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The continuous classical Heisenberg ferromagnet equation with in-plane asymptotic conditions. I. Direct and inverse scattering theory

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This paper is dedicated to Prof Tommaso Ruggeri, on the occasion of his 70th
birthday.

Abstract We develop the direct and inverse scattering theory of the linear eigen-value problem associated with the classical Heisenberg continuous equation with in-plane asymptotic conditions. In particular, analyticity of the scattering eigen-functions and scattering data, and their asymptotic behaviours are derived. The inverse problem is formulated in terms of Marchenko equations, and the reconstruction formula of the potential in terms of eigenfunctions and scattering data is provided.

1 Introduction

In this paper we study the direct and inverse scattering problems associated to the classical, continuous Heisenberg ferromagnet chain equation (*i.e.* the one-dimensional, isotropic Landau-Lifshitz equation), which is the simplest and most fundamental of the continuous, integrable models of ferromagnetism [1–4].

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Let

$$\boldsymbol{m} : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2, \quad \boldsymbol{m}(x,t) = \sum_{j=1}^{3} m_j(x,t)\,\boldsymbol{e}_j \tag{1}$$

be the magnetization vector at position x and time t, where the vectors $\boldsymbol{e}_j$, $j = 1, 2, 3$, are the standard Cartesian basis vectors for $\mathbb{R}^3$, $\mathbb{S}^2$ is the sphere in $\mathbb{R}^3$ and then $\|\boldsymbol{m}(x,t)\| = 1$. The position x is taken on the real line orientated as $\boldsymbol{e}_1$. Then, the Heisenberg ferromagnet equation reads (in non-dimensional form):

$$\boldsymbol{m}_t = \boldsymbol{m} \wedge \boldsymbol{m}_{xx}, \tag{2a}$$

to which we impose the in-plane asymptotic condition

$$\boldsymbol{m}(x) \to \cos(\gamma)\boldsymbol{e}_1 - \sin(\gamma)\boldsymbol{e}_2 \text{ as } x \to \pm\infty, \tag{2b}$$

where $\gamma \in [0, 2\pi)$ is a constant angle. Equation (2a) is the well-known continuous limit of the (quantum) ferromagnetic Heisenberg chain in a constant field when the wavelength of the excited modes is larger than the lattice distance (see, for instance, [5] for a detailed discussion, or [4] for a quick derivation; the effects of the discreteness of the lattice on the classical continuum limit of the Heisenberg chain are discussed in [6]). We observe that, in the right-hand side of (2a), one can add a term representing an external magnetic field perpendicular to the $\boldsymbol{e}_1\boldsymbol{e}_2$ plane of the form $h\,\boldsymbol{m}\wedge\boldsymbol{e}_3$, $h \in \mathbb{R}$, which can be scaled out by means of a convenient change of variables (e.g., see [4, 7]).

The boundary condition (2b) has been chosen in analogy to the boundary condition for the Landau-Lifshitz equation with easy-plane anisotropy, where the ferromagnetic chain is parallel to $\boldsymbol{e}_1$ and the direction of the spontaneous magnetization – which is absent in the isotropic case of the Heisenberg ferromagnet (2a) – lies in the $\boldsymbol{e}_1\boldsymbol{e}_2$ plane (see [8]). In particular, for the Landau-Lifshitz equation for a ferromagnet with easy-plane anisotropy, the angle γ would individuate the in-plane direction of spontaneous magnetization.

It is well known that (2) is integrable (see, for instance, [4] for a brief time-line of the early original results). In [2], Takhtajan showed that (2a) admits a Lax pair representation. Let us briefly recall here that, if V is a 2×2 invertible matrix depending on position $x \in \mathbb{R}$, time $t \in \mathbb{R}$, and a spectral parameter λ, then the Lax pair $(\boldsymbol{A}, \boldsymbol{B})$, associated to (2) is given by:

$$\begin{cases} V_x = \boldsymbol{A}\,V = [i\lambda(\boldsymbol{m} \cdot \boldsymbol{\sigma})]\,V \\[2mm] V_t = \boldsymbol{B}\,V = [-2i\lambda^2(\boldsymbol{m} \cdot \boldsymbol{\sigma}) - i\lambda(\boldsymbol{m} \wedge \boldsymbol{m}_x \cdot \boldsymbol{\sigma})]\,V\,, \end{cases} \tag{3}$$

where $\boldsymbol{\sigma}$ is the column vector with entries the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Of course, the knowledge of the Lax pair for (2a) assures that the Inverse Scattering Transform (IST) technique (see [9–11]) can be applied to solve the initial-value problem [2, 3],

$$\begin{cases} \boldsymbol{m}_t = \boldsymbol{m} \wedge \boldsymbol{m}_{xx} \\[2mm] \boldsymbol{m}(x,0) \quad \text{known}\,. \end{cases} \tag{4}$$

One of the motivations for this work is in the observation that, until now, to the best of our knowledge, the majority of papers on the inverse scattering theory for (2a) have assumed boundary conditions perpendicular to the ferromagnetic chain, *i.e.* parallel to $\boldsymbol{e}_3$,

$$\boldsymbol{m}(x) \to \boldsymbol{e}_3 \text{ as } x \to \pm\infty, \tag{5}$$

and thus mimicking the boundary conditions chosen for the Landau-Lifshitz equation with perpendicular (easy-axis) anisotropy (see [8, 12]).

We will show in Section 2 how it is possible to recast the scattering problems for (2a) with (asymptotic) boundary conditions (2b) into the scattering problems for (2a) with (asymptotic) boundary conditions (5). In particular, we have to distinguish between the auxiliary Jost functions and the Jost functions. Development of the direct and inverse scattering theory for the linear eigenvalue problem corresponding to the first of (3) with (2b) is relevant as the arguments used to afford this topic can be generalized to the Landau-Lifshitz equation with easy-plane anisotropy. Consequently, this work will pave the way to the investigation of the reflectionless solutions also for this latter equation by means of the IST machinery (which is based on the direct and inverse scattering theory) and the "triplet method" (see, for example, [13–18] for a detailed description of the triplet method). We postpone the actual development of the IST and the generation of closed-form soliton solutions for (2a) with (2b) to the second part of this work [19].

In order to develop in a rigorous way the direct scattering problem for the first of (3) we assume that the potentials satisfy the following technical conditions, which will be assumed to be valid throughout the work:

Assumption 1 *As a function of the position, the matrix $\boldsymbol{m}(x) \cdot \boldsymbol{\sigma}$ has an almost everywhere existing derivative with respect to x with entries in $L^1(\mathbb{R})$. Thus $\boldsymbol{m}(x) \cdot \boldsymbol{\sigma}$ is bounded and continuous in $x \in \mathbb{R}$.*

Assumption 2 *The inequality $1 + \cos(\gamma)\, m_1(x) - \sin(\gamma)\, m_2(x) > 0$ holds for all $x \in \mathbb{R}$.*

These conditions are less restrictive than the usual (see [11]) Schwartz class hypotheses. Moreover, it is worth observing that, under the first Assumption 1, $\boldsymbol{m}(x)$ is absolutely continuous for $x \in \mathbb{R}$; thus its *point-wise* values make sense and it makes mathematical sense to assume that, in addition, $1 + \cos(\gamma)\, m_1(x) - \sin(\gamma)\, m_2(x) > 0$ for each $x \in \mathbb{R}$.

