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Citation: Nath, Rajat Kanti, Fafous, Walaa Nabil Taha, Das, Kinkar Chandra and Shang, Yilun (2021) Common Neighborhood Energy of Commuting Graphs of Finite Groups. Symmetry, 13 (9). p. 1651. ISSN 2073-8994

Published by: MDPI

URL: <https://doi.org/10.3390/sym13091651> <<https://doi.org/10.3390/sym13091651>>


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Article

Common Neighborhood Energy of Commuting Graphs of Finite Groups

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Abstract: The commuting graph of a finite non-abelian group G with center $Z(G)$, denoted by $\Gamma_c(G)$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. Alwardi et al. (Bulletin, 2011, 36, 49-59) defined the common neighborhood matrix $CN(\mathcal{G})$ and the common neighborhood energy $E_{cn}(\mathcal{G})$ of a simple graph \mathcal{G} . A graph \mathcal{G} is called CN-hyperenergetic if $E_{cn}(\mathcal{G}) > E_{cn}(K_n)$, where $n = |V(\mathcal{G})|$ and K_n denotes the complete graph on n vertices. Two graphs \mathcal{G} and \mathcal{H} with equal number of vertices are called CN-equienergetic if $E_{cn}(\mathcal{G}) = E_{cn}(\mathcal{H})$. In this paper we compute the common neighborhood energy of $\Gamma_c(G)$ for several classes of finite non-abelian groups, including the class of groups such that the central quotient is isomorphic to group of symmetries of a regular polygon, and conclude that these graphs are not CN-hyperenergetic. We shall also obtain some pairs of finite non-abelian groups such that their commuting graphs are CN-equienergetic.

Keywords: commuting graph; CN-energy; finite group

MSC: 20D99; 05C50; 15A18; 05C25



Citation: Nath, R.K.; Fafous, W.N.T.; Das, K.C.; Shang, Y. Common Neighborhood Energy of Commuting Graphs of Finite Groups. *Symmetry* **2021**, *13*, 1651. <https://doi.org/10.3390/sym13091651>

Academic Editors: Juan Luis García Guirao and Aviv Gibali

Received: 24 June 2021

Accepted: 18 August 2021

Published: 8 September 2021

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1. Introduction

Let \mathcal{G} be a simple graph whose vertex set is $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$. The common neighborhood of two distinct vertices v_i and v_j , denoted by $C(v_i, v_j)$, is the set of vertices adjacent to both v_i and v_j other than v_i and v_j . The common neighborhood matrix of \mathcal{G} , denoted by $CN(\mathcal{G})$, is a matrix of size n whose (i, j) th entry is 0 or $|C(v_i, v_j)|$ according as $i = j$ or $i \neq j$. The common neighborhood matrix is a symmetric matrix, hence all its eigenvalues are real. The common neighborhood eigenvalues are symmetric with respect to the origin for some special class of graphs. There is a nice relation between $CN(\mathcal{G})$ and $A(\mathcal{G})$, the adjacency matrix of \mathcal{G} . More precisely, if $i \neq j$ then the (i, j) th entry of $CN(\mathcal{G})$ is same as the (i, j) th entry of $A(\mathcal{G})^2$, which is the number of 2-walks between the vertices v_i and v_j . Further, the (i, i) th entry of $CN(\mathcal{G})$ is equal to the degree of v_i . Hence, $CN(\mathcal{G}) = A(\mathcal{G})^2 - D(\mathcal{G})$, where $D(\mathcal{G})$ is the degree matrix of \mathcal{G} . Let $\text{CN-spec}(\mathcal{G})$ be the spectrum of $CN(\mathcal{G})$. Then $\text{CN-spec}(\mathcal{G})$ is the set of all the eigenvalues of $CN(\mathcal{G})$ with multiplicities. If $\alpha_1, \alpha_2, \dots, \alpha_k$ are the distinct eigenvalues of $CN(\mathcal{G})$ with multiplicities a_1, a_2, \dots, a_k , respectively, then we write $\text{CN-spec}(\mathcal{G}) = \{\alpha_1^{a_1}, \alpha_2^{a_2}, \dots, \alpha_k^{a_k}\}$. The common neighborhood energy (abbreviated as CN-energy) of the graph \mathcal{G} is given by

$$E_{cn}(\mathcal{G}) = \sum_{i=1}^k a_i |\alpha_i|.$$

The study of CN-energy of graphs was introduced by Alwardi et al. in [1]. Various properties of CN-energy of a graph can also be found in [1,2]. The motivation of studying $E_{cn}(\mathcal{G})$ comes from the study of $E(\mathcal{G})$, which is well-known as energy of \mathcal{G} , a notion introduced by Gutman [3]. Many results on $E(\mathcal{G})$, including some bounds and chemical applications, can be found in [4–15]. It is worth recalling that $E(\mathcal{G})$ is the sum of the absolute values of the eigenvalues of the adjacency matrix of \mathcal{G} . It is also interesting to note that $E(\mathcal{G})$ can be obtained if $E_{cn}(\mathcal{G})$ is known for some classes of graphs. For instance, $E(K_n) = E_{cn}(K_n)/(n-2)$ and $E(K_{m,n}) = \sqrt{E_{cn}(K_{m,n}) + 2(n+n)}$, where K_n is the complete graph on n vertices and $K_{m,n}$ is the complete bipartite graph on $(m+n)$ vertices. A graph \mathcal{G} is called CN-hyperenergetic if $E_{cn}(\mathcal{G}) > E_{cn}(K_n)$, where $n = |V(\mathcal{G})|$. It is still an open problem to produce a CN-hyperenergetic graph or to prove the non-existence of such graph (see [1] (Open problem 1)). In this paper we give an attempt to answer this problem by considering commuting graphs of finite groups.

