Non-existence of the asymptotic flocking in the Cucker−Smale model with short range communication weights

Xiuxia Yin, Zhiwei Gao, Zili Chen and Yichuan Fu

***Abstract*—For the long range communicated Cucker−Smale model, the asymptotic flocking exists for any initial condition. It is noted that, for the short range communicated Cucker−Smale model, the asymptotic flocking only holds for very restricted initial conditions. In this case, the non-existence of the asymptotic flocking has been frequently observed in numerical simulations, however, the theoretical results are far from perfect. In this note, we first point out that the non-existence of the asymptotic flocking is equivalent to the unboundedness of the second order space moment, i.e., $\sup_t \sum |x_i(t) - x_j(t)|^2 = \infty$. Furthermore, by taking the second derivative and then integrating, we establish a new and key equality about this moment. At last, we use this equality and relevant technical lemmas to deduce a general sufficient condition to the non-existence of the asymptotic flocking.**

***Index Terms*—Cucker−Smale model, multi-agent system, asymptotic flocking, communication weights.**

I. Introduction

During the recent two decades, flocking of agents has attracted much attention from various research fields [1]–[5], such as control theory, biology, robotics, smart sensor networks, etc. The Cucker−Smale (C−S) model, which was first proposed and investigated in [1] in 2007, is a classical model that captures many of the observed features of moving animals in nature, such as flocking of birds, schooling of fishes and so on. From then on, the C−S model was quickly extended in many directions [6]–[14], including the model with singular communication functions or other general communication weights, the model with stochastic noises, the kinetic description of this model, and so forth.

For the C−S model, the existence of the asymptotic flocking depends on the decay rate of the communication weight between agents. When the communication weight ϕ has a long range[1], the asymptotic flocking occurs for any initial condition. However, when the communication weight has a short range, the asymptotic flocking appears only for very restricted class of initial configurations. In this case, the non-existence of the asymptotic flocking has been observed in numerical simulations for a finite set of initial conditions see [15] for example. It is noted that the theoretical results are far from perfect. In [16], a sufficient and necessary condition to the non-existence of the asymptotic flocking was established when the particle number $N = 2$, and the emergence of the bi-cluster flocking was investigated from some well-prepared initial configurations that were already close to bi-cluster flocking configurations. To keep every agent staying in its own sub-ensemble, a sufficient condition to the non-existence of the asymptotic flocking was obtained in [17] for $N \geq 3$. It is noticed that in theory and practice it is highly possible that some agents will move from one cluster to another. Recently, based on a specific structure of C−S model in $\mathbb{R}^1$, a sufficient and necessary condition to the non-existence of the asymptotic flocking was obtained for the one dimensional case in [18]. Nevertheless, similar specific structure does not exist in higher dimensions. Therefore, for the short range communicated C−S model, it is extremely challenging to get a sufficient and necessary condition in $\mathbb{R}^d$ with $d \geq 2$ and $N \geq 3$ if it exists. In this technical note, by establishing a new equality about the second order space moment we obtain a general and brief sufficient condition to the non-existence of the asymptotic flocking. We do not need to divide the agents into the sub-ensembles beforehand, or assume the decreasing of ϕ.

The remainder of this paper is organized as follows. Section II is to address system description and some preliminaries. In Section III, we are devoted to developing a general sufficient condition to the non-existence of the asymptotic flocking. The key is that a new equality about the space moment. The technical note is ended by the conclusion in Section IV.

II. Preliminaries

Let N be the number of agents and let $(x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d$ denote the position and velocity of the ith agent at time t, where the dimension $d \geq 1$. The C−S model is described by the following dynamical system

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j \neq i} \phi(|x_j - x_i|)(v_j - v_i) \end{cases} \tag{1}$$

subjecting to the initial configuration

$$(x_i(0), v_i(0)) = (x_{i0}, v_{i0}), \tag{2}$$

where $|\cdot|$ denotes the standard Euclidean norm of $\mathbb{R}^d$. The nonnegative function ϕ denotes the communication weight.

Firstly, we give the definition of the asymptotic flocking.

Xiuxia Yin and Zili Chen are with the Department of Mathematics, School of Science, Nanchang University, Nanchang, 330031, P.R. China. e-mails: (yinxiuxia@ncu.edu.cn; chenzili@ncu.edu.cn).

