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# Wave breaking and the generation of undular bores in an integrable shallow-water system

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## Abstract

The generation of an undular bore in the vicinity of a wave-breaking point is considered for the integrable Kaup-Boussinesq shallow water system. In the framework of the Whitham modulation theory, an analytic solution of the Gurevich-Pitaevskii type of problem for a generic “cubic” breaking regime is obtained using a generalized hodograph transform, and a further reduction to a linear Euler-Poisson equation. The motion of the undular bore edges is investigated in detail.

## 1 Introduction

It is well known that the solutions to an initial value problem for the non-dispersive shallow water equations may lead to wave-breaking after a finite time, when the first spatial derivatives blow up. After the wave breaking point, a formal solution becomes multi-valued and loses its physical meaning. The divergence of the spatial derivatives at the wave-breaking point suggests that dispersion effects described by terms with higher order spatial derivatives must be taken into account. Then these small dispersion effects lead to the onset of oscillations in the vicinity of the wave breaking point followed by the development of an undular bore, or in different terminology, a dissipationless shock wave.

This physical picture has been put into mathematical form for waves described by the Korteweg-de Vries (KdV) equation by Gurevich and Pitaevskii (GP) in [1] (see also Whitham in [3]). In the GP formulation, the region of oscillations is presented as a slowly modulated periodic wave solution of the KdV equation. The parameters of the wave change little on typical wavelength/period scales which permits one to apply the Whitham modulation theory ([2, 3]). The resulting Whitham equations describe the slow evolution of the parameters in the undular bore. In the original paper [1], two typical problems were considered. One problem is concerned with the description of decay of an initial discontinuity for the KdV

equation and the exact  $x/t$ -similarity solution of this problem was constructed. Another problem corresponds to the (universal) initial stage of development of a bore when the solution of the dispersionless equation can be approximated locally by a properly chosen cubic curve. This problem was studied in [1] numerically. An exact analytic solution to this problem was later obtained by Potëmin [4] using Krichever's algebro-geometrical procedure for integration of the Whitham equations [5], [7]. Later, Potëmin's solution was put into the general context of Tsarev's generalized hodograph transform [6] in [10, 11, 12], where the hodograph equations were reduced to the classical Euler-Poisson equation.

However, the KdV equation describes unidirectional propagation of nonlinear dispersive waves. An integrable bi-directional analog of the KdV equation was derived by Kaup in [8] using the Boussinesq approximation for shallow water waves [3]. Like the KdV equation, the Kaup-Boussinesq (KB) system is completely integrable and therefore a powerful inverse scattering transform method can be applied to its investigation. In particular, the multi-phase periodic solutions of the KB system were found in [9], the Whitham theory of modulations was applied in [13] to the problem of the decay of an initial discontinuity, and a quasiclassical description of soliton trains arising from a large initial pulse was developed in [14].

In this paper, we further extend the Gurevich-Pitaevskii theory to the case of bi-directional shallow water equations using the KB system, and construct an analytic solution to the Whitham-KB equations for the regime of generation of an undular bore in the vicinity of a breaking point. The obtained solution, along with its own significance in the representation of undular bores, will serve as an intermediate asymptotic in a more general formulation we are presently undertaking, in which small dissipation is taken into account.

## 2 Periodic waves in the Kaup-Boussinesq system

In dimensionless units (see, e.g. [13]) the KB system can be written in the form

$$\begin{aligned} h_t + (hu)_x + \frac{1}{4}u_{xxx} &= 0, \\ u_t + uu_x + h_x &= 0, \end{aligned} \tag{1}$$

where  $h(x, t)$  denotes the height of the water surface above a horizontal bottom and  $u(x, t)$  is related to the horizontal velocity field (at the leading order it is the depth-averaged horizontal field).

The KB system (1) is completely integrable and can be represented as the compatibility condition of two linear equations [8]

$$\psi_{xx} = \mathcal{A}\psi, \quad \psi_t = -\frac{1}{2}\mathcal{B}_x\psi + \mathcal{B}\psi_x \tag{2}$$

with

$$\mathcal{A} = \left(\lambda - \frac{1}{2}u\right)^2 - h, \quad \mathcal{B} = -\left(\lambda + \frac{1}{2}u\right). \tag{3}$$

Thus, the inverse scattering transform method can be applied for its investigation. In particular, the periodic solution of (1) can be obtained by the well-known finite-gap integration

method (see, e.g. [15]) in the following way. Let  $\psi_+$  and  $\psi_-$  be two basis solutions of the second order spatial linear differential equation (2). Then their product

$$g = \psi_+ \psi_- \quad (4)$$

satisfies the third order equation

$$g_{xxx} - 2\mathcal{A}_x g - 4\mathcal{A}g_x = 0. \quad (5)$$

Upon multiplication by  $g$ , this equation can be integrated once to give

$$\frac{1}{2}gg_{xx} - \frac{1}{4}g_x^2 - \mathcal{A}g^2 = -P(\lambda) \quad (6)$$

where the integration constant  $P(\lambda)$  can only depend on  $\lambda$ . The time dependence of  $g(x, t)$  is determined by the equation

$$g_t = \mathcal{B}g_x - \mathcal{B}_x g. \quad (7)$$

This equation can readily be put in the form

$$\left(\frac{1}{g}\right)_t = \left(\frac{\mathcal{B}}{g}\right)_x, \quad (8)$$

which can in turn be considered as a generating function of an infinite sequence of conservation laws.

