
\begin{cases}
x = \bar{x} + v\bar{t}, \\
t = \bar{t},
\end{cases}
\] (3.36)
and the characteristic, linear velocities are
\[
c_j = \bar{c}_j + v, \quad j = 1, 2, 3.
\] (3.37)
Furthermore, by replacing the plane wave solutions in (3.35), we get
\[
\eta_j = \bar{\eta}_j + v\bar{\nu}_j, \quad \bar{\nu}_j = \nu_j, \quad \bar{a}_j = a_j, \quad \forall j = 1, 2, 3.
\] (3.38)
By the Galilean transformations for the frequencies and for the wave numbers (3.38), without loss of generality, we choose, for example, $\eta_1 = q$ and $\eta_2 = \pm q$. However, due to the resonant conditions (3.31), we can put $\eta_1 = \eta_2 = q$ only, otherwise three wave resonance does not occur.

From now on, we shall choose the reference frame in which $c_3 = 0$, and we shall fix $\eta_1 = \eta_2 = q$ and, as a consequence, $\eta_3 = -2q$. In this way, the plane waves (3.30) become
\[
u_1 = \frac{q}{c_1} + s_1 s_3 a_2^2 \frac{c_2 (c_1 - c_2)}{2qc_1},
\nu_2 = \frac{q}{c_2} + s_2 s_3 a_1^2 \frac{c_1 (c_1 - c_2)}{2qc_2}.
\] (3.40)

Indeed,
\[
\bar{u}_j(x, t) = \bar{a}_j(\bar{x}, \bar{t}),
\]
entails
\[
a_j e^{i(\eta_j t - \nu_j x)} = \bar{a}_j e^{i(\bar{\eta}_j \bar{t} - \bar{\nu}_j \bar{x})},
\]
and, by substituting (3.36),
\[
a_j e^{i(\eta_j t - \nu_j x)} = a_j e^{i((\eta_1 - \nu_1)\bar{t} - \nu_1 \bar{x})},
\]
from which the formulas (3.38).
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3.4 Lax Pair Formulation

The phenomenon of the linear stability, is analysed using the feature of the 3WRI model to be Lax-integrable. Indeed, a squared combination of fundamental solutions of the Lax pair is presented like a combination of the ‘eigenmode-solutions’ of the linearised equation [64]. In this respect, the differences of the eigenvalues of the Lax operators are necessary to write a solution of the linearised equation. In this section we give the explicit expressions of the Lax operators involved in the Lax formulation of the problem. We show how a similarity transformation allows us to break free the Lax operators from the dependence of space and time variables, such that the Lax equation is reduced to the Liouville equations, whose integration is trivial. A further gauge transformation simplifies our computations and makes our formula easier to handle. Finally, we find out that every Lax operator, obtained after such transformations, is written as a polynomial expression in the other Lax operator. Therefore, we obtain the relation between the differences of the eigenvalues of the Lax operators, and the solution of the linearised equation is provided in detail.

In the following, we apply the theory for multi-component systems provided in Chapter 2.

3.4.1 Linearised Equation

In this section we present the Lax pair and the linearised equation, following the research work [64]. Since the 3WRI is Lax-integrable, we associate to the system (3.24) the Lax operators $\tilde{X} \equiv \tilde{X}(x,t,\kappa)$ and $\tilde{T} \equiv \tilde{T}(x,t,\kappa)$

\[
\begin{align*}
\tilde{X} &= -i\kappa \tilde{C} + U, \\
\tilde{T} &= i\kappa \tilde{D} + V,
\end{align*}
\]

(3.41)

where $x$ and $t$ are the space and time variables, respectively, $\kappa$ is the spectral parameter, and $\tilde{C}$ and $\tilde{D}$ are $3 \times 3$ matrices depending only on the linear velocities $c_j$

\[
\tilde{C} = \begin{pmatrix}
-\frac{1}{2}(c_1 + c_2) & 0 & 0 \\
0 & \frac{1}{3}(2c_2 - c_1) & 0 \\
0 & 0 & \frac{1}{3}(2c_1 - c_2)
\end{pmatrix},
\]

(3.42)
\[
\hat{D} = \frac{c_1 c_2}{3} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The matrices \( U = U(x,t) \) and \( V = V(x,t) \) involve the solutions \( u_j \) of the system (3.24), the signs \( s_j \) and the velocities \( c_j \), and they are defined as follows

\[
U = \begin{pmatrix} 0 & s_1 u_1 & -s_1 s_2 s_3 u_2^* \\ s_1 s_2 s_3 u_4^* & 0 & s_3 u_3 \\ s_2 u_2 & s_1 s_2 s_3 u_4^* & 0 \end{pmatrix},
\]

\[
V = \begin{pmatrix} 0 & -s_1 c_1 u_1 & s_1 s_2 s_3 c_2 u_2^* \\ -s_1 s_2 s_3 c_1 u_1^* & 0 & 0 \\ -s_2 c_2 u_2 & 0 & 0 \end{pmatrix}.
\]

Let us introduce the Lax pair

\[
\tilde{\psi}_t = \tilde{X} \tilde{\psi}, \quad \tilde{\psi}_x = \tilde{T} \tilde{\psi},
\]

whose solution \( \tilde{\psi} = \tilde{\psi}(x,t,\kappa) \) is a \( 3 \times 3 \) matrix which satisfies the compatibility condition

\[
\tilde{\psi}_{xt} = \tilde{\psi}_{tx},
\]

that is equivalent to the equation for the Lax operators

\[
\tilde{X}_t - \tilde{T}_x + [\tilde{X}, \tilde{T}] = 0, \quad \forall \tilde{\psi},
\]

or, equivalently,

\[
U_t - V_x + [U, V] = 0, \quad \forall \tilde{\psi}.
\]

Here and thereafter the brackets \([\cdot, \cdot]\) denote the commutator between the operators.

**Proposition 3.4.1.** The following two operators

\[
X = -i\kappa C + U, \quad T = i\kappa D + V,
\]

where

\[
C = \tilde{C} + \frac{1}{3}(c_1 + c_2)I, \quad D = \hat{D} + \frac{2}{3}c_1 c_2 I,
\]

\[\text{We remind the reader that the asterisk stands for complex conjugation.}\]
and $\mathbf{I}$ is the identity matrix $3 \times 3$, constitute a Lax pair

$$
\psi_x = X\psi, \quad \psi_t = T\psi,
$$
(3.52)

whose solution transforms as

$$
\psi = \tilde{\psi}e^{\frac{i}{3}\kappa(c_1 + c_2)t - i\frac{2}{3}(c_1 + c_2)x}.
$$
(3.53)

Proof. Indeed the matrices $\tilde{C}$ and $\tilde{D}$, involved in (3.41), are

$$
\tilde{C} = -\frac{1}{3}(c_1 + c_2)\mathbf{I} + (c_1 + c_2)(\Sigma_+ + \Sigma_-),
$$
(3.54)

$$
\tilde{D} = -\frac{2}{3}c_1c_2\mathbf{I} + c_1c_2(\Sigma_+ + \Sigma_-),
$$
(3.55)

where

$$
\Sigma_+ = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}, \quad \Sigma_- = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
$$
(3.56)

This means that the Lax operators $X$ and $T$ have a diagonal part proportional to $\Sigma_+$ and $\Sigma_-$, and with entries $X_{11}$ and $T_{11}$ which are null elements. Looking at the compatibility conditions (3.48) and (3.49), we see that the terms $-i\frac{2}{3}\kappa c_1 c_2 I$ and $-i\frac{\kappa}{3}(c_1 + c_2) I$ do not affect the computation of the 3WRI equations and then the stability analysis. Moreover, by substituting $\tilde{X} = X + i\frac{\kappa}{3}(c_1 + c_2)$ and $\tilde{T} = T - i\frac{\kappa}{3}c_1c_2$ in the Lax pair (3.41), we have the equations

$$
\tilde{\psi}_x - i\frac{\kappa}{3}(c_1 + c_2)\tilde{\psi} = X\tilde{\psi},
$$
(3.57a)

$$
\tilde{\psi}_t + i\frac{2}{3}\kappa c_1 c_2 \tilde{\psi} = T\tilde{\psi},
$$
(3.57b)

and, by defining $\tilde{\psi} = \tilde{\psi}e^{i\frac{2}{3}\kappa c_1 c_2 t - i\frac{2}{3}(c_1 + c_2)x}$, the equations (3.57) become the Lax pair (3.52). $\square$

If we perturb the solutions $u_j \to u_j + \delta u_j$, we have, as a consequence, that also the Lax operators become perturbed $X \to X + \delta X$, $T \to T + \delta T$. If we set these perturbed
operators in the compatibility condition (3.48), at the first order in the perturbation, we get the equation for the perturbations \( \delta X \) and \( \delta T \):

\[
\delta X_t - \delta T_x + [\delta X, T] + [X, \delta T] = 0,
\]

(3.58)

named **linearised equation**. In order to find a solution of the linearised equation related to the solution of the Lax pair (3.46), we define the squared eigenfunction \( \Psi = \Psi(x, t, \kappa) \) by the following similarity transformation

\[
\Psi = \psi M \psi^{-1},
\]

(3.59)

where \( M = M(\kappa) \), is a constant matrix, independent on \( x \) and \( t \) and \( \psi^{-1} \) is the inverse of the matrix \( \psi = \psi(x, t, \kappa) \). After that, two propositions, stated in the paper [64], are given below.

**Proposition 3.4.2.** If the pair \( E, J \) solve the linearised equation (3.58), then also the pair

\[
F = [E, \Psi], \quad H = [J, \Psi],
\]

(3.60)

is a solution of the linearised equation (3.58), namely

\[
F_t - H_x + [F, T] + [X, H] = 0.
\]

(3.61)

**Proposition 3.4.3.** The following expressions,

\[
F = \left[ \frac{\partial X}{\partial \kappa}, \Psi \right], \quad H = \left[ \frac{\partial T}{\partial \kappa}, \Psi \right],
\]

(3.62)

provide solutions of the linearised equation (3.58).

One notes that, to obtain the solution \( F \), one needs to know the explicit expression of the squared eigenfunction (3.59) related to the solution of the Lax pair. This will be the aim of the next section.

### 3.4.2 Similarity Transformation of the Lax Pair

Once the Lax operators are provided, we see that only the matrices \( U \) and \( V \) enclose the solutions \( u_j \) and, so, the dependence on the variables \( x \) and \( t \). Below, we introduce a
similarity transformation by which the Lax pair becomes a pair of Liouville equations simple to integrate. In this way, one can obtained the squared eigenfunctions.

Let \( G = G(x, t, \kappa) \) be a matrix such that

\[
U = GU_0G^{-1}, \quad V = GV_0G^{-1},
\]

(3.63)

with \( G^{-1} \) the inverse of \( G = G(x, t, \kappa) \) and

\[
U_0 = \begin{pmatrix}
0 & s_1a_1 & -s_1s_2s_3a_2 \\
s_1s_2s_3a_1 & 0 & s_3a_3 \\
s_2a_2 & s_1s_2s_3a_3^* & 0
\end{pmatrix}, \quad V_0 = \begin{pmatrix}
0 & -s_1c_1a_1 & s_1s_2s_3c_2a_2 \\
-s_1s_2s_3c_1a_1 & 0 & 0 \\
-s_2c_2a_2 & 0 & 0
\end{pmatrix},
\]

(3.64)

so that the Lax operators are independent of \( x \) and \( t \).

More in details, the explicit expression of the matrix \( G \) is (Appendix D)

\[
G = e^{-\frac{i}{2}(\nu_1-\nu_2)x}e^{-i(qt-\nu_1x)\Sigma_+}e^{i(qt-\nu_2x)\Sigma_-},
\]

(3.65)

and \( [\Sigma_+, \Sigma_-] = 0 \). In addition, this transformation induces the similarity transformation \( \psi = G\phi \) on the solution of the Lax problem \(^{11}\) and, so, on its squared eigenfunctions (see definition in [64])

\[
\Psi = G\Phi G^{-1},
\]

(3.66)

\(^{11}\) The operator \( X \) is

\[
X = G(-i\kappa C + U_0)G^{-1};
\]

this, substituted in \( \psi_x = X\psi \), gives

\[
\psi_x = G(-i\kappa C + U_0)G^{-1}\psi,
\]

and, by multiplying to the left by \( G^{-1} \),

\[
G^{-1}\psi_x = (-i\kappa C + U_0)G^{-1}\psi.
\]

Since \( G^{-1}\psi_x = (G^{-1}\psi)_x - (G^{-1})_x\psi \) and \( (G^{-1})_xG + G^{-1}G_x = 0 \), we obtain

\[
(G^{-1}\psi)_x = (-i\kappa C + U_0 - G^{-1}G_x)(G^{-1}\psi);
\]

this, after defining \( \phi = G^{-1}\psi \) and \( iW_0 = i\kappa C + U_0 - G^{-1}G_x \), becomes

\[
\phi_x = iW_0\phi.
\]
and on the Lax operators (appendix E)

\[ iW_0 = G^{-1} XG - G^{-1} G_x, \quad -iZ_0 = G^{-1} TG - G^{-1} G_t, \]  

(3.67)

Therefore, the Lax pair becomes

\[ \Phi_x = i [W_0, \Phi], \quad \Phi_t = -i [Z_0, \Phi], \]  

(3.68)

whose solution is

\[ \Phi = e^{i(W_0 x - Z_0 t)} \Phi(0, 0) e^{-i(W_0 x - Z_0 t)}, \]  

(3.69)

with \( \Phi(0, 0) \) initial condition. Explicitly, we have

\[
W_0 = \begin{pmatrix}
\nu_1 - \nu_2 & -i s_1 a_1 & i s_1 s_2 s_3 a_2 \\
-\nu_1 s_1 s_2 s_3 a_1 & -\kappa c_2 - \frac{\nu_1 + \nu_2}{2} & -i s_3 a_3 \\
-\nu_2 a_2 & -\nu_1 s_2 s_3 a_3 & -\kappa c_1 + \frac{\nu_1 + \nu_2}{2}
\end{pmatrix}, 
\]  

(3.70a)

\[
Z_0 = \begin{pmatrix}
0 & -i s_1 c_1 a_1 & i s_1 s_2 s_3 c_2 a_2 \\
-\nu_1 s_1 s_2 s_3 c_1 a_1 & -\kappa c_1 c_2 - q & 0 \\
-\nu_2 c_2 a_2 & 0 & -\kappa c_1 c_2 + q
\end{pmatrix}, 
\]  

(3.70b)

with wave-numbers \( \nu_1 \) and \( \nu_2 \) given by (3.40) and \([W_0, Z_0] = 0\).

### 3.4.3 Gauge Transformation

In order to carry out our stability analysis, we need to know the eigenvalues of the matrices \( W_0 \) and \( Z_0 \), and so it is necessary to compute the characteristic polynomials of such matrices. However, the expressions (3.70a) and (3.70b) make the computation of their characteristic polynomials difficult. For this reason, we introduce a gauge transformation simplifying the form of the polynomials that we will introduce in the next sections.

The matrices \( G, W_0 \) and \( Z_0 \) in (3.63) and (3.70) can be generalised as follows (see formulas (D.7) in the Appendix D),

\[
G = \begin{pmatrix}
e^{i(m qt - \nu_1 x - \nu_2 x)} & 0 & 0 \\
0 & e^{i((m-1)qt - (l-1)\nu_1 x - \nu_2 x)} & 0 \\
0 & 0 & e^{i((m+1)qt - \nu_1 x - (n+1)\nu_2 x)}
\end{pmatrix}, 
\]  

(3.71)
or,
\[ G = e^{i(mq-(l\nu_1+n\nu_2)x)} I e^{-i(qt-\nu_1 x) \Sigma_+ e^{i(qt-\nu_2 x) \Sigma_-}}, \quad (3.72) \]
\[
W_0(\kappa; l, n) = \begin{pmatrix}
\nu_1 l + \nu_2 n & -is_1 a_1 & is_1 s_2 s_3 a_2 \\
-is_1 s_2 s_3 a_1 & -\kappa c_2 + \nu_1 (l - 1) + \nu_2 n & -is_3 a_3 \\
-is_2 a_2 & -is_1 s_2 s_3 a_3^* & -\kappa c_1 + \nu_1 l + \nu_2 (n + 1)
\end{pmatrix}, \quad (3.73)
\]
\[
Z_0(\kappa; m) = \begin{pmatrix}
mq & -is_1 c_1 a_1 & is_1 s_2 s_3 c_2 a_2 \\
-is_1 s_2 s_3 c_1 a_1 & -\kappa c_1 c_2 + q(m - 1) & 0 \\
-is_2 c_2 a_2 & 0 & -\kappa c_1 c_2 + q(m + 1)
\end{pmatrix}, \quad (3.74)
\]
where \( l, m \) and \( n \) are rational numbers.

They reduce to (3.65), (3.70a) and (3.70b) by setting \( m = 0, l = \frac{1}{2} \) and \( n = -\frac{1}{2} \).

Let us define the gauge transformation
\[ \bar{G} = Ge^{-i(mq-(l\nu_1+n\nu_2)x)} I, \quad (3.75) \]
such that,
\[ \bar{\psi} = \bar{G}\bar{\phi} = G\phi = \psi, \quad (3.76) \]
with \( \bar{\phi} = e^{i(mq-(l\nu_1+n\nu_2)x)} I \phi \). The Lax pair becomes \( \bar{\psi}_x = -i\bar{W}\bar{\psi}, \bar{\psi}_t = -i\bar{Z}\bar{\psi}. \)

By combining the Lax pair (3.77) with (3.76), we get
\[
\begin{cases}
\bar{\phi}_x = i\bar{W}\bar{\phi}, \\
\bar{\phi}_t = -i\bar{Z}\bar{\phi}.
\end{cases} \quad (3.78)
\]

On the other hand (Appendix G)
\[
\begin{cases}
\phi_x = iW_0\phi, \\
\phi_t = -iZ_0\phi,
\end{cases} \quad (3.79)
\]

\(^{12}\)In this case, \( X \equiv X \) and \( T \equiv T \).
and therefore,

\[ W = W_0 - (\nu_1 + \nu_2)I, \quad Z = Z_0 - mqI, \]  

(3.80)

with \( l, n \) and \( m \) which are, in general, rational numbers.

By using the gauge transformations (3.75) and (3.80), by setting \( m = 0, l = \frac{1}{2}, n = -\frac{1}{2} \) and multiplying \( W \) by \( 2c_1c_2q \), we shall work with the matrices

\[
W = 2c_1c_2q \begin{pmatrix}
0 & -is_1a_1 & is_1s_2s_3a_2 \\
-is_1s_2s_3a_1 & -\kappa c_2 - \nu_1 & -is_3c_2 \\
-is_2a_2 & -is_1s_2s_3a_3 & -\kappa c_1 + \nu_2
\end{pmatrix},
\]

(3.81)

and

\[
Z = \begin{pmatrix}
0 & -is_1c_1a_1 & is_1s_2s_3c_2a_2 \\
-is_1s_2s_3c_1a_1 & -\kappa c_1c_2 - q & 0 \\
-is_2c_2a_2 & 0 & -\kappa c_1c_2 + q
\end{pmatrix}.
\]

(3.82)

This choice will prove convenient in the light of the treatment illustrated in the next chapter.

### 3.4.4 Relations between the Transformed Lax Operators and the Differences of their Eigenvalues

The matrix \( W \) is expressible as a polynomial of the matrix \( Z \) (Appendix H) as follows

\[
W = (c_1 - c_2)Z^2 - c_1(-q - c_1c_2\kappa)Z + c_2(q - c_1c_2\kappa)Z - (c_1 - c_2)(a_1^2a_2^2s_1s_3 - a_1^2c_1s_2s_3)I,
\]

(3.83)

and vice versa, the matrix \( Z \) as function of the matrix \( W \) (Appendix H) is

\[
Z \left[ \frac{a_1^2s_1s_3(c_1 - c_2)}{2c_1q} - \frac{a_2^2s_2s_3(c_1 - c_2)}{2c_2q} - \frac{(c_2 + c_1)q}{c_1c_2(c_1 - c_2)} + \kappa \right] = \frac{W^2}{4c_1c_2q^2} - \frac{W}{2c_1c_2q} \left[ \frac{a_1^2s_2s_3c_1(c_1 - c_2)}{2c_2q} - \frac{a_2^2s_1s_3c_2(c_1 - c_2)}{2c_1q} + \frac{q(c_1 - c_2)}{c_1c_2} - (c_1 + c_2)\kappa + \frac{2q}{c_1 - c_2} \right] - (a_1^2s_1s_3 - a_2^2s_2s_3)I.
\]

(3.84)

In addition, if \( w_j \) and \( z_j, \ j = 1, 2, 3 \) are the eigenvalues of \( W \) and \( Z \), respectively, then

\[
w_j = (c_1 - c_2)z_j^2 - c_1(-q - c_1c_2\kappa)z_j + c_2(q - c_1c_2\kappa)z_j - (c_1 - c_2)(a_1^2a_2^2s_1s_3 - a_1^2c_1s_2s_3), \quad j = 1, 2, 3,
\]
and
\[ z_j \left[ \frac{a_2^2 s_1 s_3 (c_1 - c_2)}{2 c_1 q} - \frac{a_1^2 s_2 s_3 (c_1 - c_2)}{2 c_2 q} - \frac{(c_2 + c_1) q}{c_1 c_2 (c_1 - c_2)} + \kappa \right] = \frac{w_j^2}{4 c_1^2 c_2 q^2} - 
\]
\[ - \frac{w_j}{2 c_1 c_2 q} \left[ \frac{a_2^2 s_2 s_3 c_1 (c_1 - c_2)}{2 c_2 q} - \frac{a_1^2 s_1 s_3 c_2 (c_1 - c_2)}{2 c_1 q} + \frac{q (c_1 - c_2)}{c_1 c_2} - \frac{(c_1 + c_2) q}{c_1 c_2} \right] - 
\]
\[ - (a_2^2 s_1 s_3 - a_1^2 s_2 s_3), \quad j = 1, 2, 3, \]

(3.85)

hence, the differences of the eigenvalues of \( W \) are (Appendix I)
\[ w_j - w_{j+1} = (z_j - z_{j+1}) (c_1 - c_2) \left( -z_{j+2} - c_1 c_2 \kappa + q \frac{c_1 + c_2}{c_1 - c_2} \right), \quad j = 1, 2, 3, \text{ mod}(3). \]

(3.86)

and the differences between the eigenvalues of \( Z \) are (Appendix I)
\[ (z_j - z_{j+1}) \left( 2 a_2^2 s_1 s_3 c_2 (c_1 - c_2) - 2 a_1^2 s_2 s_3 c_1 (c_1 - c_2) - 2 q^2 \frac{(c_2 + c_1)}{c_1 - c_2} + 2 c_1 c_2 q \kappa \right) = 
\]
\[ = - (w_j - w_{j+1}) \left( \frac{w_{j+2}}{2 c_1 c_2 q} + \frac{2 q}{c_1 - c_2} \right). \]

(3.87)

### 3.4.5 Characteristic Polynomials and Rescaled Differences of the Eigenvalues

Although in the previous sections we use some transformations to greatly simplify the Lax operators and the Lax pair, the characteristic polynomials are still hard to handle. However, we can do some further substitutions to obtain a more elegant expression of the characteristic polynomials.

First of all, we can rescale some parameters by \( q \). Once the characteristic polynomials of \( W \) and \( Z \) are denoted with \( \tilde{P}_W(\tilde{w}; \kappa) \) and \( \tilde{P}_Z(\tilde{z}; \kappa) \), respectively, then the amplitudes, the unknowns \( \tilde{w} \) and \( \tilde{z} \), and the spectral parameter can be rescaled as follows:

\[ a_1 = q \alpha_1, \quad a_2 = q \alpha_2, \]

(3.89a)

\[ \tilde{w} = q^2 w, \quad \tilde{z} = q z, \]

(3.89b)
\[ \kappa = \frac{q\lambda}{c_1c_2}. \tag{3.89c} \]

At this point, we rename the characteristic polynomials
\[ P_W(w; \lambda) = -\frac{\tilde{P}(w; \kappa)}{q^6}, \quad P_Z(z; \lambda) = -\frac{\tilde{P}(z; \kappa)}{q^3}. \tag{3.90} \]

The coefficients of these characteristic polynomials are expressions of the rescaled amplitudes to the second and to the fourth and we benefit from further substitutions,
\[ p_1 = \frac{c_1^2s_1\alpha_1^2 + c_2^2s_2\alpha_2^2}{s_1s_2s_3}, \quad p_2 = \frac{c_1^2s_1\alpha_1^2 - c_2^2s_2\alpha_2^2}{s_1s_2s_3}, \tag{3.91} \]

and combinations of the velocities
\[ \frac{c_1 - c_2}{c_1 + c_2} = p_3, \quad c_1 + c_2 = p_4, \quad p_4 \neq 0. \tag{3.92} \]

We highlight that later in our analysis we will consider also the limiting case \( p_3 \to \infty \) (that is \( p_4 \to 0 \)).

The computations of the characteristic polynomials after the substitutions above show that, after multiplying the variable \( w \) by a factor \( p_4 \), we can rescale once more the polynomial \( P_W(w; \lambda) \) by a factor \( p_4^3 \). Finally, the characteristic polynomials become
\[ P_W(w; \lambda) = w^3 + [2\lambda - p_3(2 + p_2)] w^2 + \]
\[ + [p_2(1 + 2p_3 - p_3\lambda) + p_1p_3(-3 + p_3\lambda) - (p_3^2 - 1)(\lambda^2 - 1)] w + \]
\[ + [p_2(\lambda - p_3(-1 + p_2 + p_3(p_3 + \lambda))) - p_1^2p_3^3 + \]
\[ + p_1(-1 + p_3(p_3 + 2p_2p_3 - \lambda + p_2^3\lambda)], \tag{3.93} \]

and
\[ P_Z(z; \lambda) = z^3 + 2\lambda z^2 + (\lambda^2 + p_2 - 1)z + p_2\lambda - p_1. \tag{3.94} \]

Moreover, by setting the substitutions above, we rewrite the formulas (3.83) and (3.84) respectively as follows
\[ W = p_3Z^2 + q(1 + p_3\lambda)Z + p_3p_2q^2, \tag{3.95} \]

and
\[
\begin{align*}
Z &= -\frac{p_3}{1 - p_3(\lambda + p_3^2(p_1 - \lambda) - p_2p_3 + p_3)} W^2 + \\
&\quad + \frac{(p_2p_3^2 - 2\lambda p_3^3 + p_3^2 + 1)}{1 - p_3(\lambda + p_3^2(p_1 - \lambda) - p_2p_3 + p_3)} W + \\
&\quad + \frac{2p_1p_3^2 - p_2p_3^2 - p_2p_3}{1 - p_3(\lambda + p_3^2(p_1 - \lambda) - p_2p_3 + p_3)},
\end{align*}
\]
hence, the differences between the eigenvalues become

\[ w_j - w_{j+1} = (z_j - z_{j+1}) \left[ (-p_3 z_{j+2} + q(1 - \lambda p_3)) \right], \tag{3.97} \]

for \( j = 1, 2, 3, \mod(3) \), and

\[ z_j - z_{j+1} = \frac{(w_1 - w_2) \left( p_3 \left( -p_3(-2\lambda + p_2 + 1) + p_4 q^2 ((p_2 + 2)p_3 - 2\lambda) - w_3 \right) - 1 \right)}{1 - p_3 \left( \lambda + p_3^2(p_1 - \lambda) - p_2 p_3 + p_3 \right)}, \tag{3.98} \]

for \( j = 1, 2, 3, \mod(3) \). Furthermore, we give some useful formulas. Because the trace is invariant under cyclic permutations, the eigenvalues \( w_j \) and \( z_j \) satisfy the following relations

\[ w_1 + w_2 + w_3 = ((2 + p_2)p_3 - 2\lambda), \tag{3.99} \]

\[ z_1 + z_2 + z_3 = -2\lambda, \tag{3.100} \]

with \( p_4 = c_1 + c_2 \), and by the computation of the determinant of the matrices \( W \) and \( Z \), we get

\[ z_1 z_2 z_3 = p_1 - \lambda p_2, \tag{3.101} \]

\[ w_1 w_2 w_3 = p_1^2 p_3^3 - p_1 \left[ p_3 \left( -\lambda + 2p_2 p_3 + \lambda p_3^2 - p_3 \right) - 1 \right] + p_2 \left[ p_2 p_3 + (p_3^2 - 1) (\lambda + p_3) \right]. \tag{3.102} \]

Obviously, the formulas above are satisfied for both complex and real \( \lambda \).

Because a solution of the linearised equation can be expressed as combination of the exponentials \( e^{i(\pi(w_j - w_{j+1}) - t(z_j - z_{j+1}))} \) \[64\], we have completed the preliminary calculations in order to prepare the work for the stability analysis.
Chapter 4

Spectra and Linear Instabilities of the 3WRI Equations

In this Chapter we will follow the theory for multi-component systems provided in Chapter 2 and we will use all the preliminary computations carried out in Chapter 3.

We provide the definition of $S_x$-spectrum as composed by the values of the spectral parameter $\lambda$ which are the roots of the polynomial $P(\xi, \lambda)$ of the squares of the differences $\xi = (w_1 - w_2)^2$. The analysis of the nature of the $\lambda$-roots allows us to obtain a full topological classification of the stability spectra in the parameters space. Using a numerical routine implemented in MATLAB R2017a, for any generic choice of the physical parameters, we plot the stability spectrum and its associated gain function (see definition below). The fact that this function is always different from zero indicates that linear instability occurs for any generic choice of the parameters $p_1$, $p_2$ and $p_3$.

4.1 Spatial and Temporal Stability Spectra

Given the real parameters $p_1$, $p_2$ and $p_3$ (see formulas (3.91) and (3.92)), we are interested in finding the values of the complex spectral parameter $\lambda$ such that the plane waves are bounded in space and see if they are linearly stable or unstable in time. In other words, let $w_j$ be the eigenvalues of the matrix $W$ and $z_j$ the eigenvalues of $Z$; then we aim to search
those values of the spectral parameter $\lambda$ corresponding to real differences $k_j = w_{j+1} - w_{j+2}$, whereas the differences $\omega_j = z_{j+1} - z_{j+2}$ can be real or complex. This latter point can be understood for the differences $k_j$ are linked to the differences $\omega_j$ by the formula

$$k_j = \omega_j (-p_3 z_j + q (1 - \lambda p_3)), \quad j = 1, 2, 3. \quad (4.1)$$

Therefore, values of the spectral parameter $\lambda$ corresponding to real $k_j$ may correspond to complex $\omega_j$ and so linear instability may occur.

In the following, we will refer to the eigenvalues $w$ (resp. $z$) of the matrix $W$ (resp. $Z$) also as $w$-roots (resp. $z$-roots) of the characteristic polynomial $P_W(w; \lambda)$ (resp. $P_Z(z; \lambda)$), namely, the polynomial roots of the equation $P_W(w; \lambda) = 0$ (resp. $P_Z(z; \lambda) = 0$), solved with respect to $w$ (resp. $z$). From here on, we fix $q = 1$, without loss of generality, because of the Galilean invariance (Chapter 3).

**Definition 4.1.1.** The spatial stability spectrum $S_x$ for the plane wave solutions of the 3WRI system (3.5), is defined as the locus of the $\lambda$-plane identified with $\mathbb{C}$ such that, for fixed values of the physical parameters $p_1$, $p_2$ and $p_3$, the characteristic polynomial $P_W(w; \lambda)$, admits at least two $w$-roots such that their difference is a real number, namely, the set of the spectral parameter $\lambda$ for which there exist at least two eigenvalues $w_\ell$ and $w_m$ for the matrix $W$, for some $\ell$ and $m$, for which $(w_\ell - w_m) \in \mathbb{R}$.

**Definition 4.1.2.** The temporal stability spectrum $S_t$ for the plane wave solutions of the 3WRI system (3.5), is defined as the locus of the $\lambda$-plane identified with $\mathbb{C}$ such that, for fixed values of the physical parameters $p_1$, $p_2$ and $p_3$, the characteristic polynomial $P_Z(z; \lambda)$ admits at least two $z$-roots such that their difference is a real number, namely, the set of values of the spectral parameter $\lambda$ such that there exist at least two eigenvalues $z_\ell$ and $z_m$ for the matrix $Z$, for some $\ell$ and $m$, for which $(z_\ell - z_m) \in \mathbb{R}$.

In the following, we give the definition of the components of the stability spectrum $S_x$.

**Definition 4.1.3.** Real values of the spectral parameter $\lambda$, not belonging to $S_x$ constitute a gap ($G$). A gap, including a single isolated real point within its real endpoints, will be renamed as split gap ($SG$).

Complex values of the spectral parameter $\lambda$, belonging to $S_x$ correspond to branches ($B$) and loops ($L$), which are open and closed curves, respectively. Figure of eight loops, self-intersecting on the real axis, will be referred to as twisted loops ($TL$).
Once the parameter $p_3$ is fixed, the topological classification of the spacial stability spectra $S_x$ in the $(p_1, p_2)$ parameters space, can be obtained by means of the algebraic-geometric procedure described in the next sections.

4.1.1 Polynomials of the Squares of the Differences

Let us write the characteristic polynomials (3.93) and (3.94) in a general form

$$P_W(w; \lambda) \equiv P_W(w; p_1, p_2, p_3) = \sum_{j=0}^{3} a_j^{(W)} w^j = \prod_{j=1}^{3} (w - w_j), \quad a_3^{(W)} = 1, \quad (4.2a)$$

$$P_Z(z; \lambda) \equiv P_Z(z; p_1, p_2, p_3) = \sum_{j=0}^{3} a_j^{(Z)} z^j = \prod_{j=1}^{3} (z - z_j), \quad a_3^{(Z)} = 1. \quad (4.2b)$$

By combinations of the coefficients $\{a_j^{(W)}\}_{j=0}^{3}$, we construct the coefficients of the polynomial $P_W(\xi; \lambda)$ of degree 3 in the variable $\xi$, whose $\xi$-roots are the squares of all the possible differences of the roots of the polynomial $P_W(w; \lambda)$ (Appendix J).

$$P_W(\xi; \lambda) \equiv P_W(\xi; p_1, p_2, p_3) = \prod_{j,h=1, j<h}^{3} \left[ \xi - (w_j - w_h)^2 \right], \quad (4.3)$$

that is a 2-variate polynomial in $\xi$ and $\lambda$. For the sake of simplicity, we will refer to $P_W(\xi; \lambda)$ as the polynomial of the squares of the differences. A $\xi$-root (resp. $\lambda$-root) of $P_W(\xi; \lambda)$ is a polynomial root of the equation $P_W(\xi; \lambda) = 0$, solved with respect to $\xi$ (resp. $\lambda$).

For any fixed $p_1$, $p_2$ and $p_3$ parameters, the spectrum $S_x$ is the locus of the $\lambda$-roots of $P_W(\xi; \lambda)$ for all $\xi \in \mathbb{R}$, $\xi \geq 0$. In other words, the spatial spectrum $S_x$ can be seen as a one-parameter algebraic variety over the complex numbers, and it is defined as

$$S_x = \{ \lambda \in \mathbb{C} \mid P_W(\xi; \lambda) = 0, \xi \in \mathbb{R}, \xi \geq 0 \} \quad (4.4)$$

Similarly, by combinations of the coefficients $\{a_j^{(Z)}\}_{j=0}^{3}$, we can construct the coefficients of the polynomial $P_Z(\eta; \lambda) = P(\eta; \lambda; p_1, p_2, p_3)$, that is a polynomial of degree 3 in the

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1 An analogous procedure can be implemented for deriving the temporal stability spectra $S_t$, starting from $Z$ instead of $W$. 

---
variable \( \eta \) and whose \( \eta \)-roots are the squares of the differences of the roots of the polynomial \( P_Z(z; \lambda) \) (Appendix J),

\[
P_Z(\eta; \lambda) \equiv P_Z(\eta; \lambda; p_1, p_2, p_3) = \prod_{j,h=1 \atop j \neq h}^{3} (\eta - (z_j - z_h)^2),
\]

that is a 2-variate polynomial in \( \eta \) and \( \lambda \). A \( \eta \)-root (resp. \( \lambda \)-root) of \( P_Z(\eta; \lambda) \) is a polynomial root of the equation \( P_Z(\eta; \lambda) = 0 \), solved with respect to \( \eta \) (resp. \( \lambda \)).

For any fixed \( p_1, p_2 \) and \( p_3 \) parameters, the temporal spectrum \( S_t \) is the locus of the \( \lambda \)-roots of \( P_Z(\eta; \lambda) \) for all \( \eta \in \mathbb{R}, \eta \geq 0 \). In other words, the spectrum \( S_t \) can be seen as the one-parameter algebraic variety over the complex numbers, and it is defined as

\[
S_t = \{ \lambda \in \mathbb{C} | P_Z(\eta; \lambda) = 0, \eta \in \mathbb{R}, \eta \geq 0 \}.
\]

### 4.2 Real Spectrum

In this section we take into account only real \( \lambda \)-roots. By bearing in mind this assumption, and because \( P(w; \lambda) \) is a cubic polynomial, the existence of a real \( w \)-root implies that the other two \( w \)-roots are real too, otherwise there are two complex conjugate \( w \)-roots and a \( w \)-real root. Therefore, in the first case the differences \( k_j \) are all real and the corresponding \( \lambda \) values belong to the spectrum \( S_x \). In the second case, they are all complex and \( \lambda \) values belong to a gap.

By denoting by \( \Delta_y(P(y)) \) the discriminant with respect to \( y \) of the polynomial \( P(y) \), we observe that the polynomial of the squares of the differences \( P_W(0; \lambda) \) (resp. \( P_Z(0; \lambda) \)) is equal to the discriminant with respect to \( w \) (resp. \( z \)) of the characteristic polynomial \( P_W(w; \lambda) \) (resp. \( P_Z(z; \lambda) \)) with the opposite sign,

\[
P_W(0; \lambda) = -\Delta_w(P_W(w; \lambda)), \quad P_Z(0; \lambda) = -\Delta_z(P_Z(z; \lambda)),
\]

and they are related as follows

\[
P_W(0; \lambda) = P_Z(0; \lambda)R^2(\lambda),
\]

where

\[
-P_Z(0; \lambda) = 4\lambda^4 - 27p_1^2 - 4\lambda^3p_1 + 18\lambda p_1(p_2 + 2) + \lambda^2((p_2 - 20)p_2 - 8) - 4(p_2 - 1)^3.
\]
and
\[ R(\lambda) = \left( p_3 \left( \lambda + p_3^2 (p_1 - \lambda) - p_2 p_3 + p_3 \right) - 1 \right). \] (4.10)

Since we impose \( k_j \) to be real, we consider only \( P_W(0; \lambda) \leq 0 \).

Indeed, by assuming \( R(\lambda) \neq 0 \), \( P_Z(0; \lambda) \leq 0 \) if and only if \( P_W(0; \lambda) \leq 0 \), that is, by (4.8), if \( P_Z(0; \lambda) \) is negative or zero, then \( P_W(0; \lambda) \) is like that, and vice versa.

There is also the limiting case \( R(\lambda) = 0 \), satisfied for
\[ \lambda = \frac{p_3^2 (p_1 p_3 - p_2 + 1) - 1}{p_3 (p_3^2 - 1)}, \] (4.11)
for which \( P_W(0; \lambda) = 0 \), but \( P_Z(0; \lambda) \) can be zero, positive or negative. In the following proposition we prove that only the real part of \( S_x \) contributes to the stability, namely that, if \( S_x \) has an off-real component, then the solution is expected to be unstable.

**Proposition 4.2.1.** For a generic choice of the physical parameters, the plane wave solution of the 3WRI system is stable against perturbations \( \delta Q \) integrated only over values of \( \lambda \) in \( S_x \) strictly real, with the exception of the point separating a split gap.

**Proof.** Let us suppose \( \lambda \in \mathbb{R} \), so that \( w_j \) and \( z_j \) are roots of third degree polynomials with real coefficients. As a consequence, the characteristic polynomials can have: three real and distinct \( w \)-roots; or a real triple \( w \)-root; or a \( w \)-real double root and a real simple \( w \)-root or two complex conjugate \( w \)-roots and a real \( w \)-root. We impose that \( w_j - w_{j+1} \) be real\(^2\) thus we exclude the case in which the characteristic polynomial \( P(w; \lambda) \) has two complex conjugate \( w \)-roots and a real \( w \)-root.