Zhiwei Gao (Corresponding author) and Yichuan Fu are with the Faculty of Engineering and Environment, University of Northumbria at Newcastle, Newcastle upon Tyne, NE1 8ST, UK. emails:(zhiwei.gao@northumbria.ac.uk; yichuan.fu@northumbria.ac.uk)

[1] A long range communication weight ϕ means a nonnegative, bounded and decreasing function satisfying $\int_0^\infty \phi(r)dr = \infty$.

Definition 1: [1], [19] Model (1) exhibits the asymptotic flocking if and only if $\{(x_i(t), v_i(t))\}_{i=1}^N$ satisfies the following two conditions:

(i) The relative velocity fluctuation tends to zero as time goes to infinity, i.e.,

$$\lim_{t\to\infty}\sum_{i=1}^N |v_i(t)-v_c(t)|^2 = 0. \tag{3}$$

(ii) The diameter of the group is uniformly bounded in time $t\in[0,\infty)$, i.e.

$$\sup_{t\geq 0}\sum_{i=1}^N |x_i(t)-x_c(t)|^2 < \infty, \tag{4}$$

where $(x_c(t), v_c(t))$ is the average position and velocity at time t, i.e.

$$(x_c(t), v_c(t)) = \left(\frac{1}{N}\sum_{i=1}^N x_i(t), \frac{1}{N}\sum_{i=1}^N v_i(t)\right).$$

Note that from (1) we have that $v_c(t)\equiv v_c(0)$ and $x_c(t) = v_c(0)t + x_c(0)$. For simplicity, we use (x_c, v_c) to denote $(x_c(0), v_c(0))$.

Secondly, we state key properties of model (1).

Lemma 1: [7], [19], [20] Let $\{(x_i, v_i)\}_{i=1}^N$ be a smooth solution to model (1). Then, for any $t\geq 0$,

$$\frac{d}{dt}\sum_{i=1}^N |v_i(t)-v_c|^2 = -\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}|v_i-v_j|^2\phi(|x_i-x_j|), \tag{5}$$

and for any $\alpha>0$, $t\geq 0$,

$$\begin{aligned}&\frac{d}{dt}\Big[(t+\alpha)\sum_{i=1}^N\left|v_i-\frac{x_i}{t+\alpha}\right|^2-\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^{|x_i-x_j|} r\phi(r)dr\Big]\\ =&-\sum_{i=1}^N\left|v_i-\frac{x_i}{t+\alpha}\right|^2-\frac{t+\alpha}{N}\sum_{i=1}^N\sum_{j\neq i}|v_i-v_j|^2\phi(|x_j-x_i|),\end{aligned} \tag{6}$$

where $\int_{|x_{i0}-x_{j0}|}^{|x_i-x_j|} := -\int_{|x_i-x_j|}^{|x_{i0}-x_{j0}|}$ if $|x_i-x_j|<|x_{i0}-x_{j0}|$.

Remark 1: Let $\{(x_i, v_i)\}_{i=1}^N$ be a smooth solution to model (1). Then, $\{(x_i(t)-x_c(t), v_i(t)-v_c)\}_{i=1}^N$ is also a smooth solution to model (1). From the above lemma, equality (6) also holds for $(x_i(t)-x_c(t), v_i(t)-v_c)$. Thus,

$$\begin{aligned}&\int_0^t\sum_{i=1}^N\left|v_i(s)-v_c-\frac{x_i(s)-x_c(s)}{s+\alpha}\right|^2 ds\\ &\int_0^t\frac{s+\alpha}{N}\sum_{i=1}^N\sum_{j\neq i}|v_i-v_j|^2\phi(|x_j-x_i|)ds\\ \leq\;&\alpha\sum_{i=1}^N\left|v_{i0}-v_c-\frac{x_{i0}-x_c}{\alpha}\right|^2\\ &+\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr.\end{aligned} \tag{7}$$

Equality (5) can be used to deduce the asymptotic flocking when ϕ has a long range. Actually, without restriction of initial conditions, asymptotic flocking was firstly obtained in [1] and [19] for $\phi(r) = (1+r^2)^{-\frac{\beta}{2}}$ with $\beta\in[0,1)$ and $\beta=1$, respectively. Then, such results were generalized to the C$-$S model with long range communication weights in [7]. But, the asymptotic flocking occurs only for special initial configurations when ϕ has a short range. To be specific, we have the following lemma.

