Double-diffusive instabilities in rotating flows: Pseudo-Hermiticity and exceptional points

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Summary The Prandtl number, i.e. the ratio of the fluid viscosity to a diffusivity parameter of other physical nature such as thermal diffusivity or ohmic dissipation, plays a decisive part for the onset of instabilities in hydrodynamic and magnetohydrodynamic flows. The studies of many particular cases suggest a significant difference in stability criteria obtained for the Prandtl number equal to 1 from those for the Prandtl number deviating from 1. We demonstrate this for a circular Couette flow with a radial temperature gradient and for a differentially rotating viscous flow of electrically conducting incompressible fluid subject to an external azimuthal magnetic field. Furthermore, in the latter case we find that the local dispersion relation is governed by a pseudo-Hermitian matrix both in the case when the magnetic Prandtl number, Pm, is $\mathrm{Pm} = 1$ and in the case when $\mathrm{Pm} = -1$. This implies that the complete neutral stability surface contains three Whitney umbrella singular points and two mutually orthogonal intervals of self-intersection. At these singularities the double-diffusive system reduces to a marginally stable G-Hamiltonian system. The role of double complex eigenvalues (exceptional points) stemming from the singular points in exchange of stability between modes is demonstrated.

INSTABILITIES OF A CIRCULAR COUETTE FLOW WITH RADIAL TEMPERATURE GRADIENT

Circular Couette flow of a viscous Newtonian fluid between two coaxial differentially rotating cylinders is a canonical system for modelling instabilities leading to spatio-temporal patterns and transition to turbulence in many natural and industrial processes. Modern astrophysical applications require understanding of basic instability mechanisms in rotating flows with the Keplerian shear profile in accretion- and protoplanetary disks that are hydrodynamically stable according to the centrifugal Rayleigh criterion. Usually, these instabilities are a consequence of additional factors such as electrical conductivity of the fluid and the magnetic field of the central gravitating object. However, in the so-called dead-zones of protoplanetary disks that are characterized by high electrical resistivity due to very low ionization levels, some magnetohydrodynamic (MHD) instabilities become inefficient (e.g. the standard VelikhovChandrasekhar magnetorotational instability (MRI) in an axial magnetic field). Alternative mechanisms include MRIs caused by azimuthal or helical magnetic fields that work well also in the inductionless limit at a very low conductivity as well as pure hydrodynamic finite-amplitude nonlinear instabilities. Some recent studies indicated a possibility that Keplerian disks with strong mean radial temperature gradients can support the so-called Goldreich-Schubert-Fricke (GSF) instability, which is the instability of short-radial-wavelength inertial modes [1, 2, 3]

In [1] we applied the geometric optics stability analysis to circular Couette flow of a viscous Newtonian fluid with a temperature gradient in the absence of gravity, but retaining the term of the centrifugal buoyancy. We derived a system of characteristic equations that includes the transport equations for the lowest-order amplitude of the envelope of the localized perturbation and find a dispersion relation that takes into account the radial variation of the angular velocity and the temperature as well as the kinematic viscosity and the thermal diffusivity.

Using algebraic stability criteria for localization of the roots of polynomials in the left half of the complex plane, we obtained two stability conditions in compact and explicit form; one of them generalizes the Rayleigh discriminant for stationary axisymmetric instabilities to include viscosity effects, and the other provides a marginal stability curve in the parameter plane for oscillatory instabilities. In the case of a sole outer cylinder rotation, solid body rotation, and rotating flow with Keplerian shear we found a destabilizing effect of the inward temperature gradient that leads to oscillatory instability at small values of the Prandtl number ($\mathrm{Pr} < 1$) and to stationary instability at $\mathrm{Pr} > 1$ [1].

DOUBLE-DIFFUSIVE AZIMUTHAL MAGNETOROTATIONAL INSTABILITY

In [4, 5] we studied local instabilities of a differentially rotating viscous flow of electrically conducting incompressible fluid subject to an external azimuthal magnetic field. It was found that in the presence of the magnetic field the hydrodynamically stable flow can demonstrate non - axisymmetric azimuthal magnetorotational instability (AMRI) both in the diffusionless case and in the double-diffusive case with viscous and ohmic dissipation. It was shown in [4] that the dispersion relation for the double-diffusive AMRI is $p(\lambda) := \det(\mathbf{H}_0 + \mathbf{H}_1 - i^{-1}\mathbf{G}_1\lambda\mathbf{I}) = 0$, where $\mathbf{I}$ is the 4×4 identity matrix, $\mathbf{G}_1$ and $\mathbf{H}_0$ are Hermitian matrices and $\mathbf{H}_1$ is a complex non-Hermitian matrix of dissipative perturbations to a G-Hamiltonian (with a pseudo-Hermitian Hamiltonian) [6, 7] system of ideal MHD. Performing stability analysis based on the dispersion relation, we established that the threshold of the diffusionless AMRI via the Hamilton-Hopf bifurcation is a singular limit of the thresholds of the viscous and resistive AMRI corresponding to the dissipative Hopf bifurcation and manifests itself as the Whitney umbrella singular point. A smooth transition between the two types of instabilities is possible only if the magnetic Prandtl number is equal to unity, $\mathrm{Pm} = 1$. At a fixed $\mathrm{Pm} \neq 1$ the threshold of the double-diffusive AMRI is displaced by finite distance in the parameter space with respect to the diffusionless case even in the zero dissipation limit, see Fig. 1(left).