Let us assume \( R(\lambda) \neq 0 \) and let \( w_j \) be real and all distinct, i.e. \( P_W(0; \lambda) < 0 \). Thus, since \( p_4 \in \mathbb{R} \), all the differences \( w_j - w_{j+1} \) are also real and all distinct
\[ w_j - w_{j+1} = (z_j - z_{j+1})(-z_j p_3 - \lambda p_3 + 1), \quad j = 1, 2, 3, \mod(3), \] (4.12)
and then the product \( (z_j - z_{j+1})(-z_j p_3 - \lambda p_3 + 1) \) is real too. For \( \lambda \in \mathbb{R} \), we cannot have \( (z_j - z_{j+1}) \in \mathbb{C} \) and \( (-z_j p_3 - \lambda p_3 + 1) \in \mathbb{C} \) so that \( (z_j - z_{j+1})(-z_j p_3 - \lambda p_3 + 1) \in \mathbb{R} \) because, by the relation (4.8), we would have \( P_Z(0; \lambda) > 0 \), and so \( P_W(0; \lambda) > 0 \), in contradiction with the hypothesis. Instead, we can have \( (z_j - z_{j+1}) \in \mathbb{R} \) and \( (-z_j p_3 - \lambda p_3 + 1) \in \]

\(^2\)Since \( w_j \) are roots of a third degree polynomial with real coefficients, then the requirement to have at least a real difference implies that all the differences \( w_j - w_{j+1} \) are real too.
\(R\), so that their product is real, and then \(w_j - w_{j+1} \in R, \forall j = 1, 2, 3.\)

Moreover, we remind that if also only two roots coincide, then \(P_z(0; \lambda) = 0\), and this condition pushes \(P_w(0; \lambda) = 0\) to vanish, but we are imposing \(P_w(0; \lambda)\) to be strictly positive.

Therefore we conclude that if \(\lambda \in R\) and \(R(\lambda) \neq 0\), then \(P_w(0; \lambda) < 0\) if and only if \(P_z(0; \lambda) < 0\).

Let us suppose \(P_w(0; \lambda) = 0\) and \(P(w; \lambda)\) has a double real root and a simple real root. For the sake of simplicity, we impose \(w_1 = w_2\), while \(w_3\) is different from the other two. From the formula \((4.12)\) and by considering \(p_3 \neq 0\), we distinguish three cases corresponding to all the possibilities for which we have \(w_1 - w_2 = 0\):

1. \(z_1 - z_2 = 0\) and \(-z_3p_3 - \lambda p_3 + 1 = 0\), \((P_z(0; \lambda) = 0\) and \(R(\lambda) = 0)\);
2. \(z_1 - z_2 = 0\) and \(-z_3p_3 - \lambda p_3 + 1 \neq 0\), \((P_z(0; \lambda) = 0\) and \(R(\lambda) \neq 0)\);
3. \(z_1 - z_2 \neq 0\) and \(-z_3p_3 - \lambda p_3 + 1 = 0\), \((P_z(0; \lambda) \neq 0\) and \(R(\lambda) = 0)\).

In the following we shall discuss the reverse arguments: starting from the hypothesis 1., 2., and 3., we show that \(w_1 = w_2\), and the \(w\)-roots are all real, i.e. if \(P_z(0; \lambda) = 0\), then \(P_w(0; \lambda) = 0\).

1. If \(z_1 = z_2\), then \(z_1\) and \(z_2\) are real, then \(z_3\) is necessarily real, and for this particular case we get

\[z_3 = \frac{1}{p_3} - \lambda,\]  

\((4.13)\)

and by \((3.100)\)

\[z_1 = z_2 = -\frac{1}{2} \left( \frac{1}{p_3} + \lambda \right).\]  

\((4.14)\)

Moreover,

\[z_2 - z_3 = -\frac{1}{2} \left( \frac{3}{p_3} - \lambda \right), \quad z_3 - z_1 = \frac{1}{2} \left( \frac{3}{p_3} - \lambda \right),\]  

\((4.15)\)

so that

\[z_1 - z_2 = 0, \quad (z_2 - z_3) = -(z_3 - z_1).\]  

\((4.16)\)
By keeping in mind formulas (4.13) and (4.12), we compute the differences \( w_j - w_{j+1} \) and get

\[
w_1 - w_2 = 0, \quad w_2 - w_3 = -(w_3 - w_1) = -\frac{p_4}{2} \left( \frac{3}{p_3} - \lambda \right) (-z_1p_3 - \lambda p_3 + 1),
\]

all real, because all the parameters involved and all the three \( z_j \) are real.

Thus, we conclude that \( P_Z(0; \lambda) = 0 \) implies \( P_W(0; \lambda) = 0 \).

Furthermore, the characteristic polynomial \( P(z_3; \lambda) \) becomes proportional to \( R(\lambda) \),

\[
P(z_3; \lambda) = \frac{1 - p_3(\lambda + p_3^2(p_1 - \lambda) - p_2p_3 + p_3)}{p_3^3} = -\frac{1}{p_3^3} R(\lambda).
\]

This implies that, if \( z_3 = \frac{1}{p_3} - \lambda \) is a zero of the characteristic polynomial \( P(z; \lambda) \), then \( R(\lambda) = 0 \) and \( \lambda \) has the expression (4.11). The reverse argument is also true: if \( R(\lambda) = 0 \), then \( z_3 = \frac{1}{p_3} - \lambda \) is a zero of the characteristic polynomial \( P(z; \lambda) \).

2. We assume \( z_1 = z_2 \) and \( z_1 \) and \( z_2 \) are both real. As a consequence \( z_3 \) is real, but we impose that \( z_3 \) does not satisfy the formula (4.13) any more. Nevertheless, by the formulas (3.100) and (3.101), we have that

\[
2z_j^3 + 2\lambda z_j^2 + (1 - \lambda)p_2 = 0, \quad z_j = z_1 = z_2,
\]

and once \( z_j \) is obtained, we can use (3.100) to get \( z_3 = -2(z_j + \lambda) \). Therefore, the relations (4.17) and (4.16) are still satisfied, indeed

\[
z_1 = z_2, \quad z_3 - z_1 = -(z_2 - z_3) = -(3z_j + 2\lambda), \quad z_j = z_1 = z_2,
\]

and then,

\[
w_1 = w_2, \quad w_2 - w_3 = -(w_3 - w_1) = (3z_j + 2\lambda)(-p_3z_j - p_3\lambda + 1), \quad z_j = z_1 = z_2,
\]

and all the three differences \( (w_j - w_{j+1}) \) are real. Even for this case \( P_Z(0; \lambda) = 0 \) implies \( P_W(0; \lambda) = 0 \). In this case \( R(\lambda) \) is different from zero.

3. Let us consider the most general case in which all the three \( z_j \) are different from one another. \( P_W(0; \lambda) = 0 \) when \( R(\lambda) = 0 \). We identify two sub-cases:

\footnote{It is not the aim of our analysis to find the explicit expression of the solutions \( z_j = z_1 = z_2 \).}
3a) \( \mathcal{P}_Z(0; \lambda) < 0 \), and the characteristic polynomial \( P(z; \lambda) \) has three real \( z \)-roots all distinct;

3b) \( \mathcal{P}_Z(0; \lambda) > 0 \), and the characteristic polynomial \( P(z; \lambda) \) has two complex conjugate \( z \)-roots and a real \( z \)-root.

Furthermore, since \( R(\lambda) = 0 \), then \( \mathcal{P}_W(0; \lambda) = 0 \), and \( P(w; \lambda) \) has a) a real double \( w \)-root and a simple \( w \)-root or b) a triple real \( w \)-root.

3a) Let us suppose \( z_j \) are all different, but \( z_3 = \frac{1}{p_3} - \lambda \). By substituting \( \lambda = \frac{1}{p_3} - z_3 \), into the other two differences \( w_2 - w_3 \) and \( w_3 - w_1 \), we get

\[
   w_2 - w_3 = -p_3(z_2 - z_3)(z_1 - z_3), \quad w_3 - w_1 = p_3(z_2 - z_3)(z_1 - z_3), \tag{4.22}
\]

that is

\[
   w_2 - w_3 = -(w_3 - w_1), \tag{4.23}
\]

and in addition

\[
   w_1 - w_2 = 0. \tag{4.24}
\]

However, this case is impossible. Indeed, by using the formula (4.22) and matching the two formulas

\[
   w_2 - w_3 = (z_2 - z_3)(-p_3z_1 - p_3\lambda + 1), \tag{4.25}
\]

\[
   w_3 - w_1 = (z_3 - z_1)(-p_3z_2 - p_3\lambda + 1), \tag{4.26}
\]

via (4.23), we get the equation

\[
   (z_2 - z_3)(-p_3z_1 - p_3\lambda + 1) = (z_1 - z_3)(-p_3z_2 - p_3\lambda + 1), \tag{4.27}
\]

that is satisfied for \( z_1 = z_2 \). However, this is a contradiction to the hypothesis \( \mathcal{P}_Z(0; \lambda) < 0 \).

Thus, we cannot have \( \mathcal{P}_Z(0; \lambda) < 0 \), and instead we have \( \mathcal{P}_Z(0; \lambda) = 0 \). Note that because of the formula (4.8), we have \( R(\lambda) = 0 \).
Finally, we conclude that if \( w_1 = w_2 = w_3 \) has to be verified, we need to impose \( z_1 = z_2 = z_3 = \frac{1}{p_3} - \lambda \), that is a contradiction to the hypothesis \( P_Z(0; \lambda) < 0 \).

3b) If \( P_Z(0; \lambda) > 0 \), the characteristic polynomial \( P(z; \lambda) \) has two complex conjugate roots, say, \( z_1 \) and \( z_2 \), and a real root \( z_3 \). In order the case a) to be verified, the relation \( z_3 = \frac{1}{p_3} - \lambda \) has to be satisfied. We can not have \( (-z_j p_3 - \lambda p_3 + 1) = 0, \) for \( j = 1, 2 \), because this equation would be satisfied only for \( z \)-roots all real, but we are in the hypothesis \( P_Z(0; \lambda) > 0 \). We note that, since \( \Re(z_1) = \Re(z_2) \) and \( \Im(z_1) = -\Im(z_2) \), then \( z_2 - z_3 \) and \( z_3 - z_1 \) are complex with the same imaginary part and \( z_1 - z_2 = 2i\Im(z_1) \). Moreover, because \( w_2 - w_3 = -(w_3 - w_1) = -p_3(z_1 - z_3)(z_2 - z_3) \), the differences \( w_2 - w_3 \) and \( w_3 - w_1 \) are real, in particular

\[
- w_2 - w_3 = - (w_3 - w_1) = -p_3 \left( (\Re(z_j) - z_3)^2 + (\Im(z_j))^2 \right), \quad z_j = z_1 = z_2, \quad (4.28)
\]

and we have that if \( p_3 p_4 > 0 \), then \( w_2 - w_3 < 0 < w_3 - w_1 \), and if \( p_3 p_4 < 0 \), then \( w_3 - w_1 < 0 < w_2 - w_3 \).

Moreover, \( z_3 \) coincides with the solution of \( R(\lambda) \) and the formula (4.18) is still satisfied.

The case b) is impossible because, since we require a priori that all the roots \( z_j \) must be different from each other, we have to impose \( (-z_j p_3 - \lambda p_3 + 1) = 0, \forall j = 1, 2, 3 \) in order to have all the three \( w_j \) coinciding and real, which gives us all \( z_j \) coinciding and real, but this is a contradiction to the hypothesis \( P_Z(0; \lambda) > 0 \).

Let us suppose \( P(w; \lambda) \) has a triple real root, that is \( P_W(0; \lambda) = 0 \). We distinguish two cases:

1. \( z_1 = z_2 = z_3, \) but \( -p_3 z_j - p_3 \lambda + 1 \neq 0, \) \( (P_Z(0; \lambda) = 0 \) and \( R(\lambda) \neq 0) \);

2. \( z_1 = z_2 = z_3, \) and \( -p_3 z_j - p_3 \lambda + 1 = 0, \) \( (P_Z(0; \lambda) = 0 \) and \( R(\lambda) = 0) \).

All the cases above are trivial to show, but in the case 2., we have that, by making the sum of all the three \( z_j \) solutions of the equation \( -p_3 z_j - p_3 \lambda + 1 = 0, \forall j = 1, 2, 3 \), we get \( z_1 + z_2 + z_3 = 3 \left( \frac{1}{p_3} - \lambda \right) \) with \( p_3 \neq 0 \), in contrast with the trace (3.100), unless \( \lambda = \frac{2}{p_3} \).

Finally, we conclude that if \( R(\lambda) = 0 \) and \( P_Z(0; \lambda) > 0 \), then \( P_W(0; \lambda) = 0 \) and \( P(w; \lambda) \)
has a double real root and a simple real root.

If $P_Z(0; \lambda) > 0$, we have that all the three real differences $w_j - w_{j+1}$ correspond to the three differences $z_j - z_{j+1}$, all complex.

On the other hand, let us assume now that $\lambda$ is complex, $\lambda = \mu + i\rho$, with nonvanishing imaginary part, $\rho \neq 0$. Let $w_j = \alpha_j + i\beta_j$ be the (generically complex) roots of $P_W(w; \lambda)$.

With this notation, we have that one of the wave numbers, say $k_3 = w_1 - w_2$, will be real only if $\beta_1 = \beta_2 = \beta$. Then, from (3.98), we have that $\omega_3 = z_1 - z_2$ will also be real only if the following equation is satisfied:

$$-\beta_3 (-1 + p_3(p_3 - p_2p_3 + p_2^2(p_1 - \mu + \mu)]) + (-1 + p_3^2) (-1 + p_3^2 - p_3\alpha_3) \rho. \quad (4.29)$$

Writing the polynomial $P_W(w; \lambda)$ as $P_W(w; \lambda) = \prod_{j=1}^3(w - \alpha_j - i\beta_j)$, and comparing the real and imaginary parts of the coefficients of same powers of $w$ from this expression with those obtained from (3.93), we get, in addition to equation (4.29), six further polynomial equations, constituting overall a system of seven polynomial equations for the seven unknowns $\alpha_1, \alpha_2, \alpha_3, \beta, \beta_3, \mu, \rho$, each equation being of degree 1, 2 or 3 in the unknowns:

$$(4.30a)$$

$$\begin{align*}
p_2^2p_3 + p_1^2p_3^2 - \alpha_1\alpha_2\alpha_3 + \alpha_3\beta^2 + \alpha_1\beta_3 + \alpha_2\beta\beta_3 + p_2(-1 + p_3^2)(p_3 + \mu) - \\
p_1(-1 + p_3(p_3 + 2p_2p_3 - \mu + p_3^2\mu)),
\end{align*}$$

$$(4.30b)$$

$$- (\alpha_1 + \alpha_2)\alpha_3\beta + (-\alpha_1\alpha_2 + \beta^2)\beta_3 + (p_2 - p_1p_3)(-1 + p_3^2)\rho,$$

$$(4.30c)$$

$$\begin{align*}
1 + \alpha_1\alpha_2 + (\alpha_1 + \alpha_2)\alpha_3 - \beta(\beta + 2\beta_3) - \mu\big)^2 + p_1p_3(3 - p_3\mu) + \\
p_2(-1 + p_3(-2p_3 + \mu)] + \rho\big)^2 + p_3^2(-1 + \mu^2 - \rho^2),
\end{align*}$$

$$(4.30d)$$

$$\begin{align*}
(\alpha_1 + \alpha_2 + 2\alpha_3)\beta + (\alpha_1 + \alpha_2)\beta_3 + (-2\mu + p_3(p_2 - p_1p_3 + 2p_3\mu))\rho,
\end{align*}$$

$$(4.30e)$$

$$\begin{align*}
(2 + p_2)p_3 - \alpha_1 - \alpha_2 - \alpha_3 - 2\mu,
\end{align*}$$

$$(4.30f)$$

$$-2\beta - \beta_3 - 2\rho.$$
Using an algebraic manipulation software (like Mathematica or Maple), after a long and tedious work (we apply the method developed in [65]), one shows that, for a generic choice of the parameters $p_1$, $p_2$ and $p_3$, system (4.30) either does not have real solutions or features at most 72 complex solutions. This excludes, for a generic choice of the parameters $p_1$, $p_2$ and $p_3$, that there exist sets of non-vanishing measure off the real axis on the $\lambda$-plane for which $z_1$ and $z_2$ generate a real difference when $w_1$ and $w_2$ generate a real difference, namely for which either side of (3.98) is real. As the set of exceptional complex values of $\lambda$ for which either side of (3.98) is real has generically at most vanishing measure, it does not contribute to the integral in (2.79); therefore, for a generic choice of the parameters $p_1$, $p_2$, and $p_3$, the plane wave solution is stable against the perturbation $\delta Q$ integrated only over values of $\lambda$ in $S_x$ strictly real, with the exception of the isolated points in the split gaps. The discussion of the non-generic choices of the parameters $p_1$, $p_2$ and $p_3$, possibly allowing a set of non-vanishing measure off the real axis on the $\lambda$-plane providing a stable contribution to the integral of the perturbation, is left to future investigation.

We stress that for both the cases 1. and 2., if $P(z; \lambda)$ has a double real root and a simple real root, then it is the same also for $P(w; \lambda)$.

Whatever the labelling is, we have always three real differences $w_j - w_{j+1}$. Moreover, the relations $z_j - z_{j+1} = 0$ and $z_{j+1} - z_{j+2} = -(z_{j+3} - z_j)$ imply $w_j - w_{j+1} = 0$ and $w_{j+1} - w_{j+2} = -(w_{j+3} - w_j)$ for $j = 1, 2, 3, \mod(3)$. In particular, one can prove that, for real $\lambda$, there is an order relation for the differences $z_j - z_{j+1}$ and $w_j - w_{j+1}$, and there exist a bijective relation between the two orderings of the differences, but this argument will not be discussed in this thesis.

### 4.2.1 Gaps and Branches

In this section we give a topological description of the components of the spectrum $S_x$ which are referred to as gaps and branches. We impose that the eigenvalues of $W$ are non-simple and we take into account the polynomial $P_W(0; \lambda)$ and its relation with the polynomial $P_Z(0; \lambda)$ given by (4.8). In fact, as discussed in the previous section, by (4.8), the eigenvalues $w_j$ are non-simple if and only if the eigenvalues $z_j$ are non-simple, with the
exception of a point on the real axis of the $S_x$ spectrum\footnote{We can have at most one split-gap. This is a consequence of the formula \eqref{eq:R}, in which $R(\lambda)$ is a first degree polynomial in $\lambda$, and the only root gives the point between the two endpoints of a gap.}. Thus, one can study first the
discriminant with respect to $\lambda$ of $P_Z(0; \lambda)$ to get the main structure of the $S_x$ spectrum\footnote{One observes that the discriminant with respect to $\lambda$ of $P_W(0; \lambda)$ is always zero because the polynomial $P_W(0; \lambda)$ has a double real $\lambda$-root for any choice of the parameters $p_j$.} but then one needs to introduce a resultant, which will be defined later, to know for which
values of the physical parameters a split-gap occurs. Therefore, it is useful to introduce the
discriminant $\Delta_\lambda P_Z(0; \lambda)$ which can be factorised as
\begin{equation}
\Delta_\lambda P_Z(0; \lambda) = \prod_j D_j(p_1, p_2)^{d_j}, \quad \forall p_3 \in \mathbb{R}, \tag{4.31}
\end{equation}
and the resultant $\text{Res}_\lambda(P_Z(0; \lambda), R(\lambda))$ which can be factorised as
\begin{equation}
\text{Res}_\lambda(P_Z(0; \lambda), R(\lambda)) = \prod_j R_j(p_1, p_2, p_3)^{r_j}. \tag{4.32}
\end{equation}

Let $D_j$ be the real-analytic variety in the parameter space $(p_1, p_2)$, implicitly defined as
\begin{equation}
D_j = \{ p_1, p_2, p_3 \in \mathbb{R} : D_j(p_1, p_2) = 0, R_j(p_1, p_2, p_3) = 0 \}. \tag{4.33}
\end{equation}
Once the value of the parameter $p_3$ is fixed, a first topological classification of the possible
curves in the $\lambda$-plane in terms of the choices of the parameters $(p_1, p_2)$ can be made by
observing the nature of the $\lambda$-roots for which the matrix $Z$ (resp. $W$) has non-simple
eigenvalues. This corresponds to analysing the sign of \eqref{eq:4.31}. By imposing \eqref{eq:4.31} to be
negative, we obtain a set of regions in the $(p_1, p_2)$-plane in which $Z$ (resp. $W$) is not
diagonalisable for two real values of $\lambda$ and for a pair of complex conjugate values of $\lambda$.
This values of $\lambda$ identify the end-points of a gap and of a branch, respectively. On the
contrary, if \eqref{eq:4.31} is positive, we obtain regions in the $(p_1, p_2)$-plane in which $Z$ (resp. $W$)
is not diagonalisable either for four real values of $\lambda$ corresponding to the end-points of
two gaps, or for two pairs of complex conjugate values of $\lambda$ corresponding to the end-points
of two branches. In addition, besides the $\lambda$-roots of the polynomial $P_Z(0; \lambda)$, the polynomial
$P_W(0; \lambda)$ has also one real double $\lambda$-root, that is the $\lambda$-root of the polynomial $R(\lambda)$. The
value of this $\lambda$-root depends on the parameters $p_1, p_2$ and $p_3$, and so it varies as $p_j$ vary,
and it varies in the $(p_1, p_2)$-plane once the value of $p_3$ is fixed. Interestingly, we have found
that also the sign of the resultant \eqref{eq:4.32} is relevant for the classification of gaps and it is
a determining factor for defining the so called split-gap. In particular, a split-gap exists in
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the regions of the \((p_1, p_2)\)-plane for which the resultant is negative (see below).

Finally, the curves defined by \((4.33)\) are the boundaries of the regions in the \((p_1, p_2)\)-plane associated to different topologies of the spectra \(S_x\).

In order to define the number of gaps and branches, we compute the discriminant \(\Delta_z(P(z; \lambda)) = -P_Z(0; \lambda)\), that is a polynomial in \(\lambda\), with parameters \(p_1\), \(p_2\), and \(p_3\). The polynomial \(P_Z(0; \lambda)\) is negative whenever the three roots \(z_j\) are real, and it is positive if only one root \(z_j\) is real. As a consequence, for fixed values of \(p_1\), \(p_2\), and \(p_3\), the polynomial \(P_Z(0; \lambda)\) is positive (resp. negative) for those values of \(\lambda\) inside (resp. outside) the gap, and they become zero at the end-points of gaps. Gaps and branches appear or disappear at the multiple-zeros of the polynomial \(P_Z(0; \lambda)\) (as discussed more in details later), thus at the zeros of the discriminant \(\Delta_\lambda P_Z(0; \lambda)\), namely when

\[
\Delta_\lambda P_Z(0; \lambda) = -256(p_1 - p_2)(p_1 + p_2)(27p_1^2 - (p_2 - 1)(p_2 + 8))^3 = 0. \tag{4.34}
\]

The three polynomial factors appearing in \((4.34)\) bound the regions in the \((p_1, p_2)\)-plane characterised by different numbers of gaps and branches. We denote such curves as follows

\[
D_1 = \{(p_1, p_2) \in \mathbb{R}^2 : p_1 - p_2 = 0\}, \tag{4.35}
\]

\[
D_2 = \{(p_1, p_2) \in \mathbb{R}^2 : p_1 + p_2 = 0\}, \tag{4.36}
\]

\[
D_3 = \{(p_1, p_2) \in \mathbb{R}^2 : (27p_1^2 - (p_2 - 1)(p_2 + 8))^3 = 0\}. \tag{4.37}
\]

On the curves \((4.35)-(4.37)\), two real values of \(\lambda\)-roots collide by closing a gap, or they separate by opening a gap. Whenever two real \(\lambda\)-roots collide to close a gap, they become two complex conjugate roots, and so they identify the end-points of a gap. Vice versa, if two complex conjugate \(\lambda\)-roots become two real \(\lambda\)-roots, we expect that a gap appears. As a result, the \((p_1, p_2)\)-plane is divided in domains identified by different number of gaps and branches.

**Proposition 4.2.2.** The \(S_x\)-spectrum has the gaps and branches structure described in
Table 4.1: Gaps and branches structure.

<table>
<thead>
<tr>
<th>Regions in the ((p_1, p_2))-plane</th>
<th>#G</th>
<th>#B</th>
<th>(\lambda)-roots of (P_z(0; \lambda))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 &lt; p_2 &lt; \gamma \cap -p_2 &lt; p_1 &lt; p_2)</td>
<td>2G</td>
<td>0B</td>
<td>4 distinct and real</td>
</tr>
<tr>
<td>(\gamma &lt; p_2 &lt; p_1 \cap \gamma &lt; p_1)</td>
<td>2G</td>
<td>0B</td>
<td>4 distinct and real</td>
</tr>
<tr>
<td>(\gamma &lt; p_2 &lt; -p_1 \cap \gamma &lt; -p_1)</td>
<td>2G</td>
<td>0B</td>
<td>4 distinct and real</td>
</tr>
<tr>
<td>(p_2 &gt; \gamma \cap -p_2 &lt; p_1 &lt; p_2)</td>
<td>1G</td>
<td>1B</td>
<td>2 distinct and real, 2 complex conjugate</td>
</tr>
<tr>
<td>(p_1 &lt; p_2 &lt; -p_1 \cap p_1 &lt; 0)</td>
<td>1G</td>
<td>1B</td>
<td>2 distinct and real, 2 complex conjugate</td>
</tr>
<tr>
<td>(-p_1 &lt; p_2 &lt; p_1 \cap p_1 &gt; 0)</td>
<td>1G</td>
<td>1B</td>
<td>2 distinct and real, 2 complex conjugate</td>
</tr>
<tr>
<td>(p_2 &lt; 0 \cap p_2 &lt; p_1 &lt; -p_2)</td>
<td>0G</td>
<td>2B</td>
<td>2 pairs of complex conjugate roots</td>
</tr>
</tbody>
</table>

Table 4.1, where the polynomial \(\gamma\) is defined as

\[
\gamma = 27p_1^2 - (p_2 - 1)(p_2 + 8)^2,
\]

(4.38)

and the intervals in the parameter space are written in implicit form.

The classification of gaps and branches in the \((p_1, p_2)\)-plane has been obtained by studying simultaneously the sign of the discriminant \(\Delta_{\lambda}(P_Z(0; \lambda))\) and the signs of the following polynomials [117]

\[
\Delta_{01} = 8a_2a_4^2 - 3a_3^2,
\]

(4.39a)

\[
\Delta_{02} = 64a_0a_4^3 - 16a_1a_3a_4^2 - 16a_2a_4^2 + 16a_2a_3a_4 - 3a_3^2,
\]

(4.39b)

\[
\Delta_{03} = 8a_1a_4^2 - 4a_2a_3a_4 + a_3^3,
\]

(4.39c)

\[
\Delta_{04} = 12a_0a_4 - 3a_1a_3 + a_2^2,
\]

(4.39d)

Note that the discriminant w.r.t. \(\lambda\) of both the discriminant \(\Delta_{\lambda}(P(z; \lambda))\) and of the polynomial \(P_Z(0; \lambda)\) is actually the same, although \(\Delta_{\lambda}(P(z; \lambda))\) and \(P_Z(0; \lambda)\) have opposite signs.
where \( a_j \) are the coefficients of\(^7\)

\[
P_Z(0; \lambda) = a_4 \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0. \tag{4.40}
\]

Let us identify, case by case, the conditions on the polynomials (4.39a)-(4.39d) and on the discriminant \( \Delta_\lambda(P_Z(0; \lambda)) \) within the regions in the \((p_1, p_2)\)-plane \( [117] \).

- By imposing \( \Delta_\lambda P_Z(0; \lambda) > 0 \), \( \Delta_{01} < 0 \) and \( \Delta_{02} < 0 \), we have 4 distinct real \( \lambda \)-roots which identify the four end-points of 2 gaps and 0 branches.

- When \( \Delta_\lambda P_Z(0; \lambda) < 0 \), we have 2 distinct real \( \lambda \)-roots and 2 complex conjugate \( \lambda \)-roots. In this case we have 1 gap and 1 branch.

- The conditions to obtain the regions in the \((p_1, p_2)\)-plane corresponding two pairs of complex conjugate roots, are

1. \( \Delta_\lambda P_Z(0; \lambda) > 0, \Delta_{01} > 0 \) and \( \Delta_{02} > 0 \);
2. \( \Delta_\lambda P_Z(0; \lambda) > 0, \Delta_{01} < 0 \) and \( \Delta_{02} > \Delta_{01}^2 \);
3. \( \Delta_\lambda P_Z(0; \lambda) > 0, \Delta_{01} > 0 \) and \( \Delta_{02} < 0 \).

Thus, we have 0 gaps and 2 branches.

- The condition to have 4 real roots all coincident, that is \( \Delta_\lambda P_Z(0; \lambda) = 0, \Delta_{04} = 0 \) and \( \Delta_{02} = 0 \), is never satisfied.

- There are some extreme points of gaps domains. Indeed, the condition \( \Delta_\lambda P_Z(0; \lambda) = 0 \) and \( \Delta_{04} = 0 \) is verified at the points \((4, 4)\) and \((-4, 4)\) for which we have a triple real root and a simple real root, thus we have 1 gap. The condition \( \Delta_\lambda \Delta_z P(z) = 0, \Delta_{02} = 0 \) and \( \Delta_{01} < 0 \) is verified only at the point \((0, 0)\) for which we have two double real roots and 1 gap.

- Furthermore, there are some exceptional points for which, although they are corresponding to 0 gaps, the nature of the roots is different from that in their neighbourhood. In more detail, we have two pairs of complex conjugate \( \lambda \)-roots at the point

\(^7\)Strictly speaking, we should have written

\[
-P_Z(0; \lambda) = a_4 \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0,
\]

but the sign on front of the polynomial \( P_Z(0; \lambda) \) is completely irrelevant after the computation of the discriminant w.r.t. \( \lambda \).
(0, -8), that is when $\Delta_\lambda P_Z(0; \lambda) = 0$ and $\Delta_{02} > 0$, or a real double root and two complex conjugate roots for 1) $\Delta_\lambda P_Z(0; \lambda) = 0$ and $\Delta_{02} > 0$, or 2) $\Delta_\lambda P_Z(0; \lambda) = 0$, $\Delta_{01} > 0$ and $\Delta_{03} \neq 0$, and the corresponding domains belonging to the curves $D_1$ and $D_2$, and they are:

$$-8 + 4\sqrt{3} < p_2 < 0, \quad \{p_1 = p_2 \cup p_1 = -p_2\},$$

(4.41a)

$$p_2 < -8 - 4\sqrt{3}, \quad \{p_1 = p_2 \cup p_1 = -p_2\},$$

(4.41b)

or,

$$-4 < p_2 < -20 + 8\sqrt{6}, \quad \{p_1 = p_2 \cup p_1 = -p_2\},$$

(4.42a)

$$-20 - 8\sqrt{6} < p_2 < -4, \quad \{p_1 = p_2 \cup p_1 = -p_2\}.$$

(4.42b)

We have also 4 real distinct $\lambda$-roots for $\Delta_\lambda P_Z(0; \lambda) > 0$, $\Delta_{01} < 0$ and $0 < \Delta_{02} < \frac{\Delta_{01}^2}{4}$.

Other limiting cases can be discussed, but this is not the aim of our research work because we are interested in generic cases only.

The same classification for branches and gaps was obtained in [64] for the CNLS equation.

Split Gaps

Looking at the expressions (4.35), (4.36) and (4.37), we deduce that the gaps structure described so far can change by varying $p_1$ and $p_2$ only. Nevertheless, since $P_W(0; \lambda) = P_Z(0; \lambda)R^2(\lambda)$, $P_W(0; \lambda)$ can be zero when $R(\lambda) = 0$ also if $P_Z(0; \lambda)$ is negative or positive. Since $R(\lambda)$ is a first degree polynomial and it has real coefficients, the equality $R(\lambda) = 0$ may be verified just at one point of the real $S_x$-spectrum. Moreover, because of the expression of $R(\lambda)$, for values of $p_1$ and $p_2$ varying in an interval so that we have a fixed number of gaps, this point can move inside or can coincide with an endpoint of gap. Since $R(\lambda)$ depends on $p_3$ as well, we expect that the regions in the $(p_1, p_2)$-plane

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8This point is also a real double-zero of the polynomial $R^2(\lambda)$. 
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in which the zero of $\mathcal{R}(\lambda)$ is inside a gap move by varying $p_3$. Moreover, when $\mathcal{R}(\lambda) = 0$ then $\mathcal{P}_W(0; \lambda) = 0$ too, and for this reason we have only two possible scenarios:

1. $\mathcal{P}_Z(0; \lambda) > 0$ and the double-zero is within a gap, i.e. a split-gap;

2. $\mathcal{P}_Z(0; \lambda) = 0$ and the double-zero is a triple-zero of the polynomial $\mathcal{P}_W(0; \lambda)$, and it coincides with the end-point of a gap.

We underline that the situation in which $\mathcal{P}_Z(0; \lambda) < 0$ and $\mathcal{P}_W(0; \lambda) = 0$ never occurs because of the Proposition 4.2.1.

We define a transition in the evolution of gaps structure, also in the case in which, a split-gap appears because a point falls within an existent gap (or, vice versa, a split-gap becomes an effective gap because a point shifts from the inside of a gap to the end-point of the gap). To see for which values of the parameters $p_j$ such a transition occurs, we have to understand for which values of the parameters $p_1$, $p_2$ and $p_3$ the zero of $\mathcal{R}(\lambda)$ collides with a zero of $\mathcal{P}_Z(0; \lambda)$, or, in other words, when $\mathcal{P}_Z(0; \lambda)$ has a common root with the polynomial $\mathcal{R}(\lambda)$. This analysis is conducted by studying the resultant with respect to $\lambda$ between the two polynomials $\mathcal{P}_Z(0; \lambda)$ and $\mathcal{R}(\lambda)$. The discriminant with respect to $\lambda$ of the product $\mathcal{P}_Z(0; \lambda)\mathcal{R}^2(\lambda)$ does not give us further information about this kind of transition, since the quantity

$$
\Delta_\lambda(\mathcal{P}_Z(0; \lambda)\mathcal{R}^2(\lambda)) = \Delta_\lambda(\mathcal{P}_Z(0; \lambda)\mathcal{R}(\lambda))(\text{Res}_\lambda(\mathcal{P}_Z(0; \lambda)\mathcal{R}(\lambda), \mathcal{R}(\lambda)))^2\Delta_\lambda(\mathcal{R}(\lambda)) = \\
= \Delta_\lambda \mathcal{P}_Z(0; \lambda)(\Delta_\lambda \mathcal{R}(\lambda))^2(\text{Res}_\lambda(\mathcal{P}_Z(0; \lambda), \mathcal{R}(\lambda)))^4(\text{Res}_\lambda(\mathcal{R}(\lambda), \mathcal{R}(\lambda)))^2,
$$

(4.43)

is always zero because of the resultant $\text{Res}_\lambda(\mathcal{R}(\lambda), \mathcal{R}(\lambda)) = 0$. However, the other terms might or might not be zero. Therefore, the useful quantity to analyse is the resultant $\text{Res}_\lambda(\mathcal{P}_Z(0; \lambda), \mathcal{R}(\lambda))$: in other words, we are interested to understand for which values of the parameters $p_1$, $p_2$ and $p_3$ the resultant

$$
\text{Res}_\lambda(\mathcal{P}_Z(0; \lambda), \mathcal{R}(\lambda)) = (p_3^2 (-2p_1p_3 + (p_2 - 1)p_3^2 + p_2 + 2) - 1)^2 \\
(p_3 (p_3^3 (p_1^2 - 4p_2 + 4) - 2p_1(p_2 - 2)p_3^3 - 4p_1 + (p_2(p_2 + 4) - 8)p_3) + 4),
$$

(4.44)

is zero or not. In the discussion below, we shall prove that, in (4.44), the polynomial factor that appears squared does not correspond to any transition, while the other polynomial

\footnote{Note that $\Delta_\lambda \mathcal{R}(\lambda) = 1$.}
factor identifies a curve bounding the regions in the \((p_1, p_2)\)-plane where there exist split gaps; we define such a curve as,

\[
D_4 = \{p_1, p_2, p_3 \in \mathbb{R} : R_4(p_1, p_2, p_3) = 0\},
\]

(4.45)

where the polynomial \(R_4(p_1, p_2, p_3) = R_4\) is

\[
R_4 = p_3\left(p_3^3 - 4p_2 + 4\right) - 2p_1(p_2 - 2)p_3^2 - 4p_1 + (p_2(p_2 + 4) - 8)p_3 + 4.
\]

(4.46)

**Proposition 4.2.3.** In the \((p_1, p_2)\)-plane, the curve \(D_4\), defined in (4.45), identifies the transition curve for the existence of split gaps. In particular, once the values of \(p_3\) is fixed, the values of the parameters \(p_1\) and \(p_2\) for which the polynomial \(R_4\) is negative correspond to regions where there are split gaps.

**Proof.** Let us write \(P_Z(0; \lambda)\) and \(R(\lambda)\) in a more general form

\[
P_Z(0; \lambda) = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,
\]

(4.47)

\[
R(\lambda) = b_1\lambda + b_0,
\]

(4.48)

and let \(\tau_j\) and \(\theta_j\) be the roots of \(P_Z(0; \lambda)\) and \(R(\lambda)\) respectively, so that the resultant with respect to \(\lambda\) can be written as

\[
\text{Res}_\lambda(P_Z(0; \lambda), R(\lambda)) = a_4b_1^4 \prod_{k=1}^{4} \prod_{j=1}^{1} (\tau_k - \theta_j).
\]

(4.49)

Let us suppose that, say, \(\tau_j\) are all real and distinct, such that we have two gaps. Let us consider the initial situation in which \(\theta_1 = \tau_1 < \tau_2 < \tau_3 < \tau_4\), which corresponds to having \((4.49)\) equal to zero and the point \(\theta_1\) coinciding with an end-point of a gap. Then, we vary the values of the parameters \(p_1\) and \(p_2\) so that \(\theta_1\) increases until we have \(\tau_1 < \theta_1 < \tau_2 < \tau_3 < \tau_4\), and we have that \((4.49)\) is negative \(^{10}\) and \(\theta_1\) is inside the gap. In particular, we have the so called split-gap \(^{11}\). Thus, the resultant \((4.49)\), and so \((4.44)\), is negative for this kind of gap structure. Finally, we conclude that the relevant part of the resultant is only

\(^{10}\) \(a_4b_1^4\) is positive, because \(a_4 = 4\).

\(^{11}\) If, per absurdum, it was \(R(\lambda) = 0\) also in the case in which \(P_Z(0; \lambda) < 0\), then \((4.49)\) would be positive for the ordering \(\tau_1 < \tau_2 < \theta_1 < \tau_3 < \tau_4\).
the second polynomial. Indeed, by rescaling \( \lambda \) by \( \tilde{\lambda} = a\lambda \), with \( a \in \mathbb{R} \), in both \( P_Z(0; \lambda) \) and \( R(\lambda) \). In this way, we get the resultant \( \text{Res}_\lambda(P_Z(0; \tilde{\lambda}), R(\tilde{\lambda})) = a^4 \text{Res}_\lambda(P_Z(0; \lambda), R(\lambda)) \). Therefore, \( \lambda \) can be rescaled to eliminate the squared polynomial, or, in other words, such a polynomial is arbitrary.