Lemma 2: [1], [7], [19] (1) Let the communication weight ϕ be nonnegative, decreasing and non-integrable. Then model (1) exhibits the asymptotic flocking for any initial condition.

(2) Let the communication weight ϕ be nonnegative, decreasing and integrable. Then model (1) exhibits the asymptotic flocking if

$$\max|x_{i0}-x_c|>0, \max|v_{i0}-v_c|<\frac{1}{2}\int_{\max|x_{i0}-x_c|}^\infty\phi(2r)dr. \tag{8}$$

Condition (8) is a sufficient condition of initial conditions that model (1) exhibits the asymptotic flocking, which means that the contrary side of (8) is a necessary condition of the non-existence of the asymptotic flocking. We are devoted to finding a general sufficient condition of non-existence of the asymptotic flocking with short range communication weights. In this case, equality (6) is useful, which was established in [20] recently, see [21] for its kinetic version. Actually, from (6) we can establish that

$$\sum_{i=1}^N|v_i(t)-x_i(t)/t|^2\longrightarrow 0,\quad\text{as } t\to\infty, \tag{9}$$

for any $0\leq\phi\in L^1(\mathbb{R}^+)$. See Remark 2 (4) in [20] for the precise proof. By the definition of $x_c(t), v_c$, (9) is equivalent to

$$\sum_{i=1}^N\left|v_i(t)-v_c-\frac{x_i(t)-x_c(t)}{t}\right|^2\longrightarrow 0,\quad\text{as } t\to\infty. \tag{10}$$

Then, we have the following remark.

Remark 2: Let ϕ be nonnegative and integrable. Model (1) has an asymptotic flocking if and only if $\sup_{t\geq 0}\sum|x_i(t)-x_c(t)|^2<\infty$. In other words, there is the non-existence of the asymptotic flocking if and only if $\sup_{t\geq 0}\sum|x_i(t)-x_c(t)|^2=\infty$.

When the asymptotic flocking does not exist, from Definition 1 and the above remark there are two cases:

(1) $\sup_{t\geq 0}\sum|x_i(t)-x_c(t)|^2=\infty$ and $\sum|v_i(t)-v_c|^2\to 0$.
(2) $\sum|v_i(t)-v_c|^2\nrightarrow 0$.

Now, we show that what does $\sum|v_i(t)-v_c|^2\nrightarrow 0$ mean. By (5) we have the decreasing of $\sum|v_i-v_c|^2$, so there exists a $c_0>0$ such that $\sum|v_i(t)-v_c|^2\to c_0$. Then it follows from (10) that

$$\begin{aligned}&t^{-2}\sum|x_i(t)-x_c(t)|^2\\ \geq&\sum|v_i(t)-v_c|^2-\sum\left|v_i(t)-v_c-\frac{x_i(t)-x_c(t)}{t}\right|^2\\ \geq& c_0/2\end{aligned} \tag{11}$$

for sufficiently large t. So $\sum|v_i(t)-v_c|^2\nrightarrow 0$ can deduce $\sum|x_i(t)-x_c(t)|^2\geq Ct^2$. The reverse is similar.

Remark 3: Let ϕ be nonnegative and integrable, then $\sum |v_i(t) - v_c|^2 \nrightarrow 0$ if and only if $\sum |x_i(t) - x_c(t)|^2 \geq Ct^2$ for sufficiently large t.

III. Main results

A. Non-existence of the asymptotic flocking

Now, we use inequality (7) to deduce a sufficient condition of the non-existence of the asymptotic flocking. Firstly, we need the following lemma, which is a bridge between the space moment $\sum |x_i(t) - x_c(t)|^2$ and the velocity-position moment $\sum |v_i(t) - v_c - \frac{x_i(t)-x_c(t)}{t+\alpha}|^2$.

