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Citation: Shang, Yilun (2020) Resilient interval consensus in robust networks. International Journal of Robust and Nonlinear Control, 30 (17). pp. 7783-7790. ISSN 1049-8923

Published by: Wiley-Blackwell

URL: https://doi.org/10.1002/rnc.5153 <https://doi.org/10.1002/rnc.5153>

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DOI: xxx/xxxx

SHORT COMMUNICATION

Resilient interval consensus in robust networks

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Summary

This paper considers the resilient interval consensus problems for continuous-time time-varying multi-agent systems when a normal agent is surrounded by no more than r misbehaving agents. Each normal agent individually proposes a constraint interval which specifies their acceptable consensus range and misbehaving agents are anonymous and exert different arbitrary rules posing threats to the global performance of the systems. On the basis of our purely distributed resilient interval consensus strategy, we showed that if the network is $(2r+1)$-robust and the interval intersection is nonempty, the normal agents are able to reach an agreement with the final consensus equilibrium in the interval intersection. A numerical example is presented to illustrate the theoretical results.

KEYWORDS:
resilient consensus, interval consensus, robust network, multi-agent system

1 | INTRODUCTION

With distributed coordination and cohesive agreement, consensus protocols play a key role in a broad spectrum of areas such as formation control, sensor networks, distributed computing, swarm robotics, and social dynamics, to name but a few. In a consensus problem [1], a group of agents update their states based on local interaction with their nearest neighbors aiming to achieve some common value in a distributed manner. A variety of important advances on consensus problems have been reported in the literature over the last few decades [2,3,4].

In practice, it is uncommon to have unrestricted states of agents given the complicated operational environment and limited agent's capability. In the formation control of a group of mobile robotics, for instance, each agent has its own speed limit, which may lead to failure in generating a formation of the entire group. Similarly, for load balancing in a network of heterogenous workstations, the load constraints for each workstation have to be taken into account. Recently, interval consensus is proposed as a novel framework in the work [5] to cope with state constrained consensus, where each agent in a strongly connected network imposes an interval constraint within which agreement is acceptable with the object of reaching an agreement residing in the intersection of all intervals. To achieve such an agreement, each agent transmits to its neighbors a saturated state bounded by its admissible interval. It is shown that unique equilibrium of the multi-agent system lies in the intersection of these intervals if it is non-empty, and stability analysis is also performed for the empty intersection situation. In the work [6], interval consensus is shown to be achieved in the sense of almost sure convergence when the interaction topology can be characterized by an undirected random network. Although stated constrained consensus problems in general have been investigated extensively by using e.g., projection based protocols [7,8,9] and discarded methods [10], the interval consensus framework is unique as the constraints are not hard-coded meaning that the transient trajectories of agents are allowed to trespass the admissible intervals, and that initial configuration is not required to be confined in the interval intersection.

All the works mentioned above concerning state constrained consensus assume that agents are cooperative or not compromised. In real-world applications, however, the performance of a multi-agent system can be undermined if a subset of agents becomes misbehaving due to system level faults or malicious attacks[4]. Resilient consensus problems withstanding misbehaving agents have been studied in the work[11,12] on the basis of the graph robustness notion. A discrete-time Weighted Mean-Subsequence-Reduced (W-MSR) consensus protocol is proposed in the work[11], which is resilient against a number r of neighboring misbehaving agents if the underlying communication network is $(2r+1)$-robust. More complicated system dynamics including switched[13] and hybrid[14] systems have also been examined. In the paper[15], a resilient consensus protocol is designed when the states of agents are constrained to be integers under time-varying delays and asynchronous interactions. Quantized resilient consensus strategies have been designed for event-triggered control problems[16]. A resilient consensus strategy for state-saturated multi-agent systems has been designed based on a round-robin protocol[17]. To our knowledge, general state constraint has not been studied in resilient consensus algorithms.

In this paper we aim at solving resilient interval consensus problems in the presence of locally bounded misbehaving agents over time-varying networks. We assume that the misbehaving agents in the network are anonymous to normal agents, have no constraints on their states, and possess a complete knowledge of the network, hence posing a severe threat to the collective decision making of the whole network. The contributions of the work are as follows. First, based upon graph-theoretical properties of robustness, we develop a novel resilient interval consensus algorithm for continuous-time dynamical agents in a fully distributed manner, meaning that each normal agent only knows its own admissible interval and the states of its own and neighbors. Second, we establish sufficient conditions for reaching resilient interval consensus over bidirectional robust networks when the interval intersection is non-empty. It is worth noting that only fixed network is considered previously for interval consensus[5,6].