Let us consider another situation in which we have two distinct real roots and two complex conjugate roots (1G 1B), and we focus only on the product \( \prod_{k=1}^4 \prod_{j=1}^1 (\tau_k - \theta_j) \). One can show that the product between the terms \( (\tau_k - \theta_j) \) corresponding to the two complex conjugate \( \tau_k \) is always real and positive. Indeed, let \( \tau_1 = c + i d \) and \( \tau_2 = \tau_1^* = c - i d \) be the two complex conjugate roots. Since \( \theta \) is always real, then the following product

\[
(\tau_1 - \theta)(\tau_1^* - \theta) = (c + id - \theta)(c - id - \theta) = c^2 + d^2 + \theta^2 - 2\theta = d^2 + (\theta - c)^2, \tag{4.50}
\]

is positive. At this point, the proof above on the case for the four distinct real roots can be repeated also here, by considering only the two factors of the resultant corresponding to the two distinct real roots \( \tau_k \).

The proof for the other cases, showed in table 4.1 is straightforward.

By taking into account the regions associated to split gaps, a general classification of gaps structure is given below.

**Proposition 4.2.4.** Besides the gaps structure described in table 4.1 the \( S_x \)-spectrum may feature split gaps for the following choices of the parameters:

1) if \(-1 < p_3 < 1, \sigma_1 < p_1 < \sigma_2 \) and \( p_2 > 1 - \frac{1}{p_3} \) with \( 1 - \frac{1}{p_3} < 0 \);

2) if \( p_3 < -1 \) or \( p_3 > 1, \sigma_1 < p_1 < \sigma_2 \) and \( p_2 > 1 - \frac{1}{p_3} \) with \( 1 - \frac{1}{p_3} > 0 \);

where

\[
\sigma_1 = \frac{2}{p_3^3} + \frac{p_2 - 2}{p_3} - \frac{2|p_3^2 - 1|\sqrt{1 + (p_2 - 1)p_3^2}}{|p_3|^3}, \tag{4.51}
\]

\[
\sigma_2 = \frac{2}{p_3^3} + \frac{p_2 - 2}{p_3} + \frac{2|p_3^2 - 1|\sqrt{1 + (p_2 - 1)p_3^2}}{|p_3|^3}, \tag{4.52}
\]

and the intervals in 1) and 2) are written in implicit form.
Figure 4.1: $(p_1, p_2)$-plane, $p_3 = -0.6$. Split gaps appear inside the region bounded by the curve $D_4$.

Figure 4.2: $(p_1, p_2)$-plane, $p_3 = 2$. Split gaps appear inside the region bounded by the curve $D_4$. 
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The Figures 4.1(a) and 4.2(a) show where there exist split gaps. By overlaying the results of the Table 4.1 to the findings of the Proposition 4.2.4, the total number of gaps and split gaps is predicted. For example, looking at the Figure 4.1(a) we have a total of 1G and 1SG in the region between the curves $D_3$ and $D_4$, while we have 1G and 1SG in the region between $D_4$ and $D_3$. Instead, in Figure 4.2(a) the region below $D_3$ has 1G and 1SG, while that one above $D_3$ has 1G and 1SG.

Symmetries and Gaps

In the following discussion, we take advantage of the symmetries in the $(p_1, p_2)$-plane to classify gaps as $p_3$ changes. Indeed, since $\Delta_3 P_2(0; \lambda)$ is invariant under transformations $p_1 \to -p_1$ and $p_2 \to p_2$, the plot of the curves $D_1$, $D_2$, and $D_3$ are symmetric with respect to the $p_2$-axis. The curve $D_4$ depends also on $p_3$, and it has symmetries $p_1 \to -p_1$, $p_2 \to p_2$ and $p_3 \to -p_3$. Nevertheless, for finite values of $p_3$, such a curve is not symmetric with respect to $p_2$-axis and moves in the $(p_1, p_2)$-plane as $p_3$ varies. In particular, by changing $p_3$ to $-p_3$, the $D_4$ plot is reflected with respect to $p_2$-axis. The curve $D_4$ can be written by expressing $p_2$ as a function of $p_1$ and $p_3$

$$p_2(p_1, p_3) = -2 + p_3 p_1 + 2p_3^2 \pm \frac{2|p_3 - p_3^2|\sqrt{p_3 p_1 + p_3^2 - 1}}{p_3^2}, \quad (4.53)$$

and by changing $p_3 \to -p_3$, we get

$$p_2(p_1, -p_3) = -2 - p_3 p_1 + 2p_3^2 \pm \frac{2|p_3 - p_3^2|\sqrt{-p_3 p_1 + p_3^2 - 1}}{p_3^2}, \quad (4.54)$$

that is $p_2(p_1, -p_3) = p_2(-p_1, p_3)$, and one can define

$$\bar{p}_2(p_1, p_3) = -2 + p_3 \bar{p}_1 + 2\bar{p}_3^2 \pm \frac{2|p_3 - p_3^2|\sqrt{p_3 \bar{p}_1 + p_3^2 - 1}}{p_3^2}, \quad (4.55)$$

where $\bar{p}_1 = -p_1$ and the bar denotes the parameters $p_1$ and $p_2$ after the transformations. Thus, $\bar{p}_2(p_1, p_3)$ is the curve $p_2(p_1, p_3)$, reflected with respect to the $p_2$-axis.

On the other hand, as $p_3$ approaches infinity, $D_4$ becomes symmetric to the $p_2$-axis. In the following, we shall discuss the symmetries of $D_4$ in more detail.

Proposition 4.2.5. For $-1 < p_3 < 1$, and for $p_3 < -1$ or $p_3 > 1$, the curve $D_4$ is asymmetric with respect to the $p_2$-axis. For $p_3 \to \pm \infty$, $D_4$ is symmetric with respect to $p_2$-axis. For $p_3 = 0$, $D_4$ disappears. For $p_3 = 1$, $D_4$ coincides with $D_1$, while for $p_3 = -1$,
\( \mathcal{D}_4 \) coincides with \( \mathcal{D}_2 \). For \( p_3 = 0, p_3 = 1 \) and \( p_3 = -1 \), there are not split gaps and the gaps structure is described only by table 4.1.

**Proof.** By looking at the expression of \( \mathcal{D}_4 \) written as a function of \( p_1 \), i.e. (4.53), we see that \( p_2(p_1) \) is neither even nor odd. Nevertheless, by dividing (4.45) by the maximum power of \( p_3 \), after taking the limit \( p_3 \to \pm \infty \), we obtain

\[
p_2(p_1) = 1 - \frac{p_1^2}{4},
\]

which is clearly an even function, and so it is symmetric with respect to the \( p_2 \)-axis. Furthermore, the domain of the function (4.53) is

\[
\left\{ \forall p_3 \in \mathbb{R}/\{0\}, \ (p_1 \in \mathbb{R} : p_1 \geq \left( \frac{1}{p_3} - p_3 \right) \right\}.
\]

(4.57)

The curve \( \mathcal{D}_4 \) is not defined for \( p_3 = 0 \). In addition, once we chose \( p_3 = \pm 1 \), (4.53) can be considered as a function of \( p_1 \) only, whose explicit expression is

\[
p_2(p_1) = \pm p_1,
\]

(4.58)

that are \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) for \( p_3 = 1 \) and \( p_3 = -1 \), respectively. In these cases the curve \( \mathcal{D}_4 \) becomes \((p_1 - p_2)^2\) or \((p_1 + p_2)^2\), which are positive and, as a result, there are no split gaps, and the discussion reduces to Table 4.1 only.

Because of curve symmetries on the \((p_1, p_2)\)-plane, here and thereafter we consider only negative values of \( p_3 \).

The Figures 4.3 and 4.4 are plots of the \((p_1, p_2)\)-plane with the entire topological classification of gaps and branches components for \( p_3 = -0.6 \). We choose \( p_3 = -0.6 \) without loss of generality, indeed for \(|p_3| > 1\) the curve \( \mathcal{D}_4 \) moves in the regions with 1G and 0G by creating a split-gap, and the discussion is the same.

### 4.3 Complex Spectrum

In the previous section we have considered the characteristic polynomial \( P(w; \lambda) \) and we have analysed the situations in which two \( w_j \)-roots coincide. We can summarise the classification of \( \lambda \)-roots at which at least one difference \( w_j - w_k \) is zero, i.e. \( \xi = 0 \):

\[\text{The domain of the function } p_2(p_1) \text{ becomes } \{p_1 \in \mathbb{R}\} \text{ for this case.}\]
Let us consider the polynomial of the squares of the differences $P_W(\xi; \lambda)$, defined in (4.3), which is a sixth degree polynomial in $\lambda$ and a third degree polynomial in $\xi^2$. We construct the polynomial $Q(\xi) \equiv Q(\xi; p_1, p_2, p_3)$, that is the discriminant with respect to $\lambda$ of $P_W(\xi; \lambda)
abla$

\[ Q(\xi) = \Delta_\lambda P_W(\xi; \lambda). \quad (4.59) \]

Then, we perform the polynomial factorisation of $Q(\xi)$ with respect to $\xi$. This results in the following form

\[ Q(\xi) = \xi Q_1^2(\xi) Q_2(\xi), \quad (4.60) \]
where \( Q_1(\xi) \equiv Q_1(\xi; p_1, p_2, p_3) \) and \( Q_2(\xi) \equiv Q_2(\xi; p_1, p_2, p_3) \) are two polynomials in the variable \( \xi \), whose degree are four and six respectively, and \( Q_{1,2}(0) \neq 0 \). We will refer to \( Q_1 \) and \( Q_2 \) as the even and odd parts of \( Q(\xi) \), respectively.

**Proposition 4.3.1.** Let \( \bar{\xi} \) be the largest positive root of \( Q_2(\xi) \). Then, for \( \xi > \bar{\xi} \), all the \( \lambda \)-roots of \( P_W(\xi; \lambda) \) are real; therefore, the stability spectra always contains part of the real axis and never features a gap containing the point at infinity.

**Proof.** The polynomial of the squares of the differences is written as (Appendix J)

\[
P_W(\xi; \lambda) = \xi^3 - f_1 \xi^2 + \frac{f_1^2}{4} \xi - \frac{f_2}{6},
\]

where \( f_1 \equiv f_1(\lambda; p_1, p_2, p_3) \) and \( f_2 \equiv f_2(\lambda; p_1, p_2, p_3) \) are polynomials whose unknown is the spectral parameter \( \lambda \) and they are depending also by the physical parameters \( p_1 \), \( p_2 \) and \( p_3 \). In particular, for the 3WRI model, these polynomials have the general expressions

\[
f_1 = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2,
\]

(4.62a)
The polynomial of the squares of the differences (4.61) vanishes for any \( \lambda \)-root

\[
\xi^3 - f_1 \xi^2 + \frac{f_1^2}{4} \xi - \frac{f_2}{6} = 0
\]

(4.63)

and, by diving (4.63) by \( \xi^3 \), we get the equation

\[
1 - \frac{f_1}{\xi} + \frac{f_1^2}{4 \xi^2} - \frac{f_2}{6 \xi^3} = 0.
\]

(4.64)

Since \( \frac{1}{\xi} \to 0 \) as \( \xi \to +\infty \), we define \( \frac{1}{\xi} \equiv \epsilon \), so that the equation (4.64) becomes

\[
1 - f_1 \epsilon + \frac{f_1^2}{4} \epsilon^2 - \frac{f_2}{6} \epsilon^3 = 0.
\]

(4.65)

At this point, we note that for \( \epsilon = 0 \), we have the impossible equality \( 1 = 0 \), that means all the three roots approach infinity as \( \epsilon \) goes to zero. Thus, we deal with a singular perturbation problem and, in order to solve the equation (4.65), we set the rescaled variable

\[
\lambda = \frac{y}{\delta(\epsilon)}
\]

into the equation (4.65) and substitute \( f_1 \) and \( f_2 \), such that

\[
1 + \frac{\alpha_0^2 \epsilon^2}{4} - \alpha_0 \epsilon - \frac{\beta_0 \epsilon^3}{6} + \frac{y}{\delta} \left( \frac{\alpha_0 \alpha_1}{2} \epsilon^2 - \alpha_1 \epsilon - \frac{b_1}{6} \epsilon^3 \right) + \frac{y^2}{2 \delta^2} \alpha_0 \alpha_2 \epsilon^2 +
\]

\[
+ \frac{y^2}{4 \delta^2} \alpha_1^2 \epsilon^2 - \frac{y^2}{2 \delta} \alpha_2 \epsilon - \frac{y^2}{6 \delta^2} \beta_2 \epsilon^3 +
\]

\[
+ \frac{y^3}{2 \delta^3} \alpha_1 \alpha_2 \epsilon^2 - \frac{y^3}{6 \delta^3} \beta_3 \epsilon^3 + \frac{y^4}{4 \delta^4} \alpha_2^2 \epsilon^2 - \frac{y^4}{6 \delta^4} \beta_4 \epsilon^3 - \frac{y^5}{6 \delta^5} \beta_5 \epsilon^3 - \frac{y^6}{6 \delta^6} \beta_6 \epsilon^3 = 0.
\]

(4.66)

By using the principle of dominant balance, we require that at least two leading-order terms have the same order of magnitude. By imposing the condition

\[
\frac{\epsilon^3}{\delta^5} = \frac{\epsilon^3}{\delta^6},
\]

(4.67)

we get

\[
\delta = 1,
\]

(4.68)

that gives us solutions not approaching infinity as \( \epsilon \to 0 \), so we have to rule out this choice.

The right expression for \( \delta \) is given by the condition

\[
\frac{\epsilon^3}{\delta^6} = \frac{\epsilon^2}{\delta^4},
\]

(4.69)
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hence,
\[ \delta = \epsilon^2. \]  \hfill (4.70)

By substituting (4.70) in (4.66), we obtain
\[ 1 + \frac{\alpha_0^2 \epsilon^2}{4} - \alpha_0 \epsilon - \frac{\beta_0 \epsilon^3}{6} + \]
\[ + y^2 \left( \frac{\alpha_0 \alpha_2 \epsilon}{2} + \frac{\alpha_1^2 \epsilon}{4} - \alpha_2 - \frac{\beta_2 \epsilon^2}{6} \right) + y \left( \frac{1}{2} \alpha_0 \alpha_1 \epsilon^{3/2} - \alpha_1 \sqrt{\epsilon} - \frac{1}{6} \beta_1 \epsilon^{5/2} \right) + \]
\[ + y^3 \left( \frac{1}{2} \alpha_1 \alpha_2 \sqrt{\epsilon} - \frac{1}{6} \beta_3 \epsilon^{3/2} \right) + y^4 \left( \frac{\alpha_2^2}{4} - \frac{\beta_4 \epsilon}{6} \right) - \frac{1}{6} \beta_5 \epsilon^5 \sqrt{\epsilon} - \frac{\beta_6 y^6}{6} = 0. \]  \hfill (4.71)

Therefore, we look for solutions of the kind \[ y = y_0 + \epsilon^{1/2} y_1 + O(\epsilon), \]  \hfill (4.72)

and by putting the expansion above into the equation (4.71) and collecting the terms with respect to equal powers of \( \epsilon \)
\[ \sqrt{\epsilon} \left( \frac{1}{2} \alpha_1 \alpha_2 y_0^3 - \alpha_1 y_0 + \alpha_2 y_0^3 y_1 - 2 \alpha_2 y_0 y_1 - \frac{\beta_3 y_0^5}{6} - \beta_6 y_0^6 \right) + \]
\[ + \frac{\alpha_2 y_0^4}{4} - \frac{\alpha_2 y_0^2}{6} y_6 + 1 + ... = 0, \]  \hfill (4.73)

where we have neglected the terms \( O(\epsilon) \).

By expanding and matching the coefficients of \( \epsilon^n \) to zero, for \( n = 0, 1, \) we obtain the equations
\[ 1 - \alpha_2 y_0^2 + \frac{\alpha_2 y_0^4}{4} - \frac{\beta_3 y_0^6}{6} = 0, \]  \hfill (4.74a)
\[ \frac{1}{2} \alpha_1 \alpha_2 y_0^3 - \alpha_1 y_0 + \frac{\alpha_2 y_0^3 y_1}{4} - 2 \alpha_2 y_0 y_1 - \frac{\beta_3 y_0^5}{6} - \beta_6 y_0^6 y_1 = 0, \]  \hfill (4.74b)

whose solutions \(^{13}\) are \( y_{0j} \) with \( j = 1, ..., 6 \). To find the order of the correction, we have to substitute the solution \( y_{0j} \) into the second equations and get \( y_{1j} \). Thus the solutions are of the kind \( y_j = y_{0j} + O(\epsilon^{1/2}) \), \( j = 1, ..., 6 \). Moreover, \( y_{0j}^2 \) are all reals, in fact, if one

\(^{13}\)Because of (4.70), the corrections to the roots must be a regular perturbation expansion in powers of \( \epsilon^{1/2} \), otherwise we can not match powers of an expansion having only integral powers of \( \epsilon \).

\(^{14}\)For the aim of this discussion the explicit expressions of \( y_{0j} \) are not necessary.
considers the polynomial (4.74a) as a third degree polynomial whose unknown is $y_0^2$, the discriminant is
\begin{equation}
\Delta_{y_0} = \frac{2}{27} \beta_6^3 \left( \alpha_2^3 - 9 \beta_6 \right)^2,
\end{equation}
i.e.
\begin{equation}
\Delta_{y_0} = 6144 p_3^6 \left( p_3^2 - 1 \right)^6 \left( 1295 p_3^6 + 1298 p_3^4 + 431 p_3^2 + 48 \right)^2,
\end{equation}
and the condition $\Delta_{y_0} \geq 0$ is satisfied $\forall p_3 \in \mathbb{R}$. In addition, by the Descartes’ rule of signs, it results that $y_0^2$ are all positive, hence $y_0$ are reals.

However, we are interested in finding $\lambda$, i.e.
\begin{equation}
\lambda_j = y_0 j \epsilon^{- \frac{2}{3}} + O(1), \quad j = 1, ..., 6,
\end{equation}
and, coming back to the old variables, it turns out that
\begin{equation}
\lambda_j = y_0 j \sqrt{\xi_j} + O(1), \quad j = 1, ..., 6.
\end{equation}

Finally, keeping in mind the assumption $\xi_j \in \mathbb{R}^+$, the roots $\lambda_j$ are all reals. Furthermore, since $y_0 j$ are solutions of the polynomial (4.74a) and since $\alpha_2 = 2 \left( 3 p_3^2 + 1 \right) > 0$ and $\beta_6 = (4 p_3^6 - 8 p_3^4 + 4 p_3^2) > 0, \forall p_3 \in \mathbb{R}$, by using the Descartes’ rule for such a polynomial, we see there are exactly 3 positive and 3 negative roots.

Let us impose a difference $w_j - w_k$ to be real and strictly positive, i.e. $\xi > 0$. By the formula (4.59), the values of $\xi$ for which two $\lambda$-roots collide are those ones for which $Q_2(\xi)$ vanishes. After that, $Q_2(\xi)$ may change sign, that is, after a collision, two $\lambda$-roots may change their nature. In more detail, two real $\lambda$-roots may become complex and vice versa.

Indeed, if the polynomial of the squares of the differences is regarded as a polynomial in $\lambda$, for any fixed $\xi$, we expect one of these scenarios:

a) 6 distinct real $\lambda$-roots;

b) 4 distinct real $\lambda$-roots and 2 complex conjugate $\lambda$-roots;

c) 2 distinct real $\lambda$-roots and 2 pairs of complex conjugate $\lambda$-roots;

\footnote{That is the condition for which $y_0^2$ are three distinct real roots or multiple real roots.}
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3 pairs of complex conjugate λ-roots.

As ξ varies, two or more of the above scenarios can coexist. However, the case d) never occurs. There is no interval of ξ in which all the roots are complex. In other words, ∀ξ > 0, there are at least two real roots. This is a consequence of the fact that the polynomial of the squares of the differences \( P_W(ξ; λ) \) is equal to the square of the polynomial of the differences \( w_ℓ - w_m \) (Appendix J). Since the polynomial of the differences has only real coefficients, it can have a pair of complex conjugated λ-roots at most. As a result, we expect two pairs of complex conjugate λ-roots at most for the polynomial of the squares of the differences.

In this section we focus on the complex subset of the \( S_x \)-spectrum which may lead to instability in time (besides the point separating a split-gap). This part of the spectrum consists of open and closed continuous curves named branches and loops respectively. To understand how these curves appear, we have to imagine an initial situation in which the values of the spectral parameters \( λ \) are roots of the polynomial \( P_W(0; λ) \). Then, we impose the condition \( ξ > 0 \), so that the values of the spectral parameters \( λ \) are roots of the polynomial \( P_W(ξ; λ) \) ∀ξ ∈ ℝ. After that, one or more of the scenarios a), b) or c) occur as ξ varies. For example, at ξ = 0, let us consider the initial condition for which we have 2 distinct real λ-roots and 2 complex conjugate λ-roots. Let us suppose that the polynomial \( P_W(ξ; λ) \) has 2 distinct real λ-roots and 2 complex conjugate λ-roots for some ξ ∈ ℝ. This means that all the real roots remain on the real axis and the 2 complex conjugate roots collide at some point on the real axis and then go to infinity necessarily on the real axis (see Proposition 4.3.1). In this case, the spectrum would be composed by 1G and 1B and 0L. However, several other situations can occur as ξ varies. For instance, let b) be the next situation. We have that 2 real λ-roots become a pair of complex conjugate λ-roots. Since ∀ξ > 0 these couple of complex roots must remain conjugate (they are the roots of a polynomial with real coefficients) and, in addition, all the roots must be real as ξ → ∞, instead of having a branch we have a loop. In this case, the spectrum would be composed by 1G 1B 1L.

A loop can be regarded as a branch closed on the real axis. Sometimes, the difference with a branch is that it is created by λ-roots not corresponding to ξ = 0. In any case, a loop comes from two initial real λ-roots, instead of two complex conjugate λ-roots, unlike a
branch. Nevertheless, there are some situations in which a loop is created by starting from real $\lambda$-roots associated to $\xi = 0$. As $\xi$ varies, two real $\lambda$-roots cannot travel in a branch because, if that happens, they are forced to travel it again in order to come back on the real axis when $\xi \to \infty$ (see Proposition 4.3.1). For the same reason, two initial complex conjugate roots, starting form the ends of a branch, travel such a branch but never come back on it. Therefore, they collide on the real axis and, after that, approach infinity.

**Proposition 4.3.2.** The $\lambda$-roots cannot be periodic functions of $\xi$.

**Proof.** Since $\lambda$-roots are solutions of a polynomial, they can not be periodic. In particular, once the other parameters are fixed, let us suppose, that the $\lambda$-roots are periodic functions of $\xi$. Then the limit of $\lambda(\xi)$ as $\xi \to \infty$ is not convergent, because they are oscillating. Thus, the $\lambda(\xi)$ must be monotonic function of $\xi$. 

### 4.3.1 Loops Classification

In this subsection we give the loops classification and so the complete spectra classification. The $(p_1, p_2)$-plane is divided in regions in which the spectra have the same topology and the number of gaps and the number of branches are known in every region. After choosing the values of the parameters $p_1$, $p_2$ and $p_3$ in any of such regions, we use MATLAB codes in order to find the $\xi$-roots of the polynomial $Q_2(\xi)$. Between these roots we select only the real and positive $\xi$-roots. If we find $\bar{\xi}_1, ..., \bar{\xi}_N$ real and positive roots, we have to consider $N + 1$ intervals: from 0 to $\bar{\xi}_1$, from $\bar{\xi}_1$ to $\bar{\xi}_2$, etc... , until the last interval from $\bar{\xi}_N$ to $+\infty$. In this way, if the polynomial of the squares of the differences $P_W(\xi; \lambda)$ is meant like a polynomial in the $\lambda$ variable, every coefficient of such a polynomial is a function of the $\xi$ variable. Then, we require every coefficient to be positive within every interval $\{\bar{\xi}_j, \bar{\xi}_{j+1}\}$, with $j = 1, ..., N - 1$. After that, we see that the coefficients can change their sign inside intervals whose endpoints do not coincide with the $\xi$-roots of $Q_2(\xi)$, we denote them as $\xi_1, ..., \xi_M$, and so further intervals appear in the $\xi$ domain. We apply the Descartes rule of signs in all the intervals $\{\xi_j, \xi_{j+1}\}$, with $j = 1, ..., M - 1$, and we see how many real positive, real negative and complex conjugate roots there are. Nevertheless, different cases may be present. For example, if we have three sign changes for the polynomial in $\lambda$ and for the polynomial in $-\lambda$, then we may have 3 positive roots, 3 negative roots, or
3 positive roots, 1 negative root and 2 complex conjugate roots, or 3 positive roots, 1 negative root and 2 complex conjugate roots, or 1 positive root, 1 negative root and 2 pairs of complex conjugate roots. However, only one of these options is the real one. Therefore, we choose a generic point in every interval \( \{\xi_j, \xi_{j+1}\} \), \( j = 1, ..., M - 1 \), and compute again the coefficients of the polynomial of the squares of the differences. We count the number of sign changes in every interval. Thus, we exclude all the options not corresponding to the real one. By this method, as \( \xi \) varies, we can imagine the dynamic of the \( \lambda \)-roots on the \( S_x \) spectrum, and so suppose the creation of a new spectrum component, or when two \( \lambda \)-roots collide to return to the real axis.

Moreover, we write down a formula linking the number of branches, loops and twisted loops:

\[
\#TL + 2\#L + \#B = \#\xi^+,
\]

(4.79)

where \( \xi^+ \) stands for the positive roots of \( Q_2(\xi) \).

### 4.3.2 Spectra Classification: Descartes Rule of Signs and Sturm Chains

Here we give a detailed, but general, description of the procedure used to obtain the topological classification of the spectra in the \((p_1, p_2)\)-plane. Then, we will apply this procedure to any region in the parameter space with a particular number of gaps and branches.

Let \( P(x) \) be a polynomial in \( x \) with real coefficients, and let \( \deg(P) \) be its degree. Let us suppose \( P(x) \) is ordered by descending variable exponent, then the number of positive roots of the polynomial is equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Multiple roots are counted separately. In order to obtain the number of negative roots, we substitute \(-x\) into the polynomial \( P(x) \) to get \( Q(x) \equiv P(-x) \), and we apply the Descartes rule of sign to \( Q(x) \). If, for instance, \( P(x) \) is a third degree polynomial, and the sequence of successive signs for \( P(x) \) is \( \{+ + - -\} \), we expect 1 positive root. On the other hand, let us suppose \( Q(x) \) has the sequence of successive signs \( \{-+ ++\} \). Then, the polynomial \( Q(x) \) has 2 positive roots and the polynomial \( P(x) \) has 2 negative roots. Since the number of complex roots must be equal to \( \deg(P) \), the minimum number of strictly complex roots is \( \deg(P) - (p + n) \),
where $p$ denotes the number of positive roots, and $n$ denotes the number of negative roots. Thus, if we apply the Descartes rule of signs to both $P(x)$ and $Q(x)$, and the sum of the number of their positive roots does not match $\text{deg}(P)$, we expect the polynomial $P(x)$ to have complex roots.

We apply the Descartes rule of signs to the polynomial of the squares of the differences $P_W(\xi; \lambda)$, regarded as polynomial in $\lambda$, for any choice of the parameters $p_1$, $p_2$ and $p_3$. The polynomial $P_W(\xi; \lambda)$ is a sixth degree polynomial and the coefficients of $\lambda^6$ and of $\lambda^3$ are constants and so independent on $\xi$. In other words, their signs are the same as $\xi$ varies. However, the other coefficients are depending on $\xi$, and so their sign can change as $\xi$ varies. Thus, after choosing the values of $p_1$, $p_2$ and $p_3$, we get

$$P_W(\xi; \lambda) = g_6 \lambda^6 + g_5 \lambda^5 + g_4(\xi)\lambda^4 + g_3(\xi)\lambda^3 + g_2(\xi)\lambda^2 + g_1(\xi)\lambda + g_0(\xi), \quad (4.80)$$

where $g_6$ and $g_5$ are numbers, while $g_j(\xi)$, with $j = 0, 1, 2, 3, 4$, are polynomials in $\xi$.

The trick is to require some coefficients to be positive and see for which value of $\xi$ this condition is satisfied. For instance, let $\xi_1$ and $\xi_2$ be the two values of $\xi$ at which $Q(\xi)$, i.e. $Q_2(\xi)$, changes sign. We impose the conditions $g_j(\xi) > 0$, $\forall j = 1, 2, 3, 4$, in every interval $0 < \xi < \xi_1$, $\xi_1 < \xi < \xi_2$ and $\xi_2 < \xi < +\infty$. Sometimes, it will happen that some polynomial coefficient is negative, for example, for $0 < \xi < \xi_1 < \xi_2$ and instead positive for $\xi_1 < \xi < \xi_2$. In such case, we will split the $\xi$-domain in intervals as $[0, \xi_1]$, $[\xi_1, \xi_2]$, $[\xi_2, \xi_3]$, $[\xi_3, +\infty)$, where we have redefined $\xi_1 = \xi_2$ and $\xi_2 = \xi_3$. Then, in every of this interval we apply the Descartes rule of sign. We choose a particular value of $\xi$, say, into the interval $\{0, \xi_1\}$, substitute it in the polynomial $P_W(\xi; \lambda)$ and count the sign changes. For example, we could have the sequence of signs $\{++, --,-, -\}$, that is associated to the possibilities: $3$ positive and $1$ complex conjugate $\lambda$-roots, or $1$ positive and $2$ complex conjugate $\lambda$-roots. Whereas, for the polynomial $P_W(\xi; \lambda)$ after substituting $\lambda \rightarrow -\lambda$, we have the sequence of signs $\{++, --,-, -\}$ associated to the possibilities: $3$ negative and $0$ complex conjugate $\lambda$-roots, or $1$ negative and $2$ complex conjugate $\lambda$-root.

By combining altogether the sequences of signs we obtain the possibilities: $3$ positive and $3$ negatives $\lambda$-roots, $3$ positive (resp. $3$ negative), $1$ negative (resp. $1$ positive) and $2$ complex conjugate $\lambda$-roots. After that, we will repeat the same procedure by choosing a particular value of $\xi$ into the other intervals.

Once we have the intervals in which the structure of the algebraic curves in the $\lambda$-plane may change, we will apply the Sturm chains method to the polynomial $P_W(\xi; \lambda)$. First
of all, we choose a value of $\xi$ in every interval $[0, \xi_1], [\xi_1, \xi_2], [\xi_2, \xi_3]$ and $[\xi_3, +\infty)$, and substitute this value of $\xi$ in the polynomial of the squares of the differences which becomes dependent only on $\lambda$, i.e. $P_W(\lambda)$. From here, we construct the Sturm chain
\[
\{P_W^{(j)}(\lambda)\}_{j=0}^6 = \{P_W^{(0)}(\lambda), P_W^{(1)}(\lambda), P_W^{(2)}(\lambda), P_W^{(3)}(\lambda), P_W^{(4)}(\lambda), P_W^{(5)}(\lambda), P_W^{(6)}(\lambda)\}
\]
for every fixed $\xi$ in any interval $[0, \xi_1], [\xi_1, \xi_2], [\xi_2, \xi_3]$ and $[\xi_3, +\infty)$:
\[
\begin{align*}
P_W^{(0)}(\lambda) &= P_W(\lambda), \\
P_W^{(1)}(\lambda) &= \frac{d}{d\lambda} P_W^{(0)}(\lambda), \\
P_W^{(2)}(\lambda) &= -\text{Remainder} \left( P_W^{(0)}(\lambda), P_W^{(1)}(\lambda) \right), \\
P_W^{(3)}(\lambda) &= -\text{Remainder} \left( P_W^{(1)}(\lambda), P_W^{(2)}(\lambda) \right), \\
P_W^{(4)}(\lambda) &= -\text{Remainder} \left( P_W^{(2)}(\lambda), P_W^{(3)}(\lambda) \right), \\
P_W^{(5)}(\lambda) &= -\text{Remainder} \left( P_W^{(3)}(\lambda), P_W^{(4)}(\lambda) \right), \\
P_W^{(6)}(\lambda) &= -\text{Remainder} \left( P_W^{(4)}(\lambda), P_W^{(5)}(\lambda) \right).
\end{align*}
\]

The result of this computation is a sequence of numbers changing as $\xi$ changes, and we write down only the sign of every number in the sequence and, from such a sequence, we extract the sequence of the corresponding signs. This sequence of signs changes only if we choose values of $\xi$ from intervals different from to each other.

### 4.4 Gain Function

In this section, we present the function $\mathcal{H}(\omega, k)$ as an implicit function of the eigenwavenumber $k = w_i - w_j$ and of the eigenfrequency $\omega = z_i - z_j$. The vanishing of the this polynomial, i.e. $\mathcal{H}(\omega, k) = 0$, for fixed $k$, provides $\omega$ as a function of $\mathcal{H}$ and of the other parameters of the system $p_1, p_2$ and $p_3$, whose imaginary part is the gain function. However, we do not solve this polynomial because it is sixth degree polynomial in $\omega$, instead, we will compute numerically (via a MATLAB 2018a routine) and we will display an example of the gain function for any spectrum in the classification in the next section.

In the following, we show how the function $\mathcal{H}(\omega, k)$ is obtained.

Let us consider the two characteristic polynomials $P_Z(z; \lambda)$ and $P_W(w; \lambda)$. Because the
two associate matrices commute, i.e. $[Z, W] = 0$, they have roots in common, such that
their resultant must be zero for values of the parameters involved in their expressions. The
resultants with respect to $p_1$ and $p_2$ read, respectively,

$$\text{Res}_{p_1}(P_Z(z; \lambda); P_W(w; \lambda)) = (-p_3(p_2 + z(\lambda + z)) + w - z) \left( \lambda^2 + p_2(p_3(\lambda + z) - 1)^2 + \right.$$  
$$\left. + p_3^2(\lambda + z - 1)(\lambda + z + 1)(z(\lambda + z) - 1) + p_3 \left( w(z(\lambda + z) - 2) - z(\lambda + z)^2 + z \right) + w^2 + \right.$$  
$$\left. + 2\lambda(w + z) + wz + z^2 - 1 \right),$$

(4.82)

$$\text{Res}_{p_2}(P_Z(z; \lambda); P_W(w; \lambda)) = (-p_1p_3 - p_3z + (w - z)(\lambda + z)) \left( -\lambda + p_1(p_3(\lambda + z) - 1)^2 + \right.$$  
$$\left. + (\lambda + z)(p_3^2(-\lambda + z - 1))(\lambda + z + 1) + p_3(w(z(\lambda + z) - 2) + z(\lambda + z - 1)(\lambda + z + 1)) + \right.$$  
$$\left. + (\lambda + w)(\lambda + w + z)) \right).$$

(4.83)

Because the resultants (4.82) and (4.83) are the product of two polynomials, for each one
of them, we equal to zero the polynomial with the simplest expression, in this way we obtain
two maps between the eigenvalue $w$ and $z$,

$$w \rightarrow \frac{p_3(p_1 + z)}{\lambda + z} + z,$$

(4.84)

and

$$w \rightarrow p_3(p_2 + z(\lambda + z)) + z.$$  

(4.85)

Using the two maps (4.84) and (4.85), we define the two polynomials by taking the numer-
ators of the following expressions

$$J_1(z_1, z_2; \lambda) = 4((\lambda + z_1)(\lambda + z_2)(k - z_1 - z_2) - p_3(p_1 + z_1)(\lambda + z_2) - p_3(p_1 + z_2)(\lambda + z_1)),$$

(4.86)

and

$$J_2(z_1, z_2; \lambda) = 4(k - (p_3(p_2 + z_1(\lambda + z_1)) + z_1 - p_3(p_2 + z_2(\lambda + z_2)) + z_2)),$$

(4.87)

and, after the substitutions $\omega = z_1 - z_2$ and $\theta = z_1 + z_2$, they become

$$J_1(\theta; \lambda) = k \left( \theta^2 + 2\lambda \theta - \omega^2 \right) + \omega \left( -\theta^2 + 4p_1p_3 - 4\lambda(\theta + \lambda + p_3) + \omega^2 \right),$$

(4.88)
\[ J_2(\theta; \lambda) = k - \omega(p_3(\theta + \lambda) + 1). \] (4.89)

Moreover, we introduce another polynomial, that is the polynomial of the sums of the eigenvalues \( z_j \) (Appendix K)

\[ S_Z(\theta, \lambda) = \theta^3 + 4\lambda\theta^2 + (5\lambda^2 + p_2 - 1)\theta + p_1 + \lambda(-2 + p_2 + 2\lambda^2). \] (4.90)

Then, the Groebner basis of the three polynomial \( J_1(\theta; \lambda), J_2(\theta; \lambda) \) and \( S_Z(\theta, \lambda) \) yields a list of polynomials of which only the first is independent on \( \lambda \) and \( \theta \) and provides the gain function

\[
H(\omega, k) = k^4 (\omega^2 - 4) - 4k^3 \omega (p_1 p_3 + \omega^2 - 4) - \\
- k^2 \omega^2 (-12 p_1 p_3 + p_3^2 (p_2^2 + 4p_2 + 2\omega^2 - 8) - 6 (\omega^2 - 4)) + \\
+ 2k \omega^3 (p_1 p_3 (- (p_2 - 2)p_3^2 - 6) + p_3^2 (p_2^2 + 4p_2 + 2\omega^2 - 8) - 2\omega^2 + 8) + \\
+ \omega^4 (p_3^4 (- (p_1^2 - 4p_2 + 4)) + 2p_1 (p_2 - 2)p_3^2 + 4p_1p_3 - (p_2(p_2 + 4) - 8)p_3^2 - 4) + \\
(p_3^2 - 1)^2 \omega^6.
\] (4.91)

### 4.5 Description of the \( x \)-Stability Spectra

In this section we provide an analytical description of the spatial stability spectra obtained for any generic choice of the parameters in the \((p_1, p_2)\)-plane and we display both the \( S_x \)-spectrum and its associated gain function \( \omega_3 = \Gamma(k_3) \) in any region of the \((p_1, p_2)\)-plane (see Appendices M and L).

**Regions with 1 Gap and 1 Branch**

In the regions with 1 gap and 1 branch, we note the following correspondence between the number of loops and the number of positive \( \xi \)-roots:

- 2L: 5 positive \( \xi \)-roots;
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• 1L: 3 positive $\xi$-roots;

• 0L: 1 positive $\xi$-roots.

If we choose $p_1 = -0.8$, $p_2 = 0.4$ and $p_3 = -0.6$, we have that $Q_2(\xi)$ changes sign 5 times at any of the following values of $\xi$: $\bar{\xi}_1 = 0.028$, $\bar{\xi}_2 = 1.887$, $\bar{\xi}_3 = 2.115$, $\bar{\xi}_4 = 3.146$ and $\bar{\xi}_5 = 33.419$. After applying the Descartes rule of signs to the polynomial of the squares of the differences multiplied by minus, further intervals must be considered whose end-points are: $\xi_1 = 0.028$, $\xi_2 = 0.152$, $\xi_3 = 1.887$, $\xi_4 = 2.115$, $\xi_5 = 3.146$, $\xi_6 = 6.517$, $\xi_7 = 15.021$ and $\xi_8 = 33.419$. Then, we apply the Sturm chains technique by substituting a generic $\xi$ value on every interval $[\xi_j, \xi_{j+1}]$ into the expression of the coefficients and we count the sign changes. Finally, we can classify the nature of the $\lambda$-roots and so we can describe the whole spectrum. In particular:

• $0 < \xi < \xi_1$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots are travelling along the branch;

• $\xi_1 < \xi < \xi_2$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on the branch and 2 $\lambda$-roots on a loop;

• $\xi_2 < \xi < \xi_3$: 4 $\lambda$-roots on the real axis, and 2 $\lambda$-roots on a loop or on the branch;

• $\xi_3 < \xi < \xi_4$: 6 $\lambda$-roots on the real axis;

• $\xi_4 < \xi < \xi_5$: 6 $\lambda$-roots on the real axis;

• $\xi_5 < \xi < \xi_6$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on the second loop;

• $\xi_6 < \xi < \xi_7$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on the second loop;

• $\xi_7 < \xi < \xi_8$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on the second loop.

• $\xi > \xi_8$: 6 $\lambda$-roots on the real axis.

We conclude that 2 loops exist for these choices of the physical parameters.
Figure 4.5: Stability spectrum for $p_1 = -0.8$, $p_2 = 0.4$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$.