Lemma 3: Let $\{(x_i, v_i)\}_{i=1}^N$ be a smooth solution to model (1). Then for any $\epsilon > 0$ and $\alpha > 0$,

$$\frac{(1+\epsilon)\alpha^2}{(t+\alpha)^2}|x_i(t) - x_c(t)|^2 \geq |x_{i0} - x_c|^2 - \alpha(1+1/\epsilon)\int_0^t \left|v_i(s) - v_c - \frac{x_i(s) - x_c(s)}{s+\alpha}\right|^2 ds. \quad (12)$$

Proof. For simplicity, we denote the vector-valued function $v_i(t) - v_c - \frac{x_i(t)-x_c(t)}{t+\alpha} = g_i(t)$ for any i. By solving the ODEs

$$\frac{d}{dt}(x_i(t) - x_c(t)) - \frac{x_i(t) - x_c(t)}{t+\alpha} = g_i,$$

we obtain that

$$x_i(t) - x_c(t) = \frac{t+\alpha}{\alpha}(x_{i0} - x_c) + (t+\alpha)\int_0^t \frac{g_i(s)}{s+\alpha}ds. \quad (13)$$

Note that for any $a, b \in \mathbb{R}^d$, we have the following basic iequality

$$|a+b|^2 = |a|^2 + |b|^2 + 2a\cdot b \leq (1+\epsilon)|a|^2 + (1+\epsilon^{-1})|b|^2,$$

where ϵ can be any positive constant. So from (13) and the above inequality,

$$\begin{aligned}
&\left|\frac{t+\alpha}{\alpha}(x_{i0} - x_c)\right|^2 \\
= &\left|x_i(t) - x_c(t) - (t+\alpha)\int_0^t \frac{g_i(s)}{s+\alpha}ds\right|^2 \\
\leq &(1+\epsilon)\left|x_i(t) - x_c(t)\right|^2 \\
&+(1+1/\epsilon)\left|(t+\alpha)\int_0^t \frac{g_i(s)}{s+\alpha}ds\right|^2 \\
\leq &(1+\epsilon)\left|x_i(t) - x_c(t)\right|^2 \\
&+(1+1/\epsilon)\frac{(t+\alpha)^2}{\alpha}\int_0^t |g_i(s)|^2 ds, \quad (14)
\end{aligned}$$

where the last inequality is obtained from Hölder's inequality. By (14) we get (12).

□

Then, by choosing a sufficiently large ϵ and an appropriate α, we use inequality (7) and Lemma 3 to deduce the following result.

Theorem 1: Let $\{(x_i, v_i)\}_{i=1}^N$ be a smooth solution to model (1). Assume that ϕ is nonnegative, bounded and $r\phi(r) \in L^1$. If the initial conditions satisfy that

$$\sum_{i=1}^N (x_{i0} - x_c)\cdot(v_{i0} - v_c) > \frac{1}{2N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr, \quad (15)$$

there is non-existence of the asymptotic flocking.

Proof. It follows from Lemma 3 and inequality (7) that

$$\begin{aligned}
&\frac{(1+\epsilon)\alpha^2}{(t+\alpha)^2}\sum_{i=1}^N |x_i(t) - x_c(t)|^2 \\
\geq &\sum_{i=1}^N |x_{i0} - x_c|^2 \\
&-\alpha(1+1/\epsilon)\int_0^t \sum_{i=1}^N \left|v_i(s) - v_c - \frac{x_i(s) - x_c(s)}{s+\alpha}\right|^2 ds \\
\geq &\sum_{i=1}^N |x_{i0} - x_c|^2 - (1+1/\epsilon)\left[\alpha^2\sum_{i=1}^N\left|v_{i0} - v_c - \frac{x_{i0} - x_c}{\alpha}\right|^2\right. \\
&\left.+\frac{\alpha}{N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr\right].
\end{aligned}$$

Thus, if there exists $\alpha > 0$ such that

$$\begin{aligned}
\sum_{i=1}^N |x_{i0} - x_c|^2 > &\alpha^2\sum_{i=1}^N\left|v_{i0} - v_c - \frac{x_{i0} - x_c}{\alpha}\right|^2 \\
&+\frac{\alpha}{N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr, \quad (16)
\end{aligned}$$

then we can choose sufficiently large ϵ to get $\sum\limits_{i=1}^N |x_i(t) - x_c(t)|^2 \to \infty$. Condition (16) is equivalent to

$$\begin{aligned}
&\alpha^2\sum_{i=1}^N |v_{i0} - v_c|^2 - 2\alpha\sum_{i=1}^N (v_{i0} - v_c)\cdot(x_{i0} - x_c) \\
&+\frac{2\alpha}{2N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr < 0.
\end{aligned}$$