The commuting graph of a finite non-abelian group G with center $Z(G)$ is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if and only if $xy = yx$. We write $\Gamma_c(G)$ to denote this graph. In [16–23], various aspects of $\Gamma_c(G)$ are studied. In Section 2 of this paper, we derive an expression for computing CN-energy of a particular class of graphs and list a few already known results. In Section 3, we compute CN-energy of commuting graph of certain metacyclic group, dihedral group (which is the group of symmetries of a regular polygon), quasidihedral group, generalized quaternion group, Hanaki group etc. We also consider some generalizations of dihedral group and generalized quaternion group. Two graphs \mathcal{G} and \mathcal{H} with equal number of vertices are called CN-equienergetic if $E_{cn}(\mathcal{G}) = E_{cn}(\mathcal{H})$. In Section 3, we shall also obtain some pairs of finite non-abelian groups such that their commuting graphs are CN-equienergetic. As consequences of our results, in Section 4, we show that $\Gamma_c(G)$ for all G considered in Section 3 are not CN-hyperenergetic. We also identify some positive integers n such that $\Gamma_c(G)$ is not CN-hyperenergetic if G is an n -centralizer group. It is worth mentioning that CN-spectrums of $\Gamma_c(G)$ for certain classes of finite groups have been computed in [24] recently. However, the method adopted here, in computing CN-energy of $\Gamma_c(G)$ for various families of finite groups, is independent of $\text{CN-spec}(\Gamma_c(G))$.

Recall that an n -centralizer group G is a group such that $|\text{Cent}(G)| = n$, where $\text{Cent}(G) = \{C_G(w) : w \in G\}$ and $C_G(w) = \{v \in G : vw = wv\}$ is the centralizer of w (see [25,26]). We also identify some $r \in \mathbb{Q}^{>0}$ such that $\Gamma_c(G)$ is not CN-hyperenergetic if $\text{Pr}(G) = r$. Also recall that the commutativity degree of G , denoted by $\text{Pr}(G)$, is the probability that a randomly chosen pair of elements of G commute.

Readers may review [27–32] for the background and various results regarding this notion. Further, we show that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Gamma_c(G)$ is not planar or toroidal. Note that a graph is planar or toroidal according as its genus is zero or one respectively. Finally, we conclude the paper with a few conjectures.

2. A Useful Formula and Prerequisites

We write $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$ to denote that \mathcal{G} has two components namely \mathcal{G}_1 and \mathcal{G}_2 . Also, lK_m denotes the disjoint union of l copies of the complete graph K_m on m vertices. We begin this section with the following two key results of Alwardi et al. [1].

Theorem 1 ([1] Proposition 2.4). *If $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2 \sqcup \dots \sqcup \mathcal{G}_m$ then $E_{cn}(\mathcal{G}) = \sum_{i=1}^m E_{cn}(\mathcal{G}_i)$.*

Lemma 1 ([1] Example 2.1). *If K_n denotes the complete graph on n vertices then*

$$E_{cn}(K_n) = 2(n-1)(n-2).$$

Now we derive a formula for CN-energy of graphs which are disjoint unions of some complete graphs. The following theorem is very useful in order to compute CN-energy of commuting graphs of finite groups.

Theorem 2. Let $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \dots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $1 \leq i \leq k$. Then

$$E_{cn}(\mathcal{G}) = 2 \sum_{i=1}^k l_i(m_i - 1)(m_i - 2).$$

Proof. By Theorem 1 we have

$$E_{cn}(\mathcal{G}) = \sum_{i=1}^k l_i E_{cn}(K_{m_i}).$$

Therefore, the result follows from Lemma 1. \square

We conclude this section with the following useful results from [17,18].

Lemma 2. Let G be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to

1. The Suzuki group $Sz(2)$, presented by $\langle u, v : u^5 = v^4 = 1, v^{-1}uv = u^2 \rangle$, then $\Gamma_c(G) = 5K_{3|Z(G)|} \sqcup K_{4|Z(G)|}$.
2. $\mathbb{Z}_p \times \mathbb{Z}_p$, for any prime p , then $\Gamma_c(G) = (p + 1)K_{(p-1)|Z(G)|}$.
3. The dihedral group D_{2m} ($m \geq 2$), presented by $\langle u, v : u^m = v^2 = 1, vuv^{-1} = u^{-1} \rangle$, then $\Gamma_c(G) = K_{(m-1)|Z(G)|} \sqcup mK_{|Z(G)|}$.