The periodic solutions of the system (1) are distinguished by the condition that  $P(\lambda)$  in (6) be a polynomial in  $\lambda$ . The one-phase periodic solution, which we are interested in, corresponds to the fourth degree polynomial

$$P(\lambda) = \prod_{i=1}^4 (\lambda - \lambda_i) = \lambda^4 - s_1 \lambda^3 + s_2 \lambda^2 - s_3 \lambda + s_4. \quad (9)$$

Then we find from Eq. (6) that  $g(x, t)$  is the first-degree polynomial,

$$g(x, t) = \lambda - \mu(x, t), \quad (10)$$

where  $\mu(x, t)$  is connected with  $u(x, t)$  and  $h(x, t)$  by the relations

$$u(x, t) = s_1 - 2\mu(x, t), \quad h(x, t) = \frac{1}{4}s_1^2 - s_2 - 2\mu^2 + s_1\mu, \quad (11)$$

which in turn follow from a comparison of the coefficients of  $\lambda^i$  on both sides of Eq. (6). The spectral parameter  $\lambda$  is arbitrary and on substitution of  $\lambda = \mu$  into Eq. (6) we obtain an equation for  $\mu$ ,

$$\mu_x = 2\sqrt{P(\mu)},$$

while a similar substitution into Eq. (7) gives

$$\mu_t = -\left(\mu + \frac{1}{2}u\right)\mu_x = -\frac{1}{2}s_1\mu_x.$$

Hence,  $\mu(x, t)$  as well as  $u(x, t)$  and  $h(x, t)$  depend only on the phase

$$\theta = x - \frac{1}{2}s_1 t, \quad (12)$$

$$\text{so that } V = \frac{1}{2}s_1 = \frac{1}{2} \sum_{i=1}^4 \lambda_i \quad (13)$$

is the phase velocity of the nonlinear wave, and  $\mu(\theta)$  is determined by the equation

$$\mu_\theta = 2\sqrt{P(\mu)}. \quad (14)$$

For the fourth degree polynomial (9) the solution of this equation is readily expressed in terms of elliptic functions. Let the zeros  $\lambda_i$ ,  $i = 1, 2, 3, 4$ , of the polynomial  $P(\lambda)$  be real and ordered according to the rule

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4. \quad (15)$$

Then the real variable  $\mu$  oscillates in the interval where the expression under the square root in (14) is positive,

$$\lambda_2 \leq \mu \leq \lambda_3. \quad (16)$$

Consequently the solution of Eq. (14) with the initial condition  $\mu(0) = \lambda_3$  is given by

$$\mu(\theta) = \frac{\lambda_3(\lambda_4 - \lambda_2) - \lambda_4(\lambda_3 - \lambda_2)\text{sn}^2\left(\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}\theta, m\right)}{\lambda_4 - \lambda_2 - (\lambda_3 - \lambda_2)\text{sn}^2\left(\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}\theta, m\right)}, \quad (17)$$

where

$$m = \frac{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)}{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)} \quad (18)$$

is the modulus of the elliptic functions. An equivalent solution corresponding to the initial condition  $\mu(0) = \lambda_2$  is given by

$$\mu(\theta) = \frac{\lambda_2(\lambda_3 - \lambda_1) - \lambda_1(\lambda_3 - \lambda_2)\text{sn}^2\left(\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}\theta, m\right)}{\lambda_3 - \lambda_1 - (\lambda_3 - \lambda_2)\text{sn}^2\left(\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}\theta, m\right)}. \quad (19)$$

Substitution of (17) or (19) into (11) gives expressions for  $u(\theta)$  and  $h(\theta)$  in the periodic nonlinear wave. Its wavelength is given by

$$L = \int_{\lambda_2}^{\lambda_3} \frac{d\mu}{\sqrt{P(\mu)}} = \frac{2K(m)}{\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}}, \quad (20)$$

$K(m)$  being the complete elliptic integral of the first kind.

The soliton limit ( $m = 1$ ) is obtained either for  $\lambda_1 = \lambda_2$  or for  $\lambda_3 = \lambda_4$ . For  $\lambda_1 = \lambda_2$  Eq. (17) gives

$$\mu(\theta) = \lambda_4 - \frac{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_3)}{\lambda_4 - \lambda_1 + (\lambda_3 - \lambda_1)/\cosh^2[\sqrt{(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1)}\theta]}, \quad (21)$$

and for  $\lambda_3 = \lambda_4$  Eq. (19) gives

$$\mu(\theta) = \lambda_1 + \frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_1)}{\lambda_2 - \lambda_1 + (\lambda_4 - \lambda_2)/\cosh^2[\sqrt{(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)}\theta]}. \quad (22)$$

Their substitution into (11) yields the soliton solution of the KB system.

In the opposite limit  $\lambda_2 = \lambda_3$  ( $m = 0$ ) both expressions (17) and (19) reduce to

$$\mu = \lambda_2 = \lambda_3. \quad (23)$$

Thus the limit  $\lambda_2 \rightarrow \lambda_3$  yields sinusoidal waves.