When the initial conditions satisfy (15), the above inequality and (16) hold for some $\alpha > 0$. □

Remark 4: (1) In the above proof, we actually get that $\sum |x_i(t) - x_c(t)|^2 \geq Ct^2$ for sufficiently large t. Therefore, from Remark 3 we have that $\sum |v_i(t) - v_c|^2 \nrightarrow 0$.

(2) The non-existence of the asymptotic flocking means that there is non-flocking in the whole group $\{1, 2, \cdots, N\}$. Actually, the method in Theorem 1 can be used to show that there is non-flocking in a smaller group for some initial conditions. Firstly, we can follow the method in Lemma 3 to get that

$$\begin{aligned}
&\frac{(1+\epsilon)\alpha^2}{(t+\alpha)^2}|x_i(t) - x_j(t)|^2 \geq |x_{i0} - x_{j0}|^2 \\
&-\alpha(1+1/\epsilon)\int_0^t \left|v_i(s) - v_j(s) - \frac{x_i(s) - x_j(s)}{s+\alpha}\right|^2 ds.
\end{aligned}$$

Then, for any $S \subsetneq \{1, 2, \cdots, N\}$ we have that

$$\frac{(1+\epsilon)\alpha^2}{(t+\alpha)^2}\sum_{i,j\in S}|x_i(t) - x_j(t)|^2 \geq \sum_{i,j\in S}|x_{i0} - x_{j0}|^2$$

$$-\alpha(1+1/\epsilon)\int_0^t\sum_{i=1}^N\sum_{j\neq i}\left|v_i-v_j-\frac{x_i-x_j}{s+\alpha}\right|^2 ds.$$

Combining the above inequality with (7), for some initial conditions we can get $\sum_{i,j\in S}|x_i(t)-x_j(t)|^2\to\infty$, which means that there is non-flocking in S.

However, if we focus on the non-existence of the asymptotic flocking, condition (15) in Theorem 1 is too strong. Even the communication is very weak, which yields the right hand side of (15) is small, $\sum(v_{i0}-v_c)\cdot(x_{i0}-x_c)$ should be positive, at least. Thus, the space moment $\sum|x_i(t)-x_c(t)|^2$ increases initially, since $2(v_{i0}-v_c)\cdot(x_{i0}-x_c)$ is the derivative of this moment at $t=0$.

B. A new equality

Now, we establish a new equality about $\sum|x_i(t)-x_c(t)|^2$, from which we can first get a better understanding of condition (15).

Proposition 1: Let $\{(x_i,v_i)\}_{i=1}^N$ be a smooth solution to model (1). Then,

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^N|x_i(t)-x_c(t)|^2\right)\\ =&\int_0^t\sum_{i=1}^N|v_i(s)-v_c|^2ds+\sum_{i=1}^N(x_{i0}-x_c)\cdot(v_{i0}-v_c)\\ &-\frac{1}{2N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^{|x_i(t)-x_j(t)|}r\phi(r)dr\end{aligned}$$

Proof. We compute $\frac{d^2}{dt^2}\sum|x_i(t)-x_c(t)|^2$ and then integrate over $(0,t)$. It follows from (1) that

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\frac{1}{2}\sum_{i=1}^N|x_i(t)-x_c(t)|^2\right)\\ =&\frac{d}{dt}\sum_{i=1}^N(x_i(t)-x_c(t))\cdot(v_i(t)-v_c)\\ =&\sum_{i=1}^N|v_i(t)-v_c|^2+\\ &\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}\phi(\cdot)(x_i(t)-x_c(t))\cdot(v_j(t)-v_i(t))\\ =&\sum_{i=1}^N|v_i(t)-v_c|^2-\\ &\frac{1}{2N}\sum_{i=1}^N\sum_{j\neq i}\phi(\cdot)(x_j(t)-x_i(t))\cdot(v_j(t)-v_i(t))\end{aligned}$$

