Phase Dynamics of Periodic Wavetrains Leading to the 5th Order KP Equation

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Abstract

Using the previous approach outlined in [12, 10], a novel method is presented to derive the fifth order Kadomtsev-Petviashvili (KP) equation from periodic wavetrains. As a result, the coefficients and criterion for the fifth order KP to emerge take a universal form that can be determined *a-priori*, relating to the system's conservation laws and the termination of a Jordan chain. Moreover, the analysis reveals that generically a mixed dispersive term q_{XXXY} appears within the final phase equation. The theory presented here is complimented by an example from the context of flexural gravity waves in shallow water and a higher order Nonlinear Schrödinger model relevant in plasma physics, demonstrating how the coefficients in this model are determined via elementary calculations.

Key words: Lagrangian dynamics, nonlinear waves, Whitham modulation.

1 Introduction

The method of phase dynamics is a widely used mechanism in the derivation of nonlinear models from periodic wavetrains,

$$U(x,y,t) = \hat{U}(\theta; k, m, \omega), \quad \hat{U}(\theta + 2\pi) = \hat{U}(\theta), \quad \theta = kx + my + \omega t + \theta_0,$$

for wavenumbers k, m and frequency ω. The main principle is to use the solution's phase invariance to perturb the wave variable by a slowly varying function ϕ whose size is characterised by $\varepsilon \ll 1$, so that the solution now takes the form

$$U = \widehat{U}(\theta + \varepsilon^p \phi; \ldots).$$

The idea is to then substitute this back into the governing equation and Taylor expand this new solution around the $\varepsilon = 0$ state and solve at each order of ε, much like other classical multiple scales methods. The key difference is that this approach brings derivatives of the basic state $\hat{U}$ with respect to its variables naturally into the equations at each order.

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The fifth order KP obtained in this paper is done so through such phase dynamics arguments. Historically, such an approach in conservative settings dates back to Whitham [13, 14] although the present incarnation is inspired by the works of both Pomeau and Manneville [9] and Doelman et al. [2] and builds upon its extension by Bridges et al. [1, 10, 12]. The idea is to take a solution to the system $\widehat{Z}$ as a skeleton for a guess at a perturbed solution (an ansatz) of the form

$$Z = \widehat{Z}\big(\theta + \varepsilon^3\phi, k + \varepsilon^4 q, m + \varepsilon^6 r, \omega + \varepsilon^8\Omega\big) + \varepsilon^5 W(\theta, X, Y, T) \tag{1}$$

where ϕ, q, r and Ω depend on the slow variables

$$X = \varepsilon x,\ Y = \varepsilon^3 y,\ T = \varepsilon^5 t, \tag{2}$$

which are slow space and time variables respectively. This differs from scalings used in other contexts, such as the dispersionless,

$$Z = \widehat{Z}\big(\theta + \phi, k + \varepsilon q, m + \varepsilon^{\frac{3}{2}} r, \omega + \varepsilon^2\Omega\big) + \varepsilon^2 W(\theta, X, Y, T)\,, \quad X = \varepsilon x,\ Y = \varepsilon^{\frac{3}{2}} y,\ T = \varepsilon^2 t,$$

and third order dispersive,

$$Z = \widehat{Z}\big(\theta + \varepsilon\phi, k + \varepsilon^2 q, m + \varepsilon^3 r, \omega + \varepsilon^4\Omega\big) + \varepsilon^3 W(\theta, X, Y, T)\,, \quad X = \varepsilon x,\ Y = \varepsilon^2 y,\ T = \varepsilon^3 t,$$

KP equations in order to balance the first non-vanishing dispersion term, which in this paper is fifth order. The process for deriving the fifth order KP is to substitute the above ansatz into the Euler Lagrange equation for the problem, undertake a Taylor expansion around the $\varepsilon = 0$ state and to solve at each order, leading to the fifth order KP emerging at ninth order in ε. The reason such a form is introduced is so that various terms arising from the Taylor expansion will cancel with each other due to properties of the solution, and so the complexity of the asymptotics is reduced.

The benefit of using the phase dynamics approach in such an abstract way presents many benefits. Firstly, since such a general framework (which will be introduced within the paper) is considered, the asymptotics undertaken here apply to a multitude of systems. This means any PDE that can be cast in the form of that considered in the paper will generate the same result without the need to repeat the calculations, providing all suitable solvability requirements are met. Secondly, the form of the equations considered relate all of the coefficients to quantities associated to the basic state that can be determined *a-priori*. Moreover, as the paper will demonstrate, derivatives of the conservation laws evaluated along the basic state with respect to their parameters will form the majority of the coefficients in the emerging nonlinear PDE.