The rest of the paper is organized as follows. Section 2 presents the graph theoretical results regarding robustness and the resilient interval consensus protocol. Section 3 contains the main results for consensus analysis. Section 4 presents a simulation example and Section 5 draws the conclusion.

2 | PROBLEM FORMULATION

2.1 | Graph theory

Consider a multi-agent network represented by a bidirectional graph $G=(V,E)$ with node set $V=\{1,2,\cdots,n\}$ and edge set $E\subseteq V\times V$, where $(i,j)\in E$ means that information can be sent from agent i to agent j. The adjacency matrix, denoted by $A=(a_{ij})\in R^{n\times n}$, characterizes the topology of G with $a_{ij}>0$ when $(j,i)\in E$ and $a_{ij}=0$ otherwise. Here, by bidirectional network, we mean $a_{ij}>0$ if and only if $a_{ji}>0$ but we do not require any equivalence of the weights. To accommodate the misbehaving agents, the node set V is partitioned into $V=N\cup M$, where N contains normal agents and M misbehaving agents. The behavior of normal and misbehaving agents will be defined in Section 2.2 below. For a node i, its (in-degree) neighborhood is given by $\mathcal{N}_i=\{j\in V:(j,i)\in E\}$. A directed path of length ℓ from $i_0\in V$ to $i_\ell\in V$ is a sequence of edges $\{(i_l,i_{l+1})\}_{l=0}^{\ell-1}$ in G. We say a graph G contains a directed spanning tree with root node i if any other node in G can be connected by a path originating from the root i. For a bidirectional (or undirected) graph, having a spanning tree is equivalent to connectedness.

For a bidirectional graph G and an integer r, a subset $S\subseteq V$ is said to be r-reachable[11] if there exists $i\in S$ satisfying $|\mathcal{N}_i\backslash S|\ge r$. Building on reachability, G is called r-robust when at least one of any two disjoint subsets of V is r-reachable. The following lemma indicates robustness is a measure of graph connectivity.

Lemma 1. *If G is an r-robust bidirectional graph and G' is obtained from G by deleting up to $s<r$ incoming edges of each node in G, then G' is $(r-s)$-robust. Moreover, G is 1-robust if and only if G is connected.*

Proof. The first part of the proof follows from the directed graph version in the work[11, Lems 6 and 7]. For the second part, we first show the sufficiency. Assume that G is connected but not 1-robust, then there exist two disjoint sets S_1 and S_2, which do not have in-degree neighbors from outside. This contradicts with the connectedness of G. Hence, G must be 1-robust. For the necessity, we prove the contrapositive. Suppose G is not connected, then G can be decomposed to several connected components. Clearly, G can not be 1-robust.

2.2 | Resilient interval consensus algorithm

Consider a multi-agent system modeled by the graph $G = (V, E)$ with $V = N \cup M$. The agent $i \in V$ at time $t \geq 0$ has its state given by $x_i(t) \in R$. Each normal agent $i \in N$ proposes an interval $I_i := [l_i, u_i]$ with $l_i \leq u_i$, which specifies the range of consensus acceptable to them. We address the following resilient interval consensus, which requires all normal agents reach a common state in the intersection of admissible intervals for arbitrary initial configuration.

Definition 1 (Resilient Interval Consensus). The normal agents in G are said to achieve resilient interval consensus if there exists $y \in \cap_{i\in N} I_i$ such that $\lim_{t\to\infty} x_i(t) = y$ for all $i \in N$ and all initial conditions $\{x_i(0)\}_{i\in V}$.