Figure 4.6: Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the stability spectrum obtained at $p_1 = -0.8$, $p_2 = 0.4$, $p_3 = -0.6$.

By setting $p_1 = -4.0$, $p_2 = -3.0$ and $p_3 = -0.6$, we have that $Q_2(\xi)$ changes sign 3 times at any of the following values of $\xi$: $\bar{\xi}_1 = 0.030$, $\bar{\xi}_2 = 31.959$ and $\bar{\xi}_3 = 49.660$. By applying the Descartes rule of signs, further intervals must be considered whose end-points are: $\xi_1 = 0.026$, $\xi_2 = 0.030$, $\xi_3 = 0.525$, $\xi_4 = 31.959$, $\xi_5 = 38.834$, $\xi_6 = 49.660$ and $\xi_7 = 52.214$. Then, we apply the Sturm chains technique and we obtain the classification of the nature of the $\lambda$-roots:

- $0 < \xi < \xi_1$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on the branch;
• $\xi_1 < \xi < \xi_2$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on the branch;

• $\xi_2 < \xi < \xi_3$: 2 $\lambda$-roots on the real axis, and 2 $\lambda$-roots on the loop and 2 $\lambda$-roots on the branch;

• $\xi_3 < \xi < \xi_4$: 2 $\lambda$-roots on the real axis and 2 $\lambda$-roots on the branch and 2 $\lambda$-roots on the loop;

• $\xi_4 < \xi < \xi_5$: 2 $\lambda$-roots on the real axis and 2 $\lambda$-roots on the branch and 2 $\lambda$-roots on the loop;

• $\xi_5 < \xi < \xi_6$: 6 $\lambda$-roots on the real axis;

• $\xi > \xi_7$: 6 $\lambda$-roots on the real axis.

Therefore, 1 loop is present in this spectrum.

Figure 4.7: Stability spectrum at $p_1 = -4.0$, $p_2 = -3.0$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$. 
Figure 4.8: Gain function $\Gamma(k_3)$ with $k_3 = w_1 - w_2$ associated to the spectrum obtained at $p_1 = -4.0$, $p_2 = -3.0$, $p_3 = -0.6$.

By setting $p_1 = 1.0$, $p_2 = 3.0$ and $p_3 = -0.6$, we have that $Q_2(\xi)$ changes sign 1 time in the point $\bar{\xi}_1 = 36.911$. After applying the Descartes rule of signs, further intervals must be considered whose end-points are: $\xi_1 = 0.079$, $\xi_2 = 3.664$, $\xi_3 = 36.911$. Then, we apply the Sturm chains technique and we get the following classification:

- $0 < \xi < \xi_1$: 4 real $\lambda$-roots and 2 $\lambda$-roots on the branch;
- $\xi_1 < \xi < \xi_2$: 4 real $\lambda$-roots and 2 $\lambda$-roots on the branch;
- $\xi_2 < \xi < \xi_3$: 4 real $\lambda$-roots and 2 $\lambda$-roots on the branch;
- $\xi > \xi_3$: 6 $\lambda$-roots on the real axis.

No loops exist into the spectrum for these choices of the physical parameters.
Figure 4.9: Stability spectrum at $p_1 = 1.0$, $p_2 = 3.0$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$.

Figure 4.10: Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the spectrum obtained at $p_1 = 1.0$, $p_2 = 3.0$, $p_3 = -0.6$.

### 4.5.1 Regions with 0 Gap and 2 Branches

In the regions with 0 gap and 2 branches, there is the following correspondence between the number of loops and the number of positive $\xi$-roots:

- 2L: 6 positive $\xi$-roots;
- 1L: 4 positive $\xi$-roots;
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• 0L: 2 positive $\xi$-roots.

Let us set $p_1 = -6.2$, $p_2 = -6.3$ and $p_3 = -0.6$. For these values of the parameters the polynomial $Q_2(\xi)$ change sign 6 times at the values of $\xi$: $\tilde{\xi}_1 = 0.168$, $\tilde{\xi}_2 = 1.219$, $\tilde{\xi}_3 = 63.296$, $\tilde{\xi}_4 = 63.549$, $\tilde{\xi}_5 = 64.132$, $\tilde{\xi}_6 = 70.732$. Then, we apply the Descartes rule of signs and we find further intervals whose end-points are: $\bar{\xi}_1 = 0.168$, $\bar{\xi}_2 = 0.451$, $\bar{\xi}_3 = 0.839$, $\bar{\xi}_4 = 1.015$, $\bar{\xi}_5 = 1.219$, $\bar{\xi}_6 = 2.746$, $\bar{\xi}_7 = 63.296$, $\bar{\xi}_8 = 64.132$, $\bar{\xi}_9 = 66.267$, $\bar{\xi}_{10} = 70.732$, $\bar{\xi}_{11} = 71.817$, $\bar{\xi}_{12} = 82.022$, $\bar{\xi}_{13} = 87.860$. In this way, by the Sturm chains method, we have the classification of the $\lambda$-roots:

• $0 < \xi < \xi_1$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on the other branch;
• $\xi_1 < \xi < \xi_2$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch;
• $\xi_2 < \xi < \xi_3$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch;
• $\xi_3 < \xi < \xi_4$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch;
• $\xi_4 < \xi < \xi_5$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch;
• $\xi_5 < \xi < \xi_6$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;
• $\xi_6 < \xi < \xi_7$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;
• $\xi_7 < \xi < \xi_8$: 6 $\lambda$-roots on the real axis;
• $\xi_8 < \xi < \xi_9$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a loop;
• $\xi_9 < \xi < \xi_{10}$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a loop;
• $\xi_{10} < \xi < \xi_{11}$: 6 $\lambda$-roots on the real axis;
• $\xi_{11} < \xi < \xi_{12}$: 6 $\lambda$-roots on the real axis;
• $\xi_{12} < \xi < \xi_{13}$: 6 $\lambda$-roots on the real axis;
• $\xi > \xi_{13}$: 6 $\lambda$-roots on the real axis.
Thus, we see 2 loops in this spectrum.

![Figure 4.11: Stability spectrum with $p_1 = -6.2$, $p_2 = -6.3$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$.

If we choose $p_1 = -4.0$, $p_2 = -4.2$ and $p_3 = -0.6$, the polynomial $Q_2(\xi)$ vanishes 4 times in these points: $\xi_1 = 0.129$, $\xi_2 = 0.240$, $\xi_3 = 36.370$ and $\xi_4 = 44.444$. By using the Descartes rule of signs we have: $\xi_1 = 0.129$, $\xi_2 = 0.179$, $\xi_3 = 0.240$, $\xi_4 = 0.380$, $\xi_5 = 0.495$, $\xi_6 = 1.047$, $\xi_7 = 36.370$, $\xi_8 = 43.943$, $\xi_9 = 44.444$, $\xi_{10} = 46.439$, $\xi_{11} = 49.745$, $\xi_{12} = 57.025$. By constructing the Sturm chains method, we obtain the following $\lambda$-roots classification:
• $0 < \xi < \xi_1$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on the other branch;

• $\xi_1 < \xi < \xi_2$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch;

• $\xi_2 < \xi < \xi_3$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch;

• $\xi_3 < \xi < \xi_4$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;

• $\xi_4 < \xi < \xi_5$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;

• $\xi_5 < \xi < \xi_6$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;

• $\xi_6 < \xi < \xi_7$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;

• $\xi_7 < \xi < \xi_8$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch or on a loop;

• $\xi_8 < \xi < \xi_9$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a branch or on a loop;

• $\xi_9 < \xi < \xi_{10}$: 6 $\lambda$-roots on the real axis;

• $\xi_{10} < \xi < \xi_{11}$: 6 $\lambda$-roots on the real axis;

• $\xi_{11} < \xi < \xi_{12}$: 6 $\lambda$-roots on the real axis;

• $\xi > \xi_{12}$: 6 $\lambda$-roots on the real axis.

Only one loop is present in the spectrum.
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Figure 4.13: Stability spectrum with $p_1 = -4.0$, $p_2 = -4.2$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$.

Figure 4.14: Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the spectrum obtained at $p_1 = -4.0$, $p_2 = -4.2$, $p_3 = -0.6$.

If we choose $p_1 = 1.0$, $p_2 = -3.0$ and $p_3 = -0.6$, the polynomial $Q_2(\xi)$ vanishes 2 times in these two points: $\bar{\xi}_1 = 8.421$ and $\bar{\xi}_2 = 13.280$. By using the Descartes rule of signs we have: $\xi_1 = 0.076$, $\xi_2 = 0.769$, $\xi_3 = 2.284$, $\xi_4 = 8.421$, $\xi_5 = 11.354$, $\xi_6 = 12.477$, $\xi_7 = 13.157$, $\xi_8 = 13.280$, $\xi_9 = 13.289$. By using Sturm chains we get the following $\lambda$-roots classification:

- $0 < \xi < \xi_1$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on the other branch;
• $\xi_1 < \xi < \xi_2$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on the other branch;

• $\xi_2 < \xi < \xi_3$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on the other branch;

• $\xi_3 < \xi < \xi_4$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on the other branch;

• $\xi_4 < \xi < \xi_5$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch;

• $\xi_5 < \xi < \xi_6$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch;

• $\xi_6 < \xi < \xi_7$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch;

• $\xi_7 < \xi < \xi_8$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch;

• $\xi_8 < \xi < \xi_9$: 6 $\lambda$-roots on the real axis;

• $\xi > \xi_9$: 6 $\lambda$-roots on the real axis.

No loop is present in the spectrum.

![Graph](image_url)

Figure 4.15: Stability spectrum with $p_1 = 1.0$, $p_2 = -3.0$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$. 
Figure 4.16: Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the spectrum obtained at $p_1 = 1.0$, $p_2 = -3.0$, $p_3 = -0.6$.

4.5.2 Regions with 2 Gaps and 0 Branch

In the regions with 2 gaps and 0 branch, there is the following correspondence between the number of loops and the number of positive $\xi$-roots:

- 2L: 4 positive $\xi$-roots;
- 1L: 2 positive $\xi$-roots.

The region with 2 gaps, 0 branch and 0 loop does not exist. This is because the $\lambda$-roots which start to move in the interval between the 2 gaps would be trapped inside such interval and never would go to infinity: this is in contradiction of the Proposition 4.3.1.

Let us set $p_1 = -70.0$, $p_2 = 60.0$ and $p_3 = -0.6$. The polynomial $Q_2(\xi)$ changes sign 4 times in the points $\bar{\xi}_1 = 1.596$, $\bar{\xi}_2 = 218.582$, $\bar{\xi}_3 = 968.174$, $\bar{\xi}_4 = 3944.396$. After applying the Descartes rule of signs, the end-point of the intervals become: $\xi_1 = 1.596$, $\xi_2 = 32.536$, $\xi_3 = 218.582$, $\xi_4 = 482.289$, $\xi_5 = 968.174$, $\xi_6 = 1480.780$, $\xi_7 = 3944.396$, $\xi_8 = 4733.800$.

By using Sturm chains method, we get the following $\lambda$-roots classification:

- $0 < \xi < \xi_1$: 6 $\lambda$-roots on the real axis;
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- $\xi_1 < \xi < \xi_2$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;
- $\xi_2 < \xi < \xi_3$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;
- $\xi_3 < \xi < \xi_4$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop and 2 $\lambda$-roots on another loop;
- $\xi_4 < \xi < \xi_5$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop and 2 $\lambda$-roots on another loop;
- $\xi_5 < \xi < \xi_6$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;
- $\xi_6 < \xi < \xi_7$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;
- $\xi_7 < \xi < \xi_8$: 6 $\lambda$-roots on the real axis;
- $\xi > \xi_8$: 6 $\lambda$-roots on the real axis.

Therefore, we see 2 loops in this spectrum.

Figure 4.17: Stability spectrum with $p_1 = -70.0$, $p_2 = 60.0$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$. 
Figure 4.18: Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the spectrum obtained at $p_1 = -70.0$, $p_2 = 60.0$, $p_3 = -0.6$.

Let us set $p_1 = 0.2$, $p_2 = 0.6$ and $p_3 = -0.6$. The polynomial $Q_2(\xi)$ changes sign 2 times in the points $\bar{\xi}_1 = 2.348$ and $\bar{\xi}_2 = 18.989$. After applying the Descartes rule of signs, the end-point of the intervals become: $\xi_1 = 2.348$, $\xi_2 = 6.410$ and $\xi_3 = 18.989$. After constructing the Sturm chains, the $\lambda$-roots classification is:

- $0 < \xi < \xi_1$: 6 $\lambda$-roots on the real axis;
- $\xi_1 < \xi < \xi_2$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;
- $\xi_2 < \xi < \xi_3$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;
- $\xi > \xi_3$: 6 $\lambda$-roots on the real axis.

Therefore, there is only 1 loop in this spectrum.
4.5.3 Region with 1 Gap, 1 Split Gap and 0 Branches

There is only one possibility:

- 1L 1TL: 3 positive $\xi$-roots.

By choosing $p_1 = -90.0$, $p_2 = 60.0$ and $p_3 = -0.6$, the polynomial $Q_2(\xi)$ vanishes in the points $\bar{\xi}_1 = 63.032$, $\bar{\xi}_2 = 2891.269$ and $\bar{\xi}_3 = 4474.552$. By the Descartes rule, the $\xi$-
domain is split by these points: $\xi_1 = 63.032$, $\xi_2 = 267.710$, $\xi_3 = 1569.140$, $\xi_4 = 2891.269$, $\xi_5 = 4474.552$, $\xi_6 = 6153.740$. The $\lambda$-roots classification is obtained by the Sturm chains construction:

- $0 < \xi < \xi_1$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a loop;
- $\xi_1 < \xi < \xi_2$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop and 2 $\lambda$-roots on another loop;
- $\xi_2 < \xi < \xi_3$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop and 2 $\lambda$-roots on another loop;
- $\xi_3 < \xi < \xi_4$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop and 2 $\lambda$-roots on another loop;
- $\xi_4 < \xi < \xi_5$: 4 $\lambda$-roots on the real axis and 2 $\lambda$-roots on a loop;
- $\xi_5 < \xi < \xi_6$: 6 $\lambda$-roots on the real axis;
- $\xi > \xi_6$: 6 $\lambda$-roots on the real axis.

In this spectrum it looks like there are only two loops, but actually there are 1 loop and 1 twisted loop. Therefore, the second loop that we have found has to be counted twice. Indeed, we cannot individuate the exact value of $\xi$ in which the two $\lambda$-roots collide on the real axis to become again two complex conjugate roots travelling the second part of the twisted loop.
Figure 4.21: Stability spectrum with $p_1 = -90.0$, $p_2 = 60.0$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$.

Figure 4.22: Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the spectrum obtained at $p_1 = -90.0$, $p_2 = 60.0$, $p_3 = -0.6$.

### 4.5.4 Region with 1 Split Gap and 1 Branch

In this region we have:

- 1L 1TL: 4 positive $\xi$-roots;
- 1TL: 2 positive $\xi$-roots.
Let us fix $p_1 = -1.4$, $p_2 = -1.0$ and $p_3 = -0.6$. The polynomial $Q_2(\xi)$ changes sign 4 times in the points $\bar{\xi}_1 = 9.216$, $\bar{\xi}_2 = 11.823$, $\bar{\xi}_3 = 12.195$ and $\bar{\xi}_4 = 27.053$. By the Descartes rule of signs, we get: $\xi_1 = 0.002$, $\xi_2 = 0.421$, $\xi_3 = 9.216$, $\xi_4 = 11.823$, $\xi_5 = 12.195$, $\xi_6 = 27.053$, so that, by using the Sturm chains technique, the $\lambda$-roots are classified as follows:

- $0 < \xi < \xi_1$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;
- $\xi_1 < \xi < \xi_2$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop
- $\xi_2 < \xi < \xi_3$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop
- $\xi_3 < \xi < \xi_4$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch or on a loop;
- $\xi_4 < \xi < \xi_5$: 6 $\lambda$-roots on the real axis;
- $\xi_5 < \xi < \xi_6$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;
- $\xi > \xi_6$: 6 $\lambda$-roots on the real axis.

In this spectrum there is 1 loop and 1 twisted loop. Also in this case it is not possible distinguish the exact value of $\xi$ in which two $\lambda$-roots collide in the point separating a split gap and then they become again two complex conjugate roots.
Figure 4.23: Stability spectrum with $p_1 = -1.4$, $p_2 = -1.0$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$.

Figure 4.24: Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the spectrum at $p_1 = -1.4$, $p_2 = -1.0$, $p_3 = -0.6$.

If we choose $p_1 = -4$, $p_2 = 2$ and $p_3 = -0.6$, the polynomial $Q_2(\xi)$ vanishes 2 times in $\bar{\xi}_1 = 9.591$ and $\bar{\xi}_2 = 98.262$. By the Descartes rule of signs, we get the points $\xi_1 = 0.136$, $\xi_2 = 9.591$, $\xi_3 = 19.547$, $\xi_4 = 45.537$, $\xi_5 = 98.262$. The classification of the $\lambda$-roots is:

- $0 < \xi < \xi_1$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;
- $\xi_1 < \xi < \xi_2$: 2 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a branch and 2 $\lambda$-roots on a loop;
• $\xi_2 < \xi < \xi_3$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;

• $\xi_3 < \xi < \xi_4$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;

• $\xi_4 < \xi < \xi_5$: 4 $\lambda$-roots on the real axis, 2 $\lambda$-roots on a loop;

• $\xi > \xi_5$: 6 $\lambda$-roots on the real axis.

It looks like we have only one loop, but actually this is a twisted loop.

**Figure 4.25:** Stability spectrum with $p_1 = -4.0$, $p_2 = 2.0$, $p_3 = -0.6$. $\rho = \text{Re}(\lambda)$ and $\mu = \text{Im}(\lambda)$.

**Figure 4.26:** Gain function $\Gamma(k_3)$ where $k_3 = w_1 - w_2$ associated to the spectrum obtained at $p_1 = -4.0$, $p_2 = 2.0$, $p_3 = -0.6$.

Summarising, all the possible topologies in the $(p_1, p_2)$-plane are:
• 1G 0SG 1B 2L 0TL;
• 1G 0SG 1B 1L 0TL;
• 1G 0SG 1B 0L 0TL;
• 0G 0SG 2B 2L 0TL;
• 0G 0SG 2B 1L 0TL;
• 0G 0SG 2B 0L 0TL;
• 2G 0SG 0B 2L 0TL;
• 2G 0SG 0B 1L 0TL;
• 1G 1SG 0B 1L 1TL;
• 0G 1SG 1B 1L 1TL;
• 0G 1SG 1B 0L 1TL.

4.6 Topological Classification of the Spectra in the Parameter Space

At this point, we can obtain the entire topological classification of the spectra in the parameter space. In order to do that, we need to describe the curves in the \((p_1, p_2)\)-plane separating the regions with different number of gaps, branches and loops. First of all, we compute and factorise the polynomial \(Q_2(\xi)\) at \(\xi = 0\),

\[
Q_2(0) = \prod_{j=1}^{4} D_j(p_1, p_2, p_3)^{d_j}, \tag{4.92a}
\]

whose corresponding curves are given by (4.33), as well as the discriminant of \(Q_2(\xi)\) with respect to \(\xi\),

\[
\Delta_\xi Q_2(\xi) = \prod_{j=0}^{4} E_j(p_1, p_2, p_3)^{e_j}, \tag{4.92b}
\]
where the functions $E_j$ depend on the parameters $p_1$, $p_2$ and $p_3$.

Similarly to the definition (4.33), let $\mathcal{E}_j$ be the real-analytic variety in the parameter space, implicitly defined as

$$\mathcal{E}_j = \{ p_1, p_2, p_3 \in \mathbb{R} \mid \exists (p_1, p_2) \in \mathbb{R}^{1 \times 1} : E_j(p_1, p_2, p_3) = 0 \}.$$ \hspace{1cm} (4.93a)

The curves (4.93a) define transition regions with different topological structures. In particular, we find again the curves (4.93a), which are $D_1$, $D_2$, $D_3$ and $D_4$ coinciding with $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and $\mathcal{E}_4$, respectively, plus another curve that we denote as $\mathcal{E}_0$.

If the matrix $W$ features the parameters $p_1$, $p_2$ and $p_3$ only in polynomial form, then $D_j$, with $j = 1, 2, 3, 4$, and $\mathcal{E}_0$ are algebraic varieties over the reals.

**Definition 4.6.1.** Let

$$\mathcal{C} = \bigcup_{j=1,2,3,4} D_j \bigcup \mathcal{E}_0,$$ \hspace{1cm} (4.94)

be the set of all the varieties $D_j$ joined with $\mathcal{E}_0$. The set $\mathcal{C}$ defines the boundaries of the regions in the $(p_1, p_2)$-parameters space associated to different topologies of the spectra $S_x$.

For instance, if the (effective\textsuperscript{16}) parameter space is 2-dimensional, then $\mathcal{C}$ is a set of curves on the real plane.

Finally, in the figures 4.27 and 4.28 we show the curves $\mathcal{C}$ in the parameters space and the entire topological classification of the spectra in this space.

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\textsuperscript{16}The parameters $p_1$ and $p_2$ may appear in the definition of the varieties $D_j$ and $\mathcal{E}$ combined in a certain number of functions, whose total number can be less than the original number of parameters; then, these functions of the parameters, on which the varieties depend, play the role of “effective” parameters.
Figure 4.27: \((p_1, p_2)\)-plane, \(p_3 = -0.6\).
Figure 4.28: $(p_1, p_2)$-plane, $p_3 = -0.6$. 
Chapter 5

Classification of the Stability Spectra on the Physical Parameters Space and Numerical Simulations

Based on the results obtained in the previous chapter, in the following we discuss the classification of the stability spectra in terms of the physical parameters space. We divide the parameters space in octants and in every octant we provide the values of the physical parameters. Then, we provide some examples of numerical solutions of the original 3WRI system to show that the plane wave solution is everywhere linearly unstable and that the presence of a branch into the stability spectrum can be associated, depending on the class of the perturbation, to the onset of localised structures such as breather-like solutions and potential rogue waves.

These numerical simulations have been included mainly for the sake of complementing the analytical results: although the theory presented herein cannot quantitatively explain these numerical experiments, it is interesting to observe that perturbing plane waves associated to different spectra and integrating numerically (over a short time) the 3WRI system using the different perturbing waves yields different time evolutions and behaviours, hence suggesting a potential link, in the spirit of what has been done for the NLS system in [72]. This link may be the subject of future investigation.
Section 5.1 Classification of the Stability Spectra on the Physical Parameters Space

The formulas (3.91) can be written as follows:

\[ s_1 s_2 s_3 p_1 = c_1^2 a_1^2 + c_2^2 a_2^2, \]  
\[ (5.1a) \]

\[ s_1 s_2 s_3 p_2 = c_1^2 a_1^2 - c_2^2 a_2^2, \]  
\[ (5.1b) \]

By summing and subtracting (5.1) and (5.1b) we get, respectively,

\[ c_1^2 = \frac{s_2 s_3}{2a_1^2} (p_1 + p_2), \]  
\[ (5.2a) \]

\[ c_2^2 = \frac{s_1 s_3}{2a_2^2} (p_1 - p_2). \]  
\[ (5.2b) \]

Moreover, we know that

\[ p_3 = \frac{c_1 - c_2}{c_1 + c_2}. \]  
\[ (5.3) \]

Let us separate the parameters space \((p_1, p_2)\) in quadrants and let us enumerate clockwise the quadrants with the roman numbers I, II, III, IV. Then, let us separate further every quadrant in two parts or octants according to the following scheme:

In every octant, we have the following relation between the parameters \(p_1\) and \(p_2\):
• Octant Ia: \(|p_1| > |p_2|, p_1 > 0 \) and \( p_2 > 0 \);
• Octant Ib: \(|p_1| < |p_2|, p_1 > 0 \) and \( p_2 > 0 \);
• Octant IIa: \(|p_1| < |p_2|, p_1 < 0 \) and \( p_2 > 0 \);
• Octant IIb: \(|p_1| > |p_2|, p_1 < 0 \) and \( p_2 > 0 \);
• Octant IIIa: \(|p_1| > |p_2|, p_1 < 0 \) and \( p_2 < 0 \);
• Octant IIIb: \(|p_1| < |p_2|, p_1 < 0 \) and \( p_2 < 0 \);
• Octant IVa: \(|p_1| < |p_2|, p_1 > 0 \) and \( p_2 < 0 \);
• Octant IVb: \(|p_1| > |p_2|, p_1 > 0 \) and \( p_2 < 0 \).

In order to see which choice of signs exists in any octant, let us take into account the formulas (5.2a) and (5.2b).

In the quadrant I, we have \( p_1 > 0 \) and \( p_2 > 0 \), then, by the formula (5.2a), it results in \( s_2 s_3 = 1 \), and so \( s_2 = s_3 = \pm 1 \). In the octant Ia, we have \( p_1 > p_2 \), then, by the formula (5.2b), \( s_1 s_3 = 1 \). This yields that the choice of signs is \( S^{1+}_3 \) or \( S^{1-}_3 \), that is \( s_1 = s_2 = s_3 = \pm 1 \). In the octant Ib, we have \( p_1 < p_2 \), thus, by the relation (5.2b), the choice of signs is \( S^{2+}_3 \) or \( S^{2-}_3 \), that is \( s_1 = -1 \) and \( s_2 = s_3 = 1 \) or \( s_1 = 1 \) and \( s_2 = s_3 = -1 \), respectively.

In the quadrant II, we have \( p_1 < 0 \) and \( p_2 > 0 \), and by the formula (5.2b), it results \( s_1 s_3 = -1 \), and so \( s_1 = -s_3 \). In the octant IIa, \( p_1 > p_2 \), as a consequence of the relation (5.2a), we have \( s_2 s_3 = 1 \). Therefore, the choice of signs is \( S^{3+}_3 \) or \( S^{3-}_3 \), corresponding to \( s_1 = -1 \) and \( s_2 = s_3 = 1 \) or \( s_1 = -1 \) and \( s_2 = s_3 = -1 \), respectively. On the other hand in the octant IIb, we have \( p_1 < p_2 \). By the formula (5.2a), we obtain \( s_2 s_3 = -1 \). Thus we have the choices of signs \( s_1 = s_2 = 1 \) and \( s_3 = -1 \) or \( s_1 = s_2 = -1 \) and \( s_3 = 1 \) which are denoted as \( S^{4+}_3 \) and \( S^{4-}_3 \), respectively.

In the quadrant III, we have \( p_1 < 0 \) and \( p_2 < 0 \). By the relation (5.2a), we obtain \( s_2 s_3 = -1 \), that is \( s_2 = -1 \) and \( s_3 = 1 \) or \( s_2 = 1 \) and \( s_3 = -1 \). In the octant IIIa, \( p_1 < p_2 \), and as consequence of the formula (5.2b) we get \( s_1 s_3 = -1 \). Therefore, the combinations of signs are \( s_1 = s_2 = 1 \) and \( s_3 = -1 \) or \( s_1 = s_2 = -1 \) and \( s_3 = 1 \) corresponding to the choices \( S^{5+}_3 \) and \( S^{5-}_3 \), respectively. In the octant IIIb, we have \( p_1 > p_2 \), then \( s_1 s_3 = 1 \), that
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is \( s_1 = s_3 = 1 \) and \( s_3 = -1 \) or \( s_1 = s_3 = -1 \) and \( s_3 = 1 \). They correspond to the choices \( S^+_2 \) and \( S^-_2 \), respectively.

In the quadrant IV, we have the relations \( p_1 > 0 \) and \( p_2 < 0 \). By the relation (5.2b), it results in \( s_1 s_3 = 1 \), and so \( s_1 = s_3 = 1 \) or \( s_1 = s_3 = -1 \). Moreover, in the octant IVa, \( p_1 < -p_2 \), then, by (5.2a), \( s_2 s_3 = -1 \), we obtain \( s_1 = s_3 = 1 \) and \( s_2 = -1 \) or \( s_1 = s_3 = -1 \) and \( s_2 = 1 \). These combinations of signs are denoted as \( S^+_2 \) and \( S^-_2 \), respectively. In the octant IVb, \( p_1 > -p_2 \) and, by the formula (5.2a), this yields \( s_2 s_3 = 1 \). Thus, we have \( s_1 = s_2 = s_3 = 1 \) or \( s_2 = s_2 = s_3 = -1 \), corresponding to the choices \( S^+_4 \) or \( S^-_4 \), respectively.

In the following subsection we will provide the mapping from the parameters \( p_1, p_2, p_3 \) into the parameters \( c_1, c_2, a_1 \) and \( a_2 \).

5.1.1 Octant Ia

By looking at the formulas (5.2), if \( p_1 > p_2 > 0 \), then \( p_1 + p_2 > 0 \) and \( p_1 - p_2 > 0 \). As a consequence, \( s_1 s_3 = 1 \) and \( s_2 s_3 = 1 \) and, so, \( s_1 = s_2 = s_3 \).

By subtracting (5.2a) from (5.2b)

\[
c_1^2 - c_2^2 = \frac{1}{2a_1^2}(p_1 + p_2) - \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2 a_2^2}[p_1(a_2^2 - a_1^2) + p_2(a_1^2 + a_2^2)].
\]

For \( p_3 < 0 \), we have \( |c_2| > |c_1| \), so \( c_1^2 - c_2^2 < 0 \), and

\[
0 < \frac{p_1}{p_2} < \frac{a_2^2}{a_1^2} \quad \text{then} \quad |a_1| > |a_2|.
\]

Similarly, for \( p_3 > 0 \), we have \( |c_2| < |c_1| \), the equation \( c_1^2 - c_2^2 > 0 \) gives us the condition \( |a_1| > |a_2| \). We have \( |a_1| \neq |a_2| \), otherwise \( p_2 < 0 \).

In conclusion, for both the cases \( |c_2| > |c_1| (p_3 < 0) \) and \( |c_2| < |c_1| (p_3 > 0) \) the amplitudes satisfy the relation \( |a_1| > |a_2| \).

5.1.2 Octant Ib

By (5.2), if \( p_2 > p_1 > 0 \), then \( p_1 + p_2 > 0 \) and \( p_1 - p_2 < 0 \) and also we have \( s_2 s_3 = 1 \) and \( s_1 s_3 = -1 \). Thus, it results in \( s_3 = s_3 = -s_1 \).
By subtracting (5.2a) from (5.2b)
\[ c_1^2 - c_2^2 = \frac{1}{2a_1^2}(p_1 + p_2) + \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2a_2^2}[p_1(a_2^2 + a_1^2) + p_2(a_2^2 - a_1^2)] < 0. \tag{5.6} \]

For \( p_3 < 0 \), we have \( |c_2| > |c_1| \). Thus,
\[ 0 < \frac{p_1}{p_2} < \frac{a_2^2 - a_1^2}{a_1^2 + a_2^2} \text{ then } |a_1| > |a_2|. \tag{5.7} \]

It must be \( |a_1| \neq |a_2| \), although we have the contradiction \( p_1 < 0 \).

One can show that for both \( |c_2| > |c_1| \) (\( p_3 < 0 \)) and \( |c_2| < |c_1| \) (\( p_3 > 0 \)) the amplitudes satisfy the relation \( |a_1| > |a_2| \).

### 5.1.3 Octant IIa

By (5.2), if \( p_2 > 0 \) and \( p_1 < 0 \) and \( |p_2| > |p_1| \), then \( p_1 + p_2 > 0 \) and \( p_1 - p_2 < 0 \) and \( s_2s_3 = 1 \) and \( s_1s_3 = -1 \). This entails \( s_2 = s_3 = -s_1 \).

By summing (5.2a) and (5.2b)
\[ c_1^2 + c_2^2 = \frac{1}{2a_1^2}(p_1 + p_2) - \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2a_2^2}[p_1(a_2^2 - a_1^2) + p_2(a_2^2 + a_1^2)] > 0. \tag{5.8} \]

Thus,
\[ 0 > \frac{p_1}{p_2} > \frac{a_2^2 + a_1^2}{a_1^2 - a_2^2}, \text{ then, } |a_1| < |a_2|. \tag{5.9} \]

In this case, if \( |a_1| \neq |a_2| \), otherwise \( p_1 > 0 \).

For both \( |c_2| > |c_1| \) (\( p_3 < 0 \)) and \( |c_2| < |c_1| \) (\( p_3 > 0 \)) the amplitudes satisfy the relation \( |a_1| < |a_2| \).

### 5.1.4 Octant IIb

By (5.2), if \( p_2 > 0 \) and \( p_1 < 0 \) and \( |p_1| > |p_2| \), then we obtain \( p_1 + p_2 > 0 \) and \( p_1 - p_2 < 0 \) and \( s_2s_3 = 1 \) and \( s_1s_3 = -1 \). As a result, we get \( s_2 = s_3 = s_1 \). By summing (5.2a) and (5.2b)
\[ c_1^2 + c_2^2 = \frac{1}{2a_1^2}(p_1 + p_2) - \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2a_2^2}[p_1(a_2^2 - a_1^2) + p_2(a_2^2 + a_1^2)] > 0. \tag{5.10} \]
Therefore,

\[ 0 > \frac{p_2}{p_1} > \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2} \] then \( |a_1| < |a_2| \).

(5.11)

Also in this case \( |a_1| \neq |a_2| \), otherwise we have a contradiction. Finally, for both \( |c_2| > |c_1| \) \((p_3 < 0)\) and \( |c_2| < |c_1| \) \((p_3 > 0)\) the amplitudes satisfy the relation \( |a_1| < |a_2| \).

### 5.1.5 Octant IIIa

By (5.2), if \( p_1 < p_2 < 0 \), then \( p_1 + p_2 < 0 \) and \( p_1 - p_2 < 0 \) and \( s_2 s_3 = -1 \) and \( s_1 s_3 = -1 \). This entails \( s_1 = s_2 = -s_3 \).

By subtracting \((5.2a)\) from \((5.2b)\)

\[ c_1^2 - c_2^2 = -\frac{1}{2a_1^2}(p_1 + p_2) + \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2 a_2^2} [-p_1(a_1^2 - a_2^2) - p_2(a_2^2 + a_1^2)] < 0. \]

(5.12)

Therefore,

\[ 0 < \frac{p_1}{p_2} < \frac{a_1^2 + a_2^2}{a_1^2 - a_2^2} \] then \( |a_1| > |a_2| \).

(5.13)

If \( |a_1| = |a_2| \), we have the contradiction \( p_2 > 0 \).

For both \( |c_2| > |c_1| \) \((p_3 < 0)\) and \( |c_2| < |c_1| \) \((p_3 > 0)\) the amplitudes satisfy the relation \( |a_1| > |a_2| \).

### 5.1.6 Octant IIIb

By (5.2), if \( p_2 < p_1 < 0 \), then \( p_1 + p_2 < 0 \) and \( p_1 - p_2 > 0 \) and \( s_2 s_3 = -1 \) and \( s_1 s_3 = 1 \). Therefore, it turns out \( s_1 = s_3 = -s_2 \).

By subctring \((5.2a)\) from \((5.2b)\)

\[ c_1^2 - c_2^2 = -\frac{1}{2a_1^2}(p_1 + p_2) - \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2 a_2^2} [-p_1(a_1^2 + a_2^2) + p_2(a_1^2 - a_2^2)] < 0. \]

(5.14)

Thus,

\[ 0 < \frac{p_2}{p_1} < \frac{a_1^2 + a_2^2}{a_1^2 - a_2^2} \implies |a_1| > |a_2|. \]

(5.15)

If \( |a_1| = |a_2| \), we have the contradiction \( p_1 > 0 \). For both \( |c_2| > |c_1| \) \((p_3 < 0)\) and \( |c_2| < |c_1| \) \((p_3 > 0)\) the amplitudes satisfy the relation \( |a_1| > |a_2| \).
5.1.7 Octant IVa

By (5.2), if \( p_2 < 0 \) and \( p_1 > 0 \) and \( |p_2| > |p_1| \) entails \( p_1 + p_2 < 0 \) and \( p_1 - p_2 > 0 \). If \( s_2 s_3 = -1 \) and \( s_1 s_3 = 1 \) entails \( s_1 = s_3 = -s_2 \). By summing (5.2a) and (5.2b)

\[
\begin{align*}
c_1^2 + c_2^2 &= -\frac{1}{2a_1^2}(p_1 + p_2) + \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2 a_2^2} [p_1(a_1^2 - a_2^2) - p_2(a_2^2 + a_1^2)] > 0. & (5.16)
\end{align*}
\]

Thus,

\[
0 \geq \frac{p_1}{p_2} > \frac{a_1^2 + a_2^2}{a_1^2 - a_2^2} \implies |a_1| < |a_2|. & (5.17)
\]

If \( |a_1| = |a_2| \), then \( p_2 < 0 \). For both \( |c_2| > |c_1| \) \((p_3 < 0)\) and \( |c_2| < |c_1| \) \((p_3 > 0)\) the amplitudes satisfy the relation \( |a_1| \leq |a_2| \).

5.1.8 Octant IVb

By (5.2), if \( p_2 < 0 \) and \( p_1 > 0 \) and \( |p_1| > |p_2| \) yields \( p_1 + p_2 > 0 \) and \( p_1 - p_2 > 0 \). If \( s_2 s_3 = 1 \) and \( s_1 s_3 = 1 \), then \( s_1 = s_3 = s_2 \). By summing (5.2a) and (5.2b)

\[
\begin{align*}
c_1^2 + c_2^2 &= \frac{1}{2a_1^2}(p_1 + p_2) + \frac{1}{2a_2^2}(p_1 - p_2) = \frac{1}{2a_1^2 a_2^2} [p_1(a_1^2 + a_2^2) + p_2(a_2^2 - a_1^2)] > 0. & (5.18)
\end{align*}
\]

Thus,

\[
0 \geq \frac{p_1}{p_2} > \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2} \implies |a_1| < |a_2|. & (5.19)
\]

If \( |a_1| = |a_2| \), then \( p_1 > 0 \). Finally, for both \( |c_2| > |c_1| \) \((p_3 < 0)\) and \( |c_2| < |c_1| \) \((p_3 > 0)\) the amplitudes satisfy the relation \( |a_1| \leq |a_2| \).

5.1.9 Transformations of the Physical Parameters

In the following we provide the transformations form the parameters \( p_1, p_2, p_3 \) and \( p_4 \) to the velocities \( c_1, c_2 \), to the amplitudes \( a_1, a_2 \) and to the signs \( s_1, s_2 \) and \( s_3 \).

The velocities \( c_1 \) and \( c_2 \) can be also written as

\[
c_1 = \frac{(1 + p_3)p_4}{2}. & (5.20a)
\]
\[ c_2 = \frac{(1 - p_3)p_4}{2}. \]  

(5.20b)

In the following we provide the values of the signs \( s_j, j = 1, 2, 3, \) for different range of the parameters \( p_1 \) and \( p_2. \)

- if \( p_2 < 0 \) and \( p_1 \leq p_2: \) \( s_1 = -1, \) \( s_2 = -1, \) \( s_3 = 1; \)
- if \( p_2 < 0 \) and \( -p_2 < p_1 < p_2: \) \( s_1 = -1, \) \( s_2 = 1, \) \( s_3 = -1; \)
- if \( p_2 < 0 \) and \( p_1 \geq -p_2: \) \( s_1 = -1, \) \( s_2 = -1, \) \( s_3 = -1; \)
- if \( p_2 > 0 \) and \( p_1 \leq -p_2: \) \( s_1 = -1, \) \( s_2 = -1, \) \( s_3 = 1; \)
- if \( p_2 > 0 \) and \( -p_2 < p_1 < p_2: \) \( s_1 = -1, \) \( s_2 = 1, \) \( s_3 = 1; \)
- if \( p_2 > 0 \) and \( p_1 \geq p_2: \) \( s_1 = -1, \) \( s_2 = -1, \) \( s_3 = -1; \)
- if \( p_2 = 0 \) and \( p_1 < 0: \) \( s_1 = -1, \) \( s_2 = -1, \) \( s_3 = 1; \)
- if \( p_2 = 0 \) and \( p_1 > 0: \) \( s_1 = -1, \) \( s_2 = -1, \) \( s_3 = -1. \)

If \( p_1 = p_2 = 0, \) then \( a_1 = a_2 = 0 \) and is not considered.