The setting we consider is the general class of PDEs which are generated from a Lagrangian density, and assuming the existence of a symmetry such systems admit a conservation law of the form

$$A_t + B_x + C_y = 0,$$

for density $A(Z)$ and fluxes $B(Z), C(Z)$, and the subscripts denote partial derivatives. We can denote their evaluation along the basic state $\widehat{Z}$ as

$$\mathscr{A}(k, m, \omega) = \widehat{\mathscr{L}_\omega}, \quad \mathscr{B}(k, m, \omega) = \widehat{\mathscr{L}_k}, \quad \mathscr{C}(k, m, \omega) = \widehat{\mathscr{L}_m}$$

where $\widehat{\mathscr{L}}$ is the Lagrangian averaged over one period of θ. The method presented in this body of work links the derivatives of the above functions to three of the coefficients in the emergent fifth order KP. This remarkable property of the approach allows these coefficients of the problem to be determined prior to any analysis and presents one of the main strengths of using phase dynamics in this context.

This passage adds additional emphasis to the Jordan chain arising from the linearisation about the basic state $\widehat{Z}$. This takes the form

$$\mathbf{L}\xi_1 = 0, \quad \mathbf{L}\xi_{i+1} = \mathbf{J}\xi_i, \ i > 1,$$

for the relevant self adjoint linear operator $\mathbf{L}$ and skew-symmetric operator $\mathbf{J}$ related to the geometric structure of the system, which will be introduced within the paper. Due to the conservative nature of this system such a chain is always even. Moreover, one can show that the chain has length four when

$$\mathscr{B}_k = 0,$$

which is a key requirement for the phase dynamics to generate dispersive PDEs. This paper extends that notion and considers the case when the chain is of length six, requiring that

$$\mathscr{K}_4 = \langle\langle \mathbf{J}\xi_1, \xi_4 \rangle\rangle = 0,$$

for a θ-averaging inner product $\langle\langle \cdot, \cdot \rangle\rangle$, which if met, the phase dynamics gives the result that q must satisfy

$$\left((\mathscr{A}_k + \mathscr{B}_\omega)q_T + \mathscr{B}_{kk} q q_X + \mathscr{M}\, q_{XXY} + \mathscr{K}_6 q_{XXXXX}\right)_X + \mathscr{C}_m q_{YY} = 0. \tag{3}$$

The mixed dispersion term q_{XXY} arises from an entwining of the spatial Jordan chains with termination constant $\mathscr{M}$, whose form will be discussed within the paper. There is an additional requirement that $\mathscr{C}_k = 0$, but for systems with transverse symmetry this is automatically met when $m = 0$.

The main result of this paper is the mechanism for how the fifth order KP arises from a family of periodic waves, and more generally, relative equilibria. In previous work [1, 12, 10] this system has been shown to arise from perturbations about the trivial state using much shorter arguments than those presented here, however such a context forms but a subset of that which the theory applies to. For example, the second example of the paper demonstrates how solutions of the form $A_0 e^{i\theta}$ for suitably chosen amplitude can support the fifth order KP as a phase approximation if the conditions for emergence can be satisfied.

The examples at the end of this paper discuss how this approximation may arise in physical context. The first of these is in the environment of flexural gravity waves. Such systems have already been shown to admit solitary waves that are well described by fifth order KdV and KP models [3, 4, 5], and so one expects the theory to generate a relevant fifth order KP. We consider a simple example, in which the hydroelastic sheet resting on the flow generates Euler-Bernoulli beam terms into the shallow water model. The second example, producing a novel result, is in the setting of a Nonlinear Schrödinger model with a fourth order dispersion term. This system has been shown to arise in condensed matter and plasma physics [6, 7] where the higher order dispersion generates interesting phenomenon,

and so it is of interest to reduce such a system to the fifth order KP to attempt to describe these dynamics.

The format of the paper is natural. We first introduce the general Lagrangian density and its multisymplectic form, discussing the advantages and properties of this approach. This includes its direct relation to the system's conservation laws and the resulting Jordan chain analysis. The asymptotics are then undertaken in this framework, discussing the solvability conditions that have to be met in terms of the continuation of the Jordan chain. This then results in the fifth order KP (3). The theory is then applied to a shallow water flexural gravity waves problem, demonstrating how each of the coefficients can be determined.