Lemma 3. Let G be a non-abelian group. If G is isomorphic to

1. A group of order pq , where p and q are primes with $p \mid (q - 1)$, then $\Gamma_c(G) = K_{q-1} \sqcup qK_{p-1}$.
2. The quasidihedral group QD_{2^n} ($n \geq 4$), presented by $\langle u, v : u^{2^{n-1}} = v^2 = 1, vuv^{-1} = u^{2^{n-2}-1} \rangle$, then $\Gamma_c(G) = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$.
3. $PSL(2, 2^k)$, the projective special linear group for $k \geq 2$, then $\Gamma_c(G) = 2^{k-1}(2^k - 1)K_{2^k} \sqcup (2^k + 1)K_{2^k-1} \sqcup 2^{k-1}(2^k + 1)K_{2^k-2}$.
4. $GL(2, q)$, the general linear group where $q = p^n > 2$ and p is a prime, then $\Gamma_c(G) = \frac{q(q-1)}{2}K_{q^2-q} \sqcup \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup (q + 1)K_{q^2-2q+1}$.

Lemma 4. Let G be a non-abelian group. If G is isomorphic to

1. The Hanaki group $A(n, \sigma)$ ($n \geq 2$) of order 2^{2n} given by

$$\left\{ U(x, y) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & \sigma(x) & 1 \end{bmatrix} : x, y \in F \right\},$$

under matrix multiplication where $F = GF(2^n)$ and $\sigma \in Aut(F)$ given by $\sigma(u) = u^2$, then $\Gamma_c(G) = (2^n - 1)K_{2^n}$.

2. The Hanaki group $A(n, p)$ of order p^{3n} given by

$$\left\{ V(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} : x, y, z \in F \right\},$$

under matrix multiplication where $F = GF(p^n)$ and p is a prime, then $\Gamma_c(G) = (p^n + 1)K_{p^{2n}-p^n}$.

3. CN-Energy of Commuting Graphs

In this section, we compute $E_{cn}(\Gamma_c(G))$ for several classes of finite non-abelian groups.

Theorem 3. Let G be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to

1. The Suzuki group $Sz(2)$, then

$$E_{cn}(\Gamma_c(G)) = 2(61|Z(G)|^2 - 57|Z(G)| + 12).$$

2. $\mathbb{Z}_p \times \mathbb{Z}_p$, then

$$E_{cn}(\Gamma_c(G)) = 2(p + 1)((p - 1)|Z(G)| - 1)((p - 1)|Z(G)| - 2).$$

3. The dihedral group D_{2m} ($m \geq 2$), then

$$E_{cn}(\Gamma_c(G)) = 2((m^2 - m + 1)|Z(G)|^2 - (6m - 3)|Z(G)| + 2m + 2).$$

Proof. By Lemma 2 and Theorem 2 we have

$$E_{cn}(\Gamma_c(G)) = \begin{cases} 2(4|Z(G)| - 1)(4|Z(G)| - 2) + 10(3|Z(G)| - 1)(3|Z(G)| - 2), & \text{if } \frac{G}{Z(G)} \cong Sz(2) \\ 2(p + 1)((p - 1)|Z(G)| - 1)((p - 1)|Z(G)| - 2), & \text{if } \frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ 2((m - 1)|Z(G)| - 1)((m - 1)|Z(G)| - 2) \\ \quad + 2m(|Z(G)| - 1)(|Z(G)| - 2), & \text{if } \frac{G}{Z(G)} \cong D_{2m}. \end{cases}$$

Hence, the result follows on simplification. \square

We have the following two corollaries of Theorem 3.

Corollary 1. Let G be isomorphic to one of the following groups

1. $\mathbb{Z}_2 \times Q_8$,
2. $\mathbb{Z}_2 \times D_8$,
3. $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle u, v : u^4 = v^4 = 1, vuv^{-1} = u^{-1} \rangle$,
4. $\mathcal{M}_{16} = \langle u, v : u^8 = v^2 = 1, vuv = u^5 \rangle$,
5. $SG(16, 3) = \langle u, v : u^4 = v^4 = 1, uv = v^{-1}u^{-1}, uv^{-1} = vu^{-1} \rangle$,
6. $D_8 * \mathbb{Z}_4 = \langle u, v, w : u^4 = v^2 = w^2 = 1, uv = vu, uw = wu, vw = u^2wv \rangle$.

Then $E_{cn}(\Gamma_c(G)) = 36$.

Proof. If G is isomorphic to one of the above listed group then it is of order 16. Therefore, $|Z(G)| = 4$ and so $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, putting $p = 2$ in Theorem 3 (2) we get the required result. \square

Corollary 2. Let G be a non-abelian group.