where $\phi(\cdot)=\phi(|x_j(t)-x_i(t)|)$. Note that

$$\begin{aligned}&\int_0^t\phi(|x_j(s)-x_i(s)|)(x_j(s)-x_i(s))\cdot(v_j(s)-v_i(s))ds\\ =&\int_{|x_{i0}-x_{j0}|}^{|x_i(t)-x_j(t)|}r\phi(r)dr,\end{aligned}$$

so we obtain the conclusion. □

From this equality, we know from (15) that the space moment is on the increase all the time, and

$$\frac{d}{dt}\sum|x_i(t)-x_c(t)|^2\geq C>0 \tag{17}$$

for any $t\geq 0$. But, a more reasonable way is that (17) holds only for sufficiently large t. Based on Proposition 1 and other technical estimates we can achieve this goal and get a better sufficient condition than (15).

C. Revisit the non-existence of the asymptotic flocking

Before giving the main theorem, we need the following lemma, which was established in Remark 2 (3) in [20].

Lemma 4: [20] Let $\{(x_i,v_i)\}_{i=1}^N$ be a smooth solution to model (1). Assume that ϕ is nonnegative, bounded and $r\phi(r)\in L^1$. Then, $v_i^*:=\lim_{t\to\infty}v_i(t)$ exists for any i, and

$$\sum_{i=1}^N|v_i(t)-v_i^*|^2\leq Ct^{-1},\quad\forall\ t>0.$$

Theorem 2: Let $\{(x_i,v_i)\}_{i=1}^N$ be a smooth solution to model (1). Assume that ϕ is nonnegative, bounded and $r\phi(r)\in L^1$. If the initial conditions satisfy that

$$\begin{aligned}&\frac{1}{2\|\phi\|_{L^\infty}}\sum_{i=1}^N|v_{i0}-v_c|^2-\frac{1}{2N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr\\ &+\sum_{i=1}^N(x_{i0}-x_c)\cdot(v_{i0}-v_c)>0,\end{aligned}\tag{18}$$

there is the non-existence of the asymptotic flocking. If we further assume that

$$\begin{aligned}&\frac{1}{4\|\phi\|_{L^\infty}}\sum_{i=1}^N|v_{i0}-v_c|^2-\frac{1}{2N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr\\ &+\sum_{i=1}^N(x_{i0}-x_c)\cdot(v_{i0}-v_c)>0,\end{aligned}\tag{19}$$

then $\sum_{i=1}^N|v_i(t)-v_c|^2\nrightarrow 0$.

Proof. Firstly, we show that

$$\begin{aligned}&\int_0^t\sum_{i=1}^N|v_i(t)-v_c|^2ds\\ \geq&\sum_{i=1}^N|v_{i0}-v_c|^2\frac{1-\exp\{-2\|\phi\|_{L^\infty}t\}}{2\|\phi\|_{L^\infty}}.\end{aligned}\tag{20}$$

Following from (5) we obtain that

$$\begin{aligned}&\frac{d}{dt}\sum_{i=1}^N|v_i(t)-v_c|^2\\ \geq&-\frac{\|\phi\|_{L^\infty}}{N}\sum_{i=1}^N\sum_{j\neq i}|v_i-v_j|^2\\ =&-\frac{\|\phi\|_{L^\infty}}{N}\sum_{i=1}^N\sum_{j=1}^N|(|v_i-v_c|^2+|v_j-v_c|^2)\end{aligned}$$

$$= \quad -2\|\phi\|_{L^\infty}\sum_{i=1}^N |v_i - v_c|^2.$$

Thus, $\sum |v_i(t)-v_c|^2 \geq \exp\{-2\|\phi\|_{L^\infty}t\}\sum|v_{i0}-v_c|^2$, and then by integrating over $(0,t)$ we obtain (20). Combining Proposition 1 with (20), we can obtain that

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^N |x_i(t)-x_c(t)|^2\right)$$
$$\geq \sum_{i=1}^N |v_{i0}-v_c|^2 \frac{1-\exp\{-2\|\phi\|_{L^\infty}t\}}{2\|\phi\|_{L^\infty}} +$$
$$\sum_{i=1}^N (x_{i0}-x_c)\cdot(v_{i0}-v_c) - \frac{1}{2N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr.$$

Thus, if condition (18) holds, there exists $C>0$ such that $\frac{d}{dt}\left(\sum|x_i(t)-x_c(t)|^2\right) \geq C$ for sufficient large t, which yields that $\sum_{i=1}^N |x_i(t)-x_c(t)|^2 \to \infty$.