The amplitudes transform as follows

\[ a_1 = \frac{\sqrt{2s_2 s_3(p_1 + p_2)}}{|1 + p_3||p_4|}, \]  

(5.21a)

\[ a_2 = \frac{\sqrt{2s_2 s_3(p_1 - p_2)}}{|1 - p_3||p_4|}. \]  

(5.21b)

### 5.2 Numerical Simulations

In this section we show some numerical observations of different evolution obtained by integrating numerically the 3WRI system from a perturbed plane wave solution.

We use two kinds of perturbations: a localised perturbation and a random perturbation\(^1\)

\(^1\)In this respect, it is right to highlight that the linear stability analysis of plane wave solutions of the scalar NLS equation when these plane waves are perturbed via periodic perturbations was carried out in

[38] [39] [40] [41] [42] [43]
We fix $p_1$, $p_2$ and $p_3$ and we reconstruct all the physical parameters, $a_1$, $a_2$, $c_1$, $c_2$, $s_1$, $s_2$, $s_3$, based on the formulae in Section 5.1.9. Using these parameters, we construct the corresponding plane wave solution. We compute this plane wave solution at $t = 0$, we perturb it spatially and we use it as the initial condition for our numerical simulation. Each initial condition is perturbed alternatively in two different ways:

- through a localised perturbation, added to the initial data (individually to each $u_j$), having the form:
  \[ \epsilon \cos\left(\frac{\pi x}{2L}\right) e^{-\alpha x^2}, \]  
  \[ (5.22) \]
  where $L$ is the semi-length of the numerical integration interval, $\epsilon$ is the amplitude of the perturbation (in the following $\epsilon = 10^{-3}$, unless specified differently), and $\alpha$ is a chosen parameter, typically set to 2 (in the following $\alpha = 2$, unless specified differently);

- through a random perturbation, added as a noise to the initial data (individually to each $u_j$), having the form:
  \[ \epsilon \psi(x) \text{ with } \psi(-L) = \psi(L) = 0, \]
  \[ (5.23) \]
  where $\psi(x)$ for $x \in (-L, L)$ is a uniform distribution in the interval $[0, 1]$, $L$ is again the semi-length of the integration interval and $\epsilon$ is the amplitude of the perturbation; in the implementation, the random perturbation is smoothened by computing $\psi$ on a subset of the spatial nodes and then by using a Whittaker-Shannon interpolation formula [119] over the remaining nodes (see Appendix N).

As for the numerical scheme applied, we use the method of lines with a pseudospectral, Fourier discretisation in space and an adaptive Runge-Kutta scheme in time implemented in MATLAB R2017a (see Appendix O and Appendix P). In order to apply the pseudospectral method, it is very convenient to have initial conditions independent of the space variable. Let us provide the following transformation:

\[ \begin{cases} 
\tilde{u}_1 = e^{i\nu_1(x-c_1 t)} u_1, \\
\tilde{u}_2 = e^{i\nu_2(x-c_2 t)} u_2, \\
\tilde{u}_3 = e^{i(\nu_3 x + \omega t)} u_3,
\end{cases} \]

\[ (5.24) \]
where
\[
\nu_1 = \frac{\eta_1}{c_1} + \frac{s_1 s_3 a_2^2 c_2}{c_1} \frac{(c_1 - c_2)}{\eta_1 + \eta_2},
\] (5.25a)

\[
\nu_2 = \frac{\eta_2}{c_2} + \frac{s_2 s_3 a_1^2 c_1}{c_2} \frac{(c_1 - c_2)}{\eta_1 + \eta_2},
\] (5.25b)

\[
\nu_3 = -(\nu_1 + \nu_2),
\] (5.25c)

\[
\omega = c_1 \nu_1 + c_2 \nu_2
\] (5.25d)

then, it is straightforward to verify that \(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\) satisfy the following system of PDEs

\[
\begin{align*}
\tilde{u}_1_t &= -c_1 \tilde{u}_1 x + s_1 c_2 \tilde{u}_2^* \tilde{u}_3^*, \\
\tilde{u}_2_t &= -c_2 \tilde{u}_2 x + s_2 c_1 \tilde{u}_1^* \tilde{u}_3^*, \\
\tilde{u}_3_t &= i\omega \tilde{u}_3 + s_3 (c_1 - c_2) \tilde{u}_1^* \tilde{u}_2^*,
\end{align*}
\] (5.26)

with the following initial conditions

\[
\begin{align*}
\tilde{u}_1(x, 0) &= e^{i\nu_1 x} u_1(x, 0), \\
\tilde{u}_2(x, 0) &= e^{i\nu_2 x} u_2(x, 0), \\
\tilde{u}_3(x, 0) &= e^{i\nu_3 x} u_3(x, 0).
\end{align*}
\] (5.27)

We integrate system (5.26) with initial conditions (5.27) and the we invert (5.24) to obtain the solutions in terms of the \(u_j\)'s starting from the solutions in terms of the \(\tilde{u}_j\)'s.

Observe that, if \(u_j(x, t)\) is the plane wave solution (3.30), then \(\tilde{u}_j(x, t)\) does not depend explicitly on the space variable \(x\), indeed we have

\[
\begin{align*}
\tilde{u}_1 &= a_1 e^{i(\eta_1 - \nu_1 c_1)t}, \\
\tilde{u}_2 &= a_2 e^{i(\eta_2 - \nu_2 c_2)t}, \\
\tilde{u}_3 &= a_3 e^{i(\eta_3 + \omega)t},
\end{align*}
\] (5.28)

Although the solutions \(\tilde{u}_j\), for \(j = 1, 2, 3\) may appear simpler than those ones used in the theoretical part of the Thesis, they satisfy a system (see formula (5.26)) more complicated than that one used for the analytical computations and dealing with it would result an useless effort. On the other hand, the system (5.26) has been found useful for the numerics in this Chapter.
with \(a_3\) as in (3.32), and \(\eta_3 = -(\eta_1 + \eta_2)\).

In particular, from (5.28), we see that \(\tilde{u}_j(x,0)\) does not depend on the spatial variable \(x\), and hence the integration interval \([-L,L]\) can be taken arbitrary for system (5.26) using (5.28) at \(t = 0\) as initial conditions.

In the following we denote with

\[
\rho_j^{(0)} = \max_{x \in [-L,L]} \frac{|u_j(x,0)|}{a_j},
\]

the ratio between the initial maximum of the absolute value of the perturbed solution and \(a_j\) (that is the amplitude of the unperturbed wave); we also introduce

\[
\bar{\rho}_j = \max_{x \in [-L,L], t \in [0,T]} \frac{|u_j(x,t)|}{a_j},
\]

namely, the ratio between the maximum of the absolute value of the perturbed solution on the whole integration domain \([-L,L] \times [0,T]\) and \(a_j\).

Before we show some numeric simulations, it is right to provide the following definition of rogue wave:

"In the real ocean, rogue waves are waves that are very steep and much higher than the surrounding waves in a wave record, which is usually of 20-minute length;...There is currently no consensus on one unique definition of a rogue wave, but a common and simple approach is to define a rogue wave as a wave whose wave height or crest height exceeds some thresholds related to the significant wave height. A common definition is to apply the criteria (Haver, 2000):

\[
\frac{H_{\text{max}}}{H_s} > 2, \quad \text{and/or} \quad \frac{C_{\text{max}}}{H_s} > 1.25,
\]

where \(H_{\text{max}}\) denotes the zero-crossing wave height, \(C_{\text{max}}\) is the crest height, and \(H_s\) is the significant wave height, defined as four times the standard deviation of the surface, typically calculated from a 20-minute measurement of the surface elevation." [120].

Because of this definition, we report the value \(\bar{\rho}_{\text{max}}\) for each numerical solution and this value will give us in percentage the ratio between the maximum value obtained by the numerical simulations and the background. We will state that we have a potential rogue wave any time the maximum value of the solution exceeds at least of three times the value of

\(^3\)The range of values of \(L\) is taken as larger as possible to make sure that all the modes with higher velocities are included as well.
the background. We will state that we have a potential rogue wave any time the maximum value of the solution exceeds at least three times the value of the background.

5.2.1 \( 2G 0SG 0B 1L 0TL \)

We investigate the region \( 2G 0B 1L \) that is located near the origin in the parameter space (see Figure 4.27). We choose the parameters \( p_1 = 0.2, p_2 = 0.3, p_3 = -0.6 \) and \( p_4 = 1 \) corresponding to the velocities \( c_1 = 0.2, c_2 = 0.8 \), to the amplitudes \( a_1 = 2.5, a_2 = 0.27951 \) and to the choice of signs \( s_1 = 1, s_2 = 1, s_3 = 1 \). In particular with the localised perturbation, we observe:

- \( \rho_1^{(0)} = 0.04\% \) and \( \bar{\rho}_1 = 0.052206\% \);
- \( \rho_2^{(0)} = 0.35777\% \) and \( \bar{\rho}_2 = 0.35777\% \);
- \( \rho_3^{(0)} = 0.47703\% \) and \( \bar{\rho}_3 = 0.82379\% \).

By perturbing with random perturbation, we observe:

- \( \rho_1^{(0)} = 0.058176\% \) and \( \bar{\rho}_1 = 0.17725\% \);
- \( \rho_1^{(0)} = 0.48424\% \) and \( \bar{\rho}_2 = 0.61943\% \);
- \( \rho_1^{(0)} = 0.57518\% \) and \( \bar{\rho}_3 = 1.6717\% \).

5.2.2 \( 0G 0SG 2B 1L 0TL \)

We investigate the region \( 0G 2B 1L \) that is located near the origin on the parameter space (see Figure 4.27). We choose the parameters \( p_1 = 0.2, p_2 = -0.4, p_3 = -0.6 \) and \( p_4 = 1 \) corresponding to the velocities \( c_1 = 0.2, c_2 = 0.8 \), to the amplitudes \( a_1 = 1.5811, a_2 = 0.68465 \) and to the choice of signs \( s_1 = -1, s_2 = 1, s_3 = -1 \). In particular with the localised perturbation (5.22), we observe:
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- $\rho_1^{(0)} = 0.063246\%$ and $\bar{\rho}_1 = 333.6069\%$;
- $\rho_2^{(0)} = 0.14606\%$ and $\bar{\rho}_2 = 598.6718\%$;
- $\rho_3^{(0)} = 0.30792\%$ and $\bar{\rho}_3 = 1014.7219\%$.

By perturbing with random perturbation, we observe:

- $\rho_1^{(0)} = 0.082294\%$ and $\bar{\rho}_1 = 239.9949\%$;
- $\rho_2^{(0)} = 0.18711\%$ and $\bar{\rho}_2 = 407.9717\%$;
- $\rho_3^{(0)} = 0.52058\%$ and $\bar{\rho}_3 = 607.7822\%$.

5.2.3 1G 1SG 0B 1L 1TL

We choose the parameters $p_1 = -90.0$, $p_2 = 60.0$, $p_3 = -0.6$ and $p_4 = 1$ corresponding to the velocities $c_1 = 0.2$, $c_2 = 0.8$, to the amplitudes $a_1 = 19.3649$, $a_2 = 10.8253$ and to the choice of signs $s_1 = 1$, $s_2 = 1$, $s_3 = -1$. In this case we have explosive behaviour and, in particular, with the localised perturbation (5.22), we observe explosion at $t = 0.75$ seconds:

- $\rho_1^{(0)} = 0.005164\%$ and $\bar{\rho}_1 = 502961.1987\%$;
- $\rho_2^{(0)} = 0.0092376\%$ and $\bar{\rho}_2 = 449856.3477\%$;
- $\rho_3^{(0)} = 0.0015901\%$ and $\bar{\rho}_3 = 134122.7821\%$.

We do not report the plots for these simulations because they are all blue coloured. We run the simulations for the solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region 1G 1SG 0B 1L 1TL after localised perturbation and the simulations run over a time $0 \leq t \leq 0.7$ and a space $-30 \leq x \leq 30$. We observe the maximum for $|u_1|$ is 97416.7 on the background with intensity 18.6, the solution $|u_2|$ reaches its maximum of intensity at 48707.7 on the background with intensity 9.2 and the solution $|u_3|$ has its maximum of intensity at 84365.7 on the background of intensity 62.8.
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With the random perturbation $(5.23)$, we observe explosion at $t = 0.71375$ seconds:

- $\rho_1^{(0)} = 0.0072695\%$ and $\bar{\rho}_1 = 502961.1987\%$;
- $\rho_2^{(0)} = 0.011944\%$ and $\bar{\rho}_2 = 478.2925\%$;
- $\rho_3^{(0)} = 0.0022268\%$ and $\bar{\rho}_3 = 140.8017\%$.

We do not report the plots for these simulations because they are all blue colored. We have observed the evolutions of the solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $1G 1SG 0B 1L 1TL$ after random perturbation and the simulations run over a time $0 \leq t \leq 0.7$ and a space $-30 \leq x \leq 30$. The maximum reached by $|u_1|$ is 122.8 on the background with intensity 0.5, the solution $|u_2|$ reaches its maximum of intensity at 61.3 on the background with intensity 1.9 and the solution $|u_3|$ has its maximum of intensity at 122.3 on the background of intensity 60.7.

**Conjecture 5.2.1.** The presence of a twisted loop (or a split-gap) in a stability spectrum corresponds to a solution that is explosive in time.

**Conjecture 5.2.2.** The existence of branches in a stability spectrum is a necessary condition for the onset of rogue waves ascribable to rational or semi-rational solutions and which can be obtained by DDT.
Figure 5.1: Solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $2G$ 0SG 0B 1L 0TL after localised perturbation. The simulations run over a time $0 \leq t \leq 120$ and a space $-20 \leq x \leq 20$. 

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Figure 5.2: Zoom of the solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $2G\ 0SG\ 2B\ 1L\ 0TL$ after localised perturbation. The simulations run over a time $80 \leq t \leq 120$ and a space $5 \leq x \leq 15$. 
Figure 5.3: Solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $2G\ 0S\ 0B\ 1L\ 0TL$ after random perturbation. The simulations run over a time $0 \leq t \leq 120$ and a space $-20 \leq x \leq 20$. 

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Figure 5.4: Zoom of the solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $2G\ 0SG\ 2B\ 1L\ 0TL$ after random perturbation. The simulations run over a time $100 \leq t \leq 120$ and a space $0 \leq x \leq 20$. 
Figure 5.5: Solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $0G 0SG 2B 1L 0TL$ after localised perturbation. The simulations run over a time $0 \leq t \leq 60$ and a space $-30 \leq x \leq 30$. These localised structures resemble the breather solutions of the NLS equation \[121\]. We observe a complementarity in the pattern and in the colours between the three solutions displayed, this suggests a well defined exchange of energy between the three solutions that is interesting to be studied deeper in future works.
Figure 5.6: Zoom of the solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $0G0SG2B1L0TL$ after localised perturbation. The simulations run over a time $0 \leq t \leq 30$ and a space $0 \leq x \leq 15$. In the plots for $|u_1|$ and $|u_3|$ we have localised structures which are breathe-like solutions with a maximum of intensity of $4.2$ and $3.2$ (red colour) on a background with intensity $0$ (blue colour).
Figure 5.7: Solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $0G\ 0SG\ 2B\ 1L\ 0TL$ after random perturbation. The simulations run over a time $0 \leq t \leq 60$ and a space $-30 \leq x \leq 30$. We observe a complementarity in the pattern and in the colours between the three solutions displayed, this suggests a well defined exchange of energy between the three solutions that is interesting to be studied deeper in future works.
Figure 5.8: Zoom of the solutions $|u_1|$, $|u_2|$ and $|u_3|$ in the region $0G0SG2B1L0TL$ after random perturbation. The simulations run over a time $0 \leq t \leq 30$ and a space $0 \leq x \leq 15$. In the plots of $|u_1|$ and $|u_2|$ we have potential rogue waves with a maximum of intensity of 2.6 and 2.3 (red colour) on a background with 0 intensity (bue colour). These localised structures resemble the development of “integrable turbulence” studied for the focusing NLS equation in [39].
Chapter 6

Conclusions

6.1 Summary of the Results

The aim of this thesis has been the analysis of the spectral stability of plane wave solutions of the 3WRI model, when such solutions undergo localised perturbations. The approach is based on a spectral method recently developed in [64] to carry out the stability analysis of a nonlinear multi-component system, when the solutions have a non-vanishing background. The problem of assessing the stability of solutions of the 3WRI system had been already investigated in the literature. For instance, Kaup’s research works [57, 63, 58] had focussed on the stability analysis of the 3WRI model with soliton solutions on vanishing background, which can be obtained, in principle, by the IST method [31, 30, 29, 53]. It is well known that the IST machinery, in addition to being technically cumbersome to apply to multi-component systems, depends on the boundary conditions (e.g. by requiring the solution and its first derivative to be in the class of potentials vanishing sufficiently fast to infinity). On the contrary, the method in [64] is independent of the class of the potentials and tailor-made for the application to multi-component systems. We have obtained several results:

- By applying the method in [64] to the 3WRI model, we have provided for the first time, a comprehensive topological classification of the spatial stability spectra (as curves on the complex plane) with respect to the parameters space and the gain functions associated to any stability spectrum.
• Interestingly, we have found that all the stability spectra of the CNLS system are also enclosed in those of the 3WRI system. Indeed, the same method has already been successfully applied to the CNLS system [64]. The topological features of the CNLS stability spectra are gaps on the real axis, and branches and loops off the real axis: the gaps correspond to the solutions which are not bounded in space, whereas the branches and loops correspond to the solutions which are instead bounded in space, but which can be linearly unstable in time.

• Compared to the CNLS system, new topological features exist in the stability spectra of the 3WRI model, for instance, figure-of-eight loops that we have named twisted loops. Remarkably, the gain function associated to the twisted loops is non-zero in a whole neighbourhood of the origin (origin enclosed). This fact has been conjecturally associated to explosive instability: the solutions blow up in a finite time.

• We have observed that the gain function associated to the branches is non-zero at low wave numbers, symmetrically located with respect to the zero-value of the wave number, but it is anyway zero at the origin of the plot (linear instability of baseband-type). The gain function associated to the loops is non-zero only away from the origin (linear instability of passband-type).

• We have observed linear instability in time of plane waves for any choice of the physical parameters, except for those ones associated to the solutions that are explosive and we observe the subsequent generation of coherent localised structures, such as breather-like solutions and potential rogue waves. Some of these solutions have been observed numerically and, to the best of our knowledge, they have never been observed before in the context of the linear instability of the 3WRI system. Nevertheless, the 3WRI system is a dispersionless system with only coupling terms between the different wave components [58]. Thus, the observation of localised structures is remarkable, if one considers that MI has been observed in the context of nonlinear dispersive systems, where nonlinearity and dispersion can balance each other (see, for example, [37, 45, 71, 52, 17, 49, 50, 51, 9]).

A conclusion is that the mechanism for the onset of localised structures (e.g. potential rogue waves) in the 3WRI system, as a result of localised perturbations of plane waves, must be different. For this reason, in the context of the 3WRI system, we
have preferred to refer to this physical phenomenon as “linear instability” rather than as linear stage of MI.

6.2 Open Problems and Future Directions

There are several open problems related to the subject of this Thesis and to the results obtained so far. In the following, we provide some possible future directions for the research.

6.2.1 The Onset of Rogue Waves in the 3-Wave Resonant Interaction Model

We have conjectured that the existence of branches in the stability spectra is a necessary condition for the onset of rogue waves ascribable to rational or semi-rational solutions \[75, 76, 77\] and which can be obtained by DDT \[85\]. Indeed, the ends of a branch correspond to the vanishing of at least a difference between the eigenvalues of the spatial Lax operator after gauge transformation. Following \[75\], we see that if two eigenvalues are equal with one another, we have a necessary condition for the existence of semi-rational solitons whereas, if all the three eigenvalues are equal with one another, we have the necessary condition for the existence of rational solutions. Moreover, in the paper \[72\], it has been found that, for rational solutions in defocusing regime of the CNLS, potential rogue waves exist if and only if base-band MI exists. We refer to potential rogue waves as the rogue waves which can be modelled by rational and semi-rational solitons obtainable by DDT method.

In a future research work, we aim to write a necessary (and, possibly, also a sufficient) condition for the existence of potential rogue waves in terms of the parameters used to classify the topologies of the stability spectra, namely, we aim to understand for which values of the parameters the necessary condition is also sufficient.
6.2.2 Spectra of the Lax Operators and Stability Spectra

Following the paper [64], we have assumed that the perturbation $\delta Q$ has the integral representation
\[
\delta Q = \int d\kappa F(x, t, \kappa),
\]
which provides a solution $\delta Q$ bounded and localised in $x$ at any fixed time $t$. The boundedness condition of $F(x, t, \kappa)$ defines the spatial stability spectrum $S_x$ of the solution $Q(x, t)$.

As mentioned in [64], this spectrum depends on the behaviour of the matrix $Q(x)$ for large $|x|$. Indeed, if $Q(x)$ vanishes sufficiently fast as $|x| \to \infty$, then $S_x$ coincides with the spectrum of the operator $d/dx - i\lambda \Sigma - Q(x)$, that is defined by the spatial Lax equation. Instead, if $Q(x)$ is non-vanishing as $|x| \to \infty$, as for the case of plane waves, the spectrum $S_x$ of the solution $Q(x)$ in general may not coincide with the spectrum of the differential operator $d/dx - i\lambda \Sigma - Q(x)$, when $Q(x)$ is $N \times N$ with $N > 2$.

In [52, 71], it has been provided a spectral criterion for the occurrence of MI in the CNLS system. The authors establish a link between the eigenmode of the linearised problem with the eigenfunctions of the Lax problem which, in turn, can be used to construct the nonlocal dynamics of the system via Bäcklund transformations [122]. Nevertheless, they impose that the solutions satisfy boundary periodic conditions and the criterium for the existence of MI refers to the Floquet spectrum of the Lax operators. In the approach developed in [64] is independent of the boundary conditions for the solutions, so that, in the context of this new spectral method, one can establish a more general correspondence between the spectra of the Lax operators and the stability spectra.

6.2.3 Exchange of Energy in the Linear Instability of the 3-Wave Resonant Interaction Model

In the simulations of all the three solutions of the 3WRI system, we have always observed the presence of colours and pattern complementarity. We conjecture that this complementarity is due to the exchange of energy between the waves during their interaction such that, when one has the maximum values of the intensity, explained as the absorption of energy, the density of another wave is at its minimum, so the latter has given away its energy.
Similar investigation on nonlinear dispersive multicomponent systems has been conducted in [49 50 51], both via numerical simulations and perturbative methods.
Appendix A

Transformation Matrix $G(x, t)$ for the matrix NLS Equation

Let $G = G(x, t)$ be the matrix such that the transformation (2.15) is verified. Let us suppose that its expression is

$$G = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

(A.1)

whose entries are $g_1 = g_1(x, t)$, $g_2 = g_2(x, t)$, and its inverse matrix is

$$G^{-1} = \begin{pmatrix} g_1^{-1} & 0 \\ 0 & g_2^{-1} \end{pmatrix},$$

(A.2)

such that,

$$G \begin{pmatrix} 0 & sa \\ a & 0 \end{pmatrix} G^{-1} = \begin{pmatrix} 0 & sa g_1 g_2^{-1} \\ ag_1^{-1} g_2 & 0 \end{pmatrix}.$$  

(A.3)

Since (A.3) must be equal to $Q_0$ provided in (2.15), after the substitution of the relation dispersion (2.14) in the plane wave (2.13), we obtain

$$g_1 g_2^{-1} = e^{-i(qx-(q^2+2sa^2)t)}, \quad g_1^{-1} g_2 = e^{i(qx-(q^2+2sa^2)t)},$$

(A.4)

By (A.4) we find a relation between the entries $g_1$ and $g_2$

$$g_1 = g_2 e^{-i(qx-(q^2+2sa^2)t)}.$$  

(A.5)
Chapter A. Transformation Matrix $G(x, t)$ for the matrix NLS Equation

With the following ansatz on the expressions of such entries

\[ g_1 = e^{i m (qx - (q^2 + 2sa^2)t)}, \quad g_2 = e^{i n (qx - (q^2 + 2sa^2)t)}, \]  

the equation (A.5) reads

\[ e^{i m (qx - (q^2 + 2sa^2)t)} = e^{i (n - 1)(qx - (q^2 + 2sa^2)t)}, \]  

which gives us the condition

\[ m = n - 1, \]  

with $m$ and $n$ arbitrary rational numbers.

For the sake of simplicity, we choose $n = \frac{1}{2}$ and, finally, we get

\[ G = e^{-\frac{i}{2} (qx - (q^2 + 2sa^2)t)} \sigma_3, \]  

where $\sigma_3$ is the Pauli matrix (2.4).
Appendix B

Lax Equations of the NLS Equation

By the transformation (2.17) the right-hand sides of the Lax equations (3.41) become

\[ X\psi = XG\phi, \quad T\psi = TG\phi, \]  \hspace{1cm} (B.1)

and, on the other hand, the left-hand sides read

\[ \psi_x = G_x\phi + G\phi_x, \quad \psi_t = G_t\phi + G\phi_t, \]  \hspace{1cm} (B.2)

finally, matching (B.1) and (B.2)

\[ G_x\phi + G\phi_x = XG\phi, \]  \hspace{1cm} (B.3a)

\[ G_t\phi + G\phi_t = TG\phi. \]  \hspace{1cm} (B.3b)

Multiplying by \( G^{-1} \) from the left the equations (B.3), the PDEs for the solution \( \phi \) are

\[ \phi_x = iW\phi, \]  \hspace{1cm} (B.4a)

\[ \phi_t = -iZ\phi. \]  \hspace{1cm} (B.4b)

where we have defined the operators as follows

\[ iW = G^{-1}XG - G^{-1}G_x, \]  \hspace{1cm} (B.5a)

\[ -iZ = G^{-1}TG - G^{-1}G_t. \]  \hspace{1cm} (B.5b)
Appendix C

PDEs for the SE $\Phi(x,t)$ of the NLS Equation

Since $\psi$ is a solution of the Lax pair (2.2), the squared eigenfunction (SE) $\Psi$ defined in (2.24) satisfies the PDEs (2.25). The transformation (2.17) induces the similarity transformation which introduces another SE $\Phi$, solution of other PDEs that we shall find in the following discussion.

By looking at the expression of $G$ in (2.16), one can check that the left-hand sides of the PDEs (2.25) are

$$
\Psi_x = G_x \Phi G^{-1} + G \Phi_x G^{-1} + G \Phi G_x^{-1},
$$

(C.1a)

$$
\Psi_t = G_t \Phi G^{-1} + G \Phi_t G^{-1} + G \Phi G_t^{-1};
$$

(C.1b)

by matching the right-hand side and the left-hand side of the PDEs (2.25), it results

$$
XG \Phi G^{-1} - G \Phi G^{-1} X = G_x \Phi G^{-1} + G \Phi_x G^{-1} + G \Phi G_x^{-1},
$$

(C.2a)

$$
TG \Phi G^{-1} - G \Phi G^{-1} T = G_t \Phi G^{-1} + G \Phi_t G^{-1} + G \Phi G_t^{-1}.
$$

(C.2b)

Finally, by multiplying by $G^{-1}$ from the left and by $G$ from the right, and considering that $G_x^{-1}G = -G^{-1}G_x$, the equations (C.2) become the PDEs satisfied by $\Phi$

$$
\Phi_x = i[W, \Phi], \quad \Phi_t = -i[Z, \Phi].
$$

(C.3)
Appendix D

Similarity Transformation for the 3WRI System

The transformation

$$ U = G U_0 G^{-1}, $$

in matrix form reads

$$
\begin{pmatrix}
0 & s_1 a_1 e^{i(q t - \nu_1 x)} & -s_1 s_2 s_3 a_2 e^{i(q t - \nu_2 x)} \\
-s_1 s_2 s_3 a_1 e^{-i(q t - \nu_1 x)} & 0 & s_3 a_3 e^{-i(2 q t - (\nu_1 + \nu_2)) x} \\
s_2 a_2 e^{i(q t - \nu_2 x)} & s_1 s_2 s_3 a_3^* e^{i(2 q t - (\nu_1 + \nu_2)) x} & 0
\end{pmatrix}
\begin{pmatrix}
0 & s_1 a_1 g_2^{-1} & -s_1 s_2 s_3 a_2 g_3^{-1} \\
-s_1 s_2 s_3 a_1 g_2^{-1} & 0 & s_3 a_3 g_3^{-1} \\
s_2 a_2 g_3^{-1} & s_1 s_2 s_3 a_3^* g_3 g_2^{-1} & 0
\end{pmatrix},
$$

where $G$ is

$$
G = \begin{pmatrix} g_1 & 0 & 0 \\
0 & g_2 & 0 \\
0 & 0 & g_3 \end{pmatrix},
$$

$$
(D.1)
\quad
(D.2)
\quad
(D.3)$$
and the entries $g_1$, $g_2$ and $g_3$ are unknown.

The matrix equation (D.2) gives us a system of three equations

\[
\begin{align*}
    g_1 &= g_2 e^{i(qt - \nu_1 x)}, \\
    g_2 &= g_3 e^{-i(2qt - (\nu_1 + \nu_2))x}, \\
    g_3 &= g_1 e^{i(qt - \nu_2 x)}.
\end{align*}
\]  

Looking at (D.4), one assumes the general expression of the unknowns $g_j$

\[
\begin{align*}
    g_1 &= e^{i(m_1 qt - (l_1 \nu_1 + n_1 \nu_2)x)}, \\
    g_2 &= e^{i((m_2 - 1) qt - ((l - 1) \nu_1 + n_2 \nu_2)x)}, \\
    g_3 &= e^{i((m_3 + 1) qt - (l \nu_1 + (n + 1) \nu_2)x)},
\end{align*}
\]

with $m_j$, $n_j$ and $l_j$, $j = 1, 2, 3$, are positive or negative integers or can be zero. The expressions (D.5), substituted in the system (D.4), gives

\[
\begin{align*}
    m_1 &= m, \\
    m_2 &= m - 1, \\
    m_3 &= m + 1, \\
    l_1 &= l, \\
    l_2 &= l - 1, \\
    l_3 &= l, \\
    n_1 &= n, \\
    n_2 &= n, \\
    n_3 &= n + 1,
\end{align*}
\]

so that

\[
\begin{align*}
    g_1 &= e^{i(m qt - (\nu_1 + \nu_2)x)}, \\
    g_2 &= e^{i((m - 1) qt - (l - 1) \nu_1 + n_2 \nu_2)x)}, \\
    g_3 &= e^{i((m + 1) qt - (l \nu_1 + (n + 1) \nu_2)x)},
\end{align*}
\]

and for $m = 0$, $l = \frac{1}{2}$ and $n = -\frac{1}{2}$, we get

\[
\begin{align*}
    g_1 &= e^{-i(\frac{1}{2}(\nu_1 - \nu_2)x)}, \\
    g_2 &= e^{-i(\frac{1}{2}(\nu_1 + \nu_2)x)}, \\
    g_3 &= e^{i(qt - \frac{1}{2}(\nu_1 + \nu_2)x)}.
\end{align*}
\]

By substituting the expressions (D.8) in (D.3), the matrix $G$ (D.3) can rewritten as in formula (3.65).
Let
\[ \Psi = G\Phi G^{-1} \]  \hspace{1cm} (E.1)
be a SE defined via the solution \( \psi \) of the Lax Pair (3.41) and satisfying the differential equations
\[ \Psi_x = [X, \Psi], \quad \Psi_t = [T, \Psi]. \]  \hspace{1cm} (E.2)

By differentiating (E.1) with respect to \( x \) and \( t \), we obtain, respectively, \( ^1 \)
\[ \Psi_x = G_x\Phi G^{-1} + G\Phi_x G^{-1} + G\Phi(G^{-1})_x, \]  \hspace{1cm} (E.3a)
\[ \Psi_t = G_t\Phi G^{-1} + G\Phi_t G^{-1} + G\Phi(G^{-1})_t, \]  \hspace{1cm} (E.3b)

and, on the other hand, by substituting (E.1) in the equations (E.2)
\[ [X, \Psi] = XG\Phi G^{-1} - G\Phi G^{-1}X, \]  \hspace{1cm} (E.4a)
\[ [T, \Psi] = TG\Phi G^{-1} - G\Phi G^{-1}T. \]  \hspace{1cm} (E.4b)

\(^1\)We use the fact that \( G^{-1}G = I \) and \( (G^{-1}G)_x = 0 \) from which
\[ (G^{-1})_xG = -G^{-1}G_x. \]

The same argument holds for the differentiation w.r.t. \( t \).
By matching (E.3) with (E.4) and multiplying from the left by $G^{-1}$ and from the right by $G$, we obtain the Liouville equations

$$\Phi_x = i[W_0, \Phi], \quad \Phi_t = -i[Z_0, \Phi],$$

(E.5)

where the operators

$$iW_0 = G^{-1}XG - G^{-1}G_x, \quad -iZ_0 = G^{-1}TG - G^{-1}G_t,$$

(E.6)

are now independent of $x$ and $t$. The equations (E.2) are not simply integrable because of the dependence on $x$ and $t$ of the matrices $X$ and $T$. However, after the similarity transformation via the matrix $G$, the equations (E.6) are now simply integrable and the expression of their solution is well known and it is given in (3.69).
Appendix F

Differential equations for $\phi$

Let $\psi$ be the solution of the Lax Pair

$$\psi_x = X\psi, \quad \psi_t = T\psi. \tag{F.1}$$

If we make the transformation

$$\psi = G\phi, \tag{F.2}$$

by the differential equations (F.1), we obtain the identities

$$G_x\phi + G\phi_x = XG\phi, \tag{F.3a}$$

$$G_t\phi + G\phi_t = TG\phi, \tag{F.3b}$$

and multiplied to right by $G^{-1}$, give us the differential equations for $\phi$

$$\phi_x = (G^{-1}XG - G^{-1}G_x)\phi, \tag{F.4a}$$

$$\phi_t = (G^{-1}TG - G^{-1}G_t)\phi. \tag{F.4b}$$

If we define

$$iW_0 = G^{-1}XG - G^{-1}G_x, \quad -iZ_0 = G^{-1}TG - G^{-1}G_t \tag{F.5}$$
we have

\[ \phi_x = iW_0 \phi, \]  

\[ \phi_t = -iZ_0 \phi. \]
Appendix G

Gauges for $W$ and $Z$

We can check the expressions (3.75), by looking at the transformation

$$
\bar{\phi} = e^{i(mqt-(l\nu_1+n\nu_2)x)}\mathbf{I}\phi,
$$

that, differentiated, say, w.r.t. $x$ gives

$$
\bar{\phi}_x = (e^{i(mqt-(l\nu_1+n\nu_2)x)}\mathbf{I}\phi)_x = (-i(l\nu_1 + n\nu_2)\bar{\phi} - \phi_x)e^{i(mqt-(l\nu_1+n\nu_2)x)}\mathbf{I},
$$

on the other hand, because $\bar{\phi}_x = i\bar{W}\bar{\phi}$,

$$
i\bar{W}\bar{\phi} = (G^{-1}\bar{X}\bar{G} - G^{-1}\bar{G}_x)\bar{\phi} =
= (G^{-1}XG - G^{-1}G_x - i(l\nu_1 + n\nu_2))\bar{\phi} =
= i(W_0 - (l\nu_1 + n\nu_2))\bar{\phi} =
= i(W_0 - (l\nu_1 + n\nu_2))e^{i(mqt-(l\nu_1+n\nu_2)x)}\mathbf{I}\phi.
$$

By matching (G.2) with (G.3), we obtain the equation

$$
\phi_x = iW_0\phi,
$$

with

$$
\bar{W} = W_0 - (l\nu_1 + n\nu_2).
$$

The same argument holds for the operators $\bar{Z}$ and $Z_0$. 
Appendix H

Relation between the Lax Operators $W$ and $Z$

Let us consider the matrix

$$
\zeta = \left( \frac{c_1 - c_2}{2qc_1c_2} \right) \left( Z^2 - (a_1^2a_2^2s_1s_3 - a_1^2c_1^2s_2s_3) \mathbf{I} \right) =
$$

$$
= \begin{pmatrix}
0 & -ia_1s_1c_1(c_1 - c_2)(-q - c_1c_2\kappa) \\
-ia_2s_2s_3c_1(c_1 - c_2)(-q - c_1c_2\kappa) & - ia_1s_1c_1(c_1 - c_2)(-q - c_1c_2\kappa) \\
-ia_2s_2s_3c_1(c_1 - c_2)(-q - c_1c_2\kappa) & 2q \left( \frac{a_1a_2(c_1 - c_2)}{2q} \right)
\end{pmatrix}.
$$

\hspace{1cm} (H.1)

Let us focus on the off-diagonal-part of the matrix above. The off-block diagonal terms $\zeta_{ij}$, which are proportional to the terms $\zeta_{ji}$, can be handled as follows. For example, the numerator of the entry $\zeta_{12}$ is

$$
-ia_1c_1s_1(c_1 - c_2)(-q - c_1c_2\kappa) = -ia_1c_1^2s_1(-q - c_1c_2\kappa) + ia_1c_1c_2s_1(-q - c_1c_2\kappa) \hspace{1cm} (H.2)
$$

whose last term is, by adding and subtracting $q$ into the brackets,

$$
ia_1c_1c_2s_1(-q - c_1c_2\kappa) = -2ia_1c_1c_2s_1 + ia_1c_1c_2s_1(q - c_1c_2\kappa), \hspace{1cm} (H.3)
$$

by substituting (H.3) in (H.2), one gets

$$
-ia_1c_1^2s_1(-q - c_1c_2\kappa) - 2ia_1s_1c_1c_2q + ia_1s_1c_1c_2(q - c_1c_2\kappa) =
$$

$$
= W_{12} + Z_{12}c_1(-q - c_1c_2\kappa) - Z_{12}c_2(q - c_1c_2\kappa), \hspace{1cm} (H.4)
$$
that is
\[
\zeta_{12} = \frac{W_{12}}{2qc_1c_2} + Z_{12} \frac{c_1(-q - c_1c_2\kappa)}{2qc_1c_2} - Z_{12} \frac{c_2(q - c_1c_2\kappa)}{2qc_1c_2}.
\]  \hspace{1cm} (H.5)

Similarly, let us consider the term \( \zeta_{22} \). The first term in the sum is already in \( W_{22} \), while the numerator of the second term can be handled as follows

\[
(c_1 - c_2)(-q - c_1c_2\kappa)^2 = c_1(-q - c_1c_2\kappa)^2 - c_2(-q - c_1c_2\kappa)^2 =
\]
\[
= c_1(-q - c_1c_2\kappa)^2 - c_2(-q - c_1c_2\kappa)(-q - c_1c_2\kappa) =
\]
\[
= c_1(-q - c_1c_2\kappa)^2 - c_2(-q - c_1c_2\kappa)(-q + q - c_1c_2\kappa) =
\]
\[
= c_1(-q - c_1c_2\kappa)^2 + 2qc_2(-q - c_1c_2\kappa) - c_2(-q - c_1c_2\kappa)(q - c_1c_2\kappa) =
\]
\[
= W_{22} + Z_{22}c_1(-q - c_1c_2\kappa) - Z_{22}c_2(q - c_1c_2\kappa),
\]  \hspace{1cm} (H.6)

hence,
\[
\zeta_{22} = \frac{W_{22}}{2qc_1c_2} + Z_{22} \frac{c_1(-q - c_1c_2\kappa)}{2qc_1c_2} - Z_{22} \frac{c_2(q - c_1c_2\kappa)}{2qc_1c_2}.
\]  \hspace{1cm} (H.7)

By repeating the same calculations for the other terms of the matrix (H.1), we get the matrix polynomial
\[
W = Z^2(c_1 - c_2) - Zc_1(-q - c_1c_2\kappa) + Zc_2(q - c_1c_2\kappa) - (c_1 - c_2)(a_2^2s_1s_3 - a_1^2c_1^2s_2s_3)I,
\]  \hspace{1cm} (H.8)
i.e. the polynomial (3.83).