1. If G is of order p^3 , for any prime p , then

$$E_{cn}(\Gamma_c(G)) = 2(p + 1)(p^2 - p - 1)(p^2 - p - 2).$$

2. If G is the metacyclic group M_{2mn} ($m \geq 3$), presented by $\langle u, v : u^m = v^{2n} = 1, vuv^{-1} = u^{-1} \rangle$, then

$$E_{cn}(\Gamma_c(G)) = \begin{cases} 2((m^2 - m + 1)n^2 - (6m - 3)n + 2m + 2), & \text{if } m \text{ is odd} \\ 2((m^2 - 2m + 4)n^2 - (6m - 6)n + m + 2), & \text{if } m \text{ is even.} \end{cases}$$

3. If G is the dihedral group D_{2m} ($m \geq 3$), then

$$E_{cn}(\Gamma_c(G)) = \begin{cases} 2(m - 2)(m - 3), & \text{if } m \text{ is odd} \\ 2(m - 3)(m - 4), & \text{if } m \text{ is even.} \end{cases}$$

4. If G is the generalized quaternion group Q_{4n} ($n \geq 2$), presented by $\langle u, v : v^{2n} = 1, u^2 = v^n, uvu^{-1} = v^{-1} \rangle$, then

$$E_{cn}(\Gamma_c(G)) = 2(2n - 3)(2n - 4).$$

Proof. (1) If G is of order p^3 then $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 3 (2).

(2) We have

$$|Z(M_{2mn})| = \begin{cases} n, & \text{if } m \text{ is odd} \\ 2n, & \text{if } m \text{ is even} \end{cases} \text{ and } \frac{M_{2mn}}{Z(M_{2mn})} \cong \begin{cases} D_{2m}, & \text{if } m \text{ is odd} \\ D_m, & \text{if } m \text{ is even.} \end{cases}$$

Hence, the result follows from Theorem 3 (3).

(3) Follows from part (2), considering $n = 1$.

(4) Follows from Theorem 3 (3), since $|Z(Q_{4n})| = 2$ and $\frac{Q_{4n}}{Z(Q_{4n})} \cong D_{2n}$. \square

In the following theorems we compute $E_{cn}(\Gamma_c(G))$ for more families of groups.

Theorem 4. Let G be a non-abelian group.

1. If G is of order pq , where p and q are primes with $p \mid (q - 1)$, then

$$E_{cn}(\Gamma_c(G)) = 2(q^2 + p^2q - 5pq + q + 6).$$

2. If G is the quasidihedral group QD_{2^n} ($n \geq 4$), then

$$E_{cn}(\Gamma_c(G)) = 2(2^{n-1} - 3)(2^{n-1} - 4).$$

3. If $G = PSL(2, 2^k)$ then

$$E_{cn}(\Gamma_c(G)) = 2^{4k+1} - 4 \cdot 2^{3k+1} + 2^{2k+1} + 6 \cdot 2^{k+1} + 12.$$

4. If $G = GL(2, q)$ then

$$E_{cn}(\Gamma_c(G)) = 2q^6 - 6q^5 - 2q^4 + 10q^3 + 6q^2 + 2q.$$

Proof. (1) If G is of order pq then, by Lemma 3 (1) and Theorem 2, we have

$$E_{cn}(\Gamma_c(G)) = 2((q - 2)(q - 3) + q(p - 2)(p - 3)).$$

This gives the required result on simplification.

(2) Follows from Lemma 3 (2) and Theorem 2.

(3) By Lemma 3 (3) and Theorem 2 we have

$$\frac{E_{cn}(\Gamma_c(G))}{2} = (2^k + 1)(2^k - 2)(2^k - 3) + 2^{k-1}(2^k + 1)(2^k - 3)(2^k - 4) + 2^{k-1}(2^k - 1)(2^k - 1)(2^k - 2),$$

which gives the required result.

(4) By Lemma 3 (4) and Theorem 2 we have

$$E_{cn}(\Gamma_c(G)) = q(q + 1)(q^2 - 3q + 1)(q^2 - 3q) + q(q - 1)(q^2 - q - 1)(q^2 - q - 2) + 2(q + 1)(q^2 - 2q)(q^2 - 2q - 1),$$

which gives the required result on simplification. \square

Theorem 5. Let G be a non-abelian group.

1. If G is the Hanaki group $A(n, \sigma)$ then

$$E_{cn}(\Gamma_c(G)) = 2(2^n - 1)^2(2^n - 2).$$

2. If G is the Hanaki group $A(n, p)$ then

$$E_{cn}(\Gamma_c(G)) = 2(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2).$$

Proof. The result follows from Lemma 4 and Theorem 2. \square

Note that all the groups considered above are abelian centralizer group (in short, AC-group). Now we present a result on $E_{cn}(\Gamma_c(G))$ if G is a finite AC-group.

Theorem 6. Consider that an AC-group G has distinct centralizers X_1, \dots, X_n of non-central elements of G . Then $E_{cn}(\Gamma_c(G)) = 2 \sum_{i=1}^n (|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)| - 2)$.

Proof. We have $\Gamma_c(G) = \bigsqcup_{i=1}^n K_{|X_i| - |Z(G)|}$, by [17] (Lemma 1). Therefore, by Theorem 2, the result follows. \square

Corollary 3. Let K be a finite abelian group and H be a finite non-abelian AC-group. If $G \cong H \times K$ then

$$E_{cn}(\Gamma_c(G)) = 2 \sum_{i=1}^n (|Y_i||K| - |Z(H)||K| - 1)(|Y_i||K| - |Z(H)||K| - 2),$$

where $\text{Cent}(H) = \{H, Y_1, \dots, Y_n\}$.