In the case of $M = \emptyset$, the resilient interval consensus reduces to the ordinary interval consensus problem[5,6]. The dynamics of normal agent $i \in N$ can be delineated in general as follows

$$\dot{x}_i(t) = f_i\left(\{g_j(x_j^i(t))\}_{j\in\mathcal{N}_i}, x_i(t)\right), \tag{1}$$

where $x_j^i(t) \in R$ means the value transmitted from agent j to agent i before saturation at time t, and we assume normal agents intend to convey their true states, i.e., $x_j^i(t) = x_j(t)$ for any $j \in N$. The state saturation is implemented by enforcing the function g_j, where for $j \in N$,

$$g_j(x) := \begin{cases} u_j, & x > u_j; \\ x, & l_j \leq x \leq u_j; \\ l_j, & x < l_j. \end{cases} \tag{2}$$

and for $j \in M$, $g_j(x) \equiv x$. We aim to put forward a resilient interval consensus strategy which will specify the functions $\{f_i\}_{i\in N}$ that governs the behavior of normal agents.

Remark 1. The state saturation is a key feature of interval consensus problems, which has its origin in opinion dynamics in social consensus seeking — Individuals tend to express their opinions within a comfort interval accepted by them. This phenomenon is known as the observer bias in social activities[18]. A related direction is the study of consensus and discrepancy between expressed and private opinions[19], where expressed opinions follow a separate dynamics rather than saturated in an interval.

The misbehaving agents, nonetheless, may exert disparate strategies trying to corrupt the multi-agent network.

Definition 2 (Misbehaving Agents). An agent $i \in M$ is called misbehaving. It applies a different update strategy $\tilde{f}_i$ from the normal agents in (1), or transmits different values to different neighbors at some time $t > 0$.

The misbehaving agents are often referred to as Byzantine nodes in sensor networks[11,13,14] as they can collude with each other and are notoriously difficult to cope with. They are able to apply arbitrary update strategies without saturation and their identities are not available to the normal agents. Technically, we assume they can transmit misinformation to their neighbors following any Lipschitz continuous $\tilde{f}_i$. It is natural to bound the number of misbehaving agents in the network under consideration. We assume that each normal agent in G may have at most r misbehaving neighbors (i.e. the number of misbehaving agents is locally bounded by r). Hence, $|\mathcal{N}_i \cap M| \leq r$ for all $i \in N$.

In view of the parameter r, we propose the following resilient interval consensus strategy, which can be viewed as a generalization of W-MSR algorithms[11,12,13,14,15] by accommodating interval state constraints: Each normal agent $i \in N$ receives the states of its neighbors $\{g_j(x_j^i(t))\}_{j\in\mathcal{N}_i}$ at time t and orders these data in a descending list. Any index j of the highest r values in the above list which are higher than $x_i(t)$ is stored in a set $\mathcal{R}_i(t)$. If there are less than r such values, all these indices are put in the set $\mathcal{R}_i(t)$. Likewise, for those lowest values we conduct the same procedure and store these relevant indices in the set $\mathcal{R}_i(t)$. The control law for every $i \in N$ is proposed (by instantiating f_i in (1)) as

$$\dot{x}_i(t) = \sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_i(t)} a_{ij}\left(g_j(x_j^i(t)) - x_i(t)\right), \tag{3}$$

where g_j is given by (2) and $t \geq 0$.

Remark 2. Our proposed protocol is fully distributed as each normal agent is assumed to only have the information of the relative difference between expressed states of its neighbors and its own as well as its own admissible interval. In addition, the algorithm has low complexity. The most time consuming part is the sorting bit, which can be done using for example Quicksort with complexity $O(n \ln n)$.

3 | MAIN RESULT

In this section, we state our main convergence result under the resilient interval consensus algorithm presented above for robust networks. Noting the time-dependency of the set $\mathcal{R}_i(t)$ above, the system (3) is time-varying essentially. We assume the following.

Assumption 1. Let the time sequence $\{\tau_k\}_{k=1}^{\infty}$ be the points when $\mathcal{R}_i(t)$ alters for some $i \in N$. There exists $\tau > 0$ such that $|\tau_{k+1} - \tau_k| \geq \tau$ for all $k \geq 1$.

This assumptions implies the boundedness of the dwell time $\tau_{k+1} - \tau_k$, which is often in place for multi-agent systems with continuous-time switching dynamics[2,3]. Our main result is the following.