As a final remark, we remind that $1 - \cos(\gamma)\, m_1(x) + \sin(\gamma)\, m_2(x)$ is a conserved density for (2a) under asymptotic conditions (2b), its integral over the real line being the total spin density of the ferromagnetic chain [8]. So it is worthwhile to observe that our initial condition (2b) is strictly related to the conserved density.

2 Direct Scattering Theory

In this section we focus on the direct scattering theory associated to the first of equation (3). In particular, we study the analyticity properties and the asymptotic behaviour at large λ for the Jost solutions and the scattering data.

2.1 Auxiliary Jost Matrices.

Let us define the *auxiliary Jost matrices* $F_l(x, \lambda)$ and $F_r(x, \lambda)$ as those solutions of the linear eigenvalue problems (3) $[F_l]_x = AF_l$ and $[F_r]_x = AF_r$ satisfying

$$F_l(x, \lambda) = e^{i\lambda x[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2]}[I_2 + o(1)], \qquad x \to +\infty, \tag{6a}$$

$$F_r(x, \lambda) = e^{i\lambda x[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2]}[I_2 + o(1)], \qquad x \to -\infty, \tag{6b}$$

where $\gamma \in [0, 2\pi)$ is the constant angle in (2b), and

$$e^{i\lambda x[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2]} = \begin{pmatrix} \cos(\lambda x) & ie^{i\gamma}\sin(\lambda x) \\ ie^{-i\gamma}\sin(\lambda x) & \cos(\lambda x) \end{pmatrix}.$$

Then, it is easily verified that, for each $(x, \lambda) \in \mathbb{R}^2$, $F_l(x, \lambda)$ and $F_r(x, \lambda)$ belong to the unitary group $SU(2)$.

We can convert the differential systems $[F_l]_x = [i\lambda(\boldsymbol{m} \cdot \boldsymbol{\sigma})]F_l$ and $[F_r]_x = [i\lambda(\boldsymbol{m} \cdot \boldsymbol{\sigma})]F_r$ with the associated asymptotic conditions (2b) into the corresponding Volterra integral equations

$$F_l(x, \lambda) = e^{i\lambda x \hat{\sigma}} - i\lambda \int_x^\infty \mathrm{d}\xi\, e^{-i\lambda(\xi - x)\hat{\sigma}} \left[\boldsymbol{m}(\xi) \cdot \boldsymbol{\sigma} - \sigma_1\right] F_l(\xi, \lambda), \tag{7a}$$

$$F_r(x, \lambda) = e^{i\lambda x \hat{\sigma}} + i\lambda \int_{-\infty}^x \mathrm{d}\xi\, e^{i\lambda(x - \xi)\hat{\sigma}} \left[\boldsymbol{m}(\xi) \cdot \boldsymbol{\sigma} - \hat{\sigma}\right] F_r(\xi, \lambda). \tag{7b}$$

where

$$\hat{\sigma} = \cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2. \tag{7c}$$

2.2 Jost Functions.

Setting

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\gamma} \\ e^{-i\gamma} & 1 \end{pmatrix}, \tag{8}$$

we arrive at the unitary equivalence

$$\left(\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right) U = U\sigma_3. \tag{9}$$

Hence, the columns of U form an orthonormal basis of eigenvectors of $\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2$ corresponding to the eigenvalues 1 and -1, respectively. Using (8), let $\Psi(x, \lambda)$ and $\Phi(x, \lambda)$ be the following matrix functions:

$$\Psi(x, \lambda) = U^{-1}F_l(x, \lambda)U = \left(\psi(x, \lambda)\ \overline{\psi}(x, \lambda)\right), \tag{10a}$$

$$\Phi(x, \lambda) = U^{-1}F_r(x, \lambda)U = \left(\overline{\phi}(x, \lambda)\ \phi(x, \lambda)\right), \tag{10b}$$

where $\psi(x, \lambda)$, $\overline{\psi}(x, \lambda)$, $\overline{\phi}(x, \lambda)$, and $\phi(x, \lambda)$ are called *Jost functions*. Hereafter we use the following notations:

$$\Psi(x, \lambda) = \begin{pmatrix} \psi^{up}(x, \lambda) & \overline{\psi}^{up}(x, \lambda) \\ \psi^{dn}(x, \lambda) & \overline{\psi}^{dn}(x, \lambda) \end{pmatrix}, \quad \Phi(x, \lambda) = \begin{pmatrix} \overline{\phi}^{up}(x, \lambda) & \phi^{up}(x, \lambda) \\ \overline{\phi}^{dn}(x, \lambda) & \phi^{dn}(x, \lambda) \end{pmatrix}. \tag{11}$$

The differential equations $\Psi_x = U^{-1} \boldsymbol{A}\, U\, \Psi$ and $\Phi_x = U^{-1} \boldsymbol{A}\, U\, \Phi$ (cf. with (3)) can be written as

$$\Psi_x = i\lambda U^{-1} (\boldsymbol{m} \cdot \boldsymbol{\sigma})\, U\, \Psi, \qquad (12\mathrm{a})$$

$$\Phi_x = i\lambda U^{-1} (\boldsymbol{m} \cdot \boldsymbol{\sigma})\, U\, \Phi, \qquad (12\mathrm{b})$$

so that, for a vector function $\boldsymbol{m}(x) \in \mathbb{R}^3$ satisfying $\boldsymbol{m}(x) \to \cos(\gamma)\boldsymbol{e}_1 - \sin(\gamma)\boldsymbol{e}_2$ as $x \to \pm\infty$, the quantity $U^{-1}\boldsymbol{m}(x) \cdot \boldsymbol{\sigma} U - \cos(\gamma)\sigma_1 + \sin(\gamma)\sigma_2$ has its entries in $L^1(-\infty, x_0)$ for each $x_0 \in \mathbb{R}$. It can be easily verified that $\Psi(x, \lambda)$ and $\Phi(x, \lambda)$ belong to the group $SU(2)$. Indeed, any square matrix $I(x)$ which is a solution to the differential system $I_x = W(x)\, I$, where $W(x)$ is skew-Hermitian and traceless, satisfies $I^\dagger I$ and $\det(I)$ is independent of $x \in \mathbb{R}$. Here and thereafter the dagger denotes the complex conjugate transpose. As a result, we have

$$\psi^{up}(x, \lambda)^* = \overline{\psi}^{dn}(x, \lambda), \qquad \overline{\psi}^{up}(x, \lambda) = -\psi^{dn}(x, \lambda)(x, \lambda), \qquad (13\mathrm{a})$$

$$\phi^{up}(x, \lambda)^* = \overline{\phi}^{dn}(x, \lambda), \qquad \overline{\phi}^{up}(x, \lambda) = -\phi^{dn}(x, \lambda)(x, \lambda). \qquad (13\mathrm{b})$$