Proof. Clearly $Z(H \times K) = Z(H) \times K$ and $\text{Cent}(H \times K) = \{H \times K, Y_1 \times K, Y_2 \times K, \dots, Y_n \times K\}$. Hence, $H \times K$ is an AC-group and so, by Theorem 6, the result follows. \square

We shall conclude this section by obtaining some pairs of finite non-abelian groups such that their commuting graphs are CN-equienergetic.

Proposition 1. The commuting graphs of D_{4k} and Q_{4k} for $k \geq 2$ are CN-equienergetic.

Proof. The result follows from parts (3) and (4) of Corollary 2. \square

Using Corollary 2 (parts (3) and (4)) and Theorem 4 (2) we also have the following result.

Proposition 2. The commuting graphs of D_{2k} , Q_{2k} and QD_{2k} for $k \geq 4$ are pairwise CN-equienergetic.

4. Some Consequences

In this section we derive some consequences of the results obtained in Section 3.

Theorem 7. Let G be a finite group with center $Z(G)$. If $\frac{G}{Z(G)}$ is isomorphic to $Sz(2)$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m} (where p is any prime and $m \geq 2$) then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. If $\frac{G}{Z(G)} \cong Sz(2)$ then, by Theorem 3 (1), we have

$$E_{cn}(\Gamma_c(G)) = 2(61|Z(G)|^2 - 57|Z(G)| + 12).$$

Since $|V(\Gamma_c(G))| = 19|Z(G)|$, by Lemma 1 we have

$$E_{cn}(K_{19|Z(G)|}) = 2(19|Z(G)| - 1)(19|Z(G)| - 2) = 2(361|Z(G)|^2 - 57|Z(G)| + 2).$$

Clearly, $361|Z(G)|^2 + 2 > 61|Z(G)|^2 + 12$ which gives $E_{cn}(K_{19|Z(G)|}) > E_{cn}(\Gamma_c(G))$.

If $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then, by Theorem 3 (2), we have

$$E_{cn}(\Gamma_c(G)) = 2(p + 1)((p - 1)|Z(G)| - 1)((p - 1)|Z(G)| - 2).$$

Since $|V(\Gamma_c(G))| = (p^2 - 1)|Z(G)|$, by Lemma 1 we have

$$E_{cn}(K_{(p^2-1)|Z(G)|}) = 2((p^2 - 1)|Z(G)| - 1)((p^2 - 1)|Z(G)| - 2).$$

Clearly

$$\begin{aligned} & ((p^2 - 1)|Z(G)| - 1)((p^2 - 1)|Z(G)| - 2) \\ & > ((p^2 - 1)|Z(G)| - (p + 1))((p^2 - 1)|Z(G)| - 2(p + 1)) \\ & > (p + 1)((p - 1)|Z(G)| - 1)((p - 1)|Z(G)| - 2). \end{aligned}$$

Thus $E_{cn}(K_{(p^2-1)|Z(G)|}) > E_{cn}(\Gamma_c(G))$.

If $\frac{G}{Z(G)} \cong D_{2m}$ then we have

$$E_{cn}(\Gamma_c(G)) = 2((m^2 - m + 1)|Z(G)|^2 - (6m - 3)|Z(G)| + 2m + 2),$$

by Theorem 3 (3). Since $|V(\Gamma_c(G))| = (2m - 1)|Z(G)|$, by Lemma 1 we have

$$\begin{aligned} E_{cn}(K_{(2m-1)|Z(G)|}) &= 2(2m|Z(G)| - |Z(G)| - 1)(2m|Z(G)| - |Z(G)| - 2) \\ &= 2((4m^2 - 4m + 1)|Z(G)|^2 - (6m - 3)|Z(G)| + 2). \end{aligned}$$

Clearly $(4m^2 - 4m + 1)|Z(G)|^2 > (m^2 - m + 1)|Z(G)|^2 + 2m$. Therefore, $E_{cn}(K_{(2m-1)|Z(G)|}) > E_{cn}(\Gamma_c(G))$. This completes the proof. \square

We have the following two corollaries.

Corollary 4. *If G is isomorphic to one of the groups listed in Corollary 1, then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. Since $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the result follows from Theorem 7 considering $p = 2$. \square

Corollary 5. *Let G be a non-abelian group. If G is isomorphic to M_{2mn} , D_{2m} , Q_{4n} or a group of order p^3 then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. If G is isomorphic to M_{2mn} , D_{2m} or Q_{4n} then $\frac{G}{Z(G)}$ is isomorphic to some dihedral groups. If G is isomorphic to a group of order p^3 then $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, by Theorem 7, the result follows. \square

We have the following results regarding commuting graphs of finite n -centralizer groups.