Theorem 1. *Consider the multi-agent system (1) over a bidirectional graph G, which is $(2r+1)$-robust. Suppose that $\cap_{i\in N} I_i \neq \emptyset$ and Assumption 1 holds. Then resilient interval consensus can be achieved by using the above proposed protocol when the number of misbehaving agents is locally bounded by r.*

Proof. Define the Dini derivative of a function $\phi(t)$ as $D^+\phi(t) = \limsup_{h\to 0+}(\phi(t+h) - \phi(t))/h$. Let $\bar{l} = \max_{i\in N} l_i$ and $\underline{u} = \min_{i\in N} u_i$. Then the interval intersection becomes $\cap_{i\in N} I_i = [\bar{l}, \underline{u}]$. Define two continuous and Lipschitz functions $\overline{\rho}(t) = \max\{\max_{i\in N} x_i(t), \underline{u}\}$ and $\underline{\rho}(t) = \min\{\min_{i\in N} x_i(t), \bar{l}\}$. Let $\Gamma(t) = \overline{\rho}(t) - \underline{\rho}(t) \geq 0$. We first claim the following.

Claim 1. $D^+\Gamma(t) \leq 0$ for $t \geq 0$.

Fix $t > 0$. Suppose that $\overline{\rho}(t) > \underline{u}$. Therefore, there is some $\varepsilon > 0$ such that $\max_{i\in N} x_i(t) > \underline{u}$ on the interval $[t, t+\varepsilon)$. Define the index set $J_0(t) = \{j \in N : x_j(t) = \max_{i\in N} x_i(t)\}$. Taking the Dini derivative of $\overline{\rho}(t)$ along the trajectory of (3) gives

$$\begin{aligned} D^+\overline{\rho}(t) &= D^+ \max_{i\in N} x_i(t) = \dot{x}_{i_0}(t) \\ &= \sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\backslash\mathcal{R}_{i_0}(t)} a_{i_0 j}\big(g_j(x_j^{i_0}(t)) - x_{i_0}(t)\big), \end{aligned} \tag{4}$$

where $\dot{x}_{i_0}(t) = \max_{i\in J_0(t)} \dot{x}_i(t)$ following the property of Dini derivative[20,21]. In view of our strategy, for any $j \in (\mathcal{N}_{i_0}\cup\{i_0\})\backslash\mathcal{R}_{i_0}(t)$, $x_{i_0}(t) \geq x_j^{i_0}(t)$. Since $u_j \geq \underline{u}$ for $j \in N$, we have for any $j \in (\mathcal{N}_{i_0}\cup\{i_0\})\backslash\mathcal{R}_{i_0}(t)$, $g_j(x_j^{i_0}(t)) \leq x_j^{i_0}(t)$ (if $x_j(t) > \underline{u}$) and $g_j(x_j^{i_0}(t)) \leq \underline{u}$ (if $x_j(t) \leq \underline{u}$) using the definition of g_j and our resilient interval consensus strategy (noting that there are at most r misbehaving neighbors around agent j). Consequently, $g_j(x_j^{i_0}(t)) \leq \max\{x_j^{i_0}(t), \underline{u}\} \leq x_{i_0}(t)$. Note that the second inequality becomes equality only if $j \in J_0(t)$ since $\underline{u} < x_{i_0}(t)$. By (4), we so far have shown $D^+\overline{\rho}(t) \leq 0$ if $\overline{\rho}(t) > \underline{u}$, where $D^+\overline{\rho}(t) = 0$ holds only if $(\mathcal{N}_{i_0}\cup\{i_0\})\backslash\mathcal{R}_{i_0}(t) \subseteq J_0(t)$. Recall that $\overline{\rho}(t) \geq \underline{u}$ for all $t \geq 0$. The above result ensures that if $\overline{\rho}(s) = \underline{u}$ for some time $s > 0$, then $\overline{\rho}$ will not increase. In other words, $\overline{\rho}(t) = \underline{u}$ for any $t \geq s$.

Combining the above comments, we derive that $\overline{\rho}(t)$ is non-increasing for $t \geq 0$. In an analogous manner, we define the index set $J_1(t) = \{j \in N : x_j(t) = \min_{i\in N} x_i(t)\}$ and apply the Dini derivative along the trajectory of (3) to derive (if $\underline{\rho}(t) < \bar{l}$)

$$\begin{aligned} D^+\underline{\rho}(t) &= D^+ \min_{i\in N} x_i(t) = \dot{x}_{i_1}(t) \\ &= \sum_{j\in(\mathcal{N}_{i_1}\cup\{i_1\})\backslash\mathcal{R}_{i_1}(t)} a_{i_1 j}\big(g_j(x_j^{i_1}(t)) - x_{i_1}(t)\big), \end{aligned} \tag{5}$$

where $\dot{x}_{i_1}(t) = \max_{i\in J_1(t)} \dot{x}_i(t)$. Arguing similarly as above, we can show that $\underline{\rho}(t)$ is non-decreasing for $t \geq 0$. Hence, we arrive at Claim 1. Next, we will show the following.