Let us consider the matrix
\[
\xi = W^2 - (a_2^2s_1s_3 + a_1^2s_2s_3)I,
\]  \hspace{1cm} (H.9)
whose entries are
\[
\xi_{11} = 0,
\]  \hspace{1cm} (H.10)
\[
\xi_{22} = 4q^2c_1^2c_2^2 \left( -a_2^2s_1s_3 - a_1^2a_2s_1s_2 \left( \frac{c_1 - c_2}{2q} \right)^2 + \left( \frac{-a_2^2s_1s_3c_2(c_1 - c_2)}{2qc_1} - \frac{q}{c_1} - c_2\kappa \right)^2 \right),
\]  \hspace{1cm} (H.11)
\[
\xi_{33} = 4q^2c_1^2c_2^2 \left( a_1^2s_2s_3 - a_1^2a_2s_1s_2 \left( \frac{c_1 - c_2}{2q} \right)^2 + \left( \frac{a_1^2s_2s_3c_1(c_1 - c_2)}{2qc_2} + \frac{q}{c_2} - c_1\kappa \right)^2 \right),
\]
\[
\xi_{11} = s_2 s_3 \xi_{12} = 4q^2 c_1^2 c_2^2 \left( -ia_1 s_1 \left( \frac{a_2^2 s_1 s_3 (c_1 - c_2)^2}{2qc_1} - \frac{q}{c_1} - c_2 \kappa \right) \right), \tag{H.13}
\]

\[
\xi_{31} = -s_1 s_3 \xi_{13} = 4q^2 c_1^2 c_2^2 \left( -ia_2 s_2 \left( \frac{a_2^2 s_2 s_3 (c_1 - c_2)^2}{2qc_2} + \frac{q}{c_1} - c_1 \kappa \right) \right), \tag{H.14}
\]

\[
\xi_{23} = -s_1 s_2 \xi_{32} =
\[
= 4q^2 c_1^2 c_2^2 \left( a_1 a_2 + \frac{c_1 - c_2}{2q} \right) \left( - \frac{a_2^2 s_1 s_3 c_2 (c_1 - c_2)}{2qc_1} + \frac{a_1 a_2 + c_1 - c_2}{2q} \right) + \frac{a_2^2 s_2 s_3 c_3 (c_1 - c_2)}{2qc_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_2} - c_1 \kappa). \tag{H.15}
\]

Let us handle the entry \(\xi_{12}\), proportional to the entry \(\xi_{21}\). The part inside the brackets becomes

\[
\frac{a_2^2 s_1 s_3 (c_1 - c_2)^2}{2qc_1} - \frac{a_2^2 s_2 s_3 (c_1 - c_2)^2}{2qc_2} + \frac{a_1 a_2 + c_1 - c_2}{2q} \left( - \frac{a_2^2 s_1 s_3 c_2 (c_1 - c_2)}{2qc_1} + \frac{a_1 a_2 + c_1 - c_2}{2q} \right) = \tag{H.16}
\]

on the other hand,

\[
- \frac{q}{c_1} - c_2 \kappa = \frac{2q}{c_1 - c_2} - \frac{2q}{c_1 - c_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_2} - c_1 \kappa - \frac{q}{c_2} + c_1 \kappa = \tag{H.17}
\]

\[
\frac{2q}{c_1 - c_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_2} - \frac{2qc_1 c_2}{c_1 c_2 (c_1 - c_2)} + c_1 \kappa = \frac{2q}{c_1 - c_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_2} - c_1 \left( \frac{c_2 + c_1}{c_1 c_2 (c_1 - c_2)} - \kappa \right) .
\]

By summing the terms (H.16) and (H.17), and by putting altogether in the expression of the entry \(\xi_{12}\), we have

\[
\xi_{12} = 2qc_1 c_2 W_{12} \left( \frac{a_2^2 s_1 s_3 (c_1 - c_2)^2}{2qc_1} + \frac{a_2^2 s_2 s_3 (c_1 - c_2)^2}{2qc_2} - \frac{a_2^2 s_1 s_3 c_2 (c_1 - c_2)}{2qc_1} + \frac{a_1 a_2 + c_1 - c_2}{2q} \right) + \frac{a_2^2 s_2 s_3 c_3 (c_1 - c_2)}{2qc_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_2} - c_1 \kappa) + \tag{H.18}
\]

\[
+ 4q^2 c_1^2 c_2^2 Z_{12} \left( \frac{a_2^2 s_1 s_3 (c_1 - c_2)^2}{2qc_1} - \frac{a_2^2 s_2 s_3 (c_1 - c_2)^2}{2qc_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_1 c_2 (c_1 - c_2) + \kappa} .
\]
Appendix H. Relation between the Lax Operators $W$ and $Z$

By following a similar procedure, we get

$$\xi_{23} = 2qc_1c_2 W_{23}. \tag{H.19}$$

In matrix form, it turns out that

$$W^2 - (a_2^2 s_1 s_3 + a_1^2 s_2 s_3)I =$$

$$= 2qc_1c_2 W \left( \frac{a_2^2 s_1 s_3 (c_1 - c_2)^2}{2qc_1} + \frac{a_1^2 s_2 s_3 (c_1 - c_2)^2}{2qc_2} - \frac{a_2^2 s_1 s_3 c_1 (c_1 - c_2)}{2qc_1} + \frac{a_1^2 s_2 s_3 c_1 (c_1 - c_2)}{2qc_2} \right) +$$

$$+ \frac{a_1^2 s_2 s_3 c_2 (c_1 - c_2)}{2qc_2} + \frac{2q}{c_1 - c_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_2} - c_1 \kappa \right) +$$

$$+ 4Zq^2 c_1^2 c_2^2 \left( \frac{a_2^2 s_1 s_3 (c_1 - c_2)}{2qc_1} - \frac{a_1^2 s_2 s_3 (c_1 - c_2)}{2qc_2} - \frac{q}{c_1 c_2 (c_1 - c_2)} + \kappa \right), \tag{H.20}$$

and finally,

$$Z \left( \frac{a_2^2 s_1 s_3 (c_1 - c_2)}{2qc_1} - \frac{a_1^2 s_2 s_3 (c_1 - c_2)}{2qc_2} - \frac{q}{c_1 c_2 (c_1 - c_2)} + \kappa \right) =$$

$$= \frac{W^2}{4q^2 c_1^2 c_2^2} - \frac{W}{2qc_1 c_2} \left( \frac{a_2^2 s_1 s_3 (c_1 - c_2)^2}{2qc_1} + \frac{a_1^2 s_2 s_3 (c_1 - c_2)^2}{2qc_2} \right) +$$

$$- \frac{a_2^2 s_1 s_3 c_1 (c_1 - c_2)}{2qc_1} + \frac{a_1^2 s_2 s_3 c_2 (c_1 - c_2)}{2qc_2} +$$

$$+ \frac{2q}{c_1 - c_2} - \frac{q}{c_1} - c_2 \kappa + \frac{q}{c_2} - c_1 \kappa \right) - (a_2^2 s_1 s_3 + a_1^2 s_2 s_3)I. \tag{H.21}$$
Appendix I

Relation between the differences of the Eigenvalues of the Lax Operators $W$ and $Z$

From the polynomial (3.83) we can write

$$w_j = (c_1 - c_2)z_j^2 - c_1(-q - c_1c_2\kappa)z_j + c_2(q - c_1c_2\kappa)z_j - (c_1 - c_2)(a_2^2c_2^2s_1s_3 - a_1^2c_1^2s_2s_3)$$  \hspace{1cm} (I.1)

$\forall j = 1, 2, 3$, and the difference between two eigenvalues is, for example,

$$w_1 - w_2 = (c_1 - c_2)(z_1^2 - z_2^2) - c_1(-q - c_1c_2\kappa)(z_1 - z_2) + c_2(q - c_1c_2\kappa)(z_1 - z_2).$$  \hspace{1cm} (I.2)

Furthermore, because $\text{Tr}(Z) = -2c_1c_2\kappa$ and $z_1^2 - z_2^2 = (z_1 - z_2)(z_1 + z_2)$, we substitute in the previous expression

$$z_1 + z_2 = -z_3 - 2c_1c_2\kappa,$$  \hspace{1cm} (I.3)

and, as a consequence,

$$w_1 - w_2 = (c_1 - c_2)(z_1 - z_2)(-z_3 - 2c_1c_2\kappa) - c_1(-q - c_1c_2\kappa)(z_1 - z_2) + c_2(q - c_1c_2\kappa)(z_1 - z_2).$$  \hspace{1cm} (I.4)

The difference of the eigenvalues (I.4) can be also written in a different fashion after adding and subtracting the terms $qc_1$ and $qc_2$. Indeed, it reads

$$w_1 - w_2 = (z_1 - z_2)(-z_3(c_1 - c_2) + q(c_1 + c_2) - c_1c_2\kappa(c_1 - c_2)).$$  \hspace{1cm} (I.5)
To compute the differences $z_j - z_{j+1}$, we repeat all the calculations above, but we need to make the substitution $w_1 + w_2 = -w_3 - \text{Tr}(W)$, where

$$\text{Tr}(W) = 2qc_1c_2 \left( \frac{a_1^2s_2s_3c_1(c_1 - c_2)}{2qc_2} - \frac{a_2^2s_1s_3c_2(c_1 - c_2)}{2qc_1} - \frac{q}{c_1}c_2\kappa + \frac{q}{c_2}c_1\kappa \right).$$

(I.6)
Appendix J

Characteristic Polynomial and the Associated Polynomial of the Squares of the Differences

Let \( \tilde{W} \) be the diagonalised matrix of the matrix \( W \). The eigenvalues \( w_\ell \), for \( \ell = 1, 2, 3 \), are the roots of the characteristic polynomial

\[
P(w) = w^3 - \text{Tr}(\tilde{W})w^2 + \frac{1}{2} \left( \text{Tr}^2(\tilde{W}) - \text{Tr}(\tilde{W}^2) \right) w - \text{Det}(\tilde{W}).
\]  

(J.1)

We denote the trace and the determinant of matrix with \( \text{Tr}(\cdot) \) and \( \text{Det}(\cdot) \), respectively. In (J.1) we have replaced \( \text{Tr}(W) \) with \( \text{Tr}(\tilde{W}) \) by using the property of the trace to be invariant under cyclic permutations and \( \text{Det}(W) \) with \( \text{Det}(\tilde{W}) \) because of the Binet theorem in order to simplify the computations.

We take advantage from the property of trace and determinant to be invariant under similarity transformation, and also from the property of the coefficients of the polynomial (J.1) to be invariants. In this respect, let us introduce the Vandermonde matrix

\[
\Upsilon = \begin{pmatrix}
1 & w_1 & w_1^2 \\
1 & w_2 & w_2^2 \\
1 & w_3 & w_3^2
\end{pmatrix},
\]  

(J.2)
Appendix J. Characteristic Polynomial and the Associated Polynomial of the Squares of the Differences

whose determinant is

\[ \text{Det}(\Upsilon) = w_1^2(w_3-w_2) + w_2^2(w_1-w_3) + w_3^2(w_2-w_1) = (w_1-w_2)(w_2-w_3)(w_3-w_1), \quad (J.3) \]

and we note that \( (\text{Det}(\Upsilon))^2 = \Delta_W P(w) \), where \( \Delta_W P(w) \) is the discriminant of the polynomial (J.1) with respect to \( w \). In some computation, we benefit from the Cayley-Hamilton Theorem to write the determinant as

\[ \text{Det}(\tilde{W}) = \frac{1}{6} \left( \text{Tr}^3(\tilde{W}) + 2\text{Tr}(\tilde{W}^3) - 3\text{Tr}(\tilde{W})\text{Tr}(\tilde{W}^2) \right). \quad (J.4) \]

Below, we give some useful formulas

\[ 2(w_1w_2 + w_2w_3 + w_3w_1) = \text{Tr}^2(\tilde{W}) - \text{Tr}(\tilde{W}^2), \quad (J.5) \]

and

\[ (w_1^2w_2^2 + w_2^2w_3^2 + w_3^2w_1^2 + 2w_1^2w_2w_3 + 2w_1w_2^2w_3 + 2w_1w_2w_3^2) = \]

\[ = (w_2^2w_3^2 + w_3^2w_1^2 + w_1^2w_2^2 + 2w_1w_2w_3^3) = \]

\[ = \frac{1}{4} \left( \text{Tr}^2(\tilde{W}) - \text{Tr}(\tilde{W}^2) \right)^2, \quad (J.6) \]

or

\[ (w_1^2w_2^2 + w_2^2w_3^2 + w_3^2w_1^2) = \]

\[ = \frac{1}{4} \left( \text{Tr}^2(\tilde{W}) - \text{Tr}(\tilde{W}^2) \right)^2 - \frac{1}{3} \text{Tr}(\tilde{W}) \left( \text{Tr}^3(\tilde{W}) + 2\text{Tr}(\tilde{W}^3) - 3\text{Tr}(\tilde{W})\text{Tr}(\tilde{W}^2) \right), \quad (J.7) \]

where in the last equality we use the formula (J.4).

In the next subsections we shall show the connection between the characteristic polynomial (J.1) and the polynomial of the differences \( w_\ell - w_m \) and the polynomial of the squares of the differences, that is \( (w_\ell - w_m)^2 \).

### J.0.4 Polynomial of the Differences

Let

\[ P(k) = k^3 - \text{Tr}(\tilde{W}_1)k^2 + \frac{1}{2} \left( \text{Tr}^2(\tilde{W}_1) - \text{Tr}(\tilde{W}_1^2) \right) k - \text{Det}(\tilde{W}_1), \quad (J.8) \]

be the characteristic polynomial whose roots are the differences \( w_\ell - w_m = k_{\ell m} \), with \( \ell \neq m \), and \( \ell, m = 1, 2, 3 \), so the matrix \( \tilde{W}_1 \) is

\[ \tilde{W}_1 = \begin{pmatrix} k_{12} & 0 & 0 \\ 0 & k_{23} & 0 \\ 0 & 0 & k_{31} \end{pmatrix}. \quad (J.9) \]
Appendix J. Characteristic Polynomial and the Associated Polynomial of the Squares of the Differences

**Proposition J.0.1.** Every coefficient of the polynomial (J.8) can be expressed by the coefficients of the polynomial (J.1) plus the determinant of the Vandermonde matrix \( \Upsilon \). In particular, the polynomial (J.8) can be written as

\[
P\tilde{W}_1 = k^3 - \frac{1}{2} \left( 3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W}) \right) k - \text{Det}(\Upsilon). \tag{J.10}
\]

**Proof.** By looking at the matrix \( \tilde{W}_1 \), we have

\[
\text{Tr}(\tilde{W}_1) = 0, \tag{J.11}
\]

and, as a consequence,

\[
\text{Tr}^n(\tilde{W}_1) = 0, \quad n = 1, 2, 3, \ldots. \tag{J.12}
\]

On the other hand,

\[
\text{Tr}(\tilde{W}_1^2) = k_{12}^2 + k_{23}^2 + k_{31}^2 = 2\text{Tr}(\tilde{W}^2) - 2(w_1w_2 + w_2w_3 + w_3w_1), \tag{J.13}
\]

which becomes, by the relation (J.5),

\[
\text{Tr}(\tilde{W}_1^2) = 3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W}). \tag{J.14}
\]

It is trivial to see that

\[
\text{Det}(\tilde{W}_1) = \text{Det}(\Upsilon). \tag{J.15}
\]

\[\square\]

**Proposition J.0.2.** The eigenvalues \( w_\ell \) are the roots of the characteristic polynomial (J.1) if and only if the differences \( k_{\ell m} \) are roots of the characteristic polynomial (J.10).

**Proof.** First of all we prove that if \( w_\ell \) are roots of the polynomial (J.1), then the differences \( k_{\ell m} \) are roots of the polynomial (J.10).

Let us substitute \( k = w_\ell - w_m \) into the polynomial (J.10), we have

\[
(w_\ell - w_m)^3 - \frac{1}{2} \left( 3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W}) \right) (w_\ell - w_m) - \text{Det}(\Upsilon) =
\]

\[
= w_\ell^3 - w_m^3 - 3w_\ell^2w_m + 3w_\ell w_m^2 + \frac{1}{2} \left( \text{Tr}^2(\tilde{W}) - \text{Tr}(\tilde{W}^2) \right) w_\ell -
\]

\[
- \frac{1}{2} \left( \text{Tr}^2(\tilde{W}) - \text{Tr}(\tilde{W}^2) \right) w_m - \text{Tr}(\tilde{W}^2)w_\ell + \text{Tr}(\tilde{W}^2)w_m - \text{Det}(\Upsilon), \tag{J.16}
\]
we can identify the first and the fifth term in the polynomial satisfied by \( w_\ell \), while the second and the sixth term in the polynomial satisfied by \( w_m \). Leaving aside for a moment \( \text{Det}(\Upsilon) \), the remaining terms are handled to give

\[
-3w_\ell^2w_m + 3w_\ell w_m^2 - \text{Tr}(\tilde{W}^2)w_\ell + \text{Tr}(\tilde{W}^2)w_m =
\]

\[
= -w_\ell^2\text{Tr}(\tilde{W}) + w_m^2\text{Tr}(\tilde{W}) + w_\ell^2(w_k - w_m) + w_m^2(w_\ell - w_k) + w_\ell^2(w_m - w_\ell) = (J.17)
\]

\[
= -w_\ell^2\text{Tr}(\tilde{W}) + w_m^2\text{Tr}(\tilde{W}) + \text{Det}(\Upsilon).
\]

The first two terms to the left-hand side of the equation above correspond to the quadratic terms of the polynomial for \( w_\ell \) and \( w_m \), respectively. The term \( \text{Det}(\Upsilon) \) cancels out once substituted into the polynomial \( (J.10) \). However, we can identify the difference of the known terms of the polynomials for \( w_\ell \) and \( w_m \) within the trivial difference \( \text{Det}(\Upsilon) - \text{Det}(\Upsilon) = \text{Det}(\tilde{W}) - \text{Det}(\tilde{W}) \). To show that if the difference \( w_\ell - w_m \) are roots of the polynomial \( (J.10) \), then \( w_\ell \) are roots of the polynomial \( (J.1) \), we need to subtract the polynomial calculated in \( w_m \) from the polynomial calculated in \( w_\ell \)

\[
P(w_\ell) - P(w_m) = w_\ell^3 - w_m^3 - \text{Tr}(\tilde{W})w_\ell^2 + \text{Tr}(\tilde{W})w_m^2 + \frac{1}{2} \left( -\text{Tr}(\tilde{W})^2 \right) (w_\ell - w_m).
\]

By using the formula \( (J.17) \) we replace the terms \( -\text{Tr}(\tilde{W})w_\ell^2 + \text{Tr}(\tilde{W})w_m^2 \) and write down the polynomial \( (J.10) \).

\[\square\]

Note that both \( w_\ell - w_m = k_{\ell m} \) and \( w_\ell - w_m = k_{m\ell} \) are roots of the polynomial \( (J.10) \).

### J.0.5 Polynomial of the Squares of the Differences

Let

\[
PW_2 = \xi^3 - \text{Tr}(\tilde{W}_2)\xi^2 + \frac{1}{2} \left( \text{Tr}^2(\tilde{W}_2) - \text{Tr}(\tilde{W}_2^2) \right) \xi - \text{Det}(\tilde{W}_2).
\]

be the characteristic polynomial whose roots are the differences \( (w_\ell - w_m)^2 = k_{\ell m}^2 = \xi_{\ell m} \), and the matrix \( \tilde{W}_2 \) is

\[
\tilde{W}_2 = \begin{pmatrix}
\xi_{12} & 0 & 0 \\
0 & \xi_{23} & 0 \\
0 & 0 & \xi_{31}
\end{pmatrix}.
\]

\[\text{(J.21)}\]
Appendix J. Characteristic Polynomial and the Associated Polynomial of the Squares of the Differences

Proposition J.0.3. Every coefficient of the polynomial \(J.20\) can be expressed by the coefficients of the polynomial \(J.1\) plus the square of the determinant of the Vandermonde matrix \(Υ\). In particular, the polynomial \(J.20\) can be written as
\[
P(\tilde{W}) = ξ^3 - \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right) ξ^2 + \frac{1}{4} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(W)\right)^2 ξ^1 - (\text{Det}(Υ))^2. \tag{J.22}
\]

Proof. By definition of trace and by using the formula \(J.5\), it results in
\[
\text{Tr}(\tilde{W}^2) = 3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W}), \tag{J.23}
\]
that is the coefficient of the second power of \(ξ\) in \(J.22\). In general,
\[
\text{Tr}^n(\tilde{W}^2) = (3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W}))^n, \quad n = 1, 2, 3,... \tag{J.24}
\]
On the other hand, by using the formula \(J.6\), we have
\[
\text{Tr}(\tilde{W}^4) = 6\text{Tr}(W^4) - 4\text{Tr}(W)\text{Tr}(W^3) + \frac{3}{2} \left(\text{Tr}^2(W) - \text{Tr}(W^2)\right)^2 - 2\text{Tr}(W) \left(\text{Tr}^3(W) + 2\text{Tr}(W^3) - 3\text{Tr}(W)\text{Tr}(W^2)\right), \tag{J.25}
\]
and since
\[
6\text{Tr}(W^4) = -2\text{Tr}^4(W) + 3 \left(\text{Tr}^2(W) - \text{Tr}(W^2)\right)^2 + 8\text{Tr}(W)\text{Tr}(W^3), \tag{J.26}
\]
we have
\[
\text{Tr}(\tilde{W}^2) = \frac{1}{2} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right)^2, \tag{J.27}
\]
so that the coefficient of the first power of \(ξ\) is
\[
\frac{1}{2} \left(\text{Tr}^2(\tilde{W}^2) - \text{Tr}(\tilde{W}^2)^2\right) = \frac{1}{4} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(W)\right)^2. \tag{J.28}
\]
Finally, the constant term is
\[
(\text{Det}(Υ))^2 = (w_1 - w_2)^2(w_2 - w_3)^2(w_3 - w_1)^2, \tag{J.29}
\]
i.e. \(\text{Det}(\tilde{W})\).

Note that \((\text{Det}(Υ))^2 = Δ_W P(w)\) and when two eigenvalues \(w_ℓ\) coincide this term is zero.

Before we move on, we focus on the characteristic polynomial of the matrix \(\tilde{W}^2 = Y\)
\[
Y = \begin{pmatrix}
y_1 & 0 & 0 \\
0 & y_2 & 0 \\
0 & 0 & y_3
\end{pmatrix}, \quad y_ℓ = w_ℓ^2 \tag{J.30}
\]
Appendix J. Characteristic Polynomial and the Associated Polynomial of the Squares of the Differences

that is

\[ P(y) = y^3 - \text{Tr}(Y)y^2 + \frac{1}{2}(\text{Tr}^2(Y) - \text{Tr}(Y^2))y - \text{Det}(Y). \]  

(J.31)

The polynomial (J.31) can be written in terms of the eigenvalues \( w_\ell \) as

\[ P(w^2) = w^6 - \text{Tr}(\tilde{W}^2)w^4 + \frac{1}{2}(\text{Tr}^2(\tilde{W}^2) - \text{Tr}(\tilde{W}^4))w^2 - \text{Det}(\tilde{W}^2), \]  

(J.32)

and, from the Fundamental Theorem of Algebra, we expect six roots of the polynomial (J.32). However, the polynomial (J.31) is a third degree polynomial for the variable \( y \), and this means that, for \( \ell = 1, 2, 3 \), we have three roots \( w^2 = y_\ell \), but actually they are corresponding to six roots \( w_\ell = \pm \sqrt{y_\ell} \) of the polynomial (J.32). As a direct result, at first sight, it looks like that between these six roots, everyone satisfying the polynomials (J.31) and (J.32), only three roots are solutions of the polynomial (J.1). In particular, we are interested only on those ones that satisfy the conditions

\[ w_1 + w_2 + w_3 = \text{Tr}(\tilde{W}) \quad \text{and} \quad w_1w_2w_3 = \text{Det}(\tilde{W}), \]

also if all the six solutions of (J.32) satisfy the conditions \( w_1^2 + w_2^2 + w_3^2 = \text{Tr}(\tilde{W}^2) \) and \( w_1^2w_2^2w_3^2 = \text{Det}(\tilde{W}^2) \).

Thus, the roots are of the polynomial (J.1) are the six roots: \( \pm(w_1 - w_2) \), \( \pm(w_2 - w_3) \), \( \pm(w_3 - w_1) \). One can ask: which of these differences correspond to those one for the roots of the polynomial (J.1)?

Let us consider the following diagonal matrices

\[
W_1 = \begin{pmatrix}
  w_1 & 0 & 0 \\
  0 & w_2 & 0 \\
  0 & 0 & w_3
\end{pmatrix}, \quad W_2 = \begin{pmatrix}
  w_2 & 0 & 0 \\
  0 & w_1 & 0 \\
  0 & 0 & w_3
\end{pmatrix}, \quad (J.33)
\]

it is simple to check that the two matrices above have the same characteristic polynomial and they are connected by a similarity transformation. Indeed, let \( W \) be the non-diagonalised matrix, we have

\[ W = U_1^{-1}W_1U_1, \quad W = U_2^{-1}W_2U_2, \]  

(J.34)

hence

\[ W_1 = U_3^{-1}W_2U_3, \]  

(J.35)
Appendix J. Characteristic Polynomial and the Associated Polynomial of the Squares of the Differences

where $U_3 = U_2 U_1^{-1}$. If the matrix of the squares of the differences associated to $W_1$ is

$$W_{12} = \begin{pmatrix}(w_1 - w_2)^2 & 0 & 0 \\
0 & (w_2 - w_3)^2 & 0 \\
0 & 0 & (w_3 - w_1)^2\end{pmatrix}, \quad (J.36)$$

then, by following the same algorithm for the construction of the matrix above, we get the matrix associated to the matrix $W_2$, that is

$$W_{22} = \begin{pmatrix}(w_2 - w_1)^2 & 0 & 0 \\
0 & (w_1 - w_3)^2 & 0 \\
0 & 0 & (w_3 - w_2)^2\end{pmatrix}. \quad (J.37)$$

The characteristic polynomial for the matrix $W_{12}$ has the roots: $\pm(w_1 - w_2), \pm(w_2 - w_3), \pm(w_3 - w_1)$. The roots of the characteristic polynomial of the matrix $W_{22}$ are: $\pm(w_2 - w_1), \pm(w_1 - w_3), \pm(w_3 - w_2)$. Let us suppose, that we are working with the matrix $W_{12}$ and the right triplet is the differences $+(w_1 - w_2), +(w_2 - w_3), +(w_3 - w_1)$ that coincide with the same differences of the matrix $W_{22}$ but with the reversed sign. This means that choosing the other combinations of signs in front of the differences corresponds to choose another eigenspace connected to the first eigenspace by a similarity transformation. This means that the polynomial (J.22) encloses all the possible differences associated to any possible eigenspace.

**Proposition J.0.4.** The differences $w_\ell - w_m = k_{\ell m}$ are roots of the characteristic polynomial (J.10) if and only if $(w_\ell - w_m)^2 = \xi_{\ell m}$ are roots of the characteristic polynomial (J.22).

**Proof.** Let us compute the square of the polynomial $P\tilde{W}_1$ (J.10)

$$(P\tilde{W}_1)^2 = \left(k^3 - \frac{1}{2} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right)\right) k - \text{Det}(\Upsilon)^2 = $$

$$= k^6 + \frac{1}{4} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right)^2 k^2 +$$

$$+ (\text{Det}(\Upsilon))^2 - k^4 \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right) - 2\text{Det}(\Upsilon)k^3 + \text{Det}(\Upsilon) \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right) k, \quad (J.38)$$
that can be also written as

\[(P\tilde{W}_1)^2 = k^6 + \frac{1}{4} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right)^2 k^2 - k^4 \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right) - \text{Det}(\Upsilon) \left(k^3 - \frac{1}{2} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right) k\right) - \text{Det}(\Upsilon) \left(k^3 - \frac{1}{2} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right) k - \text{Det}(\Upsilon)\right), \tag{J.39}\]

and we note that the term within the parenthesis multiplied by $\text{Det}(\Upsilon)$ in the last line is $P\tilde{W}_1$ and it is zero when the roots are the eigenvalues $k_{\ell m} = w_\ell - w_m$, and, in the second line, the term multiplied by $\text{Det}(\Upsilon)$ is still $\text{Det}(\Upsilon)$ for the same roots. Moreover, if we do the substitution $k^2 = \xi$, the last equation is $P\tilde{W}_2$

\[(P\tilde{W}_1)^2 = P\tilde{W}_2 = \xi^3 - \xi^2 \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right) + \frac{1}{4} \left(3\text{Tr}(\tilde{W}^2) - \text{Tr}^2(\tilde{W})\right)^2 \xi - (\text{Det}(\Upsilon))^2. \tag{J.40}\]

On the other hand,

\[P\tilde{W}_1 = \pm \sqrt{P\tilde{W}_2}. \tag{J.41}\]

**Lemma J.0.5.** The roots $w_\ell$ are solutions of the characteristic polynomial (J.1) if and only if the roots $(w_\ell - w_m)^2 = \xi_{\ell m}$ are solutions of the polynomial (J.22).

**Lemma J.0.6.** is a consequence of the Propositions J.0.2 and J.0.4

Every result obtained in this appendix is general and can be applied to any matrix, and so to both the matrices $W$ and $Z$. 

In this appendix we show the construction of the polynomial $S_Z(\theta; \lambda)$ of the sums of the eigenvalues $z_j$.

The characteristic polynomial of the matrix $Z$ is the polynomial whose roots are the eigenvalues $z_j$, with $j = 1, 2, 3$, and, so, it takes the expression

$$P_Z(z; \lambda) = (z - z_1)(z - z_2)(z - z_3). \quad (K.1)$$

On the other hand, we can construct the polynomial of the sums of the eigenvalues $z_j$, defined as the polynomial whose roots are $z_1 + z_2$, $z_2 + z_3$ and $z_3 + z_1$, that is

$$S_Z(x; \lambda) = (x - (z_1 + z_2))(x - (z_2 + z_3))(x - (z_3 + z_1)). \quad (K.2)$$

that is

$$S_Z(x; \lambda) = x^3 - 2x^2Tr(Z) + \frac{x}{2}(3Tr^2(Z) - Tr(Z^2)) - \frac{1}{3}(Tr^3(Z) - Tr(Z^3)). \quad (K.3)$$

Hence, the coefficients of the polynomial $S_Z(x; \lambda)$ in (K.3) can be written in terms of the coefficients of the polynomial $P_Z(z; \lambda)$ in (K.1), and, so, we obtain

$$S_Z(\theta; \lambda) = \theta^3 + 4\lambda\theta^2 + (5\lambda^2 + p_2 - 1)\theta + p_1 + \lambda(-2 + p_2 + 2\lambda^2). \quad (K.4)$$

that is the polynomial of the sums of the eigenvalues $z_j$. 

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**Appendix K**

**Polynomial $S_Z(\theta; \lambda)$ of the sums of the eigenvalues $z_j$**

In this appendix we show the construction of the polynomial $S_Z(\theta; \lambda)$ of the sums of the eigenvalues $z_j$.

The characteristic polynomial of the matrix $Z$ is the polynomial whose roots are the eigenvalues $z_j$, with $j = 1, 2, 3$, and, so, it takes the expression

$$P_Z(z; \lambda) = (z - z_1)(z - z_2)(z - z_3). \quad (K.1)$$

On the other hand, we can construct the polynomial of the sums of the eigenvalues $z_j$, defined as the polynomial whose roots are $z_1 + z_2$, $z_2 + z_3$ and $z_3 + z_1$, that is

$$S_Z(x; \lambda) = (x - (z_1 + z_2))(x - (z_2 + z_3))(x - (z_3 + z_1)). \quad (K.2)$$

that is

$$S_Z(x; \lambda) = x^3 - 2x^2Tr(Z) + \frac{x}{2}(3Tr^2(Z) - Tr(Z^2)) - \frac{1}{3}(Tr^3(Z) - Tr(Z^3)). \quad (K.3)$$

Hence, the coefficients of the polynomial $S_Z(x; \lambda)$ in (K.3) can be written in terms of the coefficients of the polynomial $P_Z(z; \lambda)$ in (K.1), and, so, we obtain

$$S_Z(\theta; \lambda) = \theta^3 + 4\lambda\theta^2 + (5\lambda^2 + p_2 - 1)\theta + p_1 + \lambda(-2 + p_2 + 2\lambda^2). \quad (K.4)$$

that is the polynomial of the sums of the eigenvalues $z_j$. 

---
Appendix L

Space Stability Spectra

% Parameters to be assigned
p1 = -0.8; p2 = 0.4; p3 = -0.6; %1G 1B 2L
p1 = -4.0; p2 = -3.0; p3 = -0.6; %1G 1B 1L
p1 = 1.0; p2 = 3.0; p3 = -0.6; %1G 0B 0L
p1 = -6.2; p2 = -6.3; p3 = -0.6; %0G 2B 2L
p1 = -4.0; p2 = -4.2; p3 = -0.6; %0G 2B 1L
p1 = 1.0; p2 = -3.0; p3 = -0.6; %0G 2B 0L
p1 = -70.0; p2 = 60.0; p3 = -0.6; %2G 0B 2L
p1 = 0.2; p2 = 0.6; p3 = -0.6; %2G 0B 1L
p1 = -90.0; p2 = 60.0; p3 = -0.6; %1G 1SG 0B 1L 1TL
p1 = -1.4; p2 = -1.0; p3 = -0.6; %0G 1SG 1B 1L 1TL
p1 = -4.0; p2 = 2.0; p3 = -0.6; %0G 1SG 1B 0L 1TL
p1 = 2; p2 = -4; p3 = -0.6;
Nx = 2*1e3;

% Saveflag. If 'saveflag=0' no figure is saved. If 'saveflag =1', all
% figures are saved.
saveflag = 0;
% Dataflag. If 'dataflag=0' no data is generated. If 'dataflag =1', data is
% generated.
dataflag = 1;

% Computation of the curve on the lambda–plane, parametrized
% as a function
% of x = k3^2;
if dataflag==1
   [x,lambda] = branchsolver1(p1,p2,p3,Nx);
end

% NB: If the curves are not well centered in the lambda–plane,
then the
% instruction for the array AXISLAMBDA below has to be
modified.

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% Roots of Q2(x)
%
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% For a given choice of r and p, the roots of the polynomial
Q2(x) are
% evaluated. Let Q(x) be the discriminant of the polynomial
whose roots are
% the squares of the differences of the roots of the
characteristic
% polynomial PW; then Q(x) = Q1(x)^2 * Q2(x); thus Q2(x) is
the polynomial
% that describes the changes of sign of $Q(x)$.