### 3 Whitham modulation equations for the Kaup-Boussinesq system

The Whitham modulation equations describe the slow evolution of the parameters  $\lambda_i$ ,  $i = 1, 2, 3, 4$ , of a modulated nonlinear wave. They are

$$\frac{\partial \lambda_i}{\partial t} + v_i(\lambda) \frac{\partial \lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, 4, \quad (24)$$

where the Whitham velocities  $v_i(\lambda)$  can be expressed in the form

$$v_i(\lambda) = \left(1 - \frac{L}{\partial_i L} \partial_i\right) V, \quad \partial_i \equiv \frac{\partial}{\partial \lambda_i}, \quad i = 1, 2, 3, 4, \quad (25)$$

where the phase velocity  $V$  and the wavelength  $L$  are given correspondingly by (13) and (20). A simple calculation yields the explicit expressions [13],

$$\begin{aligned} v_1 &= \frac{1}{2} \sum \lambda_i - \frac{(\lambda_4 - \lambda_1)(\lambda_2 - \lambda_1)K(m)}{(\lambda_2 - \lambda_1)K(m) + (\lambda_4 - \lambda_2)E(m)}, \\ v_2 &= \frac{1}{2} \sum \lambda_i - \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)K(m)}{(\lambda_2 - \lambda_1)K(m) - (\lambda_3 - \lambda_1)E(m)}, \\ v_3 &= \frac{1}{2} \sum \lambda_i + \frac{(\lambda_4 - \lambda_3)(\lambda_3 - \lambda_2)K(m)}{(\lambda_4 - \lambda_3)K(m) - (\lambda_4 - \lambda_2)E(m)}, \\ v_4 &= \frac{1}{2} \sum \lambda_i + \frac{(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_1)K(m)}{(\lambda_4 - \lambda_3)K(m) + (\lambda_3 - \lambda_1)E(m)}, \end{aligned} \quad (26)$$

where  $K(m)$  and  $E(m)$  are complete elliptic integrals of the first and second kind, respectively.

In the limit  $\lambda_1 = \lambda_2$  ( $m = 1$ ) the Whitham velocities reduce to

$$v_1 = v_2 = \frac{1}{2} \sum \lambda_i, \quad v_3 = \frac{1}{2}(3\lambda_3 + \lambda_4), \quad v_4 = \frac{1}{2}(\lambda_3 + 3\lambda_4); \quad (27)$$

in the limit  $\lambda_3 = \lambda_4$  ( $m = 1$ ) they reduce to

$$v_1 = \frac{1}{2}(3\lambda_1 + \lambda_2), \quad v_2 = \frac{1}{2}(\lambda_1 + 3\lambda_2), \quad v_3 = v_4 = \frac{1}{2} \sum \lambda_i; \quad (28)$$

and in the limit  $\lambda_2 = \lambda_3$  ( $m = 0$ ) they reduce to

$$v_1 = \frac{1}{2}(3\lambda_1 + \lambda_4), \quad v_2 = v_3 = \frac{1}{2} \sum \lambda_i + \frac{2(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}{\lambda_4 + \lambda_1 - 2\lambda_2}, \quad v_4 = \frac{1}{2}(\lambda_1 + 3\lambda_4). \quad (29)$$

Next we shall apply the Whitham theory to the description of the undular bore forming in the vicinity of a wave-breaking singularity.

## 4 The Gurevich-Pitaevskii problem for the KB system

### 4.1 Wave breaking in the dispersionless limit

In the dispersionless limit, the KB system (1) reduces to well-known shallow water equations

$$h_t + (hu)_x = 0, \quad u_t + uu_x + h_x = 0, \quad (30)$$

which can be transformed to the diagonal form

$$\frac{\partial \lambda_+}{\partial t} + \frac{1}{2}(3\lambda_+ + \lambda_-)\frac{\partial \lambda_+}{\partial x} = 0, \quad \frac{\partial \lambda_-}{\partial t} + \frac{1}{2}(\lambda_+ + 3\lambda_-)\frac{\partial \lambda_-}{\partial x} = 0, \quad (31)$$

$$\text{where } \lambda_{\pm} = \frac{u}{2} \pm \sqrt{h} \quad (32)$$

are the Riemann invariants of Eqs. (30).

Initial data are given by two functions  $\lambda_+(x, 0)$  and  $\lambda_-(x, 0)$  determined by the initial distributions  $h_0(x)$  and  $u_0(x)$ . The system (31) has two families of characteristics in the  $(x, t)$  plane along which one of two Riemann invariants (either  $\lambda_+$  or  $\lambda_-$ ) is constant. The wave-breaking point corresponds to the moment when characteristics of one of the families begin to intersect, so that the corresponding Riemann invariant becomes a three-valued function in the physical plane. Let such an intersection occur for the characteristics transferring the values of  $\lambda_+$ . Then at the wave-breaking point the profile of  $\lambda_+$  as a function of  $x$  has a vertical tangent line and, hence, in vicinity of this point it varies very fast, whereas the second Riemann invariant varies with  $x$  more slowly and may be considered here as a constant parameter:

$$\lambda_- = \lambda_0 = \text{const.} \quad (33)$$

Thus, in the vicinity of the breaking point we are dealing with a simple wave. The second equation in (31) is identically satisfied by Eq. (33). The first equation in (31) then has the well-known solution

$$x - \frac{1}{2}(3\lambda_+ + \lambda_-)t = f(\lambda_+), \quad (34)$$

where  $f(\lambda_+)$  is an inverse function to an initial profile  $\lambda_+(x, 0)$ . At the wave-breaking time, normalized here to be  $t = 0$ , the function  $x = f(\lambda_+)$  must have an inflexion point with a vertical tangent line. In the vicinity of this point  $f(\lambda_+)$  can be approximated by a cubic function,

$$x - \frac{1}{2}(3\lambda_+ + \lambda_-)t = -C(\lambda_+ - \bar{\lambda}_1)^3. \quad (35)$$

The KB system (1) is invariant with respect to Galilean transformation

$$x' = x - u_0 t, \quad t' = t, \quad h = h', \quad u = u' + u_0, \quad \lambda_{\pm} = \lambda'_{\pm} + \frac{u_0}{2}, \quad (36)$$

and scaling transformation

$$x = ax', \quad t = a^2 t', \quad h = h'/a^2, \quad u = u'/a, \quad \lambda_{\pm} = \lambda'_{\pm}/a. \quad (37)$$

With the aid of these transformations Eq. (35) can be cast into the form

$$x - \frac{1}{2}(3\lambda_+ + \lambda_-)t = -\lambda_+^3, \quad \lambda_- = \lambda_0. \quad (38)$$

where we have omitted the prime superscripts for notational convenience. It corresponds to the wave breaking picture shown in Fig. 1.