Claim 2. $\bar{l} \leq x_i(t) \leq \underline{u}$ for any $i \in N$ as $t \to \infty$.

By Assumption 1, we choose any $0 < \varepsilon < \tau/2$ and consider the time interval $[\tau_k + \varepsilon, \tau_{k+1} - \varepsilon]$ for a given $k \geq 1$. Define a set $\Xi_k = \{x = (x_1, x_2, \cdots, x_n) \in R^n : D^+\Gamma(x) = 0\}$, where $D^+\Gamma(x) := D^+\Gamma(t)|_{t=t_0}$ if $x_i(t_0) = x_i$ for some $t_0 \in [\tau_k + \varepsilon, \tau_{k-1} - \varepsilon]$ and all $i \in V$. In fact, $D^+\Gamma(x)$ can be viewed as the Dini derivative of Γ along the solution of system (3)[22, p. 353]. We will first show that $x_i \in [\bar{l}, \underline{u}]$ for any $i \in N$ by using the method of contradiction. In fact, if this statement is false, then there exists $x_{i_0}^* \notin [\bar{l}, \underline{u}]$ for some $i_0 \in N$, where $x^* = (x_1^*, x_2^*, \cdots, x_n^*) \in \Xi_k$. Without loss of generality, we assume $x_{i_0}^* = \max_{i\in N} x_i^* > \underline{u}$. Consider a solution $x(t) = (x_1(t), \cdots, x_n(t))$ of (3) starting with $x(\tau_k + \varepsilon) = x^*$. Reset $J_0 = \{j \in N : x_j^* = x_{i_0}^*\}$. Since G is $(2r+1)$-robust, by Lemma 1 the communication topology under (3) is connected. Any agent $j \in J_0$ will be attracted by those in $N\backslash J_0$ (having values less than $x_{i_0}^*$) or $\underline{u}$[13,14]. Hence, there exists $s \in (\tau_k + \varepsilon, \tau_{k+1} - \varepsilon]$ such that $x_j(s) < x_{i_0}^*$ for all $j \in N$. This means $\overline{\rho}(s) < \overline{\rho}(\tau_k + \varepsilon)$ and hence

$\Gamma(s) < \Gamma(\tau_k + \varepsilon)$, which contradicts with the assumption $x^* \in \Xi_k$. Therefore, we have managed to show $x_i \in [\bar{l}, \underline{u}]$ for all $i \in N$. Involving the LaSalle invariance principle of switching system[23, Thm. 1], the positive limit set of the system is contained in $\limsup_{k\to\infty} \Xi_k$. By continuity of the solution and letting $\varepsilon \to 0$, we know that $x_i(t) \in [\bar{l}, \underline{u}]$ for $i \in N$ as $t \to \infty$ and Claim 2 is proved. Next, we will show that the maximum and minimum of normal agents admit finite limits. Namely, we have Claim 3 as follows.

Claim 3. $\lim_{t\to\infty} D^+\Lambda(t) = 0$, where $\Lambda(t) := \overline{\lambda}(t) - \underline{\lambda}(t)$, $\overline{\lambda}(t) := \max_{i\in N} x_i(t)$ and $\underline{\lambda}(t) := \min_{i\in N} x_i(t)$.