% Let $X$ be the set of the real, non-negative zeros of $Q_2(x)$. $X$
% has at most
% four elements and it is shown that it has at least one
% element. Let $X_j$ be
% its elements, where $j$ spans between 1 and the total number
% of real
% non-negative roots of $Q_2(x)$, sorted in ascending order.
% This values are utilised for plotting the regions of $x$ where
% $Q$ is
% positive (green color) and where $Q$ is negative (red color).

coefdiscr = qcoef(p1,p2,p3);
xr = roots(coefdiscr); % roots of $Q_2$
xr_real = sort(real(xr( (abs(imag(xr))<1e-10) & (real(xr) >=0) ) ) );
xspan = [0 4*max(xr_real)];
disp(['
   ******************************************
   '])
disp(['
Roots of $Q_2(x)$ for p1 = ','num2str(1),', p2 = ','num2str
(p2),', p3 = ','num2str(p3)])
xr(:) % displays the roots of $Q_2$
disp(['
   ******************************************
   '])
% Columnization of lambda and x
%

```
lambdacol = lambda (:);
lambdacol2 = conj(lambdacol).*lambdacol;
xcol = [x(:); x(:); x(:); x(:); x(:); x(:)];
%
```

% Stereographic projection
%
```
Sy = 2*real(lambdacol)./(1+lambdacol2);
Sz = 2*imag(lambdacol)./(1+lambdacol2);
Sx = (1-lambdacol2)./(1+lambdacol2);

lambda2_1 = conj(lambda (:,1)).*lambda (:,1);
Sy1 = 2*real(lambda (:,1))./(1+lambda2_1);
Sz1 = 2*imag(lambda (:,1))./(1+lambda2_1);
Sx1 = (1-lambda2_1)./(1+lambda2_1);

lambda2_2 = conj(lambda (:,2)).*lambda (:,2);
Sy2 = 2*real(lambda (:,2))./(1+lambda2_2);
Sz2 = 2*imag(lambda (:,2))./(1+lambda2_2);
Sx2 = (1-lambda2_2)./(1+lambda2_2);

lambda2_3 = conj(lambda (:,3)).*lambda (:,3);
Sy3 = 2*real(lambda (:,3))./(1+lambda2_3);
Sz3 = 2*imag(lambda (:,3))./(1+lambda2_3);
```
Sx3 = (1−\lambda_2^3)/(1+\lambda_2^3);

lambda2_4 = \text{conj}(\lambda(:,4))\ast\lambda(:,4);
Sy4 = 2\ast\text{real}(\lambda(:,4))/(1+\lambda2_4);
Sz4 = 2\ast\text{imag}(\lambda(:,4))/(1+\lambda2_4);
Sx4 = (1−\lambda2_4)/(1+\lambda2_4);

lambda2_5 = \text{conj}(\lambda(:,5))\ast\lambda(:,5);
Sy5 = 2\ast\text{real}(\lambda(:,5))/(1+\lambda2_5);
Sz5 = 2\ast\text{imag}(\lambda(:,5))/(1+\lambda2_5);
Sx5 = (1−\lambda2_5)/(1+\lambda2_5);

lambda2_6 = \text{conj}(\lambda(:,6))\ast\lambda(:,6);
Sy6 = 2\ast\text{real}(\lambda(:,6))/(1+\lambda2_6);
Sz6 = 2\ast\text{imag}(\lambda(:,6))/(1+\lambda2_6);
Sx6 = (1−\lambda2_6)/(1+\lambda2_6);

% Omega3 and Gain (gamma)
% Omega3, k3 and the gain are computed only for the values of lambda (and
% the corresponding values of x) for which lambda has a non-zero imaginary
% part. k3 = sqrt(x). gain = imag(omega3).

lambdacom = lambdacol(abs(imag(lambdacol))\"=0);
xcom = xcol(abs(imag(lambdacol))\"=0);
Appendix L. Space Stability Spectra

112 \( k_3 = \sqrt{x_{com}} \);
113 \textbf{if} dataflag==1
114 \hspace{1em} \text{omega3} = \text{omegaextractor2(p1,p2,p3,k3,lambdacom)};
115 \textbf{end}
116 gamma = \text{imag(omega3)};
117
118 \( \kappa_3 = \sqrt{x} \);
119 \textbf{if} dataflag==1
120 \hspace{1em} \text{omega3H} = \text{omegaextractorH(p1,p2,p3,\kappa_3)};
121 \textbf{end}
122 \text{omega3Hcol} = \text{omega3H(:)};
123 gammaH = \text{imag(omega3Hcol)};
124
125 \%
126 \% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
127 \%
128 \% Optimizing the axes ranges in order to centre the lambda--
129 \% curves for the
130 \% user (AXISLAMBDAA)
131 \%
132 \% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
133 \%
134 lambdacol_nonzeroimag = lambdacol((\text{imag(}\text{lambdacol})^\text{=0})&(\text{abs(}
135 \hspace{1em} \text{real(}\text{lambdacol}))>1e-9));
136 \text{maxlambdare} = \text{max(\text{real(lambdacol_nonzeroimag)})};
137 \text{minlambdare} = \text{min(\text{real(lambdacol_nonzeroimag)})};
138 \text{maxlambdamaim} = \text{max(\text{imag(lambdacol_nonzeroimag)})};
139 \text{minlambdamaim} = \text{min(\text{imag(lambdacol_nonzeroimag)})};
140 \text{maxlambdare} = \text{max([maxlambdare}\ max(\text{real(lambda(1,:))})])};
141 \text{minlambdare} = \text{min([minlambdare}\ min(\text{real(lambda(1,:))})])};
142 \textbf{if} \text{maxlambdamaim==0}
143 \hspace{1em} \text{maxlambdaim} = 1;
end

if minlambdaim==0
    minlambdaim = -1;
end

axislambda = [minlambdare maxlambdare minlambdaim maxlambdaim]*15/10;
axislambda = 2*[−3.0, 3.0, −5.5, 5.5];

% % Computation of s1, s2 and a1, a2 corresponding to the given values of r,p
% %

% if (abs(p)>=abs(r))&&(r>=0)&&(p>=0)
% s1 = 1; s2 = 1;
% elseif (abs(p)<abs(r))&&(r>=0)&&(p>=0)
% s1 = 1; s2 = -1;
% elseif (abs(p)>=abs(r))&&(r>0)&&(p<0)
% s1 = -1; s2 = -1;
% elseif (abs(p)<abs(r))&&(r>0)&&(p<0)
% s1 = 1; s2 = -1;
% elseif (abs(p)>=abs(r))&&(r<0)&&(p>=0)
% s1 = 1; s2 = 1;
% elseif (abs(p)<abs(r))&&(r<0)&&(p>=0)
% s1 = -1; s2 = 1;
% elseif (abs(p)>=abs(r))&&(r<0)&&(p<0)
% s1 = -1; s2 = -1;
% elseif (abs(p)<abs(r))&&(r<0)&&(p<0)
% s1 = -1; s2 = 1;
% end

% a1 = sqrt(s1*(p+r)/2); a2 = sqrt(s2*(p-r)/2);
% disp([s1 s2; a1 a2]);

% Computation of the two critical curves on the (r,p)-plane
%
% delta2 = 2^(2/3); delta4 = delta2^2;
% pden = @(x) (x+sqrt(4+x.^2)).^(2/3);
% pcurve1 = @(x) -5 + 3*(delta2./pdex(x) + pdex(x)/delta2);
% pcurve2 = @(x) -sqrt(16-12*delta2*((x.^2).^(1/3))+3*delta4
% *((x.^4).^(1/3)));
% if r<=4
% rmin = 0;
% rmax = 4;
% else
% rmin = 4;
% rmax = 2*r-8;
% end
% dr = (rmax-rmin)/1e3;
% rvec = [rmin:dr:rmax];
% pvec1 = pcurve1(rvec);
% pvec2 = pcurve2(rvec);
% Computation of the critical curves on the \((a_1, a_2)\)-plane

\begin{verbatim}
% rval = linspace(-100,100,1e4);
% pval1 = pcurve1(rval);
% pval2 = pcurve2(rval);
% a1val1 = sqrt(s1*(pval1+rval)/2); a2val1 = sqrt(s2*(pval1-rval)/2);
% a1val2 = sqrt(s1*(pval2+rval)/2); a2val2 = sqrt(s2*(pval2-rval)/2);
% a1vec1 = a1val1((imag(a1val1)==0)&(imag(a2val1)==0));
% a2vec1 = a2val1((imag(a1val1)==0)&(imag(a2val1)==0));
% a1vec2 = a1val2((imag(a1val2)==0)&(imag(a2val2)==0));
% a2vec2 = a2val2((imag(a1val2)==0)&(imag(a2val2)==0));

% PLOTS

redfac = 0.85;
hf1 = figure(1);
clf
subplot(6,9,[1 2 10 11])
% plot(rvec,pvec1,'k')
% hold on
% plot(rvec,-pvec2,'r')
% plot(rvec,-pvec2,'r')
% plot(rvec,rvec,'b')
\end{verbatim}
Appendix L. Space Stability Spectra

% plot(rvec,-rvec,'b')
% plot(r,p,'ko','Marker','o','MarkerFaceColor','c','
MarkerEdgeColor','k','MarkerSize',4)
% hold off
% axis([rmin rmax -rmax rmax])
% xlabel('$r$','interpreter','latex')
% ylabel('$p$','interpreter','latex')
% title(['$$(r,p)$$-plane, $r=$',num2str(r),', $p=$',num2str(p)],'interpreter','latex')
% tmp = get(gca,'position');
% set(gca,'position',[tmp(1) tmp(2) redfac*tmp(3) redfac*
  tmp(4)]);

subplot(6,9,[19 20 28 29])
% plot(rvec,pvec1,'k')
% hold on
% plot(rvec,pvec2,'r')
% plot(rvec,-pvec2,'r')
% plot(rvec,rvec,'b')
% plot(rvec,-rvec,'b')
% plot(r,p,'ko','Marker','o','MarkerFaceColor','c','
MarkerEdgeColor','k','MarkerSize',4)
% hold off
% axis([rmin 4 -4 4])
% xlabel('$r$','interpreter','latex')
% ylabel('$p$','interpreter','latex')
% title(['$$(r,p)$$-plane','interpreter','latex'])
% tmp = get(gca,'position');
% set(gca,'position',[tmp(1) tmp(2) redfac*tmp(3) redfac*
  tmp(4)]);

subplot(6,9,[37 38 46 47])
% plot(rvec,pvec1,'k')
% hold on
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240  \% \%plot(rvec,pvec2,'r')
241  \% plot(rvec,-pvec2,'r')
242  \% plot(rvec,rvec,'b')
243  \% plot(rvec,-rvec,'b')
244  \% plot(r,p,'ko','Marker','o','MarkerFaceColor','c','
245     MarkerEdgeColor','k','MarkerSize',4)
246  \% hold off
247  \% axis([4 rmax -rmax rmax])
248  \% xlabel('$r$','interpreter','latex')
249  \% ylabel('$p$','interpreter','latex')
250  \% title('$r,p$–plane','interpreter','latex')
251  \% tmp = get(gca,'position');
252  \% subplot(6,9,[3:6])
253  \% Nxx = 1e4; xx = linspace(x(1),x(end),Nxx);
254  \% imgc = zeros(2,Nxx,3);
255  \% for k=1:Nxx
256     if polyval(coefdiscr,(xx(k)))>0
257         imgc(:,k,1) = 0;
258         imgc(:,k,2) = 1;
259         imgc(:,k,3) = 0;
260     else
261         imgc(:,k,1) = 1;
262         imgc(:,k,2) = 0;
263         imgc(:,k,3) = 0;
264     end
265  \% end
266  imagesc(xx,[-1 1],imgc)
267  hold on
268  plot(xx,zeros(size(xx)),'k','MarkerSize',3)
269  y = linspace(-1,1,10);
for k=1:length(xr_real)
    plot(xr_real(k)*ones(size(y)), y, 'k', 'MarkerSize', 3)
end

% xpl = plot(x(1), 0, 'o', 'MarkerFaceColor', 'k', 'MarkerSize', 5);
hold off

axis([x(1) x(end) -1 1])
xlabel('$k^3$', 'Interpreter', 'latex')
title(['Sign of $\Delta$ for squares of differences: green
      -pos, red-neg'], 'Interpreter', 'latex')
tmp = get(gca, 'position');
set(gca, 'position', [tmp(1) tmp(2) redfac*tmp(3) redfac*tmp(4)]);

subplot(6,9,[12:15,21:24,30:33,39:42,48:51])
plot(real(lambda(:,1)), imag(lambda(:,1)), 'b.', 'MarkerSize', 2)
hold on
plot(real(lambda(:,2)), imag(lambda(:,2)), 'b.', 'MarkerSize', 2)
plot(real(lambda(:,3)), imag(lambda(:,3)), 'b.', 'MarkerSize', 2)
plot(real(lambda(:,4)), imag(lambda(:,4)), 'b.', 'MarkerSize', 2)
plot(real(lambda(:,5)), imag(lambda(:,5)), 'b.', 'MarkerSize', 2)
plot(real(lambda(:,6)), imag(lambda(:,6)), 'b.', 'MarkerSize', 2)

% pl1 = plot(real(lambda(1,1)), imag(lambda(1,1)), 'o', 'MarkerFaceColor', 'red', 'MarkerSize', 4);
% pl2 = plot(real(lambda(1,2)), imag(lambda(1,2)), 'o', 'MarkerFaceColor', 'red', 'MarkerSize', 4);
% pl3 = plot(real(lambda(1,3)),imag(lambda(1,3)),'o','MarkerFaceColor','red','MarkerSize',4);
% pl4 = plot(real(lambda(1,4)),imag(lambda(1,4)),'o','MarkerFaceColor','red','MarkerSize',4);
% pl5 = plot(real(lambda(1,5)),imag(lambda(1,5)),'o','MarkerFaceColor','red','MarkerSize',4);
% pl6 = plot(real(lambda(1,6)),imag(lambda(1,6)),'o','MarkerFaceColor','red','MarkerSize',4);
hold off
xlabel(’$\mu$’,'Interpreter','latex')
ylabel(’$\rho$’,'Interpreter','latex')
title(['$\lambda$ ',num2str(p1),', $p_2=$',num2str(p2),', $p_3=$',num2str(p3)],'Interpreter','latex')
axis(axislambda)
tmp = get(gca, ’position’);
set(gca,’position’, [tmp(1) tmp(2) redfac*tmp(3) redfac*tmp(4)]);
subplot(6,9,[7:9,16:18,25:27])
set(gcf,’color’,’w’);
[sph1,sph2,sph3] = sphere(64);
hs = surfl(sph1,sph2,sph3);
set(hs,’FaceAlpha’,0.6)
shading interp
colormap(bone)
hold on
plot3(Sx,Sy,Sz,’k.’,’MarkerSize’,2,’LineWidth’,2)
plot3(1,0,0,’r.’,’MarkerSize’,15)
plot3(-1,0,0,’g.’,’MarkerSize’,15)
axis equal % or square
box off
grid off
axis off
view(70,5)

spl1 = plot3(Sx1(1),Sy1(1),Sz1(1),'o','MarkerFaceColor','b','MarkerSize',3);

spl2 = plot3(Sx2(1),Sy2(1),Sz2(1),'o','MarkerFaceColor','b','MarkerSize',3);

spl3 = plot3(Sx3(1),Sy3(1),Sz3(1),'o','MarkerFaceColor','b','MarkerSize',3);

spl4 = plot3(Sx4(1),Sy4(1),Sz4(1),'o','MarkerFaceColor','b','MarkerSize',3);

spl5 = plot3(Sx5(1),Sy5(1),Sz5(1),'o','MarkerFaceColor','b','MarkerSize',3);

spl6 = plot3(Sx6(1),Sy6(1),Sz6(1),'o','MarkerFaceColor','b','MarkerSize',3);

hold off

title(['$',\lambda$', '$p_1=$', num2str(p1), ',$p_2=$', num2str(p2), ',$p_3=$', num2str(p3)],'interpreter','latex')

subplot([6,9,[34:36,43:45,52:54]])

plot(k3,abs(gamma),'b.','MarkerSize',3)

hold on

plot(-k3,abs(gamma),'b.','MarkerSize',3)

plot(sqrt(xcol),abs(gammaH),'r.','MarkerSize',3)

plot(-sqrt(xcol),abs(gammaH),'r.','MarkerSize',3)

% gpl1 = plot(sqrt(x(1)),abs(gamma1(1)),'o','MarkerFaceColor','red','MarkerSize',4);

% gpl2 = plot(sqrt(x(1)),abs(gamma2(1)),'o','MarkerFaceColor','red','MarkerSize',4);

% gpl3 = plot(sqrt(x(1)),abs(gamma3(1)),'o','MarkerFaceColor','red','MarkerSize',4);

% gpl4 = plot(sqrt(x(1)),abs(gamma4(1)),'o','MarkerFaceColor','red','MarkerSize',4);

% gpl5 = plot(sqrt(x(1)),abs(gamma5(1)),'o','MarkerFaceColor','red','MarkerSize',4);
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% gpl6 = plot(sqrt(x(1)).abs(gamma6(1)),'o', 'MarkerFaceColor','red','MarkerSize',4);

% axis([-k3max k3max 0 gammamax]*(12/10))
hold off
xlabel('$k_3$', 'interpreter','latex')
ylabel('$|\Gamma(k_3)|$', 'interpreter','latex')
title(['Modulus of Gain $p_1=$', num2str(p1), ', $p_2=$', num2str(p2), ', $p_3=$', num2str(p3)], 'interpreter','latex')
tmp = get(gca,'position');
set(gca,'position',[tmp(1) tmp(2) redfac*tmp(3) redfac*tmp(4)]);

%filename = ['GBL_r=', num2str(r), ',p=', num2str(p)];
%print(hf1,[pwd '/Figures/' filename '.jpeg'],'-djpeg')

% for j=2:100:length(x)
% pl1.XData = real(lambda(j,1)); pl1.YData = imag(lambda(j,
% ,1));
% pl2.XData = real(lambda(j,2)); pl2.YData = imag(lambda(j,
% ,2));
% pl3.XData = real(lambda(j,3)); pl3.YData = imag(lambda(j,
% ,3));
% pl4.XData = real(lambda(j,4)); pl4.YData = imag(lambda(j,
% ,4));
% pl5.XData = real(lambda(j,5)); pl5.YData = imag(lambda(j,
% ,5));
% pl6.XData = real(lambda(j,6)); pl6.YData = imag(lambda(j,
% ,6));
% xpl.XData = x(j);
% gpl1.XData = sqrt(x(j)); gpl1.YData = abs(gamma1(j));
% gpl2.XData = sqrt(x(j)); gpl2.YData = abs(gamma2(j));
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```matlab
% gpl3.XData = sqrt(x(j)); gpl3.YData = abs(gamma3(j));
% gpl4.XData = sqrt(x(j)); gpl4.YData = abs(gamma4(j));
% gpl5.XData = sqrt(x(j)); gpl5.YData = abs(gamma5(j));
% gpl6.XData = sqrt(x(j)); gpl6.YData = abs(gamma6(j));
% spl1.XData = Sx1(j); spl1.YData = Sy1(j); spl1.ZData = Sz1(j);
% spl2.XData = Sx2(j); spl2.YData = Sy2(j); spl2.ZData = Sz2(j);
% spl3.XData = Sx3(j); spl3.YData = Sy3(j); spl3.ZData = Sz3(j);
% spl4.XData = Sx4(j); spl4.YData = Sy4(j); spl4.ZData = Sz4(j);
% spl5.XData = Sx5(j); spl5.YData = Sy5(j); spl5.ZData = Sz5(j);
% spl6.XData = Sx6(j); spl6.YData = Sy6(j); spl6.ZData = Sz6(j);
% drawnow
% end

figure(2)
subplot(1,3,1)
plot(k3,abs(gamma),'b.','MarkerSize',3)
hold on
plot(-k3,abs(gamma),'b.','MarkerSize',3)
plot(sqrt(xcol),abs(gammaH),'r.','MarkerSize',3)
plot(-sqrt(xcol),abs(gammaH),'r.','MarkerSize',3)
hold off
xlabel('$k_{3}$','interpreter','latex')
ylabel('$|\Gamma (k_{3})|$','interpreter','latex')
title(['Modulus of Gain, $p_{1}=$',num2str(p1),'$p_{2}=$',num2str(p2),'$p_{3}=$',num2str(p3)],'interpreter','')
```
### Appendix L. Space Stability Spectra

```latex
\begin{lstlisting}

\begin{verbatim}
latex')
    tmp = get(gca, 'position');
    set(gca, 'position', [tmp(1) tmp(2) redfac*tmp(3) redfac*
                        tmp(4)]);

subplot(1,3,2)
    plot(k3, abs(gamma), 'b.', 'MarkerSize', 3)
    hold on
    plot(-k3, abs(gamma), 'b.', 'MarkerSize', 3)
    hold off
    xlabel('$k_\{3\}$', 'interpreter', 'latex')
    ylabel('$|\Gamma(k_\{3\})|$', 'interpreter', 'latex')
    title(['Modulus of Gain from numerics, $p_1=$', num2str(p1), ...]
          ', 'interpreter', 'latex')
    tmp = get(gca, 'position');
    set(gca, 'position', [tmp(1) tmp(2) redfac*tmp(3) redfac*
                        tmp(4)]);

subplot(1,3,3)
    plot(sqrt(xcol), abs(gammaH), 'r.', 'MarkerSize', 3)
    hold on
    plot(-sqrt(xcol), abs(gammaH), 'r.', 'MarkerSize', 3)
    hold off
    xlabel('$k_\{3\}$', 'interpreter', 'latex')
    ylabel('$|\Gamma(k_\{3\})|$', 'interpreter', 'latex')
    title(['Modulus of Gain from $H$, $p_1=$', num2str(p1), ...]
          ', 'interpreter', 'latex')
    tmp = get(gca, 'position');
    set(gca, 'position', [tmp(1) tmp(2) redfac*tmp(3) redfac*
                        tmp(4)]);
\end{verbatim}
\end{lstlisting}
```
Appendix L. Space Stability Spectra

406  
407  % % PLOTS
408  %
409  
410  fslabels = 28;
411  fsticks = 24;
412  
413  hf1 = figure(1);
414  clf
415  
416  rvec = linspace(0, 2*pi, 100);
417  pvec1 = rvec * cos(rvec);
418  pvec2 = rvec * sin(rvec);
419  plot(rvec, rvec, 'k', 'LineWidth', 2)
420  hold on
421  plot(rvec, -rvec, 'k', 'LineWidth', 2)
422  plot(rvec, pvec1, 'r', 'LineWidth', 2)
423  plot(rvec, pvec2, 'b', 'LineWidth', 2)
424  plot(r, p, 'ko', 'Marker', 'o', 'MarkerFaceColor', 'g', 'MarkerEdgeColor', 'k', 'MarkerSize', 9)
425  hold off
426  
427  hf2 = figure(2);
428  clf
429  
430  if max(a1, a2) < 3
431  a1max = 3; a2max = a1max;
432  else
433  a1max = 2; a2max = a1max;
434  end
435  
436  ax = gca; ax.FontSize = fsticks;
431 \% \% else
432 \% \% \quad a1max = 5.1.5*max(a1,a2);
433 \% \% \quad a2max = a1max;
434 \% \% end
435 \% \% if s1==1&&s2==1
436 \% \% \quad plot(a1vec1,a2vec1,'r','LineWidth',2)
437 \% \% \quad hold on
438 \% \% elseif s1==−1&&s2==1
439 \% \% \quad plot(a1vec2,a2vec2,'b','LineWidth',2)
440 \% \% \quad hold on
441 \% \% elseif s1==−1&&s2==−1
442 \% \% \quad plot(a1vec2,a2vec2,'b','LineWidth',2)
443 \% \% \quad hold on
444 \% \% elseif s1==1&&s2==−1
445 \% \% \quad plot(a1vec2,a2vec2,'b','LineWidth',2)
446 \% \% \quad hold on
447 \% \% \quad plot(a1vec1,a2vec1,'r','LineWidth',2)
448 \% \% \quad end
449 \% \% plot(a1,a2,'ko','Marker','o','MarkerFaceColor','g','
450 \quad MarkerEdgeColor','k','MarkerSize',9)
451 \% \% hold off
452 \% \% axis([0 a1max 0 a2max])
453 \% \% axis square
454 % \% xlabel('$a_{1}$','interpreter','latex','fontsize',fslabs 
455 \quad ,'fontweight','bold')
456 % \% ylabel('$a_{2}$','interpreter','latex','fontsize',fslabs 
457 \quad ,'fontweight','bold')
458 \% % ax = gca; ax.FontSize = fsticks;
459 \% % ax.XTick = [0:a1max]; ax.YTick = [0:a2max];
460 %
461 hf3 = figure(3);
Appendix L. Space Stability Spectra

460 clf
461 plot(real(lambda(:,1)), imag(lambda(:,1)), 'b.', 'MarkerSize', 2)
462 hold on
463 plot(real(lambda(:,2)), imag(lambda(:,2)), 'b.', 'MarkerSize', 2)
464 plot(real(lambda(:,3)), imag(lambda(:,3)), 'b.', 'MarkerSize', 2)
465 plot(real(lambda(:,4)), imag(lambda(:,4)), 'b.', 'MarkerSize', 2)
466 plot(real(lambda(:,5)), imag(lambda(:,5)), 'b.', 'MarkerSize', 2)
467 plot(real(lambda(:,6)), imag(lambda(:,6)), 'b.', 'MarkerSize', 2)
468 %plot(real(lambda(1,1)), imag(lambda(1,1)), 'ko', 'Marker', 'o', '
469 %MarkerFaceColor', 'r', 'MarkerEdgeColor', 'k', 'MarkerSize', 5)
470 %plot(real(lambda(1,2)), imag(lambda(1,2)), 'ko', 'Marker', 'o', '
471 %MarkerFaceColor', 'r', 'MarkerEdgeColor', 'k', 'MarkerSize', 5)
472 %plot(real(lambda(1,3)), imag(lambda(1,3)), 'ko', 'Marker', 'o', '
473 %MarkerFaceColor', 'r', 'MarkerEdgeColor', 'k', 'MarkerSize', 5)
474 %plot(real(lambda(1,4)), imag(lambda(1,4)), 'ko', 'Marker', 'o', '
475 %MarkerFaceColor', 'r', 'MarkerEdgeColor', 'k', 'MarkerSize', 5)
476 %plot([r/(2*p) , 0], 'ko', 'Marker', 'o', 'MarkerFaceColor', 'b', '
477 %MarkerEdgeColor', 'k', 'MarkerSize', 5)
478 hold off
479 xlabel('$\mu$', 'interpreter', 'latex', 'fontsize', fslables, '
480 %fontweight', 'bold')
481 ylabel('$\rho$', 'interpreter', 'latex', 'fontsize', fslables, '
482 %fontweight', 'bold')
483 axis(axislambda)
484 ax = gca;
485 ax.FontSize = fsticks;
486 ax.FontWeight = 'normal';
487 ax.TickLabelInterpreter = 'latex';
488 outerpos = ax.OuterPosition;
489 ti = ax.TightInset;
490 left = outerpos(1) + ti(1);
491 bottom = outerpos(2) + ti(2);
\begin{verbatim}
ax_width = outerpos(3) - ti(1) - ti(3);
ax_height = outerpos(4) - ti(2) - ti(4);
ax.Position = [left bottom ax_width ax_height];
fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

%% hf4 = figure(4);
%% clf
%% plot(k3, abs(gamma), 'b.', 'MarkerSize', 4)
%% hold on
%% plot(-k3, abs(gamma), 'b.', 'MarkerSize', 4)
%% hold off
%% xlabel('$k_{3}$', 'interpreter', 'latex', 'fontsize', fslables
         , 'fontweight', 'bold')
%% ylabel('$|\Gamma(k_{3})|$', 'interpreter', 'latex', 'fontsize
         , fslables, 'fontweight', 'bold')
%% axis([-max(k3) max(k3) 0 max(abs(gamma))] * 1.05)
%% axis square
%% ax = gca; ax.FontSize = fsticks;
%%
%% hf5 = figure(5);
%% clf
%% set(gcf, 'color', 'w');
%% [sph1,sph2,sph3] = sphere(64);
%% hs = surfl(sph1,sph2,sph3);
%% set(hs, 'FaceAlpha', 0.6)
%% shading interp
%% colormap(bone)
%% hold on
%% plot3(Sx,Sy,Sz,'k.', 'MarkerSize', 1, 'LineWidth', 1)
\end{verbatim}
Appendix L. Space Stability Spectra

515  \% \% plot3 (1,0,0,'co','Marker','o','MarkerFaceColor','c','
     MarkerEdgeColor','c','MarkerSize',5)
516  \% \% plot3 (-1,0,0,'yo','Marker','o','MarkerFaceColor','y','
     MarkerEdgeColor','y','MarkerSize',5)
517  \% \% axis equal \% or square
518  \% \% box off
519  \% \% grid off
520  \% \% axis off
521  \% \% view(70,5) %view(70,5)
522  \% \% hold off
523  \\
524  ds = 5;
525  hf6 = figure(6);
526  clf
527  plot( real(lambda(1:ds:end,1)),imag(lambda(1:ds:end,1)), 'b .','
     MarkerSize',2)
528  hold on
529  plot( real(lambda(1:ds:end,2)),imag(lambda(1:ds:end,2)), 'b .','
     MarkerSize',2)
530  plot( real(lambda(1:ds:end,3)),imag(lambda(1:ds:end,3)), 'b .','
     MarkerSize',2)
531  plot( real(lambda(1:ds:end,4)),imag(lambda(1:ds:end,4)), 'b .','
     MarkerSize',2)
532  plot( real(lambda(1:ds:end,5)),imag(lambda(1:ds:end,5)), 'b .','
     MarkerSize',2)
533  plot( real(lambda(1:ds:end,6)),imag(lambda(1:ds:end,6)), 'b .','
     MarkerSize',2)
534  \%plot(real(lambda(1,1)),imag(lambda(1,1)), 'ko','Marker','o','
     MarkerFaceColor','r','MarkerEdgeColor','k','MarkerSize',5)
535  \%plot(real(lambda(1,2)),imag(lambda(1,2)), 'ko','Marker','o','
     MarkerFaceColor','r','MarkerEdgeColor','k','MarkerSize',5)
Appendix L. Space Stability Spectra

536 \%plot( real(lambda(1,3)), imag(lambda(1,3)), 'ko', 'Marker', 'o', 'MarkerFaceColor', 'r', 'MarkerEdgeColor', 'k', 'MarkerSize', 5)
537 \%plot( real(lambda(1,4)), imag(lambda(1,4)), 'ko', 'Marker', 'o', 'MarkerFaceColor', 'r', 'MarkerEdgeColor', 'k', 'MarkerSize', 5)
538 \%plot([r/(2*p),0], 'ko', 'Marker', 'o', 'MarkerFaceColor', 'b', 'MarkerEdgeColor', 'k', 'MarkerSize', 5)

539 hold off
540 xlabel('$\mu$', 'interpreter', 'latex', 'fontsize', fslabels, 'fontweight', 'bold')
541 ylabel('$\rho$', 'interpreter', 'latex', 'fontsize', fslabels, 'fontweight', 'bold')

542 axis(axislambda)
543 ax = gca;
544 ax.FontSize = fsticks;
545 ax.FontWeight = 'normal';
546 ax.TickLabelInterpreter = 'latex';
547 outerpos = ax OUTERPOSITION;
548 ti = ax.TIGHTINSET;
549 left = outerpos(1) + ti(1);
550 bottom = outerpos(2) + ti(2);
551 ax_width = outerpos(3) - ti(1) - ti(3);
552 ax_height = outerpos(4) - ti(2) - ti(4);
553 ax.Position = [left bottom ax_width ax_height];
554 fig = gcf;
555 fig.PAPERPOSITIONMODE = 'auto';
556 fig_pos = fig.PAPERPOSITION;
557 fig.PAPERSIZE = [fig_pos(3) fig_pos(4)];

558 \% \% hf7 = figure(7);
559 \% clf
% % plot(k3(1:ds:end),abs(gamma(1:ds:end)),'b.','MarkerSize','4')
% % hold on
% % plot(-k3(1:ds:end),abs(gamma(1:ds:end)),'b.','MarkerSize','4')
% % hold off
% % xlabel('$k_{3}$','interpreter','latex','fontsize',fslabs,'fontweight','bold')
% % ylabel('$|\Gamma(k_{3})|$','interpreter','latex','fontsize',fslabs,'fontweight','bold')
% % axis([-max(k3) max(k3) 0 max(abs(gamma))]*1.05)
% % axis square
% % ax = gca; ax.FontSize = fsticks;

hf7 = figure(7);
clf
plot(sqrt(xcol),abs(gammaH),'b.','MarkerSize','2')
hold on
plot(-sqrt(xcol),abs(gammaH),'b.','MarkerSize','2')
hold off
xlabel('$k_{3}$','interpreter','latex','fontsize',fslabs,'fontweight','bold')
ylabel('$|\Gamma(k_{3})|$','interpreter','latex','fontsize',fslabs,'fontweight','bold')
axis([-max(k3) max(k3) 0 max(abs(gamma))]*1.05)
ax = gca;
ax.FontSize = fsticks;
ax.FontWeight = 'normal';
ax.TickLabelInterpreter = 'latex';
outerpos = ax.OuterPosition;
ti = ax.TightInset;
left = outerpos(1) + ti(1);
bottom = outerpos(2) + ti(2);
ax_width = outerpos(3) - ti(1) - ti(3);
ax_height = outerpos(4) - ti(2) - ti(4);
ax.Position = [left bottom ax_width ax_height];
fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

hf8 = figure(8);
clf
plot(sqrt(xcol(1:ds:end)), abs(gammaH(1:ds:end)), 'b.', 'MarkerSize', 2)
hold on
plot(-sqrt(xcol(1:ds:end)), abs(gammaH(1:ds:end)), 'b.', 'MarkerSize', 2)
hold off
xlabel('$k\{3}\$','interpreter','latex','fontsize',fslabels,'fontweight','bold')
ylabel('$|\Gamma(k\{3})|$','interpreter','latex','fontsize',fslabels,'fontweight','bold')
axis([-max(k3) max(k3) 0 max(abs(gamma))]*1.05)
ax = gca;
ax.FontSize = fsticks;
ax.FontWeight = 'normal';
ax.TickLabelInterpreter = 'latex';
outerpos = ax.OuterPosition;
ti = ax.TightInset;
left = outerpos(1) + ti(1);
bottom = outerpos(2) + ti(2);
ax_width = outerpos(3) - ti(1) - ti(3);
ax_height = outerpos(4) - ti(2) - ti(4);
ax.Position = [left bottom ax_width ax_height];
fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

if saveflag==1
    file_name = ['spectrum_p1=',num2str(p1),'_p2=',num2str(p2),
                '_p3=',num2str(p3)];
    print(hf3,[pwd '/Figures/' file_name '_lambda.jpe', '−djpeg '])
    print(hf6,[pwd '/Figures/' file_name '_lambda.eps'], '−dep', '−tiff ')
    print(hf3,[pwd '/Figures/' file_name '_lambda.pdf'], '−dpdf ')
    print(hf7,[pwd '/Figures/' file_name '_gain.jpe'], '−djpe '
    print(hf8,[pwd '/Figures/' file_name '_gain.eps'], '−dep', '−tiff ')
    print(hf7,[pwd '/Figures/' file_name '_gain.pdf'], '−dpdf ')
end
Appendix M

Gain Function

```matlab
function omega=omegaextractor2(p1,p2,p3,k,lambda)

omega = zeros(size(k));
jmax = length(k);