Theorem 8. *If G is a finite 4-centralizer group then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, by [25] (Theorem 2). Hence, using Theorem 7 for $p = 2$, the result follows. \square

Theorem 9. *Let G be a finite $(p + 2)$ -centralizer p -group. Then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, by [33] (Lemma 2.7). Hence, by Theorem 7, the result follows. \square

Theorem 10. *If G is a finite 5-centralizer group then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or D_6 , by [25] (Theorem 4). Hence, by Theorem 7, the result follows. \square

As a corollary to Theorems 8 and 10 we have the following result.

Corollary 6. *Let G be a finite non-abelian group and $\{x_1, x_2, \dots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then $\Gamma_c(G)$ is not CN-hyperenergetic if $r = 3, 4$.*

Proof. By [34] (Lemma 2.4), we have that G is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4. Hence the result follows from Theorems 8 and 10. \square

Theorem 11. *Let G be a non-abelian group. If G is isomorphic to QD_{2^n} , $PSL(2, 2^k)$, $A(n, \sigma)$, $GL(2, q)$, $A(n, p)$ or a group of order pr , where p and r are primes with $p \mid (r - 1)$ and $q = p^m > 2$, then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. If G is isomorphic to QD_{2^n} then, by Theorem 4, we have $E_{cn}(\Gamma_c(G)) = 2(2^{n-1} - 3)(2^{n-1} - 4)$. Since $|V(\Gamma_c(G))| = 2^n - 2$, by Lemma 1 we have

$$E_{cn}(K_{2^n-2}) = 2(2^n - 3)(2^n - 4).$$

Clearly, $(2^n - 3)(2^n - 4) > (2^{n-1} - 3)(2^{n-1} - 4)$. Hence, $E_{cn}(K_{2^n-2}) > E_{cn}(\Gamma_c(G))$. If G is isomorphic to $PSL(2, 2^k)$ then, by Theorem 4 (3), we have

$$E_{cn}(\Gamma_c(G)) = 2^{4k+1} - 4 \cdot 2^{3k+1} + 2^{2k+1} + 6 \cdot 2^{k+1} + 12.$$

Since $|V(\Gamma_c(G))| = 2^k(2^{2k} - 1) - 1 = 2^{3k} - 2^k - 1$, by Lemma 1 we have

$$\begin{aligned} E_{cn}(K_{2^{3k}-2^k-1}) &= 2(2^{3k} - 2^k - 1)(2^{3k} - 2^k - 3) \\ &= 2^{6k+1} - 2 \cdot 2^{4k+1} - 3 \cdot 2^{3k+1} + 2^{2k+1} + 5 \cdot 2^{k+1} + 12. \end{aligned}$$

Therefore,

$$\begin{aligned} E_{cn}(K_{2^{3k}-2^k-1}) - E_{cn}(\Gamma_c(G)) &= 2^{6k+1} - 3 \cdot 2^{4k+1} + 2^{3k+1} - 2^{k+1} \\ &= 2^{4k+1}(2^{2k} - 3) + 2^{k+1}(2^{2k} - 1). \end{aligned}$$

Since $2^{2k} - 3 > 0$ and $2^{2k} - 1 > 0$ we have $E_{cn}(K_{2^{3k}-2^k-1}) - E_{cn}(\Gamma_c(G))$ is positive. Hence, the result follows.

If G is isomorphic to $GL(2, q)$ then, by Theorem 4 (4), we have

$$E_{cn}(\Gamma_c(G)) = 2q^6 - 6q^5 - 2q^4 + 10q^3 + 6q^2 + 2q.$$

Since $|V(\Gamma_c(G))| = (q^2 - 1)(q^2 - q) - (q - 1) = q^4 - q^3 - q^2 + 1$, by Lemma 1 we have

$$E_{cn}(K_{q^4-q^3-q^2+1}) = 2(q^4 - q^3 - q^2)(q^4 - q^3 - q^2 - 1) = 2q^8 - 4q^7 - 2q^6 + 4q^5 + 2q^3 + 2q^2.$$

Therefore,

$$\begin{aligned} E_{cn}(K_{q^4-q^3-q^2+1}) - E_{cn}(\Gamma_c(G)) &= 2q^8 - 4q^7 - 4q^6 + 10q^5 + 2q^4 - 8q^3 - 4q^2 - 2q \\ &= 2q^6(q^2 - 2q - 2) + 2q^2(5q^3 - 4q - 2) + 2q(q^3 - 2). \end{aligned}$$

We have $q^2 - 2q - 2 = q(q - 2) - 2 > 0$, $5q^3 - 4q - 2 = q(5q^2 - 4) - 2 > 0$ and $q^3 - 2 > 0$ since $q = p^m > 2$ for some prime p . Therefore, $E_{cn}(K_{q^4 - q^3 - q^2 + 1}) - E_{cn}(\Gamma_c(G))$ is positive and hence the result follows.