We will use $\Lambda(t)$ as a Lyapunov-like function similar to the idea in[24]. To begin with, we rewrite (3) as

$$\dot{x}_i(t) = \sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_i(t)} a_{ij}\left(x_j^i(t) - x_i(t)\right) + z_i(t), \tag{6}$$

where $z_i(t) = \sum_{j\in(\mathcal{N}_i\cup\{i\})\setminus\mathcal{R}_i(t)} a_{ij}\big(g_j(x_j^i(t)) - x_j^i(t)\big)$ for $i \in N$. In view of Claim 2, definition of admissible intervals and the fact that G is a finite graph, we know that for any $\varepsilon > 0$, there is some time $t_\varepsilon > 0$ satisfying $|z_i(t)| \le \varepsilon$ for all $t \ge t_\varepsilon$ and all $i \in N$. Along the trajectory of (6), taking the Dini derivatives gives us the following:

$$D^+\overline{\lambda}(t) = \dot{x}_{i_0}(t) = \sum_{j\in(\mathcal{N}_{i_0}\cup\{i_0\})\setminus\mathcal{R}_{i_0}(t)} a_{i_0 j}\big(x_j^{i_0}(t) - x_{i_0}(t)\big) + z_{i_0}(t) \tag{7}$$

and

$$D^+\underline{\lambda}(t) = \dot{x}_{i_1}(t) = \sum_{j\in(\mathcal{N}_{i_1}\cup\{i_1\})\setminus\mathcal{R}_{i_1}(t)} a_{i_1 j}\big(x_j^{i_1}(t) - x_{i_1}(t)\big) + z_{i_1}(t), \tag{8}$$

where $\dot{x}_{i_0}(t) = \max_{i\in K_0(t)} \dot{x}_i(t)$, $\dot{x}_{i_1}(t) = \max_{i\in K_1(t)} \dot{x}_i(t)$, $K_0(t) = \{i : x_i(t) = \overline{\lambda}(t), i \in N\}$, and $K_1(t) = \{i : x_i(t) = \underline{\lambda}(t), i \in N\}$ similarly as in Claim 1. Note that $x_j^{i_0}(t) \le x_{i_0}(t)$ in (7) and $x_j^{i_1}(t) \ge x_{i_1}(t)$ in (8). For any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ satisfying $D^+\overline{\lambda}(t) \le \varepsilon/2$ and $D^+\underline{\lambda}(t) \ge -\varepsilon/2$ for all $t \ge t_\varepsilon$. Thus, $D^+\Lambda(t) \le \varepsilon$ for all $t \ge t_\varepsilon$.

In the sequel, we show $\lim_{t\to\infty} D^+\Lambda(t) = 0$ by contradiction. Assume that it is not true. There are two constants $\varepsilon_0 > 0$, $\delta_0 > 0$, and a time sequence $\{s_p\}_{p=1}^\infty$ satisfying $\lim_{p\to\infty} s_p = \infty$, $D^+\Lambda(s_p) \le -\varepsilon_0$ and $|s_{p+1} - s_p| > \delta_0$ for all $p \ge 1$. Consider any time interval $J \subseteq [t_\varepsilon, \infty)$ satisfying $J \cap \{\tau_k\}_{k=1}^\infty = \emptyset$. It can be seen that $D^+\Lambda(t)$ is uniformly continuous in J because it is continuous in J and $\dot{x}_i(t)$ is bounded for each $i \in N$. There is $\delta_1 > 0$ satisfying $|D^+\Lambda(t^1) - D^+\Lambda(t^2)| < \varepsilon_0/2$ for all $t^1, t^2 \in J$ and $|t^1 - t^2| < \delta_1$. Due to Assumption 1, we take $0 < \delta_2 < \delta_1$ such that for any $p \ge 1$, the interval $[s_p - \delta_2, s_p + \delta_2] \subseteq J$ for some J delineated above. Therefore,

$$D^+\Lambda(t) = -\left|D^+\Lambda(s_p) - (D^+\Lambda(s_p) - D^+\Lambda(t))\right| \le -(|D^+\Lambda(s_p)| - |D^+\Lambda(s_p) - D^+\Lambda(t)|) \le -\frac{\varepsilon_0}{2} \tag{9}$$

for every $t \in [s_p - \delta_2, s_p + \delta_2]$. Since $|s_{p+1} - s_p| > \delta_0$ for every $p \ge 1$, we select $0 < \delta < \delta_2$ to mark for a pairwise disjoint family $\{[s_p - \delta, s_p + \delta]\}_{p=1}^\infty$. Using (9) we obtain

$$\int_{t_\varepsilon}^{\infty} D^+\Lambda(t)dt \le \lim_{Q\to\infty} \sum_{p=1}^{Q} \int_{s_p-\delta}^{s_p+\delta} D^+\Lambda(t)dt \le -\lim_{Q\to\infty} \sum_{p=1}^{Q} \int_{s_p-\delta}^{s_p+\delta} \frac{\varepsilon_0}{2}dt = -\lim_{Q\to\infty} Q\delta\varepsilon_0 = -\infty. \tag{10}$$

This conflicts with the fact $\Lambda(t) = \overline{\lambda}(t) - \underline{\lambda}(t) \ge 0$ for any t. Hence, we proved Claim 3. Finally, what remains to show is Claim 4 below.