The actual solution of the KB system now consists of two parts. Following Gurevich and Pitaevskii [1], we suppose that the region of oscillations can be approximated by a modulated periodic solution of the KB system. Its evolution is determined by the Whitham equations

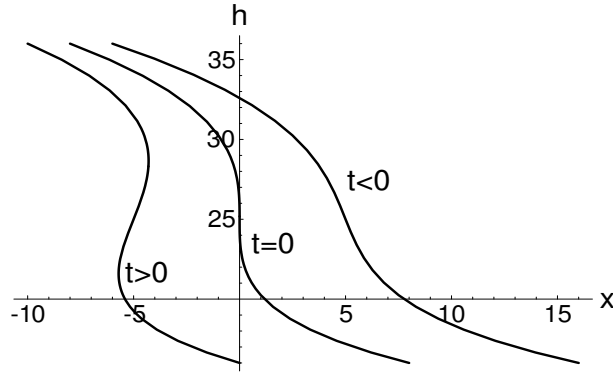


Figure 1: Wave breaking of the water elevation in the dispersionless limit;  $\lambda_-$  is taken equal to -10.

(24) and we have to find that solution which matches the solution (38) at the end points of the oscillatory region. One may say that this oscillatory region (the undular bore) “replaces” a non-physical multi-valued region of the solution (38). One should emphasize, however, that the boundaries of the undular bore *do not coincide* with the boundaries of formal multi-valued solution. Outside these boundaries, the solution approaches the dispersionless solution (38).

## 4.2 Undular bore solution

We look for the solution of the Whitham equations (24) in the form

$$x - v_i(\lambda)t = w_i(\lambda), \quad i = 1, 2, 3, 4, \quad (39)$$

where  $v_i$  are the Whitham velocities (26). Since we consider the breaking of the Riemann invariant  $\lambda_+$  and  $\lambda_- < \lambda_+$ , we take

$$\lambda_1 = \lambda_- = \lambda_0 = \text{const.} \quad (40)$$

Then the limiting formulas (27)-(29) show that if we find  $w_i(\lambda)$  such that

$$\begin{aligned} w_4 &= -\lambda_4^3 & \text{at } \lambda_2 &= \lambda_3 & (m = 0), \\ w_2 &= -\lambda_2^3 & \text{at } \lambda_4 &= \lambda_3 & (m = 1), \end{aligned} \quad (41)$$

then Eqs. (39) determine the Riemann invariants in such a way that  $\lambda_4 = \lambda_+$  at the trailing edge  $x^-(t)$  where  $m = 0$ ,  $\lambda_2 = \lambda_+$  at the leading edge  $x^+(t)$  where  $m = 1$ , and  $\lambda_1 = \lambda_- = \lambda_0$  everywhere. Thus, the plots of Riemann invariants  $\lambda_2, \lambda_3, \lambda_4$  as functions of  $x$  are joined into continuous curve whose upper and lower branches match with the solution (38) of the dispersionless equations (see Fig. 2). In the region  $x^-(t) < x < x^+(t)$  there are four Riemann invariants which determine the modulated periodic solution representing the undular bore. At its trailing edge  $x \rightarrow x^-(t)$  the amplitude of oscillations vanishes and at the leading edge  $x \rightarrow x^+(t)$  the periodic solution transforms into a soliton train.

According to the generalized hodograph method [6, 16], Eqs. (39) satisfy the Whitham equations (24) provided  $w_i(\lambda)$  are the velocities of the flows

$$\frac{\partial \lambda_i}{\partial \tau} + w_i(\lambda) \frac{\partial \lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, 4, \quad (42)$$



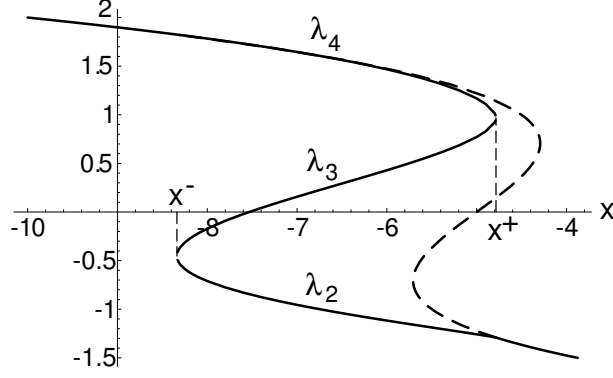


Figure 2: Dependence of Riemann invariants  $\lambda_2, \lambda_3, \lambda_4$  on  $x$  at fixed value of time  $t = 1$  and with  $\lambda_1 = -10$ . The dashed line shows the corresponding dependence of  $\lambda_+$  for the solution of the KB equations in the dispersionless limit.

commuting with (24), i.e.  $\partial_{t\tau}\lambda_i = \partial_{\tau t}\lambda_i$ . If we represent  $w_i(\lambda)$  in the form analogous to Eqs. (25),

$$w_i(\lambda) = \left(1 - \frac{L}{\partial_i L} \partial_i\right) W, \quad i = 1, 2, 3, 4, \quad (43)$$

then the condition of commutativity of the flows (24) and (43) reduces to the system of Euler-Poisson equations, exactly as happens in the KdV [10, 11, 12] and NLS [17] cases,

$$\partial_i \partial_j W - \frac{1}{2(\lambda_i - \lambda_j)} (\partial_i W - \partial_j W) = 0, \quad i \neq j. \quad (44)$$