2.3 Analiticity of the Jost Functions.

We can straightforwardly write the Volterra equations for the Jost functions introduced in (10) as follows:

$$\Psi(x, \lambda) = e^{i\lambda x \sigma_3} - i\lambda \int_x^\infty \mathrm{d}\hat{z}\, e^{-i\lambda(\hat{z}-x)\sigma_3} \left[U^{-1}\boldsymbol{m}(\hat{z}) \cdot \boldsymbol{\sigma} U - \sigma_3 \right] \Psi(\hat{z}, \lambda), \qquad (14\mathrm{a})$$

$$\Phi(z, \lambda) = e^{i\lambda x \sigma_3} + i\lambda \int_{-\infty}^z \mathrm{d}\hat{z}\, e^{i\lambda(z-\hat{z})\sigma_3} \left[U^{-1}\boldsymbol{m}(\hat{z}) \cdot \boldsymbol{\sigma} U - \sigma_3 \right] \Phi(\hat{z}, \lambda). \qquad (14\mathrm{b})$$

Equations (14) are the same Volterra equations which appear in the study of the Heisenberg equation with "easy-axis" conditions (5). For such equations the following result can be proved:

Proposition 1 *Suppose that $U^{-1}\boldsymbol{m}(x) \cdot \boldsymbol{\sigma} U - \sigma_3$ have their entries in $L^1(\mathbb{R})$. Then, the so-called Faddeev functions*

$$e^{-i\lambda x}\, \psi^{up}(x, \lambda),\ e^{-i\lambda x}\, \psi^{dn}(x, \lambda),\ e^{i\lambda x}\, \phi^{up}(x, \lambda),\ \text{and}\ e^{i\lambda x}\, \phi^{dn}(x, \lambda)$$

are analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \overline{\mathbb{C}}^+$, while the Faddeev functions

$$e^{i\lambda x}\, \overline{\psi}^{up}(x, \lambda),\ e^{i\lambda x}\, \overline{\psi}^{dn}(x, \lambda),\ e^{-i\lambda x}\, \overline{\phi}^{up}(x, \lambda),\ \text{and}\ e^{-i\lambda x}\, \overline{\phi}^{dn}(x, \lambda)$$

are analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \overline{\mathbb{C}}^-$. Here and thereafter, $\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and lower half-planes, respectively, whereas $\overline{\mathbb{C}}^+ = \mathbb{C}^+ \cup \mathbb{R}$ and $\overline{\mathbb{C}}^- = \mathbb{C}^- \cup \mathbb{R}$ denote the closure of $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

The proof of this proposition is identical to the proof of the analogous proposition given in [12, 20]. We observe that, as a consequence of Gronwall's inequality (see Appendix of [20]) we get for $(x, \lambda) \in \mathbb{R}^2$

$$\|\Psi(x, \lambda)\| \le \exp\left(|\lambda| \int_x^\infty \mathrm{d}\xi\, \|U^{-1}\boldsymbol{m}(\xi) \cdot \boldsymbol{\sigma} U - \sigma_3\| \right), \qquad (15\mathrm{a})$$

$$\|\Phi(x, \lambda)\| \le \exp\left(|\lambda| \int_{-\infty}^x \mathrm{d}\xi\, \|U^{-1}\boldsymbol{m}(\xi) \cdot \boldsymbol{\sigma} U - \sigma_3\| \right), \qquad (15\mathrm{b})$$

where we have to assume that $U^{-1}\boldsymbol{m}(x)\cdot\boldsymbol{\sigma}U-\sigma_3$ has its entries in $L^1(\mathbb{R})$. By using the Volterra integral equations (14), nothing can be said about the asymptotic behavior of the Jost solutions as $\lambda \to \pm\infty$. In order to get such information, let us derive a different set of Volterra integral equations. To do so, we need Assumptions 1 and 2, namely that $\boldsymbol{m}(x) \cdot \boldsymbol{\sigma}$ has an almost everywhere existing derivative $\boldsymbol{m}'(x) \cdot \boldsymbol{\sigma}$ with respect to x which has its entries in $L^1(\mathbb{R})$, and that $1 + \cos(\gamma)\, m_1(x) - \sin(\gamma)\, m_2(x) > 0$ for all $x \in \mathbb{R}$. Here and thereafter the prime indicates the total derivative with respect to the spatial variable x.

We observe that recently the study of the long time behavior of the Volterra equations in a different, but significative, context has been performed in [21].

Under Assumption 1, we can apply partial integration to (14a) obtaining

$$\Psi(x,\lambda) = e^{i\lambda x\sigma_3} + \left[e^{-i\lambda(\xi-x)\sigma_3}\sigma_3(\boldsymbol{m}^0(\xi)\cdot\boldsymbol{\sigma})\Psi(\xi,\lambda)\right]_{\xi=x}^{\infty}$$
$$- \int_x^\infty d\xi\, e^{-i\lambda(\xi-x)\sigma_3}\sigma_3\left[U^{-1}(\boldsymbol{m}'(\xi)\cdot\boldsymbol{\sigma})U\Psi(\xi,\lambda) + (\boldsymbol{m}^0(\xi)\cdot\boldsymbol{\sigma})\frac{\partial\Psi}{\partial\xi}(\xi,\lambda)\right].$$

where we have introduced the notation $\boldsymbol{m}^0(x)\cdot\boldsymbol{\sigma} = U^{-1}\boldsymbol{m}(x)\cdot\boldsymbol{\sigma}U - \sigma_3$ and used (12a). Taking into account (9) along with the following relations

$$\frac{\partial\Psi}{\partial\xi}(\xi,\lambda) = i\lambda U_+^{-1}\boldsymbol{m}(\xi)\cdot\boldsymbol{\sigma}\, U_+\Psi\,,\quad (\boldsymbol{m}(\xi)\cdot\boldsymbol{\sigma})^2 = I_2\,,$$
$$I_2 - \sigma_3 U_+^{-1}\boldsymbol{m}(\xi)\cdot\boldsymbol{\sigma}\, U_+ = -\sigma_3(\boldsymbol{m}^0(\xi)\cdot\boldsymbol{\sigma})\,,$$

we arrive at the equation

$$\sigma_3(U^{-1}\boldsymbol{m}(x)\cdot\boldsymbol{\sigma}U)\Psi(x,\lambda) = e^{i\lambda x\sigma_3} - \int_x^\infty d\xi\, e^{-i\lambda(\xi-x)\sigma_3}\sigma_3(U^{-1}\boldsymbol{m}'(\xi)\cdot\boldsymbol{\sigma})U\Psi(\xi,\lambda)$$
$$+ i\lambda \int_x^\infty d\xi\, e^{-i\lambda(\xi-x)\sigma_3}(\boldsymbol{m}^0(\xi)\cdot\boldsymbol{\sigma})\Psi(\xi,\lambda).$$