Now, we prove the second conclusion. On the one hand, from inequality (7) we have that

$$\begin{aligned}&\int_0^t \frac{s+\alpha}{N}\sum_{i=1}^N\sum_{j\neq i}|v_i-v_j|^2\phi(|x_j-x_i|)ds\\ \leq\ & \alpha\sum_{i=1}^N \left|v_{i0}-v_c-\frac{x_{i0}-x_c}{\alpha}\right|^2\\ &+\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr.\end{aligned}$$

On the other hand, from (5) we can take the derivative of $(t+\alpha)\sum|v_i-v_c|^2$, and then

$$\begin{aligned}&(t+\alpha)\sum_{i=1}^N|v_i-v_c|^2 - \int_0^t\sum_{i=1}^N|v_i-v_c|^2ds\\ &+\int_0^t\frac{s+\alpha}{N}\sum_{i=1}^N\sum_{j\neq i}|v_i-v_j|^2\phi(|x_j-x_i|)ds\\ =\ & \alpha\sum_{i=1}^N|v_{i0}-v_c|^2.\end{aligned}$$

Combining the above two inequalities together, we have from inequality (20) that

$$\begin{aligned}&(t+\alpha)\sum_{i=1}^N|v_i-v_c|^2\\ \geq\ & \int_0^t\sum_{i=1}^N|v_i-v_c|^2ds+\alpha\sum_{i=1}^N|v_{i0}-v_c|^2\\ &-\alpha\sum_{i=1}^N\left|v_{i0}-v_c-\frac{x_{i0}-x_c}{\alpha}\right|^2\\ &-\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr\\ \geq\ & \frac{1}{2\|\phi\|_{L^\infty}}\sum_{i=1}^N|v_{i0}-v_c|^2+2\sum_{i=1}^N(v_{i0}-v_c)\cdot(x_{i0}-x_c)\\ &-\frac{\exp\{-2\|\phi\|_{L^\infty}t\}}{2\|\phi\|_{L^\infty}}\sum_{i=1}^N|v_{i0}-v_c|^2-\frac{\sum_{i=1}^N|x_{i0}-x_c|^2}{\alpha}\\ &-\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr\end{aligned}$$

Thus, if condition (19) holds, we can choose a sufficiently large α such that

$$\sum_{i=1}^N|v_i(t)-v_c|^2\geq C(t+\alpha)^{-1}$$

for sufficiently large t. Then we use Proposition 1 to get that

$$\sum_{i=1}^N|x_i(t)-x_c(t)|^2\geq Ct\log(t+\alpha) \tag{21}$$

for sufficiently large t.

However, if we assume that $v_i(t)\to v_c$ for any i, then from Lemma 4 we have that $|v_i(t)-v_c|\leq Ct^{-\frac{1}{2}}$. By integrating we obtain that $|x_i(t)-x_c(t)|\leq Ct^{\frac{1}{2}}+|x_{i0}-x_c|$ for any i, which is equivalent to

$$\sum_{i=1}^N|x_i(t)-x_c(t)|^2\leq C(t+1). \tag{22}$$

Obviously, inequality (22) is in contradiction with inequality (21). So the assumption, i.e. $v_i(t)\to v_c$ for any i, does not hold. That is, $\sum_{i=1}^N|v_i(t)-v_c|^2\nrightarrow 0$. □

Remark 5: (18) is still a sufficient condition of the non-existence of the asymptotic flocking. By a simple computation, (18) is equivalent to

$$\begin{aligned}&\frac{1}{N(N-1)}\sum_{i=1}^N\sum_{j\neq i}(x_{i0}-x_{j0})\cdot(v_{i0}-v_{j0})\\ >&\frac{1}{N(N-1)}\sum_{i=1}^N\sum_{j\neq i}\int_{|x_{i0}-x_{j0}|}^\infty r\phi(r)dr\\ &-\frac{1}{2N(N-1)\|\phi\|_{L^\infty}}\sum_{i=1}^N\sum_{j\neq i}|v_{i0}-v_{j0}|^2.\end{aligned}$$