It is easy to check that this equation has a particular solution  $W = \text{const}/\sqrt{P(\lambda)}$ ,  $P(\lambda) = \prod(\lambda - \lambda_i)$ , which is sufficient for our purpose. We choose the normalization factor so that the coefficient before  $\lambda^{-1}$  in the series expansion of  $W$  in powers of  $\lambda^{-1}$  be equal to the phase velocity of the periodic wave  $s_1/2 = V$ . Thus, we obtain the sequence of  $W^{(k)}$  defined by the generating function

$$W = \frac{\lambda^2}{\sqrt{P(\lambda)}} = \sum \frac{W^{(k)}}{\lambda^k} = 1 + \frac{1}{2}s_1 \cdot \frac{1}{\lambda} + \left(\frac{3}{8}s_1^2 - \frac{1}{2}s_2\right) \cdot \frac{1}{\lambda^2} + \left(\frac{5}{16}s_1^3 - \frac{3}{4}s_1s_2 + \frac{1}{2}s_3\right) \cdot \frac{1}{\lambda^3} + \dots \quad (45)$$

Next a sequence of velocities of the commuting flows is given by

$$w_i^{(k)}(\lambda) = \left(1 - \frac{L}{\partial_i L} \partial_i\right) W^{(k)}, \quad i = 1, 2, 3, 4, \quad (46)$$

where  $w_i^{(1)} = v_i$  coincide with the Riemann velocities (26). It is not difficult to find the limiting formulas analogous to (27)-(29). In particular, we get at  $\lambda_3 = \lambda_4$  ( $m = 1$ )

$$\begin{aligned} w_1^{(1)} \Big|_{\lambda_3=\lambda_4} &= w_2^{(1)} \Big|_{\lambda_3=\lambda_4} = \frac{1}{2}(\lambda_1 + 3\lambda_2), \\ w_1^{(2)} \Big|_{\lambda_3=\lambda_4} &= w_2^{(2)} \Big|_{\lambda_3=\lambda_4} = \frac{3}{8}(\lambda_1^2 + 2\lambda_1\lambda_2 + 5\lambda_2^2), \\ w_1^{(3)} \Big|_{\lambda_3=\lambda_4} &= w_2^{(3)} \Big|_{\lambda_3=\lambda_4} = \frac{1}{16}(5\lambda_1^3 + 9\lambda_1^2\lambda_2 + 15\lambda_1\lambda_2^2 + 35\lambda_2^3). \end{aligned} \quad (47)$$

Now we take such the linear combination

$$w_2 = a_0 + a_1 w_2^{(1)} + a_2 w_2^{(2)} + a_3 w_2^{(3)}$$

so that  $w_2$  satisfies the condition (41). The coefficients  $a_1, a_2, a_3, a_4$  depend on the constant Riemann invariant  $\lambda_1 = \lambda_- = \lambda_0$  and their values found in this way yield the required solution of the Whitham equations:

$$\begin{aligned} x - v_i(\lambda)t &= -\frac{16}{35}w_i^{(3)}(\lambda) + \frac{8}{35}\lambda_0 w_i^{(2)} + \frac{2}{35}\lambda_0^2 v_i(\lambda) + \frac{1}{35}\lambda_0^3, \quad i = 2, 3, 4; \\ \lambda_1 &= \lambda_0 = \text{const.} \end{aligned} \quad (48)$$

These formulas define  $\lambda_2, \lambda_3, \lambda_4$  implicitly as functions of  $x$  and  $t$  and give the solution of the Gurevich-Pitaevskii problem for the KB-Whitham system. It is interesting to note that, unlike the KdV case, this solution is not scale-invariant (we recall that in the counterpart KdV solution  $\lambda_i = t^{-1/2}l_i(x/t^{3/2}), i = 1, 2, 3$ ). This happens due to the presence of the fourth Riemann invariant  $\lambda_1$  in the Whitham equations which is constant ( $\lambda_0$ ) for the obtained solution and cannot be eliminated by simple Galilean transform. As a result, the solution (48) does not possess the scaling invariance required for a generalized similarity behaviour of  $\lambda_j, j = 2, 3, 4$ . The only family of admissible such similarity ( $x/t$ ) solutions is realized in the simplest case of the decay of an initial discontinuity studied in [13].

### 4.3 Laws of motion at the trailing and leading edges of the oscillatory region

Let us find the laws of motion at the leading and trailing edges of the undular bore. First we consider the leading edge  $\lambda_3 = \lambda_4$  ( $m = 1$ ), and define the small deviations  $\lambda'_3$  and  $\lambda'_4$  from the value  $\lambda_3 = \lambda_4 = \lambda_4^+$ :

$$\lambda_3 = \lambda_4^+ + \lambda'_3, \quad \lambda_4 = \lambda_4^+ + \lambda'_4. \quad (49)$$