Summing the latter equation to (14a) and taking half of the sum we get

$$D(x)\Psi(x,\lambda) = e^{i\lambda x\sigma_3} - \int_x^\infty d\xi\, e^{-i\lambda(\xi-x)\sigma_3}D'(\xi)\Psi(\xi,\lambda), \tag{16}$$

where

$$D(x) = \frac{1}{2}\left[I_2 + \sigma_3 U_-^{-1}(\boldsymbol{m}(x)\cdot\boldsymbol{\sigma})U_-\right]$$
$$= \frac{1}{2}\begin{pmatrix} 1 + \frac{e^{-i\gamma}}{2}m_-(x) + \frac{e^{i\gamma}}{2}m_+(x) & -e^{i\gamma}m_3 - \frac{e^{2ic}}{2}m_+(x) + \frac{1}{2}m_-(x) \\ e^{-i\gamma}m_3 + \frac{e^{-2ic}}{2}m_-(x) - \frac{1}{2}m_+(x) & 1 + \frac{e^{-i\gamma}}{2}m_-(x) + \frac{e^{i\gamma}}{2}m_+(x) \end{pmatrix}. \tag{17}$$

Here $m_-(x) = m_1(x) - im_2(x)$ and $m_+(x) = m_1(x) + im_2(x)$. We easily compute $\det D(x) = \frac{1}{2}(1 + \cos(\gamma)m_1 - \sin(\gamma)m_2)$, thus, under Assumption 2, the matrix $D(x)$ is invertible and the norm of $D(x)^{-1}$ is

$$\|D(x)^{-1}\| = \frac{2\sqrt{2}}{\sqrt{1 + \cos(\gamma)m_1 - \sin(\gamma)m_2}}\,.$$

Consequently, $D(x)$ and $D(x)^{-1}$ are bounded in $x \in \mathbb{R}$. We may therefore apply Gronwall's inequality to (16) and find that

$$\|\Psi(x,\lambda)\| \leq \|D(x)^{-1}\| \exp\left[\frac{\|D(x)^{-1}\|}{2} \int_x^\infty \mathrm{d}\xi \, \|U^{-1}(\boldsymbol{m}'(\xi) \cdot \boldsymbol{\sigma})U\|\right].$$

In the same way and under Assumptions 1 and 2, adapting the procedure presented above to the Jost matrix $\Phi(x,\lambda)$, we get

$$D(x)\Phi(x,\lambda) = e^{i\lambda x \sigma_3} + \int_{-\infty}^x \mathrm{d}\xi \, e^{i\lambda(x-\xi)\sigma_3} D'(\xi)\Phi(\xi,\lambda), \tag{18}$$

where $D(x)$ is defined by (17). For the same reasons discussed above we can apply Gronwall's inequality to (18), obtaining

$$\|\Phi(x,\lambda)\| \leq \|D(x)^{-1}\| \exp\left[\frac{\|D(x)^{-1}\|}{2} \int_{-\infty}^x \mathrm{d}\xi \, \|U^{-1}(\boldsymbol{m}'(\xi) \cdot \boldsymbol{\sigma})U\|\right].$$

2.4 Triangular Representations for the Jost Functions.

Equations (16) and (18) allow us to prove that the analyticity and the continuity properties of the Jost solutions extend to the closed upper and lower half-planes. In other words, the Jost functions have a finite limit as $\lambda \to \infty$ from within the closure of its half-plane of analyticity. In order to prove these results we need to find a "suitable" triangular representation for the Jost solutions. We have the following:

Proposition 2 *There exists an auxiliary matrix function $\boldsymbol{K}^{up}(x,y)$ such that*

$$\Psi(x,\lambda) = \boldsymbol{H}^{up}(x)e^{i\lambda x \sigma_3} + \int_x^\infty \mathrm{d}\xi \, \boldsymbol{K}^{up}(x,\xi)e^{i\lambda\xi\sigma_3}, \tag{19}$$

where $\boldsymbol{H}^{up}(x)$ is a matrix function satisfying $\boldsymbol{H}^{up}(x) = \sigma_2 \, \boldsymbol{H}^{up}(x)^* \, \sigma_2$ and the integral $\int_x^\infty \mathrm{d}\xi \, \|\boldsymbol{K}^{up}(x,\xi)\|$ converges uniformly in $x \in \mathbb{R}$.

Proof The proof of this proposition is standard and can be obtained, via Gronwall's identity, by proceeding as explained in [12, 22, 23].

2.5 Aymptotic behavior and domains of analticity of the Jost solutions.

From (19), it is immediate to see that $e^{-i\lambda x}\psi(x,\lambda)$ is continuous in $\lambda \in \overline{\mathbb{C}^+}$, is analytic in $\lambda \to \mathbb{C}^+$, and tends to the first column of $\boldsymbol{H}^{up}(x)$ as $\lambda \to \infty$ from within $\overline{\mathbb{C}^+}$. Analogously, $e^{i\lambda x}\overline{\psi}(x,\lambda)$ is continuous in $\lambda \in \overline{\mathbb{C}^-}$, is analytic in $\lambda \in \mathbb{C}^-$, and tends to the second column of $\boldsymbol{H}^{up}(x)$ as $\lambda \to \infty$ from within $\overline{\mathbb{C}^-}$. Consequently, since $\boldsymbol{H}(x)$ is the limit of $\Psi(x,\lambda)$ as $\lambda \to \pm\infty$, we have $\boldsymbol{H}^{up}(x) \in SU(2)$. We have established the invertibility of $\boldsymbol{H}^{up}(x)$, provided $\det D(x) = \frac{1}{2}(1 + \cos(\gamma)m_1 - \sin(\gamma)m_2) \neq 0$ which is guaranteed by Assumption 2. Let us also remark that from the symmetry relations

$$\Psi(x,\lambda)^* = \sigma_2 \, \Psi(x,\lambda) \, \sigma_2 \,,$$
$$\boldsymbol{H}^{up}(x) = \sigma_2 \, \boldsymbol{H}^{up}(x)^* \, \sigma_2 \,,$$

we derive the following structure of the matrix function $\boldsymbol{K}^{up}(x, y)$

$$\boldsymbol{K}^{up}(x, y) = \begin{pmatrix} K_1^{up}(x, y) & -K_2^{up}(x, y)^* \\ K_2^{up}(x, y) & K_1^{up}(x, y)^* \end{pmatrix}, \tag{20}$$

where $K_1^{up}(x, y)$ and $K_2^{up}(x, y)$ are scalar functions.