If G is isomorphic to $A(n, \sigma)$ then, by Theorem 5 (1), we have $E_{cn}(\Gamma_c(G)) = 2(2^n - 1)^2(2^n - 2)$. Since $|V(\Gamma_c(G))| = 2^n(2^n - 1) = 2^{2n} - 2^n$, by Lemma 1 we have

$$E_{cn}(K_{2^{2n} - 2^n}) = 2(2^{2n} - 2^n - 1)(2^{2n} - 2^n - 2).$$

Clearly, $2^{2n} - 2^n - 1 > 2^{2n} - 2 \cdot 2^n - 1 = (2^n - 1)^2$ and $2^{2n} - 2^n - 2 > 2^n - 2$. Therefore, $E_{cn}(K_{2^{2n} - 2^n}) > E_{cn}(\Gamma_c(G))$.

If $G \cong A(n, p)$ then, by Theorem 5 (2), we have $E_{cn}(\Gamma_c(G)) = 2(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2)$. Since $|V(\Gamma_c(G))| = (p^n + 1)(p^{2n} - p^n)$, by Lemma 1 we have

$$E_{cn}(K_{(p^n + 1)(p^{2n} - p^n)}) = 2((p^n + 1)(p^{2n} - p^n) - 1)((p^n + 1)(p^{2n} - p^n) - 2).$$

We have

$$\begin{aligned} & (p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2) \\ & < (p^n + 1)(p^{2n} - p^n - 1)(p^n + 1)(p^{2n} - p^n - 2) \\ & = ((p^n + 1)(p^{2n} - p^n) - (p^n + 1))((p^n + 1)(p^{2n} - p^n) - 2(p^n + 1)) \\ & < ((p^n + 1)(p^{2n} - p^n) - 1)((p^n + 1)(p^{2n} - p^n) - 2). \end{aligned}$$

Hence, $E_{cn}(\Gamma_c(G)) < E_{cn}(K_{(p^n + 1)(p^{2n} - p^n)})$.

If G is isomorphic to a non-abelian group of order pr then, by Theorem 4 (1), we have

$$E_{cn}(\Gamma_c(G)) = 2(r^2 + p^2r - 5pr + r + 6).$$

Since $|V(\Gamma_c(G))| = pr - 1$, by Lemma 1 we have

$$E_{cn}(K_{pr - 1}) = 2(pr - 2)(pr - 3) = 2(p^2r^2 - 5pr + 6).$$

Since $r + 1 \leq 2(r - 1) < p^2(r - 1)$ we have $r^2 + p^2r + r < p^2r^2$. Hence, $E_{cn}(K_{pr - 1}) > E_{cn}(\Gamma_c(G))$. This completes the proof. \square

It is already mentioned that $\text{Pr}(G)$, the commutativity degree of a group G , is the probability that a randomly chosen pair of elements of G commute. Therefore, it measures the abelianness of a group. For any finite group G , its commutativity degree can be computed using the formula

$$\text{Pr}(G) = \frac{1}{|G|^2} \sum_{w \in G} |C_G(w)| \text{ or } \text{Pr}(G) = \frac{k(G)}{|G|},$$

where $k(G)$ is the number of conjugacy classes in G . In finite group theory, it is an interesting problem to find all the rational numbers $r \in (0, 1]$ such that $\text{Pr}(G) = r$ for some finite group G . Over the decades, many values of such r have obtained and characterized finite groups such that $\text{Pr}(G) = r$. In the following theorem we list some values of r such that $\Gamma_c(G)$ is not CN-hyperenergetic if $\text{Pr}(G) = r$.

Theorem 12. *If $\text{Pr}(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{7}{16}, \frac{1}{2}, \frac{5}{8}\}$ then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. If $\text{Pr}(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{7}{16}, \frac{1}{2}, \frac{5}{8}\}$ then $\frac{G}{Z(G)}$ is isomorphic to the groups in $\{D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$ (by [35] (p. 246) and [36] (p. 451)). Hence, the result follows from Theorem 7. \square

Theorem 13. Let G be a finite group and $\Pr(G) = \frac{p^2+p-1}{p^3}$, where p is the smallest prime divisor of $|G|$. Then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, by [37] (Theorem 3). Hence the result follows from Theorem 7. \square

Theorem 14. If G is a finite non-solvable group and $\Pr(G) = \frac{1}{12}$ then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. We have $G \cong A_5 \times K$ for some abelian group K , by [27] (Proposition 3.3.7). It can be seen that $\Gamma_c(G) = 5K_{3|K|} \sqcup 10K_{2|K|} \sqcup 6K_{4|K|}$. Therefore, by Theorem 2, we have

$$\begin{aligned} E_{cn}(\Gamma_c(G)) &= 2(5(3|K| - 1)(3|K| - 2) + 10(2|K| - 1)(2|K| - 2)) + 6(4|K| - 1)(4|K| - 2) \\ &= 2(181|K|^2 - 177|K| + 42). \end{aligned}$$

Additionally, by Lemma 1, we have $E_{cn}(K_{59|K|}) = 2(3481|K|^2 - 177|K| + 2)$. Therefore

$$E_{cn}(K_{59|K|}) - E_{cn}(\Gamma_c(G)) = 2(3300|K|^2 - 40) > 0.$$

This completes the proof. \square

The following three theorems show that $\Gamma_c(G)$ is not CN-hyperenergetic if $\Gamma_c(G)$ is planar/toroidal or the complement of $\Gamma_c(G)$ is planar.