Then we seek the asymptotic expressions for formulas (48) with  $i = 3, 4$  for small  $|\lambda'_3|, |\lambda'_4|$ :

$$x^+ + x' - (v_3^+ + v'_3)t = w_3^+ + w'_3, \quad x^+ + x' - (v_4^+ + v'_4)t = w_4^+ + w'_4, \quad (50)$$

where  $x'$  denotes the space coordinate reckoned from its limiting value  $x^+$  and

$$v_3^+ = v_4^+ = \frac{1}{2}(\lambda_1 + \lambda_2 + 2\lambda_4), \quad (51)$$

$$v'_3 = -v'_4 = -\frac{1}{2} \left\{ \lambda'_3 \ln \left[ \frac{-(\lambda_2 - \lambda_1)\lambda'_3}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)} \right] - \lambda'_4 \ln \left[ \frac{(\lambda_2 - \lambda_1)\lambda'_4}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)} \right] \right\}, \quad (52)$$

$$w_3^+ = w_4^+ = -\frac{1}{35}(5\lambda_2^3 + 6\lambda_2^2\lambda_4 + 8\lambda_2\lambda_4^2 + 16\lambda_4^2), \quad (53)$$

$$\begin{aligned} w'_3 = -w'_4 &= \frac{1}{35}(3\lambda_2^2 + 8\lambda_2\lambda_4 + 24\lambda_4^2) \times \\ &\left\{ \lambda'_3 \ln \left[ \frac{-(\lambda_2 - \lambda_1)\lambda'_3}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)} \right] - \lambda'_4 \ln \left[ \frac{(\lambda_2 - \lambda_1)\lambda'_4}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)} \right] \right\}, \end{aligned} \quad (54)$$

while  $\lambda_1, \lambda_2, \lambda_4$  denote here their limiting values  $\lambda_1^+, \lambda_2^+, \lambda_4^+$ , correspondingly. Then subtraction of one equation (50) from the other yields at once the relationship

$$t = \frac{2}{35}(3\lambda_2^2 + 8\lambda_2\lambda_4 + 24\lambda_4^2). \quad (55)$$

On the other hand, the limiting formulas

$$x^+ - v_2^+ t = w_2^+, \quad x^+ - v_3^+ t = w_3^+,$$

with

$$v_2^+ = \frac{1}{2}(\lambda_1 + 3\lambda_2), \quad w_2^+ = -\lambda_2^3 \quad (56)$$

and  $v_3^+, w_3^+$  given by (51) and (53) give after subtraction the relationship

$$t = \frac{2}{35}(15\lambda_2^2 + 12\lambda_2\lambda_4 + 8\lambda_4^2). \quad (57)$$

Then equating of the right hand sides of (55) and (57) shows that at the leading edge we have

$$\lambda_4^+ = -\frac{3}{4}\lambda_2^+. \quad (58)$$

Substitution of this relation into (55) or (57) gives the dependence of  $\lambda_2^+$  on  $t$ :

$$\lambda_2^+ = -\left(\frac{5t}{3}\right)^{1/2}. \quad (59)$$

At last, the formula  $x^+ - v_2^+ t = -(\lambda_2^+)^3$  yields the law of motion for the leading edge:

$$x^+(t) = \frac{1}{2}\lambda_0 t + \frac{1}{6}\sqrt{\frac{5}{3}}t^{3/2}. \quad (60)$$

In a similar way we can consider the trailing edge  $\lambda_3 = \lambda_2 = \lambda_2^-$  ( $m = 0$ ) where we define

$$\lambda_2 = \lambda_2^- + \lambda_2', \quad \lambda_3 = \lambda_2^- + \lambda_3' \quad (61)$$

and Eqs. (48) with  $i = 2, 3$  reduce to

$$x^- + x' - (v_2^+ + v_2')t = w_2^+ + w_2', \quad x^- + x' - (v_3^+ + v_3')t = w_3^+ + w_3', \quad (62)$$

where

$$v_2^- = v_3^- = \frac{\lambda_1^2 + 4\lambda_1\lambda_2 - 8\lambda_2^2 - 2\lambda_1\lambda_4 + 4\lambda_2\lambda_4 + \lambda_4^2}{2(\lambda_1 - 2\lambda_2 + \lambda_4)}, \quad (63)$$

$$v_2' = -v_3' = -\frac{\lambda_2' + 3\lambda_3'}{2(\lambda_1 - 2\lambda_2 + \lambda_4)}(3\lambda_1^2 - 8\lambda_1\lambda_2 + 8\lambda_2^2 + 2\lambda_1\lambda_4 - 8\lambda_2\lambda_4 + 3\lambda_4^2), \quad (64)$$

$$w_2^- = w_3^- = [128\lambda_2^4 - 64\lambda_2^3\lambda_4 - 16\lambda_2^2\lambda_4^2 - 8\lambda_2\lambda_4^3 - 5\lambda_4^4 - 7\lambda_1(16\lambda_2^3 - 8\lambda_2^2\lambda_4 - 2\lambda_2\lambda_4^2 - \lambda_4^3)] / (35(\lambda_1 - 2\lambda_2 + \lambda_4)) \quad (65)$$

$$w_2' = -w_3' = \frac{\lambda_2' + 3\lambda_3'}{70(\lambda_1 - 2\lambda_2 + \lambda_4)^2} [-384\lambda_2^4 + 384\lambda_2^3\lambda_4 - 80\lambda_2^2\lambda_4^2 - 16\lambda_2\lambda_4^3 - 9\lambda_4^4 + 7\lambda_1^2(-24\lambda_2^2 + 8\lambda_2\lambda_4 + \lambda_4^2)]. \quad (66)$$

Then subtraction of one equation (62) from the other gives

$$t = \frac{2}{35}[-384\lambda_2^4 + 384\lambda_2^3\lambda_4 - 80\lambda_2^2\lambda_4^2 - 16\lambda_2\lambda_4^3 - 9\lambda_4^4 + 7\lambda_1^2(-24\lambda_2^2 + 8\lambda_2\lambda_4 + \lambda_4^2)]/(3\lambda_1^2 - 8\lambda_1\lambda_2 + 8\lambda_2^2 + 2\lambda_1\lambda_4 - 8\lambda_2\lambda_4 + 3\lambda_4^2). \quad (67)$$