For later convenience we now define the matrix of functions $\boldsymbol{L}(x, \xi)$ in the following way

$$\boldsymbol{L}(x, \hat{z}) = \boldsymbol{H}^{up}(x)^{-1} \boldsymbol{K}(x, \hat{z}). \tag{21}$$

Analogously to Proposition 2, we have the following

Proposition 3 *There exists an auxiliary matrix function $\boldsymbol{K}^{dn}(x, y)$ such that*

$$\Phi(x, \lambda) = \boldsymbol{H}^{dn}(x)e^{i\lambda x \sigma_3} + \int_{-\infty}^{x} d\xi \, \boldsymbol{K}^{dn}(x, \xi)e^{i\lambda \xi \sigma_3}, \tag{22}$$

where $\boldsymbol{H}^{dn}(x)$ is a matrix function satisfying $\boldsymbol{H}^{dn}(x) = \sigma_2 \boldsymbol{H}^{dn}(x)^* \sigma_2$ and the integral $\int_{-\infty}^{x} d\xi \, \|\boldsymbol{K}^{dn}(x, \xi)\|$ converges uniformly in $x \in \mathbb{R}$.

Proof The proof is analogous to the proof of Proposition 2 and can be obtained, via Gronwall's identity, by proceeding as explained in [12, 22, 23].

As before, from (22) and taking into account the finiteness of the integral $\int_{-\infty}^{x} d\hat{z} \, \|\boldsymbol{K}^{dn}(x, \hat{z})\|$, it is immediate to see that $e^{i\lambda x}\phi(x, \lambda)$ is continuous in $\lambda \in \overline{\mathbb{C}^+}$, is analytic in $\lambda \to \mathbb{C}^+$, and tends to the second column of $\boldsymbol{H}^{dn}(x)$ as $\lambda \to \infty$ from within $\overline{\mathbb{C}^+}$. Analogously, $e^{-i\lambda x}\overline{\phi}(x, \lambda)$ is continuous in $\lambda \in \overline{\mathbb{C}^-}$, is analytic in $\lambda \in \mathbb{C}^-$, and tends to the first column of $\boldsymbol{H}^{dn}(x)$ as $\lambda \to \infty$ from within $\overline{\mathbb{C}^-}$. Consequently, since $\boldsymbol{H}^{dn}(x)$ is the limit of $\tilde{\Phi}(x, \lambda)$ as $\lambda \to \pm\infty$, we have $\boldsymbol{H}^{dn}(x) \in SU(2)$. We have thus established the invertibility of $\boldsymbol{H}^{dn}(z)$, provided $\det D(x) = \frac{1}{2}[1 + \cos(\gamma) \, m_1(x) + \sin(\gamma) \, m_2(x)] \neq 0$ which is guaranteed by Assumption 2. Finally we remark that, because of the symmetry relations

$$\Phi(x, \lambda)^* = \sigma_2 \, \Phi(x, \lambda) \, \sigma_2,$$
$$\boldsymbol{H}^{dn}(x) = \sigma_2 \, \boldsymbol{H}^{dn}(x)^* \, \sigma_2$$

the matrix $\boldsymbol{K}^{dn}(x, y)$ has the following structure

$$\boldsymbol{K}^{dn}(x, y) = \begin{pmatrix} K_1^{dn}(x, y)^* & K_2^{dn}(x, y) \\ -K_2^{dn}(x, y)^* & K_1^{dn}(x, y) \end{pmatrix}, \tag{23}$$

where $K_1^{dn}(x, y)$ and $K_2^{dn}(x, y)$ are scalar functions.

Analogously to what has been done for the matrix $\boldsymbol{K}^{up}(x, \xi)$ above, we introduce the matrix of functions $\bar{\boldsymbol{L}}(x, \xi)$ as

$$\bar{\boldsymbol{L}}(x, \xi) = \boldsymbol{H}^{dn}(x)^{-1} \boldsymbol{K}^{dn}(x, \xi). \tag{24}$$

2.6 Transition Matrix.

After having introduced the Jost solutions and understood their analytic properties, we are ready to study the direct scattering problem associated to the first equation in system (3). We remind that the direct scattering problem consists of constructing the scattering matrix $\boldsymbol{S}(\lambda)$, which contains part of the scattering data. In turn, to this purpose, we need to introduce the so-called transition matrix.

Since the two auxiliary Jost matrices are both solutions to the same first order linear homogeneous differential system, there exists a so-called *transition matrix* $\boldsymbol{T}_0(\lambda)$, independent of x and belonging to $SU(2)$, such that

$$F_l(x,\lambda) = F_r(x,\lambda)\boldsymbol{T}_0(\lambda), \qquad \lambda \in \mathbb{R}. \tag{25}$$

It thus appears that, for $x \in \mathbb{R}$, the columns of $U\Psi(x,\lambda)$ are linear combinations (with coefficients not depending on $z \in \mathbb{R}$) of the columns of $U\Phi(z,\lambda)$, and vice versa. Therefore, we can write (25) in the form

$$U\Psi(x,\lambda) = U\Phi(x,\lambda)\boldsymbol{T}(\lambda), \qquad \lambda \in \mathbb{R}. \tag{26}$$

where $\boldsymbol{T}(\lambda) = U^{-1}\boldsymbol{T}_0(\lambda)U \in SU(2)$ for all $\lambda \in \mathbb{R}$. Therefore,

$$\boldsymbol{T}(\lambda) = \begin{pmatrix} a(\lambda) & -b(\lambda) \\ b(\lambda)^* & a(\lambda)^* \end{pmatrix}, \qquad \boldsymbol{T}(\lambda)^{-1} = \begin{pmatrix} a(\lambda)^* & b(\lambda) \\ -b(\lambda)^* & a(\lambda) \end{pmatrix},$$

where $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$ for all $\lambda \in \mathbb{R}$. Also,

$$U\Phi(x,\lambda) = U\Psi(x,\lambda)\boldsymbol{T}(\lambda)^{-1}, \qquad \lambda \in \mathbb{R}. \tag{27}$$

Thereafter, we **assume** that $a(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$. In other words, we assume the nonexistence of spectral singularities.