2 The Euler-Lagrange equation, θ-averaging and wave action

The starting point for the theory is the class of nonlinear wave equations generated from a Lagrangian,

$$\mathscr{L}(U) = \iiint \mathcal{L}(U, U_t, U_x, U_y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,,$$

for some coordinate U. It is advantageous to recast the Lagrangian in *multisymplectic form*, for which there are a variety of processes to do so. This is not discussed here and instead turn the reader to the discussion in [12, §2] for a full discussion on the possible transformation techniques. The result, however, is the multsymplectic Hamiltonian formulation

$$\mathscr{L}^S(Z) = \iiint \left[\tfrac{1}{2}\langle \mathbf{M}Z_t + \mathbf{J}Z_x + \mathbf{K}Z_y, Z\rangle - S(Z)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,, \tag{4}$$

for new coordinates Z and skew-symmetric matrices $\mathbf{M}$, $\mathbf{J}$ and $\mathbf{K}$. The primary benefit of utilising this form is that this relates the conservation laws to the structure of the Lagrangian, which will be made clear in §2.1.

Herein it will be assumed that the Lagrangian is in the canonical multisymplectic form (4) so that the Euler-Lagrange equation is

$$\mathbf{M}Z_t + \mathbf{J}Z_x + \mathbf{K}Z_y = \nabla S(Z)\,, \tag{5}$$

where $S(Z)$ is the Hamiltonian function and $\mathbf{M}$, $\mathbf{J}$ and $\mathbf{K}$ are skew-symmetric operators. The standard inner product, denoted by $\langle\cdot,\cdot\rangle$, will be chosen.

Now assume that there exists a periodic travelling wave solution in x with wavenumbers k, m and frequency ω of the form

$$Z(x,y,t) = \widehat{Z}(\theta)\,, \quad \widehat{Z}(\theta+2\pi) = \widehat{Z}(\theta)\,, \quad \theta = kx + my + \omega t + \theta_0\,,$$

with arbitrary phase shift θ_0. The modulation theory makes the dependence on k, m and ω explicit so that $\widehat{Z}(\theta, k, m, \omega)$. There is the usual assumption on existence and smoothness of this solution so that the necessary differentiation in θ, k, m and ω is meaningful. This basic state satisfies the ODE

$$\big(\omega\mathbf{M} + k\mathbf{J} + m\mathbf{K}\big)\widehat{Z}_\theta = \nabla S(\widehat{Z})\,. \tag{6}$$

The modulation approach will make use of this wavetrain solution as the primary construct in the ansatz (1), with the perturbation of the dependent variables of this solution leading to multiple simplifications of the asymptotics and the ability to relate solvability conditions to properties of the basic state, such as the criticality of conservation law components.

2.1 Conservation of wave action

The reduction of this paper highlights that the majority of the coefficients in (3) are related to derivatives of the conservation of wave action, and so the details of these are discussed here To obtain components of the conservation law for wave action one averages the Lagrangian evaluated on the family of travelling waves over θ,

$$\widehat{\mathscr{L}}(k,m,\omega) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\omega}{2}\langle \mathbf{M}\widehat{Z}_\theta, \widehat{Z}\rangle + \frac{k}{2}\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}\rangle + \frac{m}{2}\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}\rangle - S(\widehat{Z})\right]\mathrm{d}\theta\,,$$

and differentiate with respect to ω, k, m, giving

$$\left.\begin{aligned} \mathscr{A}(k,m,\omega) &= \widehat{\mathscr{L}_\omega} = \tfrac{1}{2}\langle\!\langle \mathbf{M}\widehat{Z}_\theta, \widehat{Z}\rangle\!\rangle, \\ \mathscr{B}(k,m,\omega) &= \widehat{\mathscr{L}_k} = \tfrac{1}{2}\langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}\rangle\!\rangle\,, \\ \mathscr{C}(k,m,\omega) &= \widehat{\mathscr{L}_m} = \tfrac{1}{2}\langle\!\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}\rangle\!\rangle\,, \end{aligned}\right\} \tag{7}$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is defined as the standard inner product averaged over a period in θ,

$$\langle\!\langle U, V\rangle\!\rangle := \frac{1}{2\pi}\int_0^{2\pi}\langle U, V\rangle\,\mathrm{d}\theta\,.$$

Note that the symplectic matrices $\mathbf{M}$, $\mathbf{J}$ and $\mathbf{K}$ are present in the Euler-Lagrange equations, (5), in (6) and in the components of the conservation law (7).