Theorem 15. Let G be a finite non-abelian group. If $\Gamma_c(G)$ is planar then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. If $G \cong D_{12}, D_{10}, D_8, D_6, Q_8$ or Q_{12} then, by Corollary 5, we have that $\Gamma_c(G)$ is not CN-hyperenergetic.

If G is isomorphic to one of the groups listed in Corollary 1 then, by Corollary 4, it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong A_4$ then it can be seen that $\Gamma_c(G) = K_3 \sqcup 4K_2$. Using Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 4$. Also, by Lemma 1, we have $E_{cn}(K_{11}) = 180$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong Sz(2)$ then $\frac{G}{Z(G)} \cong Sz(2)$. Therefore, by Theorem 7, it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong SL(2, 3)$ then it can be seen that $\Gamma_c(G) = 3K_2 \sqcup 4K_4$. Therefore, by Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 48$. Also, by Lemma 1, we have $E_{cn}(K_{22}) = 840$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic.

We have $PSL(2, 4) \cong A_5$. Therefore, if $G \cong A_5$ then it follows that $\Gamma_c(G)$ is not CN-hyperenergetic (follows from Theorem 11).

If $G \cong S_4$ then the characteristic polynomial of $CN(\Gamma_c(G))$ is given by $x^8(x-3)^2(x+1)^{11}(x^2-5x-30)$ and so

$$CN\text{-spec}(\Gamma_c(G)) = \left\{ 0^8, 3^2, (-1)^{11}, \left(\frac{5 + \sqrt{145}}{2}\right)^1, \left(\frac{5 - \sqrt{145}}{2}\right)^1 \right\}.$$

Therefore, $E_{cn}(\Gamma_c(G)) = 17 + \sqrt{145}$. Additionally, by Lemma 1, we have $E_{cn}(K_{23}) = 924$. Therefore, $\Gamma_c(G)$ is not CN-hyperenergetic. Hence, the result follows from [38] (Theorem 2.2). \square

Theorem 16. Let G be a finite non-abelian group. If $\Gamma_c(G)$ is toroidal then $\Gamma_c(G)$ is not CN-hyperenergetic.

Proof. If $G \cong D_{14}, D_{16}$ or Q_{16} then by Corollary 5 it follows that $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong QD_{16}$ then, by Theorem 11, we have that $\Gamma_c(G)$ is not CN-

hyperenergetic. If G is isomorphic to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ then $\Gamma_c(G)$ is not CN-hyperenergetic, follows from Theorem 11 considering $p = 3$ and $r = 7$. If $G \cong D_6 \times \mathbb{Z}_3$ then $\frac{G}{Z(G)} \cong D_6$. Therefore, by Theorem 7, $\Gamma_c(G)$ is not CN-hyperenergetic. If $G \cong A_4 \times \mathbb{Z}_2$ then it can be seen that $\Gamma_c(G) = K_6 \sqcup 4K_4$. Therefore, by Theorem 2, we have $E_{cn}(\Gamma_c(G)) = 2(5 \cdot 4 + 4 \cdot 3 \cdot 2) = 88$. Also, by Lemma 1, we have $E_{cn}(K_{22}) = 2 \cdot 21 \cdot 20 = 840$. Hence, $\Gamma_c(G)$ is not CN-hyperenergetic. Hence, the result follows from [39] (Theorem 6.6). \square

We also have the following result.

Theorem 17. *Let G be a finite non-abelian group. If the complement of $\Gamma_c(G)$ is planar then $\Gamma_c(G)$ is not CN-hyperenergetic.*

Proof. The result follows from [40] (Proposition 2.3) and Corollary 5. \square

In view of the above results we conclude this paper with a few conjectures.

Conjecture 1. *A planar or toroidal graph is not CN-hyperenergetic.*

Conjecture 2. $\Gamma_c(G)$ is not CN-hyperenergetic.

Conjecture 3. *If $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \dots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $1 \leq i \leq k$, then it is not CN-hyperenergetic.*

Author Contributions: Conceptualization, R.K.N., W.N.T.F., K.C.D. and Y.S.; investigation, R.K.N., W.N.T.F., K.C.D. and Y.S.; writing—original draft preparation, R.K.N., W.N.T.F., K.C.D. and Y.S.; writing—review and editing, R.K.N., W.N.T.F., K.C.D. and Y.S.; funding acquisition, W.N.T.F., K.C.D. and Y.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Research Foundation fund from the Korean government, Grant No. 2021R1F1A1050, and UoA Flexible Fund from Northumbria University, Grant No. 201920A1001.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data is created in this paper.

Acknowledgments: The authors are grateful to the referees for their valuable comments and suggestions. W. N. T. Fasfous is thankful to Indian Council for Cultural Relations for the ICCR Scholarship. K. C. Das is supported by National Research Foundation funded by the Korean government (Grant No. 2021R1F1A1050). Y. Shang was supported by UoA Flexible Fund No. 201920A1001 from Northumbria University.

Conflicts of Interest: The authors declare no conflict of interest.

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