2.7 Riemann-Hilbert Problem and Scattering Matrix.

Assuming that $a(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$, we can write (26) and (27) as the following Riemann-Hilbert problems:

$$\left(U\overline{\phi}(x,\lambda)U\overline{\psi}(x,\lambda)\right) = \left(U\psi(x,\lambda)U\phi(x,\lambda)\right)\begin{pmatrix} T(\lambda) & -R(\lambda) \\ -L(\lambda) & T(\lambda) \end{pmatrix}, \tag{28a}$$

$$\left(U\psi(x,\lambda)U\phi(x,\lambda)\right) = \left(U\overline{\phi}(x,\lambda)U\overline{\psi}(x,\lambda)\right)\begin{pmatrix} T(\lambda)^* & L(\lambda)^* \\ R(\lambda)^* & T(\lambda)^* \end{pmatrix}, \tag{28b}$$

where $T(\lambda)$ is the *transmission coefficient*, $R(\lambda)$ is the *reflection coefficient from the right*, and $L(\lambda)$ is the *reflection coefficient from the left* defined as

$$T(\lambda) = \frac{1}{a(\lambda)}, \qquad R(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \qquad L(\lambda) = \frac{b(\lambda)^*}{a(\lambda)}.$$

Consequently, the *scattering matrix* $\boldsymbol{S}(\lambda) = \begin{pmatrix} T(\lambda) & R(\lambda) \\ L(\lambda) & T(\lambda) \end{pmatrix}$ satisfies the symmetry relation

$$\boldsymbol{S}(\lambda)^{-1} = \sigma_3\boldsymbol{S}(\lambda)^{\dagger}\sigma_3, \qquad \lambda \in \mathbb{R}. \tag{29}$$

Clearly, we also have

$$\det \boldsymbol{S}(\lambda) = \frac{a(\lambda)^*}{a(\lambda)} = \frac{T(\lambda)}{T(\lambda)^*}.$$

Equations (19), (22), and (26) imply that

$$\lim_{\lambda \to \pm\infty} e^{i\lambda x\sigma_3} \boldsymbol{T}(\lambda) e^{-i\lambda x\sigma_3} = \boldsymbol{H}^{dn}(x)^{-1}\boldsymbol{H}(x) \in SU(2) \tag{30}$$

for every $x \in \mathbb{R}$ for which $\det D(x) \neq 0$. This means that $\lim_{\lambda \to \pm\infty} e^{2i\lambda x} b(\lambda)$ does not depend on $x \in \mathbb{R}$ and hence it must vanish. Thus, the expression (30) is a diagonal matrix not depending on $x \in \mathbb{R}$, and there exists $\alpha \in \mathbb{R}$ such that $a(\lambda) \to e^{i\alpha}$ as $\lambda \to \pm\infty$. This limit is also valid as $\lambda \to \infty$ from within $\overline{\mathbb{C}^+}$. Consequently, $\boldsymbol{S}(\lambda) \to e^{-i\alpha} I_2$ as $\lambda \to \pm\infty$.

The functions $a(\lambda) - e^{i\alpha}$ and $b(\lambda)$ are Fourier transforms of functions belonging to $L^1(\mathbb{R})$, while $a(\lambda)$ is assumed not to have any real zeros. This entails that there exist $\rho, \ell \in L^1(\mathbb{R})$ such that

$$R(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}y\, e^{-i\lambda y} \rho(y)\,, \qquad L(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}y\, e^{i\lambda y} \ell(y)\,. \tag{31}$$

2.8 Scattering Data.

The scattering data associated with the first equation in system (3) are:

1. one of the reflection coefficients;
2. the poles of the transmission coefficient $T(\lambda)$ (respectively, $T(\lambda^*)^*$); we call such poles *the discrete eigenvalues* in the upper half-plane $\mathbb{C}^+$ (respectively, in the lower half-plane $\mathbb{C}^-$) and denote them by ia_j (respectively, by $-ia_j^*$) for $j = 1, \ldots, n$, with $\mathrm{Re}(a_j) > 0$;
3. a set of constants N_j ($\overline{N}_j$) for $j = 1, \ldots, n$ associated to the discrete eigenvalues ia_j ($-ia_j^*$) $j = 1, \ldots, n$ in the upper half-plane (respectively, lower half-plane); these constants are called the *norming constants*.

It is well known that, if there are no spectral singularities, then the number of discrete eigenvalues is finite [11]. At this stage, it is crucial to observe that, in general, the poles of the transmission coefficient $T(\lambda)$ are not necessarily simple and may have multiplicity larger than one. However, for the sake of simplicity, unless explicitly indicated differently, here and thereafter we assume that each pole of the transmission coefficient has multiplicity equal to one, as this is not restrictive when proving the symmetry of the norming constants. The same relations can be established when the multiplicity is greater than one by following the procedure illustrated in [23].

The construction of the norming constants follows a standard procedure (see [9–11]). To this aim, let us assume that there are finitely many simple poles $ia_1, \ldots, ia_n$ of the transmission coefficient $T(\lambda)$ in the upper half-plane $\mathbb{C}^+$. Following [9–11], let θ_j be the residue of $T(\lambda)$ at $\lambda = ia_j$, *i.e.*

$$\theta_s = \lim_{\lambda \to ia_s} (\lambda - ia_s) T(\lambda) = \lim_{\lambda \to ia_s} \frac{\lambda - ia_s}{a(\lambda) - a(ia_s)} = \frac{1}{\dot{a}(ia_s)}\,.$$

We then introduce the *norming constants* N_s such that

$$\theta_s U\phi(z, ia_s) = iN_s U\psi(z, ia_s), \qquad s = 1, 2, \ldots, n. \tag{32a}$$

Similarly, $T(\lambda^*)^*$ has the simple poles $-ia_1^*, \ldots, -ia_N^*$ in $\mathbb{C}^-$, all of them simple. The corresponding norming constants $\overline{N}_s$ are defined by

$$\theta_s^* \, U\overline{\phi}(z, -ia_s^*) = -i\overline{N}_s U\overline{\psi}(z, -ia_s^*), \qquad s = 1, 2, \ldots, n. \tag{32b}$$

The next proposition shows how the norming constants introduced in the upper half-plane are related to those defined in the lower half-plane.

Proposition 4 *The norming constants satisfy the following relations:*

$$\overline{N}_j = -(N_j)^*.$$

Proof The proof of this proposition can be obtained by repeating (*verbatim*) the proof of the analogous Proposition for the symmetry of the norming constants in [12, 22, 23].

3 Time Evolution of the Scattering Data

We now derive the time evolution of the scattering data introduced above. We shall arrive at the same time evolution as for the NLS equation.

Let $(\boldsymbol{A}, \boldsymbol{B})$ be the Lax pair as given by (3). Suppose that $V(x, t; \lambda)$ is a nonsingular 2×2 matrix function satisfying

$$V_x = \boldsymbol{A}\,V, \qquad V_t = \boldsymbol{B}\,V,$$

where V needs not to be one of the Jost matrices. Then, there exist two invertible matrices Z_{F_l} and Z_{F_r}, depending on (t, λ) but not on x, such that $F_l = V Z_{F_l}^{-1}$ and $F_r = V Z_{F_r}^{-1}$. Then

$$[F_l]_t = V_t Z_{F_l}^{-1} - V Z_{F_l}^{-1} [Z_{F_l}]_t Z_{F_l}^{-1} = \boldsymbol{B} V Z_{F_l}^{-1} - V Z_{F_l}^{-1} [Z_{F_l}]_t Z_{F_l}^{-1}$$
$$= \boldsymbol{B} F_l - F_l [Z_{F_l}]_t Z_{F_l}^{-1},$$

implying

$$[Z_{F_l}]_t Z_{F_l}^{-1} = F_l^{-1} \boldsymbol{B} F_l - F_l^{-1} [F_l]_t. \tag{33a}$$