for j=1:jmax
    kk = k(j); ll = lambda(j);
    pw_coeff = flipr([(-1).*p1.^2.*p3.^3+p1.*((-1)+p3.*(p3
    +2.*p2.*p3+(-1).*ll+p3.^2.*ll))+p2.*(-1).*p3.*(((-1)
    +p2+p3.*(p3+ll)))+p2.*((1+2.*p3.^2+(-1).*p3.*ll)+p1.*p3.*((1)
    *ll)+(-1).*(((-1)+p3.^2)+(((-1)+ll).^2)
    ,((-2)+(-1).*p2).*p3+2.*ll ,1]);
pz_coeff = [1,2*ll ,p2-1+ll ^2,p2+ll -p1];
    wroots = roots(pw_coeff);
    %zroots = roots(pz_coeff);
    wdiff = wroots-wroots([2,3,1]);
    [foo, mind] = min(abs(abs(wdiff)-kk));
    wroots0 = [wroots; wroots(1)];
    w1 = wroots0(mind); w2 = wroots0(mind+1);
```
Appendix M. Gain Function

\[ cz1 = [p3,1+p3*l1,p2*p3-w1]; \]
\[ cz2 = [p3,1+p3*l1,p2*p3-w2]; \]
\[ z1roots = \text{roots}(cz1); \]
\[ z1val = \text{abs}(\text{polyval}(pz\_coeff,z1roots)); \]
\[ [fooz1,indz1] = \text{min}(z1val); z1 = z1roots(\text{indz1}); \]
\[ z2roots = \text{roots}(cz2); \]
\[ z2val = \text{abs}(\text{polyval}(pz\_coeff,z2roots)); \]
\[ [fooz2,indz2] = \text{min}(z2val); z2 = z2roots(\text{indz2}); \]
\[ \theta = z1+z2; \]
\[ \omega(j) = \frac{kk}{1+p3*(\theta+l1)}; \]
\[ \text{disp}([''omega extraction: p1 = ',num2str(p1),', p2 = ',num2str(p2),', p3 = ',num2str(p3),...'
', 'diff = ',num2str(foo),' , k3 = ',num2str(kk),...
', ': j = ',int2str(j),', final j = ',int2str(jmax)]; \]
end

function omega = omegaextractorH(p1,p2,p3,k)

jmax = \text{length}(k);

omega = \text{zeros}(jmax,6);

for j=1:jmax

kk = k(j);

polH = \text{fliplr}([(-4).*kk.^4,(-4).*kk.^3 *((-4)+p1.*p3),kk .^2* ((-24)+kk.^2+12.*p1.* ... p3+(-1).* ((-8)+p2.*(4+p2)).*p3.^2),2.*kk.*(8+(-2).*kk .^2+((-8)+p2.*(... 4+p2)).*p3.^2*p1.*p3.*((-6)+(-1).*((-2)+p2).*p3.^2)) ,(-4)+4.*p1.* ... p3+(-1).* ((-8)+p2.*(4+p2)).*p3.^2+2.*p1.*((-2)+p2).*p3 .^3+(-1).*(...
}
\[ 4 + p_1^2 + (-4) \cdot p_2 \cdot p_3^4 + (-2) \cdot kk^2 \cdot ((-3) + p_3^2) \cdot 4 \cdot \]
\[ kk \cdot ((-1)+ \ldots \]
\[ p_3^2 \cdot ((-1)+p_3^2)^2 \] \}

\[ \omega(j,:) = \text{roots(polH)}; \]
\[ \text{disp}(['\omega_{\text{extraction}}H: p1 = ' , num2str(p1) , ', p2 = ', num2str(p2) , ', p3 = ', num2str(p3) , ':
', 'j = ', int2str(j) , ', final j = ', int2str(jmax) ]); \]
\[ \text{return} \]
Appendix N

Sinc Interpolation

function y_new = sincinterp(y_old, x_old, x_new)
    % if nargin==3
    %dx = x_old(2)−x_old(1);
    %end
dxold = x_old(2)−x_old(1);
dxnew = x_new(2)−x_new(1);
size_x_old = size(x_old);
size_x_new = size(x_new);
size_y_old = size(y_old);

    % rotates input arrays conveniently
    if min(size_x_old)~1||min(size_y_old)~1||min(size_x_new)~1
        error([ 'sincinterp error: input arguments of sincinterp
            must be vectors' ])
    end

    if max(size_x_old)~max(size_y_old)
        error([ 'sincinterp error: dimension mismatch in input
            arguments' ])
    end

    if size_x_old(1)~min(size_x_old)
        X_old = x_old.';
else
    X_old = x_old;
end
if size_x_new(1)==min(size_x_new)
    X_new = x_new';
else
    X_new = x_new;
end
if size_y_old(1)==min(size_y_old)
    Y_old = y_old;
    rot_flag = 0;
else
    Y_old = y_old';
    rot_flag = 0;
end

%shift
%[Y_old(1) Y_old(end)]
[Y_min_ext, ind_min_ext] = min([abs(Y_old(1)) abs(Y_old(end))]);
if ind_min_ext==1
    sig_min_ext = sign(Y_old(1));
else
    sig_min_ext = sign(Y_old(end));
end
Y_old = Y_old-sig_min_ext*Y_min_ext;
%[Y_old(1) Y_old(end)]

%flip
if y_old(1)==y_old(end)
    % flip is needed
    if ind_min_ext==1
\[ X_{\text{old}}_c = X_{\text{old}} + (X_{\text{old}}(\text{end}) - X_{\text{old}}(1) + \text{dxold}); \]
\[ X_{\text{old}} = [X_{\text{old}}, X_{\text{old}}_c]; \]
\%sizeXold = size(X_{\text{old}})

\[ X_{\text{new}}_c = X_{\text{new}} + (X_{\text{new}}(\text{end}) - X_{\text{new}}(1) + \text{dxnew}); \]
\[ X_{\text{new}} = [X_{\text{new}}, X_{\text{new}}_c]; \]
\%sizeXnew = size(X_{\text{new}})

\[ Y_{\text{old}} = [Y_{\text{old}}; \text{flipud}(Y_{\text{old}})]; \]
\%sizeYold = size(Y_{\text{old}})

else
\[ X_{\text{old}}_c = X_{\text{old}} + (X_{\text{old}}(1) - X_{\text{old}}(\text{end}) - \text{dxold}); \]
\[ X_{\text{old}} = [X_{\text{old}}_c, X_{\text{old}}]; \]
\%sizeXold = size(X_{\text{old}})

\[ X_{\text{new}}_c = X_{\text{new}} + (X_{\text{new}}(1) - X_{\text{new}}(\text{end}) - \text{dxnew}); \]
\[ X_{\text{new}} = [X_{\text{new}}_c, X_{\text{new}}]; \]
\%sizeXnew = size(X_{\text{new}})

\[ Y_{\text{old}} = [\text{flipud}(Y_{\text{old}}); Y_{\text{old}}]; \]
\%sizeYold = size(Y_{\text{old}})

end

\% if Y_{\text{old}}(1) = Y_{\text{old}}(\text{end})
\% if ind_min_ext == 1
\% \[ Y_{\text{old}} = [Y_{\text{old}}; Y_{\text{old}}(1)]; \]
\% \[ X_{\text{old}} = [X_{\text{old}}, X_{\text{old}}(\text{end}) + \text{dxold}]; \]
\% else
\% \[ Y_{\text{old}} = [Y_{\text{old}}(\text{end}); Y_{\text{old}}]; \]
\% \[ X_{\text{old}} = [X_{\text{old}}(1) - \text{dxold}, X_{\text{old}}]; \]
\% end
% end
%S = @(xold,xnew) sinc((pi/dx)*xnew - pi*(floor((xold-xold(1)))/dx)));
%S = @(xshift,xnew) sinc((pi/dx)*xnew - pi*xshift);
%[Xshift,Xnew] = meshgrid([0:N-1],X_new);
[Xnew,Xold] = ndgrid(X_new,X_old);
SS = sinc((Xnew-Xold)/dxold);
%[Xnewshift,Xoldshift] = ndgrid([0:length(X_new)-1],[0:length(X_old)-1]);
%SS = sinc(Xnewshift-Xoldshift);
%SSsize = size(SS)
%Yoldsize = size(Y_old)
Y_new = SS*Y_old;

%reshift
Y_new = Y_new+sig_min_ext*Y_min_ext;
% if Y_old(1)==Y_old(end)
% if ind_min_ext==1
% Y_new = Y_new(1:end-1);
% else
% Y_new = Y_new(2:end);
% end
% end
if ind_min_ext==1
  Y_new = Y_new(1:length(x_new));
else
  Y_new = Y_new(length(x_new)+1:end);
end
if rot_flag==1
  y_new = Y_new.';
else
  y_new = Y_new;
end
end

return
Appendix O

Pseudospectral Fourier Discretization

\texttt{function [x,t,u1,u2,u3] = twri\_solver(c,s,a,eta,u0,L,Nx,T,Nt)}

% pseudo-spectral code for the 3WRI
%
% c = [c1,c2] is the array of the speeds
% s = [s1,s2,s3] is the array of the signs
% a = [a1,a2] is the array of the amplitudes
% eta = [eta1,eta2] is the array of the frequencies
%
% u0 is the initial condition, which has to be provided as Nx-by-3 array,
% with the first column being the values of u1 at t=0, the second column
% being the values of u2 at t=0, and the third column being the values of u3 at t=0
%
% [-L:dx:L-dx] is the range of integration, where dx=2*L/Nx
% and Nx is the number of spatial nodes
Appendix O. Pseudospectral Fourier Discretization

17 % T is the time range of the integration, including Nt time points
18 %
19 % x is the vector of the nodes
20 % t is the time vector
21 % u1, u2, u3 is the solution in the form of three Nx-by-3 arrays
22 %
23 %
24 %
25 % setting global variables
26 %
27 global time_start tout
28 % ridefines the final time as a global variable
29 tout = T;
30 %
32 %
33 % computational grid in Fourier space
34 %
35 %
36 % setup grid
37 dx = 2*L/Nx;
38 x = [-L:dx:L-dx];
% fourier wavenumbers
dk = pi/L;
k = fftshift([-Nx/2:(Nx/2)-1]*dk);
k = k .';

% parameters

% Given speeds, signs, amplitudes and frequencies, reconstruct the
% parameters in the equation and in the plane wave solution.
aq = a.^2;
r = (c(1)-c(2))/(eta(1)+eta(2));
u1 = (eta(1)/c(1))+s(1)*s(3)*aq(2)*(c(2)/c(1))*r;
u2 = (eta(2)/c(2))+s(2)*s(3)*aq(1)*(c(1)/c(2))*r;
u3 = -(nu1+nu2);
omega = c(1)*nu1+c(2)*nu2;

% initial condition

U1 = u0(:,1); U2 = u0(:,2); U3 = u0(:,3);
V1 = \exp(1i*(\nu1*x(:))).*U1;
V2 = \exp(1i*(\nu2*x(:))).*U2;
V3 = \exp(1i*(\nu3*x(:))).*U3;
V0 = [V1(:); V2(:); V3(:)];

% parameters for the ODE solver
%

% sets appropriate ODE solver options
opts = odeset('OutputFcn', @twri_output, ....
              'Refine', 1, ....
              'Stats', 'on', ....
              'RelTol', 1e-9, 'AbsTol', 1e-6, ....
              'MaxStep', 1e-3, ....
              'InitialStep', 1e-9, ....
              'NormControl', 'on');

% integration
%

% starts the clock
time_start = tic;

disp(['pseudo-spectral code for HF: calculation started'])

% performs the integration in time
dt = T/Nt;
[t,y] = ode45(@(tt,yy) twri_rhs(tt,yy,Nx,k,c,s,omega),[0:dt:T],V0,opts);

% recovering output solution from integration

v1 = y(:,1:Nx);
v2 = y(:,Nx+1:2*Nx);
v3 = y(:,2*Nx+1:3*Nx);

[MX,MT] = meshgrid(x,t);
u1 = exp(-1i*nu1*(MX-c(1)*MT)).*v1;
u2 = exp(-1i*nu2*(MX-c(2)*MT)).*v2;
u3 = exp(-1i*(nu3*MX+omega*MT)).*v3;
disp(['pseudo-spectral code for 3WRI: calculation completed'])

% computing the time elapsed since the beginning of the integration


total_time_elapsed = toc(time_start);
total_days = datenum([0 0 0 0 total_time_elapsed]);
time_left = datevec(total_days - floor(total_days));
disp(['total time taken for the integration = ' int2str(floor(total_days)), 'd ', int2str(time_left(4)), 'h'])
Appendix O. Pseudospectral Fourier Discretization

112 \texttt{int2str(time\_left(5)), 'm ',...}
113 \texttt{num2str(time\_left(6)), 's']}
114 \texttt{return}
115
116 \%
117 \% twri\_output function, called at each time step
118 \%
119
120 \texttt{function status = twri\_output(tt,yy,flag)}
121 \texttt{global time\_start tout}
122 \texttt{status = 0;}
123 \texttt{if (strcmp(flag,''))}
124 \texttt{time\_elapsed = toc(time\_start);}
125 \texttt{seconds\_left = ((tout/tt(end))-1)*time\_elapsed;}
126 \texttt{days\_left = datenum([0 0 0 0 0 seconds\_left]);}
127 \texttt{time\_left = datevec(days\_left - floor(days\_left));}
128 \texttt{disp(['3wri\_solver: time = ', num2str(tt(end)),...}
129 \texttt{('',num2str(tout),') computation time left = ',...}
130 \texttt{int2str(floor(days\_left)), 'd ',...}
131 \texttt{int2str(time\_left(4)), 'h ',...}
132 \texttt{int2str(time\_left(5)), 'm ',...}
133 \texttt{num2str(time\_left(6)), 's'])}
134 \texttt{end}
135 \texttt{return}
136
137 \%
138 \%
Appendix O. Pseudospectral Fourier Discretization

138  % twri_rhs, the right-hand side of the 3WRI system
139  %

140  function dvdt = twri_rhs(tt,vv,Nx,k,c,s,omega)
141  %sizevv = size(vv)
142  v = reshape(vv,Nx,3);
143  v1 = v(:,1); v2 = v(:,2); v3 = v(:,3);
144  v1c = conj(v1); v2c = conj(v2); v3c = conj(v3);
145  v1x = ifft(1i*k.*fft(v1));
146  v2x = ifft(1i*k.*fft(v2));
147  v3x = ifft(k.*fft(v3));
148  %v3x = ifft(k.*fft(v3));
149
150  v1t = -c(1)*v1x+c(2)*s(1)*v2c.*v3c;
151  v2t = -c(2)*v2x+c(1)*s(2)*v1c.*v3c;
152  v3t = 1i*omega*v3+(c(1)-c(2))*s(3)*v1c.*v2c;
153
154  dvdt = [v1t; v2t; v3t];
155  return
Appendix P

Numerical Integration

% Flags for saving figures and generating data
%
% Dataflag. If 'dataflag=0' no data is generated. If 'dataflag =1', data is
% generated.
dataflag = 0;
% Saveflag. If 'saveflag=0' no figure is saved. If 'saveflag =1', all
% figures are saved.
saveflag = 1;
% Extractionflag. If 'extractionflag=0' no interpolated plot
% is extracted.
extractionflag = 1;
% If 'extractionflag=1', an interpolated plot is extracted at
% the time and
% space specified.
extractionflag = 1;
Appendix P. Numerical Integration

% Parameters for the plane wave and perturbation
% if c1 = -c2, then use the following syntax:
% epsilon = 1e-3; % epsilon approaches zero
% [c, s, a] = twri_from_p([p1 p2 p3/epsilon epsilon])
% for some values p1, p2, p3
p = [0.2 0.3 -0.6 1];
[c, s, a] = twri_from_p(p);
spectrumtopology = [2 0 0 1 0]; % G SG B L TL
experiment_number = ['01']; % update with the experiment number, namely,
% specify if this is the first, second, third, ... experiment with the same
topology of the spectrum.

% do not modify eta
eta = [1,1];

% Computational grid
L = 20;
numx = $2^{10} \times 5 \times 2^{7} \times 2^{9}$
dx = $2L/nux$; x = $[-L:dx:L-dx]$;
xspan = [x(1) x(end)];

T = 120;
numt = 600;
tspan = [0 T];
dt = (tspan(2) - tspan(1))/numt;
t = [tspan(1):dt:tspan(2)];

%[XX,TT] = meshgrid(x, t);

% values of the subgrid from the zoom-in
xmin = 0; xmax = 20;
tmin = 100; tmax = 120;
if tmax > T; tmax = T; end
if xmax > L - dx; xmax = L - dx; end
if xmin < -L; xmin = -L; end

% value of the time and spatial range for extracting an interpolated plot
t_extraction = 120;
x_extraction_min = 0; x_extraction_max = 20;
um_x_extraction = 1e4; % interpolation points
if t_extraction > T; t_extraction = T; end
if x_extraction_max > L - dx; x_extraction_max = L - dx; end
if x_extraction_min < -L; x_extraction_min = -L; end

% Perturbation
Appendix P. Numerical Integration

% amplitude of the perturbation in each component
epsilon1 = 1e-3;
epsilon2 = 1e-3;
epsilon3 = 1e-3;

% Perturbation mode. If pert_mode=0, then localised
% perturbation. If
% pert_mode=1, then random pertubration.
pert_mode = 1;

if pert_mode==0
    pert = @(x) cos(pi*x/(2*L)).*exp(-(2*(x).^2));
elseif pert_mode==1
    pert = @(x) 2*rand(size(x))-1;
end

% Plane wave solution

p1 = (cq(1)*alphaq(1)*s(1)+cq(2)*alphaq(2)*s(2))/(s(1)*s(2)*s(3));
Appendix P. Numerical Integration

88 \% p2 = (cq(1) * alphaq(1) * s(1) - cq(2) * alphaq(2) * s(2)) / (s(1) * s(2) * s(3));
89 \% p3 = (c(1) - c(2)) / (c(1) + c(2));
90
91 aq = a.ˆ2;
92 r = (c(1) - c(2)) / (eta(1) + eta(2));
93 nu1 = (eta(1) / c(1)) + s(1) * s(3) * aq(2) * (c(2) / c(1)) * r;
94 nu2 = (eta(2) / c(2)) + s(2) * s(3) * aq(1) * (c(1) / c(2)) * r;
95 nu3 = -(nu1 + nu2);
96 eta3 = -(eta(1) + eta(2));
97 \% omega = c(1) * nu1 + c(2) * nu2;
98
99 u1ex = @(x, t) a(1) * exp(1i * (eta(1) * t - nu1 * x));
100 u2ex = @(x, t) a(2) * exp(1i * (eta(2) * t - nu2 * x));
101 u3ex = @(x, t) 1i * s(3) * a(1) * a(2) * r * exp(1i * (eta3 * t - nu3 * x));
102
103 U10 = u1ex(x, 0) + epsilon1 * pert(x) * exp(1i * (-nu1 * x));
104 U20 = u2ex(x, 0) + epsilon2 * pert(x) * exp(1i * (-nu2 * x));
105 U30 = u3ex(x, 0) + 1i * epsilon3 * pert(x) * exp(1i * (-nu3 * x));
106
107 U0 = [U10(:, :) U20(:, :) U30(:, :)];
108
109 \%
110 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
111 % Numerical integration
112 \%
113 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
114
115 if dataflag == 1
116 [xx, tt, u1, u2, u3] = twri_solver(c, s, a, eta, U0, L, numx, T, numt);
117 \;
end

% disp ([ ' p1 = ', num2str ( p1 ) ])
% disp ([ ' p2 = ', num2str ( p2 ) ])
% disp ([ ' p3 = ', num2str ( p3 ) ])

reu1min = floor (10 * min (min (real (u1)))) / 10; reu1max = ceil (10 * max (max (real (u1)))) / 10; reu1ticks = linspace (reu1min, reu1max, 5);
imu1min = floor (10 * min (min (imag (u1)))) / 10; imu1max = ceil (10 * max (max (imag (u1)))) / 10; imu1ticks = linspace (imu1min, imu1max, 5);
absu1min = floor (10 * min (min (abs (u1)))) / 10; absu1max = ceil (10 * max (max (abs (u1)))) / 10; absu1ticks = linspace (absu1min, absu1max, 5);

reu2min = floor (10 * min (min (real (u2)))) / 10; reu2max = ceil (10 * max (max (real (u2)))) / 10; reu2ticks = linspace (reu2min, reu2max, 5);
imu2min = floor (10 * min (min (imag (u2)))) / 10; imu2max = ceil (10 * max (max (imag (u2)))) / 10; imu2ticks = linspace (imu2min, imu2max, 5);
absu2min = floor (10 * min (min (abs (u2)))) / 10; absu2max = ceil (10 * max (max (abs (u2)))) / 10; absu2ticks = linspace (absu2min, absu2max, 5);

reu3min = floor (10 * min (min (real (u3)))) / 10; reu3max = ceil (10 * max (max (real (u3)))) / 10; reu3ticks = linspace (reu3min, reu3max, 5);
imu3min = floor (10 * min (min (imag (u3)))) / 10; imu3max = ceil (10 * max (max (imag (u3)))) / 10; imu3ticks = linspace (imu3min, imu3max, 5);
absu3min = floor (10 * min (min (abs (u3)))) / 10; absu3max = ceil (10 * max (max (abs (u3)))) / 10; absu3ticks = linspace (absu3min,
absu3max, 5);

% Exact solution

[XX, TT] = meshgrid(xx, tt);
U1 = u1ex(XX, TT); U2 = u2ex(XX, TT); U3 = u3ex(XX, TT);
reU1min = floor(10 * min(min(real(U1)))) / 10; reU1max = ceil(10 * max(max(real(U1)))) / 10; reU1ticks = linspace(reU1min, reU1max, 5);
imU1min = floor(10 * min(min(imag(U1)))) / 10; imU1max = ceil(10 * max(max(imag(U1)))) / 10; imU1ticks = linspace(imU1min, imU1max, 5);
absU1min = floor(10 * min(min(abs(U1)))) / 10; absU1max = ceil(10 * max(max(abs(U1)))) / 10; absU1ticks = linspace(absU1min, absU1max, 5);
reU2min = floor(10 * min(min(real(U2)))) / 10; reU2max = ceil(10 * max(max(real(U2)))) / 10; reU2ticks = linspace(reU2min, reU2max, 5);
imU2min = floor(10 * min(min(imag(U2)))) / 10; imU2max = ceil(10 * max(max(imag(U2)))) / 10; imU2ticks = linspace(imU2min, imU2max, 5);
absU2min = floor(10 * min(min(abs(U2)))) / 10; absU2max = ceil(10 * max(max(abs(U2)))) / 10; absU2ticks = linspace(absU2min, absU2max, 5);
Appendix P. Numerical Integration

reU3min = floor(10*min(min(real(U3))))/10; reU3max = ceil(10*max(max(real(U3))))/10; reU3ticks = linspace(reU3min, reU3max, 5);

imU3min = floor(10*min(min(imag(U3))))/10; imU3max = ceil(10*max(max(imag(U3))))/10; imU3ticks = linspace(imU3min, imU3max, 5);

absU3min = floor(10*min(min(abs(U3))))/10; absU3max = ceil(10*max(max(abs(U3))))/10; absU3ticks = linspace(absU3min, absU3max, 5);

% Parameters

disp([' '])
disp(['
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  '])
disp([' parameters:   p1=', num2str(p(1)), ', p2=', num2str(p(2)), ', p3=', num2str(p(3)), ', p4=', num2str(p(4))]);
disp([' velocities:   c1=', num2str(c(1)), ', c2=', num2str(c(2))]);
disp([' signs:       s1=', int2str(s(1)), ', s2=', int2str(s(2)), ', s3=', int2str(s(3))]);
disp([' amplitudes:  a1=', num2str(a(1)), ', a2=', num2str(a(2))]);
disp([' '])

%
% Differences

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

disp(['

if pert_mode==0
    disp(['localized perturbation: ' , char(pert)])
else if pert_mode==1
    disp(['random perturbation: ' , char(pert)])
end

initrelerru1 = 100*max(max(abs(U1(1,:)-u1(1,:))))/max(max(abs(U1(1,:))));
initrelerru2 = 100*max(max(abs(U2(1,:)-u2(1,:))))/max(max(abs(U2(1,:))));
initrelerru3 = 100*max(max(abs(U3(1,:)-u3(1,:))))/max(max(abs(U3(1,:))));
maxrelerru1 = 100*max(max(abs(U1-u1)))/max(max(abs(U1)));
maxrelerru2 = 100*max(max(abs(U2-u2)))/max(max(abs(U2)));
maxrelerru3 = 100*max(max(abs(U3-u3)))/max(max(abs(U3)));

disp(['deviation from unperturbed solution ']);
disp(['U1>> initial: ' , num2str(initrelerru1), '% —— maximum: ' , num2str(maxrelerru1), '%']);
disp(['U2>> initial: ' , num2str(initrelerru2), '% —— maximum: ' , num2str(maxrelerru2), '%']);
disp(['U3>> initial: ' , num2str(initrelerru3), '% —— maximum: ' , num2str(maxrelerru3), '%']);
disp([''])

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% Extracting a subplot (zoom-in) in the region [xmin xmax tmin tmax]

% 
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 

[ Nr, Nc ] = size(XX);
YY = (XX>=xmin) & (XX<=xmax) & (TT>=tmin) & (TT<=tmax);
colind = ceil(find(YY)/Nr);
rowind = find(YY)-(colind-1)*Nr;
colind = unique(colind);
rowind = unique(rowind);
sX = XX(rowind,colind);
sT = TT(rowind,colind);
su1 = u1(rowind,colind);
su2 = u2(rowind,colind);
su3 = u3(rowind,colind);

% 
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 

% Extracting a subplot at a given specific time in a given spatial range
% with interpolation of the result for high-definition plotting
%
% 
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 

if extractionflag==1
    dxinterp = (x_extraction_max-x_extraction_min)/num_x_extraction;
end
x_interp = x_extraction_min : dx_interp : x_extraction_max;
[-, ind_extraction] = min(abs(t - t_extraction)); t_extraction_eff = t(ind_extraction);
x_extraction = xx;
u1_extraction = u1(ind_extraction, :);
u2_extraction = u2(ind_extraction, :);
u3_extraction = u3(ind_extraction, :);
%s_size_u1_extraction = size(u1_extraction)
%s_size_x_extraction = size(x_extraction)
x_extraction_range = (x_extraction >= x_extraction_min) & (x_extraction <= x_extraction_max);
x_extraction = x_extraction(x_extraction_range);
u1_extraction = u1_extraction(x_extraction_range);
u2_extraction = u2_extraction(x_extraction_range);
u3_extraction = u3_extraction(x_extraction_range);
%s_size_u1_extraction = size(u1_extraction)
%s_size_x_extraction = size(x_extraction)
u1_interp = sincinterpr(u1_extraction, x_extraction, x_interp);
u2_interp = sincinterpr(u2_extraction, x_extraction, x_interp);
u3_interp = sincinterpr(u3_extraction, x_extraction, x_interp);
end

%%
% General plotting parameters
%%
% load colormaps
% load('redblue')
% load('blueredyellow')
% load('blueredblue')
% load('redbluered')
% load('redbluered2')

% options for plotting
linwidth = 1.5; % Line Width
fnsztl = 16; % Font Size for Plot Titles
fnwgtl = 'bold'; % Font Weight for Plot Titles
fnszlb = 14; % Font Size for Plot Axis Labels
fnwglb = 'bold'; % Font Weight for Plot Axis Labels
fnsztk = 14; % Font Size for Plot Ticks
fnwgtk = 'normal'; % Font Weight for Plot Ticks
fnszcb = 14; % Font Size color bar
fnwgcbr = 'normal'; % Font Weight color bar
fnmncb = 'Serif'; % Font Name color bar

% Plotting the output

hf1 = figure(1);
clf
% colormap(jet) % use this map as an alternative colormap
```matlab
fig1a = subplot(2,3,1);
pcolor(xx, tt, abs(u1))
xlabel('\textbf{$x$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
ylabel('\textbf{$t$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
title('\textbf{|$u_1|$} (numerical)', 'interpreter', 'Latex', 'fontsize', fnstl, 'fontweight', fnwgtl)
colormap(fig1a, jet)
shading flat
axis([-L L linspace(1) linspace(2)])
ax = gca;
ax.FontSize = fnstkb;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu1min absu1max]);
hBarhf1a = colorbar('location', 'southoutside', 'xtick', absu1ticks);
labelshf1a = {absu1min:(absu1max-absu1min)/4:absu1max};
set(hBarhf1a, 'XTickLabel', labelshf1a, 'fontsize', fnscb, 'fontweight', fnwgtcb, 'fontname', fnncb);

hf1b = subplot(2,3,2);
pcolor(xx, tt, abs(u2))
xlabel('\textbf{$x$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
ylabel('\textbf{$t$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
title('\textbf{|$u_2|$} (numerical)', 'interpreter', 'Latex', 'fontsize', fnstl, 'fontweight', fnwgtl)
colormap(hf1b, jet)
shading flat
axis([-L L linspace(1) linspace(2)])
```

*Appendix P. Numerical Integration*
ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu2min absu2max]);
hBarhf1b = colorbar('location','southoutside','xtick',
    absu2ticks);
labelsf1b = {absu2min:(absu2max–absu2min)/4:absu2max};
set(hBarhf1b,'XTickLabel',labelsf1b,'fontsize',fnszcb,'
    fontweight',fnwgcbb,'fontname',fnncbb);
fig1c = subplot(2,3,3);
pcolor(xx,tt,abs(u3))
xlabel('$\mathbf{x}\$','interpreter','LaTeX','fontsize',
    fnszlb,'fontweight',fnwglb)
ylabel('$\mathbf{t}\$','interpreter','LaTeX','fontsize',
    fnszlb,'fontweight',fnwglb)
title('$\mathbf{|u_{\{3\}}|}$ (numerical),'interpreter','
    LaTeX','fontsize',fnsztl,'fontweight',fnwgtlb)
colormap(fig1c,jet)
shading flat
axis([-L L tspan(1) tspan(2)])
ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu3min absu3max]);
hBarhflc = colorbar('location','southoutside','xtick',
    absu3ticks);
labelsf1c = {absu3min:(absu3max–absu3min)/4:absu3max};
set(hBarhflc,'XTickLabel',labelsf1c,'fontsize',fnszcb,'
    fontweight',fnwgcbb,'fontname',fnncbb);
fig1d = subplot(2,3,4);
% Figure 1d: Numerical integration of function u(x, t)

colorbar('location', 'southoutside', 'xtick', absu1ticks);

labelshf1d = {absu1min: (absu1max - absu1min) / 4: absu1max};

set(hBarhf1d, 'XTickLabel', labelshf1d, 'fontsize', fnszcb, 
    'fontweight', fnwgbcb, 'fontname', fnnmcb);

fig1e = subplot(2,3,5);

colorbar(fig1e, jet)

shading flat

axis([xmin xmax tmin tmax])

ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu2min absu2max]);
hBarh1e = colorbar('location','southoutside','xtick',
   absu2ticks);
labelshf1e = {absu2min:(absu2max−absu2min)/4:absu2max};
set(hBarh1e,'XTickLabel',labelshf1e,'fontsize',fnszcb,'
   fontweight',fnwgc,'fontname',fnmcb);
fig1f = subplot(2,3,6);
pcolor(sX,sT,abs(su3))
xlabel('\boldmath{$x$}','interpreter','Latex','fontsize',
   fnszlb,'fontweight',fnwglb)
ylabel('\boldmath{$t$}','interpreter','Latex','fontsize',
   fnszlb,'fontweight',fnwglb)
title('\boldmath{$|u_{\{3\}}|$} (numerical)','interpreter','
   Latex','fontsize',fnsztl,'fontweight',fnwglt)
colormap(fig1f,jet)
shading flat
axis([xmin xmax tmin tmax])
ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu3min absu3max]);
hBarh1f1f = colorbar('location','southoutside','xtick',
   absu3ticks);
labelshf1f = {absu3min:(absu3max−absu3min)/4:absu3max};
set(hBarh1f1f,'XTickLabel',labelshf1f,'fontsize',fnszcb,'
   fontweight',fnwgc,'fontname',fnmcb);
fig = gcf;
fig.PaperPositionMode = 'auto';
Appendix P. Numerical Integration

351 \texttt{fig\_pos = fig\_PaperPosition;} \\
352 \texttt{fig\_PaperSize = [fig\_pos(3) \ fig\_pos(4)];} \\
353 %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
354 \\
355 hf2 = \texttt{figure(2);} \\
356 \texttt{clf} \\
357 jtn = 5; \\
358 \texttt{for \ jtn = 1:jtn} \\
359 \texttt{ind = floor(numt*jt/(jtn-1)+1-numt/(jtn-1));} \\
360 \texttt{subplot(jtn,3,3*(jt-1)+1)} \\
361 \texttt{plot(xx,abs(U1(ind,:)),'b')} \\
362 \texttt{hold on} \\
363 \texttt{plot(xx,abs(u1(ind,:)),'r')} \\
364 \texttt{hold off} \\
365 \texttt{xlabel(‘\boldmath$x$’,'interpreter','Latex','}
366 \texttt{fontsize’,fnszl,’fontweight’,fnwglb)} \\
367 \texttt{ylabel(‘\boldmath{$|u_{1}\{1\}|$’,'interpreter','Latex’,'}
368 \texttt{fontsize’,fnszl,’fontweight’,fnwglb)} \\
369 \texttt{title([’t=’,num2str(tt(ind)),’,(ex/b, num/r)’],’}
370 \texttt{interpreter','Latex','fontsize’,fnsztl,’fontweight’}
371 \texttt{,fnwgtl)} \\
372 \texttt{axis([-L L absu1min absu1max])} \\
373 \texttt{axis tight} \\
374 \texttt{subplot(jtn,3,3*(jt-1)+2)} \\
375 \texttt{plot(xx,abs(U2(ind,:)),’b’)} \\
376 \texttt{hold on} \\
377 \texttt{plot(xx,abs(u2(ind,:)),’r’)} \\
378 \texttt{hold off}
\begin{verbatim}
xlabel('$x$', 'interpreter', 'Latex', 'fontsize', fnszlb, 'fontweight', fnwglb)
ylabel('$|u_2|$', 'interpreter', 'Latex', 'fontsize', fnszlb, 'fontweight', fnwglb)
title(['$t=$', num2str(tt(ind)), ', (ex/b, num/r)'], 'interpreter', 'Latex', 'fontsize', fnsztl, 'fontweight', fnwgtl)
%axis([-L L absu2min absu2max])
axis tight

subplot(jtn, 3, 3*(jt-1)+3)
plot(xx, abs(U3(ind,:)), 'b')
hold on
plot(xx, abs(u3(ind,:)), 'r')
hold off
xlabel('$x$', 'interpreter', 'Latex', 'fontsize', fnszlb, 'fontweight', fnwglb)
ylabel('$|u_3|$', 'interpreter', 'Latex', 'fontsize', fnszlb, 'fontweight', fnwglb)
title(['$t=$', num2str(tt(ind)), ', (ex/b, num/r)'], 'interpreter', 'Latex', 'fontsize', fnsztl, 'fontweight', fnwgtl)
%axis([-L L absu3min absu3max])
axis tight

end
fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

%%%%%%%%%%%%%%%%%
\end{verbatim}
if extractionflag==1
    hf3 = figure(3);
    clf
    subplot(3,1,1)
        hold on
        plot(x_extraction, abs(u1_extraction), 'r.' )
        plot(x_interp, abs(u1_interp), 'b')
        hold off
        xlabel('$\text{boldmath}$\{x\}$','interpreter','Latex',''
               fontsize',fnszlb,'fontweight',fnwglb)
        ylabel('$\text{boldmath}$\{|u\{1\}|}$','interpreter','Latex',''
               fontsize',fnszlb,'fontweight',fnwglb)
        title([ ActionControllerSource] 't=',num2str(t_extraction_eff),' , (num/r ,
               interp/b)' ],'interpreter','Latex','fontsize',fnstl
               ,'fontweight',fnwtl)
        axis tight
    subplot(3,1,2)
        hold on
        plot(x_extraction, abs(u2_extraction), 'r.' )
        plot(x_interp, abs(u2_interp), 'b')
        hold off
        xlabel('$\text{boldmath}$\{x\}$','interpreter','Latex',''
               fontsize',fnszlb,'fontweight',fnwglb)
        ylabel('$\text{boldmath}$\{|u\{2\}|}$','interpreter','Latex',''
               fontsize',fnszlb,'fontweight',fnwglb)
        title([ ActionControllerSource] 't=',num2str(t_extraction_eff),' , (num/r ,
               interp/b)' ],'interpreter','Latex','fontsize',fnstl
               ,'fontweight',fnwtl)
        axis tight
    subplot(3,1,3)
        hold on
plot(x_extraction, abs(u3_extraction), 'r.')
plot(x_interp, abs(u3_interp), 'b')
hold off
xlabel('$\textbf{x}$', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwglb)
ylabel('$|u_3|$', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwglb)
title(['t=', num2str(t_extraction_eff), ', (num/r, interp/b)'], 'interpreter', 'Latex', 'fontsize', fnsltl, 'fontweight', fnwgtl)
axis tight
fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];
end

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% Saving the figures
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

if saveflag==1
    if pert_mode==0
        pert_tag = ['localised'];
    elseif pert_mode==1
        pert_tag = ['random'];
    end
file_name = ['twri_planewave_', ....
    int2str(spectrumtopology(1)), 'G', ....
    int2str(spectrumtopology(2)), 'SG', ....
    int2str(spectrumtopology(3)), 'B', ....
    int2str(spectrumtopology(4)), 'L', ....
    int2str(spectrumtopology(5)), 'TL', ....
    pert_tag, '_', 'exp', experiment_number];

fig1a = figure(4);
clf
pcolor(xx, tt, abs(u1))
xlabel('\boldmath{$x$}', 'interpreter', 'Latex', 'fontsize',
    fnszlb, 'fontweight', fnwglb)
ylabel('\boldmath{$t$}', 'interpreter', 'Latex', 'fontsize',
    fnszlb, 'fontweight', fnwglb)
%title('\boldmath{|\mathbf{u}_{}|} (numerical)', 'interpreter', 'Latex', 'fontsize',
    fntsll, 'fontweight', fnwgtl)
colormap(fig1a, jet)
shading flat
axis([-L L tspan(1) tspan(2)])
ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu1min absu1max]);
hBarhfl1a = colorbar('location', 'southoutside', 'xtick',
    absulticks);
labelshfl1a = {absu1min:(absu1max–absu1min)/4:absu1max};
set(hBarhfl1a, 'XTickLabel', labelshfl1a, 'fontsize', fnszcb, 
    'fontweight', fnwgc, 'fontname', fnmcb);
fig = gcf;
fig.PaperPositionMode = 'auto';
Appendix P. Numerical Integration

```matlab
fig_pos = fig.PaperPosition;
fig.PageSize = [fig_pos(3) fig_pos(4)];

fig1b = figure(5);
clf
pcolor(xx, tt, abs(u2))
xlabel('\textbf{$x$}', 'interpreter', 'Latex', 'fontsize', fnszlb, 'fontweight', fnwglb)
ylabel('\textbf{$t$}', 'interpreter', 'Latex', 'fontsize', fnszlb, 'fontweight', fnwglb)
%title('\textbf{\mid |u_2|} \text{(numerical)}', 'interpreter', 'Latex', 'fontsize', fnsztl, 'fontweight', fnwgtl)

colormap(fig1b, jet)
shading flat
axis([-L L tspan(1) tspan(2)])
ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu2min absu2max]);

hBarhf1b = colorbar('location', 'southoutside', 'xtick', absu2ticks);
labelshf1b = {absu2min:(absu2max–absu2min)/4:absu2max};
set(hBarhf1b, 'XTickLabel', labelshf1b, 'fontsize', fnszcb, 'fontweight', fnwgcbl, 'fontname', fnmcb);

fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PageSize = [fig_pos(3) fig_pos(4)];

fig1c = figure(6);
clf
```

Appendix P. Numerical Integration

\begin{verbatim}
500   pcolor(xx, tt, abs(u3))
501   xlabel('\textbf{$x$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
502   ylabel('\textbf{$t$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
503   title('\textbf{\{u_{\{3\}}\}}$ (numerical)', 'interpreter', 'Latex', 'fontsize', fnstl, 'fontweight', fnwgtl)
504   colormap(fig1c, jet)
505   shading flat
506   axis([−L L tspan(1) tspan(2)])
507   ax = gca;
508   ax.FontSize = fnstk;
509   ax.FontWeight = fnwgtk;
510   ax.TickLabelInterpreter = 'latex';
511   caxis([absu3min absu3max]);
512   hBarh1c = colorbar('location', 'southoutside', 'xtick', absu3ticks);
513   labelsH1c = {absu3min:(absu3max−absu3min)/4:absu3max};
514   set(hBarh1c, 'XTickLabel', labelsH1c, 'fontsize', fnszcb, 'fontweight', fnwgb, 'fontname', fnnmc);
515   fig = gcf;
516   fig.PaperPositionMode = 'auto';
517   fig.pos = fig.PaperPosition;
518   fig.PaperSize = [fig.pos(3) fig.pos(4)];
519
520   fig1d = figure(7);
521   clf
522   pcolor(sX, sT, abs(su1))
523   xlabel('\textbf{$x$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
524   ylabel('\textbf{$t$}', 'interpreter', 'Latex', 'fontsize', fnslb, 'fontweight', fnwlb)
\end{verbatim}
%title('\textbf{\textdollar u\{1\}\textdollar} (\textit{numerical}), 'interpreter', 'Latex', 'fontsize', fnszt1, 'fontweight', fnwtlt)

colormap(fig1d, jet)
shading flat
axis([xmin xmax tmin tmax])
ax = gca;
ax.FontSize = fnszt1;
ax.FontWeight = fnwtlt;
ax.TickLabelInterpreter = 'latex';
caxis([absu1min absu1max]);
hBarhf1d = colorbar('location','southoutside','xtick',
absulticks);
labelshf1d = {absu1min:(absu1max-absu1min)/4:absu1max};
set(hBarhf1d, 'XTickLabel', labelshf1d, 'fontsize', fnszcb, 'fontweight', fnwgcgb, 'fontname', fnnmcb);

fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

fig1e = figure(8);
clf
pcolor(sX,sT,abs(su2))
xlabel('\textbf{\textdollar x\textdollar}', 'interpreter', 'Latex', 'fontsize',
fnszlb, 'fontweight', fnwglb)
ylabel('\textbf{\textdollar t\textdollar}', 'interpreter', 'Latex', 'fontsize',
fnszlb, 'fontweight', fnwglb)
%title('\textbf{\textdollar u\{2\}\textdollar} (\textit{numerical}), 'interpreter', 'Latex', 'fontsize', fnszt1, 'fontweight', fnwtlt)
colormap(fig1e, jet)
shading flat
axis([xmin xmax tmin tmax])
ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu2min absu2max]);

hBarhf1e = colorbar('location','southoutside','xtick',
    absu2ticks);

labels1f = {absu2min:(absu2max-absu2min)/4:absu2max};

set(hBarhf1e,'XTickLabel',labels1f,'fontsize',fnszcb,'
    fontweight',fnwgc,'fontname',fnmcb);

fig = gcf;

fig.PaperPositionMode = 'auto';
fig.pos = fig.PaperPosition;
fig.PaperSize = [fig.pos(3) fig.pos(4)];

fig1f = figure(9);
clf

pcolor(sX,sT,abs(su3))

xlabel('mathx$','interpreter','LaTeX','fontsize',
    fnszlb,'fontweight',fnwglb)

ylabel('math{t}','$','interpreter','LaTeX','fontsize',
    fnszlb,'fontweight',fnwglb)

title('math{|u_\{3\}|}$ (numerical)','interpreter',''
    'LaTeX','fontsize',fnsztl,'fontweight',fnwgtl)

colormap(fig1f.jet)

shading flat

axis([xmin xmax tmin tmax])
ax = gca;
ax.FontSize = fnsztk;
ax.FontWeight = fnwgtk;
ax.TickLabelInterpreter = 'latex';
caxis([absu3min absu3max]);
hBarhlf1 = colorbar('location','southoutside','xtick',
    absu3ticks);
lableshlf1 = {absu3min:(absu3max-absu3min)/4:absu3max};
set(hBarhlf1,'XTickLabel',lableshlf1,'fontsize',fnszcb, 
    'fontweight',fnwgb, 'fontname',fnnmb);
fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

fig3a = figure(10);
clf
hold on
plot(x_extraction,abs(u1_extraction),'r.'
    plot(x_interp,abs(u1_interp),'b')
hold off
xlabel('\textbf{$x$}', 'interpreter','Latex','fontsize',
    fnszlb, 'fontweight',fnwglb)
ylabel('\textbf{$|u_1|}$', 'interpreter','Latex','
    fontsize',fnszlb, 'fontweight',fnwglb)
%title(['t=',num2str(t_extraction_eff),' ',(num/r, interp/b 
    )']),'interpreter','Latex','fontsize',fnsztl, 
    'fontweight',fnwgtl)
axis tight
fig = gcf;
fig.PaperPositionMode = 'auto';
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

fig3b = figure(11);
clf
hold on
\begin{lstlisting}[language=Matlab]
plot(x_extraction, abs(u2_extraction), 'r.' )
plot(x_interp, abs(u2_interp), 'b')
hold off
xlabel('\textbf{\textit{x}}', 'interpreter', 'LaTeX', 'fontsize', fnszlb, 'fontweight', fnwglb)
ylabel('\textbf{\textit{|u_2|}}', 'interpreter', 'LaTeX', 'fontsize', fnszlb, 'fontweight', fnwglb)
%title(['t=', num2str(t_extraction_eff), ', (num/r, interp/b)'], 'interpreter', 'LaTeX', 'font size', fnsztl, 'fontweight', fnwgtl)
axis tight
fig = gcf;
fig . PaperPositionMode = 'auto';
fig_pos = fig . PaperPosition;
fig . PaperSize = [ fig_pos(3) fig_pos(4) ];

fig3c = figure(12);
clf
hold on
plot(x_extraction, abs(u3_extraction), 'r.' )
plot(x_interp, abs(u3_interp), 'b')
hold off
xlabel('\textbf{\textit{x}}', 'interpreter', 'LaTeX', 'fontsize', fnszlb, 'fontweight', fnwglb)
ylabel('\textbf{\textit{|u_3|}}', 'interpreter', 'LaTeX', 'fontsize', fnszlb, 'fontweight', fnwglb)
%title(['t=', num2str(t_extraction_eff), ', (num/r, interp/b)'], 'interpreter', 'LaTeX', 'font size', fnsztl, 'fontweight', fnwgtl)
axis tight
fig = gcf;
fig . PaperPositionMode = 'auto';
\end{lstlisting}
fig_pos = fig.PaperPosition;
fig.PaperSize = [fig_pos(3) fig_pos(4)];

print(fig1a,[pwd '/Figures/' file_name '_u1.jpeg'],'-djpeg')
print(fig1b,[pwd '/Figures/' file_name '_u2.jpeg'],'-djpeg')
print(fig1c,[pwd '/Figures/' file_name '_u3.jpeg'],'-djpeg')
print(fig1d,[pwd '/Figures/' file_name '_u1_zoom.jpeg'],'-djpeg')
print(fig1e,[pwd '/Figures/' file_name '_u2_zoom.jpeg'],'-djpeg')
print(fig1f,[pwd '/Figures/' file_name '_u3_zoom.jpeg'],'-djpeg')
print(fig3a,[pwd '/Figures/' file_name '_u1_(t',num2str(t_extraction),').jpeg'],'-djpeg')
print(fig3b,[pwd '/Figures/' file_name '_u2_(t',num2str(t_extraction),').jpeg'],'-djpeg')
print(fig3c,[pwd '/Figures/' file_name '_u3_(t',num2str(t_extraction),').jpeg'],'-djpeg')

print(fig1a,[pwd '/Figures/' file_name '_u1.eps'],'-depsc','-tif','-r600')
print(fig1b,[pwd '/Figures/' file_name '_u2.eps'],'-depsc','-tif','-r600')
print(fig1c,[pwd '/Figures/' file_name '_u3.eps'],'-depsc','-tif','-r600')
print(fig1d,[pwd '/Figures/' file_name '_u1_zoom.eps'],'-depsc','-tif','-r600')
Appendix P. Numerical Integration

print(fig1e,[pwd '/Figures/' file_name '_u2_zoom.eps'],'-depsc','-tiff','-r600')
print(fig1f,[pwd '/Figures/' file_name '_u3_zoom.eps'],'-depsc','-tiff','-r600')
print(fig3a,[pwd '/Figures/' file_name '_u1_(',t_,',num2str(t_extraction),').eps'],'-depsc','-tiff','-r600')
print(fig3b,[pwd '/Figures/' file_name '_u2_(',t_,',num2str(t_extraction),').eps'],'-depsc','-tiff','-r600')
print(fig3c,[pwd '/Figures/' file_name '_u3_(',t_,',num2str(t_extraction),').eps'],'-depsc','-tiff','-r600')
print(fig1a,[pwd '/Figures/' file_name '_u1.pdf'],'-dpdf','-r600') % painters
print(fig1b,[pwd '/Figures/' file_name '_u2.pdf'],'-dpdf','-r600')
print(fig1c,[pwd '/Figures/' file_name '_u3.pdf'],'-dpdf','-r600')
print(fig1d,[pwd '/Figures/' file_name '_u1_zoom.pdf'],'-dpdf','-r600')
print(fig1e,[pwd '/Figures/' file_name '_u2_zoom.pdf'],'-dpdf','-r600')
print(fig1f,[pwd '/Figures/' file_name '_u3_zoom.pdf'],'-dpdf','-r600')
print(fig3a,[pwd '/Figures/' file_name '_u1_(',t_,',num2str(t_extraction),').pdf'],'-dpdf','-r600')
print(fig3b,[pwd '/Figures/' file_name '_u2_(',t_,',num2str(t_extraction),').pdf'],'-dpdf','-r600')
print(fig3c,[pwd '/Figures/' file_name '_u3_(',t_,',num2str(t_extraction),').pdf'],'-dpdf','-r600')
end
Bibliography