Claim 4. There exists some number $y \in [\bar{l}, \underline{u}]$ such that $\lim_{t\to\infty} x_i(t) = y$ for any $i \in N$.

In view of the analysis on (7), (8) and $\lim_{t\to\infty} D^+\Lambda(t) = 0$, there exist constants $\overline{y} \ge \underline{y}$ satisfying $\lim_{t\to\infty} \overline{\lambda}(t) = \lim_{t\to\infty} x_{i_0}(t) = \overline{y}$ and $\lim_{t\to\infty} \underline{\lambda}(t) = \lim_{t\to\infty} x_{i_1}(t) = \underline{y}$. Assume that $\overline{y} > \underline{y}$. Since G is $(2r+1)$-robust, the underlying topology has a spanning tree by Lemma 1. There exists time t^* and $\varepsilon > 0$ satisfying $x_{i_0}(t) > \overline{y} - \varepsilon > \underline{y} + \varepsilon > x_{i_1}(t)$ for $t \ge t^*$. Since $\lim_{t\to\infty} \dot{x}_{i_0}(t) = 0$, we have $\lim_{t\to\infty} x_j^{i_0}(t) - x_{i_0}(t) = 0$ for all $j \in (\mathcal{N}_{i_0} \cup \{i_0\})\setminus\mathcal{R}_{i_0}(t)$ by utilizing (3) and our resilient interval consensus strategy. Similarly, we obtain $\lim_{t\to\infty} x_j^{i_1}(t) - x_{i_1}(t) = 0$ for all $j \in (\mathcal{N}_{i_1} \cup \{i_1\})\setminus\mathcal{R}_{i_1}(t)$ through the convergence $\lim_{t\to\infty} \dot{x}_{i_1}(t) = 0$. As G is finite, there exists $t' \ge t^*$ satisfying two things: (i) There exist two paths, one starting from the root node k_0 to i_0 and the other from k_0 to i_1, in the communication network at t' and (ii) $x_{k_0}(t') > \overline{y} - \varepsilon$ and $x_{k_0}(t') < \underline{y} + \varepsilon$. Apparently, (ii) yields a contradiction. Hence, $\overline{y} = \underline{y} := y$. By Claim 2, $y \in [\bar{l}, \underline{u}]$. The proof is complete. □

Define the switching signal $\sigma : [0, \infty) \longrightarrow Q$, where Q is a finite set of bidirectional graphs. The network $G_{\sigma(t)} = (V, E_{\sigma(t)})$ is time-varying, and we have the adjacency matrix $A(t) = (a_{ij}(t)) \in R^{n\times n}$, where $t \ge 0$. We assume the following:

Assumption 2. There exist $\alpha_2 > \alpha_1 > 0$ such that $\alpha_1 < a_{ij}(t) < \alpha_2$ for $(j,i) \in E_{\sigma(t)}$ for all $t \geq 0$.
Assumption 1'. Let the time sequence $\{\tau_k\}_{k=1}^{\infty}$ be the points when $\mathcal{R}_i(t)$ or $\sigma(t)$ alter for some $i \in N$. There exists $\tau > 0$ such that $|\tau_{k+1} - \tau_k| \geq \tau$ for all $k \geq 1$.

Similarly as Theorem 1, we can show the following result.

Theorem 2. *Consider the multi-agent system (1) over a time varying bidirectional graph $G_{\sigma(t)}$, which is $(2r+1)$-robust for all t. Suppose that $\cap_{i\in N} I_i \neq \emptyset$ and Assumptions 1' and 2 hold. Then resilient interval consensus can be achieved by using the above proposed protocol when the number of misbehaving agents is locally bounded by r.*

4 | SIMULATIONS

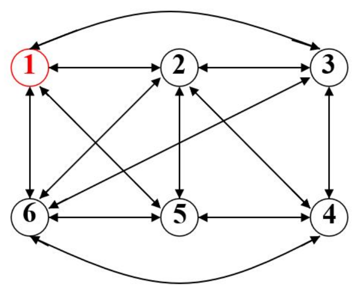

FIGURE 1 A 3-robust bidirectional graph G with $N = \{2,3,4,5,6\}$ and $M = \{1\}$.