Analogously, for the other Jost matrix $F_r(x, \lambda)$ we get

$$[Z_{F_r}]_t Z_{F_r}^{-1} = F_r^{-1} \boldsymbol{B} F_r - F_r^{-1} [F_r]_t. \tag{33b}$$

Here the left-hand side does not depend on x, whereas the right-hand side only seemingly depends on x. We may therefore allow x to tend to $+\infty$ without losing the validity of (33a), as well as to $-\infty$ without losing the validity of (33b). Since

$$\boldsymbol{B} \simeq -2i\,\lambda^2 \left[(\cos c)\sigma_1 - (\sin c)\sigma_2\right] \quad \text{and} \quad F_l \simeq e^{i\lambda x[(\cos c)\sigma_1 - (\sin c)\sigma_2]} \quad \text{as } x \to \pm\infty,$$

from (33a) we obtain

$$[Z_{F_l}]_t Z_{F_l}^{-1} = -2i\,\lambda^2 \left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right]. \tag{34a}$$

Similarly, for the other Jost matrix $F_r(x, \lambda)$ we get

$$[Z_{F_r}]_t \, Z_{F_r}^{-1} = -2i\,\lambda^2 \left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right]. \tag{34b}$$

From (25), for the transmission coefficient we get

$$
\begin{aligned}
[\boldsymbol{T}_0]_t &= \left(F_r^{-1}\, F_l\right)_t = F_r^{-1}\,[F_l]_t - F_r^{-1}\,[F_r]_t\,F_r^{-1}\,F_l \\
&= F_r^{-1}\left(\boldsymbol{B}\,F_l - F_l[Z_{F_l}]_t\,Z_{F_l}^{-1}\right) - F_r^{-1}\left(\boldsymbol{B}\,F_r - F_r\,[Z_{F_r}]_t\,Z_{F_r}^{-1}\right)F_r^{-1}\,F_l \\
&= F_r^{-1}\,\boldsymbol{B}\,F_l - [\boldsymbol{T}_0]\,[Z_{F_l}]_t\,Z_{F_r}^{-1} - F_r^{-1}\,\boldsymbol{B}\,F_l + [Z_{F_r}]_t\,Z_{F_r}^{-1}\,[\boldsymbol{T}_0] \\
&= 2i\,\lambda^2\left(\boldsymbol{T}_0\left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right] - \left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right]\boldsymbol{T}_0\right),
\end{aligned}
$$

Since $\boldsymbol{T}(\lambda) = U^{-1}\boldsymbol{T}_0(\lambda)U$, we arrive at the following equation describing the evolution of the matrix $\boldsymbol{T}(\lambda)$:

$$[\boldsymbol{T}]_t = 2i\,\lambda^2\left(U^{-1}\left[\boldsymbol{T}_0\left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right] - \left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right]\boldsymbol{T}_0\right]U\right). \tag{35}$$

Since a straightforward computation shows that

$$U^{-1}\left[\boldsymbol{T}_0\left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right] - \left[\cos(\gamma)\sigma_1 - \sin(\gamma)\sigma_2\right]\boldsymbol{T}_0\right]U = \begin{pmatrix} 0 & 2b(\lambda, t) \\ 2b^*(\lambda, t) & 0 \end{pmatrix},$$

then from (35) we obtain

$$a(\lambda, t)_t = 0 \quad b(\lambda, t)_t = -4i\lambda^2 b(\lambda, t). \tag{36}$$

Consequently, $T(\lambda)$ does not depend on t, whereas

$$R(\lambda, t) = e^{-4i\lambda^2 t}\,R(\lambda, 0), \qquad L(\lambda, t) = e^{4i\lambda^2 t}\,L(\lambda, 0). \tag{37}$$

It remains to discover the time evolution of the norming constants. Differentiating (32a) with respect to t we obtain

$$\theta_j\,U\phi_t(x, ia_j) = iN_j\,U\psi_t(x, ia_j) + i[N_j]_t\,U\psi(x, ia_j).$$

Taking into account the following relations

$$
\begin{aligned}
\psi_t(x, \lambda) &= U^{-1}\boldsymbol{B}U\psi(x, \lambda) + 2i\lambda^2\psi(x, \lambda) \\
\phi_t(x, \lambda) &= U^{-1}\boldsymbol{B}U\phi(x, \lambda) - 2i\lambda^2\phi(x, \lambda)
\end{aligned}
$$

we get

$$
\begin{aligned}
\theta_j\left\{\boldsymbol{B}(ia_j)\,U\phi(x, ia_j) + 2ia_j^2\,U\phi(x, ia_j)\right\} &= i\,N_j\left\{\boldsymbol{B}(ia_j)\,U\psi(x, ia_j) + 2ia_j^2\,U\psi(x, ia_j)\right\} \\
&\quad + i[N_j]_t\,U\psi(x, ia_j).
\end{aligned}
$$

Using (32a) again we obtain

$$[N_j]_t = -4\,i\,a_j^2\,N_j.$$

Remembering that $\overline{N}_j = -N_j^*$ (see Proposition 4), finally we obtain the time evolution of the norming constants

$$N_j(t) = e^{-4ia_j^2 t}\,N_j(0), \qquad \overline{N}_j(t) = e^{4ia_j^{*2} t}\,\overline{N}_j(0). \tag{38}$$

4 Inverse Scattering Theory

The inverse scattering problem consists of the reconstruction of the (unique) magnetization vector $\boldsymbol{m}(x)$ once the scattering data are given. Following, for instance, [11, 14], we formulate and solve this problem by using the Marchenko method.

First of all we prove the following

Theorem 3 *The auxiliary function $\boldsymbol{K}^{up}(x,y)$ which appears in (19) has to satisfy the following integral Marchenko equations:*

$$\boldsymbol{K}^{up}(x,y) + \boldsymbol{H}^{up}(x)\,\boldsymbol{\Omega}(x+y) + \int_x^\infty \mathrm{d}\xi\,\boldsymbol{K}^{up}(x,\xi)\,\boldsymbol{\Omega}(\xi+y) = 0_{2\times2}\,, \qquad (39)$$

where

$$\boldsymbol{\Omega}(x) = \begin{pmatrix} 0 & \Omega(x) \\ -\Omega(x)^* & 0 \end{pmatrix}, \quad with \quad \Omega(x) = \rho(x) + \sum_{j=1}^n N_j\,e^{-a_j x}\,, \qquad (40)$$

and $\rho(x)$ is the Fourier transform of the reflection coefficient (see (31)).

We omit the proof of this theorem as it is analogous to the proof of Theorem 2.8 in [12] (see also [22, 23]).