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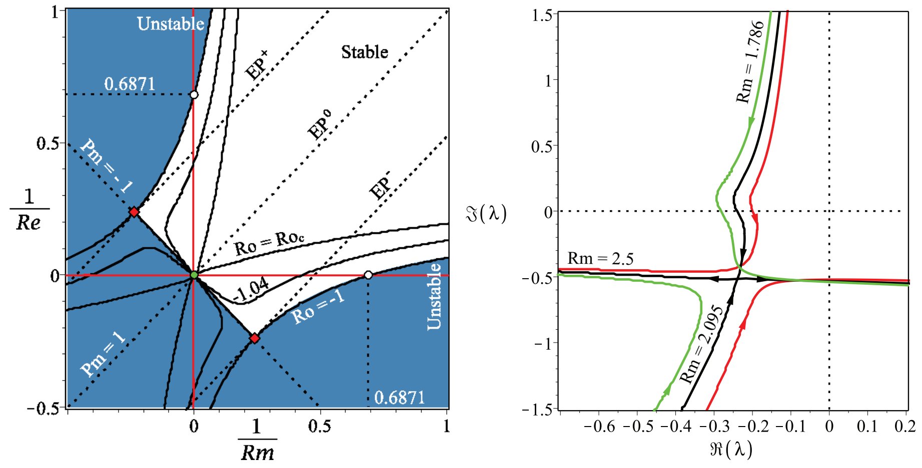

Figure 1: (Left) Contour plots of the neutral stability surface in the plane of inverse Reynolds numbers at (cuspidal curve) $\mathrm{Ro} = \mathrm{Ro}_c \approx -1.07855$, (filled area) $\mathrm{Ro} = -1$, and (intermediate curve) $\mathrm{Ro} = -1.04$. Two singular Whitney umbrella points (filled diamonds) exist at the intersection of the line $\mathrm{Pm} = -1$ and the neutral stability curve at $\mathrm{Ro} = -1$ and another one exists at the origin when $\mathrm{Ro} = \mathrm{Ro}_c$. From these singularities the lines $\mathrm{EP}^{\pm}, \mathrm{EP}^0$ of exceptional points are stemming that govern the transfer of modes shown in the right panel. (Right) For $\mathrm{Rb} = -1$, $\mathrm{S} = 1$, $n = \sqrt{2}$, and $\mathrm{Re} = 1000$ the movement of eigenvalues with decreasing Ro at various Rm chosen such that $\mathrm{Pm} < 1$. At $\mathrm{Rm} < 1000$ and up to $\mathrm{Rm} = \mathrm{Rm}_{\mathrm{EP}^-} \approx 2.095$ it is the branch corresponding to perturbed imaginary eigenvalues with positive Krein sign that causes instability. When $\mathrm{Rm} = \mathrm{Rm}_{\mathrm{EP}^-}$ two simple eigenvalues approach each other to merge exactly at $\mathrm{Ro} = -1$ into a double eigenvalue whose corresponding matrix is a Jordan block, $\lambda_{\mathrm{EP}^-} \approx -i0.5086 - 0.2391$. At $\mathrm{Rm} < \mathrm{Rm}_{\mathrm{EP}^-}$ the instability shifts to the branch of perturbed imaginary eigenvalues with negative Krein sign.

Let $\Omega(r)$ be the radial profile of the angular velocity of the fluid, $\mathrm{Ro} = \frac{r\partial_r \Omega}{2\Omega}$ the hydrodynamic Rossby number, $\mathrm{S} = \frac{\omega_{A_\phi}}{\Omega}$ the Alfvén angular velocity in the units of Ω, $\mathrm{Rb} = \frac{r\partial_r \omega_{A_\phi}}{2\omega_{A_\phi}}$ the magnetic Rossby number, Re the hydrodynamic Reynolds number, Rm the magnetic Reynolds number, and $\mathrm{Pm} = \mathrm{Rm}/\mathrm{Re}$ the magnetic Prandtl number.

Consider the following particular case: $\mathrm{Ro} = \mathrm{Rb} = -1$, $\mathrm{S} = 1$, and $\mathrm{Re} = -\mathrm{Rm}$ (that is $\mathrm{Pm} = -1$). Then, the dispersion relation for the double-diffusive AMRI takes the form $p(\lambda) := \det(\mathbf{H} - i^{-1}\mathbf{G}\lambda\mathbf{I}) = 0$, where $\mathbf{H}$ and $\mathbf{G}$ are Hermitian matrices

$$\mathbf{H} = \begin{pmatrix} n & 0 & \frac{i}{\mathrm{Rm}} - n & 0 \\ 0 & \alpha^2 n & -\frac{2\alpha}{n\mathrm{Rm}} & \frac{i\alpha^2}{\mathrm{Rm}} - n\alpha^2 \\ -\frac{i}{\mathrm{Rm}} - n & -\frac{2\alpha}{n\mathrm{Rm}} & n & 0 \\ 0 & -\frac{i\alpha^2}{\mathrm{Rm}} - n\alpha^2 & 0 & \alpha^2 n \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2i\alpha}{n} & \alpha^2 \\ 1 & -\frac{2i\alpha}{n} & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 \end{pmatrix}$$

with $n = \frac{m}{\alpha}$ the modified azimuthal wavenumber and α the coefficient depending on wavenumbers of the perturbation. Hence, in the case of $\mathrm{Pm} = -1 < 0$ the dispersion relation is governed by a G-Hamiltonian system, which implies neutral stability on an interval in the plane of inverse Reynolds numbers, see Fig. 1(left). The loss of the neutral stability is accompanied by the Krein collision of two modes with the opposite Krein/action signs when the parameters correspond to singular Whitney umbrella points. The sets of parameters stemming form the singular points and corresponding to complex double eigenvalues determine transfer of stability between modes, see Fig. 1(right). This picture provides the detailed mechanism of the destabilization of the Chandrasekhar equipartition solution in the case when $\mathrm{Pm} \neq 1$ [4, 8, 9, 10].

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