Now the limiting formulas

$$x^- - \frac{1}{2}(\lambda_1 + 3\lambda_4)t = -\lambda_4^3, \quad x^- - v_3^- t = -w_3^- \quad (68)$$

give

$$t = \frac{2}{35} \frac{(8\lambda_2 - 7\lambda_1)(8\lambda_2^2 + 4\lambda_2\lambda_4 + 3\lambda_4^2) - 15\lambda_4^3}{4\lambda_2 - 3\lambda_1 - \lambda_4}. \quad (69)$$

Equating the right hand sides of (67) and (69), we find the relationship between the values of  $\lambda_2^-$  and  $\lambda_4^-$  at the trailing edge:

$$21\lambda_1^2(\lambda_4 + 4\lambda_2) - 10\lambda_1(20\lambda_2^2 + 2\lambda_2\lambda_4 - \lambda_4^2) + 16(8\lambda_2^3 - \lambda_2^2\lambda_4 - \lambda_2\lambda_4^2) + 9\lambda_4^3 = 0. \quad (70)$$

Given  $t$  and  $\lambda_1 = \lambda_0 = \text{const}$ , we find  $\lambda_2 = \lambda_2^-$  and  $\lambda_4 = \lambda_4^-$  from (69) and (70) and then the law of motion of the trailing edge follows from

$$x^- = \frac{1}{2}(\lambda_1 + 3\lambda_4)t - \lambda_4^3. \quad (71)$$

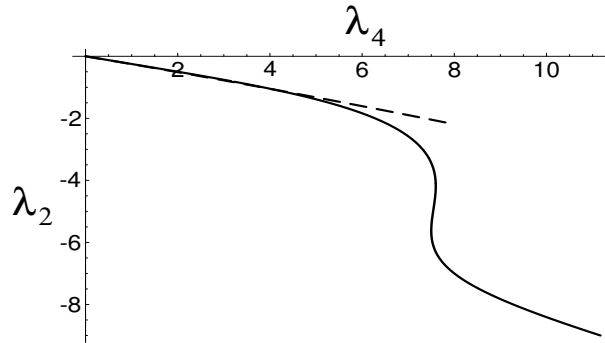


Figure 3: Riemann invariant  $\lambda_2$  as a function of  $\lambda_4$  defined implicitly by Eq. (70). The plot corresponds to a fixed value of  $\lambda_1 = -10$ . Dashed line shows the dependence according to asymptotic formula (74).

It is worth noticing that Eq. (58) coincides with the corresponding relation between the Riemann invariants at the leading edge in the solution of the Gurevich-Pitaevskii problem in the KdV equation case (see, e.g. [15]). A similar relation  $\lambda_2^- = -\lambda_4^-/4$  between the Riemann invariants of the KdV theory follows from (70) in the limit  $|\lambda_1| = |\lambda_0| \rightarrow \infty$ . In the next approximation we obtain

$$\lambda_2 = \lambda_3 \cong -\frac{1}{4}\lambda_4 - \frac{5}{168} \frac{\lambda_4^2}{\lambda_1}, \quad (72)$$

and similar expansion of Eq. (69) in powers of  $1/\lambda_1$ ,

$$t \cong \frac{1}{3}\lambda_4^2 \left(1 + \frac{10}{21} \frac{\lambda_4}{\lambda_1}\right), \quad (73)$$

yields with the same accuracy

$$\lambda_4 \cong \sqrt{3t} - \frac{5}{7} \frac{t}{\lambda_1}, \quad \sqrt{3t} \ll |\lambda_1|. \quad (74)$$

Dependence of  $\lambda_2$  on  $\lambda_4$  given by Eq. (70) with fixed value of  $\lambda_1$  is illustrated in Fig. 3.

Substitution of Eq. (73) into Eq. (71) yields an approximate expression for the law of motion of the trailing edge;

$$x^- \cong \frac{1}{2}\lambda_0 t - \frac{3\sqrt{3}}{2}t^{3/2} + \frac{75}{14} \frac{t^2}{\lambda_0}, \quad \sqrt{3t} \ll |\lambda_0|. \quad (75)$$

Thus, the analytic formulas (48) for the solution of the Whitham equations allowed us to find the main characteristics of the dissipationless shock. With the use of Eqs. (48) we can find  $\lambda_2, \lambda_3, \lambda_4$  as functions of  $x$  at given  $t$ , and their substitution into (17) and (11) yields the profiles of  $u(x)$  and  $h(x)$  in the undular bore. An example of such a profile of the water elevation is shown in Fig. 4. As we see, the non-physical solution obtained in

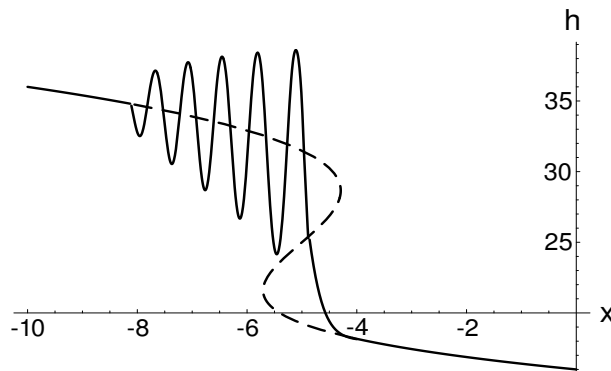


Figure 4: Undular bore for the KB equations. The plot corresponds to the time  $t = 1$  and  $\lambda_1 = -10$ . Dashed line shows the solution in the dispersionless limit.

the dispersionless limit is replaced by an undular bore. Its end points move according to the laws found above, so that the oscillatory region expands with time and its width grows mainly as  $t^{3/2}$ . Small amplitude oscillations are generated at the trailing edge (actually these oscillations represent gravity waves propagating into an undisturbed smooth region), and they transform gradually into solitons at the leading edge.