We recall that $\boldsymbol{H}^{up}(x) \in SU(2)$ and that in (21) we have set

$$\boldsymbol{K}^{up}(x,y) = \boldsymbol{H}^{up}(x)\,\boldsymbol{L}(x,y)\,.$$

This allow us to convert (39) into the ("usual") Marchenko integral equation:

$$\boldsymbol{L}(x,y) + \boldsymbol{\Omega}(x+y) + \int_x^\infty \mathrm{d}\xi\,\boldsymbol{L}(x,\xi)\,\boldsymbol{\Omega}(\xi+y) = 0_{2\times2}. \qquad (41)$$

By following the same proof as in the focusing AKNS case [22–24], we find that equation (41) is uniquely solvable on the space $L^1(x,+\infty)^{2\times2}$.

Analogously, for $\boldsymbol{K}^{dn}$ and $\boldsymbol{H}^{dn}$, one can prove that

$$\boldsymbol{K}^{dn}(x,y) + \boldsymbol{H}^{dn}(x)\,\overline{\boldsymbol{\Omega}}(x+y) + \int_{-\infty}^x \mathrm{d}\xi\,\boldsymbol{K}^{dn}(x,\xi)\overline{\boldsymbol{\Omega}}(\xi+y) = 0_{2\times2}\,, \qquad (42a)$$

with

$$\overline{\boldsymbol{\Omega}}(x) = \begin{pmatrix} 0 & \overline{\Omega}(x) \\ -\overline{\Omega}(x)^* & 0 \end{pmatrix} \quad and \quad \overline{\Omega}(x) = \ell(x) + \sum_{j=1}^n \overline{N}_j\,e^{-a_j^* x}\,, \qquad (42b)$$

where $\ell(x)$ is defined as in (31). By using (24) and stripping off the common factor $\boldsymbol{H}^{dn}(x)$, we get

$$\bar{\boldsymbol{L}}(x,y) + \overline{\boldsymbol{\Omega}}(x+y) + \int_x^\infty \mathrm{d}\xi\,\bar{\boldsymbol{L}}(x,\xi)\,\overline{\boldsymbol{\Omega}}(\xi+y) = 0_{2\times2}. \qquad (43)$$

Finally, we observe that since $\overline{N}_s = -[N_s]^*$ $(s = 1,2,\ldots,N)$, we have the symmetry relations

$$\overline{\Omega}(w) = -\Omega(w)^*, \qquad \Omega(w)^\dagger = -\Omega(w)\,. \qquad (44)$$

4.1 Relationship between the magnetization vector $\boldsymbol{m}(x,\lambda)$ and the auxiliary matrix $\boldsymbol{K}^{up}(x)$.

Substituting (19) into (12a) we obtain

$$0_{2\times2} = \frac{\partial\Psi}{\partial x}(x,\lambda) - i\lambda\,U^{-1}\left(\boldsymbol{m}(x)\cdot\boldsymbol{\sigma}\right)U\Psi(x,\lambda)$$

$$= \left[G_1(x) + i\lambda\,G_2(x)\right]e^{i\lambda x\sigma_3} + \int_x^\infty \mathrm{d}\xi\,G_3(\xi)\,e^{i\lambda\xi\sigma_3}$$

with

$$G_1(x) = \left[\boldsymbol{H}^{\mathrm{up}\prime}(x) - \boldsymbol{H}^{\mathrm{up}}(x)\tilde{\boldsymbol{K}}(x,x)\right],$$

$$G_2(x) = \left[\boldsymbol{H}^{up}(x)\sigma_3 - U^{-1}(\boldsymbol{m}(x)\cdot\boldsymbol{\sigma})U\boldsymbol{H}^{\mathrm{up}}(x)\right],$$

$$G_3(x) = \boldsymbol{H}^{\mathrm{up}}(x)\frac{\partial\boldsymbol{L}}{\partial x}(x,\xi) - i\lambda U^{-1}(\boldsymbol{m}(x)\cdot\boldsymbol{\sigma})U\boldsymbol{H}^{\mathrm{up}}(x)\boldsymbol{L}(x,\xi) + \boldsymbol{H}^{\mathrm{up}\prime}(x)\boldsymbol{L}(x,\xi),$$

so that

$$\boldsymbol{m}(x)\cdot\boldsymbol{\sigma} = U\,\boldsymbol{H}^{up}(x)\,\sigma_3\,\boldsymbol{H}^{up}(x)^{-1}\,U^{-1}. \tag{45}$$

Moreover, from (19) we find

$$I_2 = \Psi(x,0) = \boldsymbol{H}^{up}(x) + \int_x^\infty \mathrm{d}\xi\,\boldsymbol{K}^{up}(x,\xi) = \boldsymbol{H}^{up}(x)\left[I_2 + \int_x^\infty \mathrm{d}\xi\,\boldsymbol{L}(x,\xi)\right],$$

which implies

$$\boldsymbol{H}^{up}(x)^{-1} = I_2 + \tilde{\boldsymbol{L}}(x), \tag{46}$$

where we defined $\tilde{\boldsymbol{L}}(x) = \int_x^\infty \mathrm{d}\xi\,\boldsymbol{L}(x,\xi)$ Since $\boldsymbol{H}^{up}(x)$ and $\boldsymbol{K}^{up}(x)$ belong to $SU2$, it is immediate to verify that

$$\left(I_2 + \tilde{\boldsymbol{L}}(x)\right)^{-1} = \left(I_2 + \tilde{\boldsymbol{L}}(x)\right)^\dagger = I_2 + \tilde{\boldsymbol{L}}(x)^\dagger.$$

Combining the latter equation with (46) and (45) we finally arrive at the relevant formula

$$\boldsymbol{m}(x)\cdot\boldsymbol{\sigma} = U\left(I_2 + \tilde{\boldsymbol{L}}(x)^\dagger\right)\sigma_3\left(I_2 + \tilde{\boldsymbol{L}}(x)\right)U^{-1}, \tag{47}$$

which allows one to find the magnetization vector $\boldsymbol{m}(x)$, solution to (2a) with (2b), once the matrix function $\tilde{\boldsymbol{L}}(x)$ is known. Thus, if one is able to solve the Marchenko equation (41) then the magnetization vector can be explicitly computed by using (47).

We remark that a formula similar to (45) can be obtained from the Jost matrix $\Phi(x,\lambda)$. Indeed, by proceeding analogously to what we have done above for $\Psi(x,\lambda)$, we get

$$\boldsymbol{m}(x)\cdot\boldsymbol{\sigma} = U\boldsymbol{H}^{dn}(x)\sigma_3\boldsymbol{H}^{dn}(x)^{-1}U^{-1}. \tag{48}$$

where $\boldsymbol{H}^{dn}(x)^{-1} = I_2 + \int_x^\infty \mathrm{d}\xi\,\bar{\boldsymbol{L}}(x,\xi)$ and the function $\bar{\boldsymbol{L}}(x,y)$ is the unknown of the Marchenko equation (43). Therefore, if one is able to solve the Marchenko equation (43), then the magnetization vector can be explicitly computed by using (48).

Acknowledgments

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