We consider a network G with $n = 6$ agents having normal agents $N = \{2,3,4,5,6\}$ and misbehaving agent $M = \{1\}$; see Fig. 1. When $(j,i) \in E$, the adjacency matrix element is set as $a_{ij} = 0.1$ for $i < j$ and $a_{ij} = 0.2$ for $i > j$. It is direct to check that G is $(2r+1)$-robust with $r{=}1$. For each normal agent, its admissible interval is chosen as follows: $I_2 = [-4,2]$, $I_3 = [0,3]$, $I_4 = [-2,4]$, $I_5 = [-3,1]$, $I_6 = [-1,5]$, with the intersection $\cap_{i\in N} I_i = [0,1]$; see Fig. 2(b).

The initial condition is taken as $x_1(0) = -1$, $x_2(0) = 3$, $x_3(0) = -4$, $x_4(0) = -2$, $x_5(0) = 4$, $x_6(0) = -3$, which are outside their intervals. The misbehaving node 1 follows its own dynamics $\dot{x}_1(t) = -2x_1(t) + t/10$. By using our resilient interval consensus strategy, the state trajectories are shown in Fig. 2(a). We observe that all normal agents are able to reach an equilibrium at around $0.2 \in [0,1]$ in line with our theoretical prediction presented in Theorem 1.

5 | CONCLUSION

In this paper, we have investigated the problem of reaching agreement in the framework of resilient interval consensus over bidirectional graphs. A purely distributed resilient interval consensus strategy has been developed to withstand locally bounded misbehaving nodes for robust networks. On the basis of robust consensus analysis, we proved the convergence of states of normal nodes to the intersection of their admissible intervals for nonlinear multi-agent systems. The effectiveness of our algorithm is validated through a numerical example.

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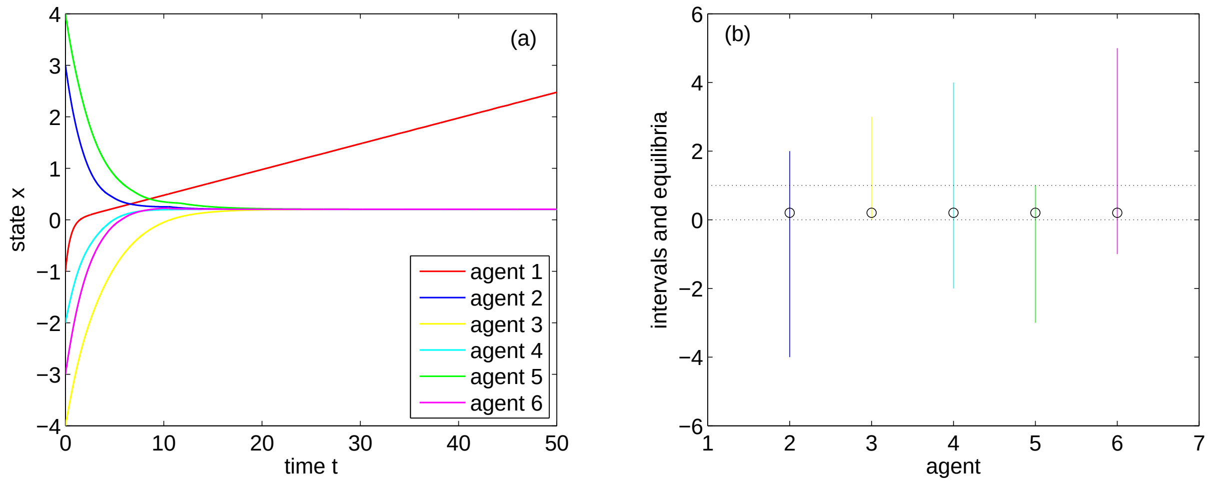

FIGURE 2 (a) State trajectories of the agents following the resilient interval strategy with agent 1 as a misbehaving node. (b) Admissible intervals I_i for $2 \leq i \leq 6$ and the final consensus equilibria at around $y = 0.2$.

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