## 5 Conclusion

In this article, the wave breaking (Gurevich-Pitaevskii) problem is solved for the shallow water waves described by the Kaup-Boussinesq system. This theory generalizes the KdV

model to the case of a bi-directional wave propagation model, and so allows for the effects of wave interaction. As a result, the dispersionless theory includes two Riemann invariants and the breaking of the wave means breaking of one of these two invariants. Thus, the wave breaking picture depends on the value of an additional parameter: the value of the non-breaking Riemann invariant. Correspondingly, the Whitham theory for the KB system contains four Riemann invariants (instead of three in the KdV case) and, as a result, the solution of the Whitham equations for the Gurevich-Pitaevskii problem of the wave front breaking is parameterized by a constant value  $\lambda_0$ .

The analytic solution of the Whitham equations obtained here allows one to investigate the form of an undular bore as a function of the parameter  $\lambda_0$ . In particular, the velocity of the trailing edge increases with decrease of  $|\lambda_0|$  and in the limit  $|\lambda_0| \rightarrow \infty$  the present theory reduces to known results of the KdV model.

Our obtained solution can also be viewed as an intermediate asymptotic in a more general problem of the description of frictional shallow water undular bores where small dissipation is taken into account (see [18, 19, 20] for the results relevant to the KdV equation with weak dissipation). This problem will be the subject of a separate study.

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## References

- [1] A.V. Gurevich and L.P. Pitaevskii, “Nonstationary structure of a collisionless shock wave”, *Zh. Eksp. Teor. Fiz.* **65**, 590 (1973) [*Soviet Physics JETP*, **38**, 291 (1973)].
- [2] G.B. Whitham, “Non-linear dispersive waves”, *Proc. Roy. Soc. London*, **283**, 238 (1965).
- [3] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley–Interscience, New York, 1974.
- [4] G.V. Potëmin, “Algebraic-geometrical construction of self-similar solutions of Whitham’s equations”, *Usp. Mathem. Nauk* **43**, 211 (1988) [*Russian Math. Surveys* **43**, 252 (1988)].
- [5] I.M. Krichever, The method of averaging for two-dimensional “integrable” equations. *Funct. Anal. Appl.* **22**, 200-213, (1988).
- [6] S.P. Tsarev, On Poisson brackets and one-dimensional systems of hydrodynamic type *Soviet Math. Dokl.* **31**, 488-491 (1985).
- [7] B.A. Dubrovin and S.P. Novikov, Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory. *Russian Math. Surveys* **44**, 35–124 (1989).
- [8] D.J. Kaup, A higher order water-wave equation and method for solving it, *Progr. Theor. Phys.* **54**, 396–408 (1976).

- [9] V.B. Matveev and M.I. Yavor, Almost periodical solutions of nonlinear hydrodynamic equation of Kaup, *Ann. Inst. H. Poincaré*, Sect. A **31**, 25–41 (1979).
- [10] V.R. Kudashev and S.E. Sharapov, Inheritance of KdV symmetries under Whitham averaging and hydrodynamic symmetries of the Whitham equations. *Theor. Math. Phys.* **87**, 358–363 (1991).
- [11] A.V.Gurevich, A.L. Krylov, and G.A. El, Riemann wave breaking in dispersive hydrodynamics. *JETP Lett.* **54** 102-107; Evolution of a Riemann wave in dispersive hydrodynamics. *Sov. Phys. JETP* **74**, 957–962 (1991).
- [12] F.R. Tian, Oscillations of the zero dispersion limit of the Korteweg – de Vries equation. *Comm. Pure Appl. Math.* **46**, 1093–1129 (1993).
- [13] G.A. El, R.H.J. Grimshaw, and M.V. Pavlov, Integrable shallow-water equations and undular bores, *Stud. Appl. Math.* **106**, 157–186 (2001).
- [14] A.M. Kamchatnov, R.A. Kraenkel, and B.A. Umarov, Asymptotic soliton train solutions of Kaup-Boussinesq equations, *Wave Motion*, **38**, 355–365 (2003).
- [15] A.M. Kamchatnov, *Nonlinear Periodic Waves and Their Modulations—An Introductory Course*, World Scientific, Singapore, 2000.
- [16] S.P. Tsarev, “The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method,” *Izv. Akad. Nauk*, **54**, 1048 (1990) [*Math. USSR Izvestia*, **37**, 397 (1991)].
- [17] G.A. El and A.L. Krylov, General solution of the Cauchy problem for the defocusing NLS equation in the Whitham limit, *Phys. Lett. A* **203**, 77–82 (1995).
- [18] A.V. Gurevich and L.P. Pitaevskii, Averaged description of waves in the Korteweg-de Vries-Burgers equation, *Zh. Eksp. Teor. Fiz.* **93**, 871 (1987) [*Soviet Physics JETP*, **66**, 490 (1987)].
- [19] V.V. Avilov, I.M. Krichever, and S.P. Novikov, Evolution of Whitham zone in the theory of Korteweg-de Vries, *Dokl. Akad. Nauk SSSR*, **295**, 345-349 (1987) [*Sov. Phys. Dokl.*, **32**, 564-566 (1987)].
- [20] S. Myint and R.H.J. Grimshaw, The modulation of nonlinear periodic wavetrains by dissipative terms in the Korteweg-de Vries equation, *Wave Motion*, **22**, 215-238 (1995).