Roughly speaking, it means that the average of $(x_{i0}-x_{j0})\cdot(v_{i0}-v_{j0})$ has a lower bound, which can be negative and independent of N. Thus, (18) is a rather general sufficient condition. Another sufficient condition was obtained in Theorem 3.1 of [17]. However, all agents should be divided into sub-ensembles at $t=0$, and the minimum of initial velocity differences between agents with different sub-ensembles should be very large, especially when $N\gg 1$. Actually, in [17] the authors further proved that every agent will stay in its original sub-ensemble under this strong sufficient condition. So, this condition is more useful in the multi-cluster problem.

Finally, we point out that there could be the asymptotic flocking when the left hand side of (18) is negative or even zero. Now, we give an example. Let $\phi=\chi_{[0,4\sqrt{3}]}$ [2] and $N=3$,

[2] $\chi_{[0,R]}$ is an indicator function such that $\chi_{[0,R]}(r)=1$ if $r\in[0,R]$ and $\chi_{[0,R]}(r)=0$ if $r>R$.

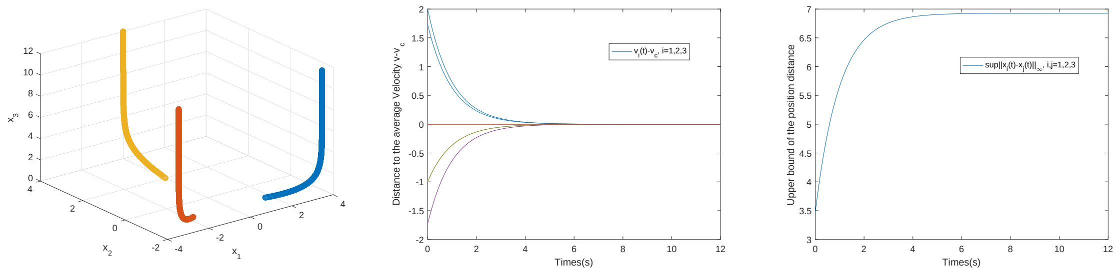

Fig. 1. Positions $x_i(t)$, velocity differences $v_i(t) - v_c$ and upper bound of $|x_i(t) - x_c(t)|$.

$d = 3$,

$$\begin{aligned} x_{10} &= (\sqrt{3}, -1, 0), \quad & v_{10} &= (\sqrt{3}, -1, 1), \\ x_{20} &= (-\sqrt{3}, -1, 0), \quad & v_{20} &= (-\sqrt{3}, -1, 1), \\ x_{30} &= (0, 2, 0), \quad & v_{30} &= (0, 2, 1). \end{aligned}$$

The above initial condition satisfies

$$\frac{1}{2\|\phi\|_{L^\infty}} \sum_{i=1}^{N} |v_{i0} - v_c|^2 - \frac{1}{2N} \sum_{i=1}^{N} \sum_{j \neq i} \int_{|x_{i0} - x_{j0}|}^{\infty} r\phi(r)dr$$
$$+ \sum_{i=1}^{N} (x_{i0} - x_c) \cdot (v_{i0} - v_c) = 0, \tag{23}$$

and model (1) exhibits the asymptotic flocking. See also Fig. 1.

IV. Conclusion

This paper has investigated the non-existence of the asymptotic flocking for the Cucker$-$Smale model with short range communication weights. First, we have pointed out that the second order space moment $\sum |x_i(t) - x_c(t)|^2$ is the key to the non-existence of the asymptotic flocking. Then, we have deduced an inequality between $\sum |x_i(t) - x_c(t)|^2$ and the velocity-position moment $\sum |v_i(t) - v_c - \frac{x_i(t) - x_c(t)}{t+\alpha}|^2$, based on which a sufficient condition of the initial conditions for the non-existence of the asymptotic flocking has been developed in Theorem 1. Furthermore, we have established a new equality about $\sum |x_i(t) - x_c(t)|^2$, from which we have deduced some more general and novel sufficient conditions for the non-existence of the asymptotic flocking in Theorem 2.

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