Key derivatives that appear as coefficients in the modulation theory are

$$\left.\begin{aligned} \mathscr{A}_k &= \langle\!\langle \mathbf{M}\widehat{Z}_\theta, \widehat{Z}_k\rangle\!\rangle\,, \quad \mathscr{B}_\omega = \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_\omega\rangle\!\rangle\,, \quad \mathscr{B}_k = \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_k\rangle\!\rangle\,, \\ \mathscr{B}_m &= \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_m\rangle\!\rangle\,, \quad \mathscr{C}_k = \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_k\rangle\!\rangle\,, \quad \mathscr{C}_m = \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_m\rangle\!\rangle\,. \end{aligned}\right\} \tag{8}$$

Although the remaining derivatives can also be computed they do not appear in the theory and so are omitted here. The second k derivative in $\mathscr{B}$ also appears at the end of the modulational analysis, taking the form

$$\mathscr{B}_{kk} = \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_{kk}\rangle\!\rangle + \langle\!\langle \mathbf{J}\widehat{Z}_{\theta k}, \widehat{Z}_k\rangle\!\rangle.$$

We also note the cross derivative identities appearing from (7),

$$\mathscr{B}_\omega = \widehat{\mathscr{L}_{k\omega}} = \widehat{\mathscr{L}_{\omega k}} = \mathscr{A}_k \quad \text{and} \quad \mathscr{B}_m = \widehat{\mathscr{L}_{km}} = \widehat{\mathscr{L}_{mk}} = \mathscr{C}_k. \tag{9}$$

3 Linearisation about the periodic basic state

It is convenient for the modulation analysis to define the linear operator

$$\mathbf{L}f = \left[\mathrm{D}^2 S(\widehat{Z}) - \big(k\mathbf{J} + \omega\mathbf{M} + m\mathbf{K}\big)\frac{d}{d\theta}\right] f\,, \tag{10}$$

where D denotes the directional derivative, obtained through linearising (6) about $\widehat{Z}$. By differentiating (6) with respect to the numerous wave variables provides the relations

$$\begin{aligned}
D^2S(\widehat{Z})\widehat{Z}_\theta &= \big(k\mathbf{J} + \omega\mathbf{M} + m\mathbf{K}\big)\widehat{Z}_{\theta\theta}\,,\\
D^2S(\widehat{Z})\widehat{Z}_k &= \big(k\mathbf{J} + \omega\mathbf{M} + m\mathbf{K}\big)\widehat{Z}_{\theta k} + \mathbf{J}\widehat{Z}_\theta\,,\\
D^2S(\widehat{Z})\widehat{Z}_m &= \big(k\mathbf{J} + \omega\mathbf{M} + m\mathbf{K}\big)\widehat{Z}_{\theta m} + \mathbf{K}\widehat{Z}_\theta\,,\\
D^2S(\widehat{Z})\widehat{Z}_\omega &= \big(k\mathbf{J} + \omega\mathbf{M} + m\mathbf{K}\big)\widehat{Z}_{\theta\omega} + \mathbf{M}\widehat{Z}_\theta\,,
\end{aligned}$$

or equivalently

$$\mathbf{L}\widehat{Z}_\theta = 0\,,\quad \mathbf{L}\widehat{Z}_k = \mathbf{J}\widehat{Z}_\theta\,,\quad \mathbf{L}\widehat{Z}_m = \mathbf{K}\widehat{Z}_\theta\,,\quad \text{and}\quad \mathbf{L}\widehat{Z}_\omega = \mathbf{M}\widehat{Z}_\theta\,, \tag{11}$$

with other derivatives following a similar pattern. The first equation of (11) shows that $\widehat{Z}_\theta$, the Goldstone mode, is in the kernel of $\mathbf{L}$. It is natural to assume that the kernel is no larger so that

$$\mathrm{Kernel}(\mathbf{L}) = \mathrm{span}\big\{\widehat{Z}_\theta\big\}\,. \tag{12}$$

The second through fourth equations of (11) show that there are non-trivial Jordan chains associated with the zero eigenvalue of $\mathbf{L}$ with geometric eigenvector $\widehat{Z}_\theta$. There are three Jordan chains of length two in (11)

$$\begin{aligned}
\mathbf{L}\widehat{Z}_\theta &= 0 \quad\text{and}\quad \mathbf{L}\widehat{Z}_k = \mathbf{J}\widehat{Z}_\theta,\\
\mathbf{L}\widehat{Z}_\theta &= 0 \quad\text{and}\quad \mathbf{L}\widehat{Z}_m = \mathbf{K}\widehat{Z}_\theta,\\
\mathbf{L}\widehat{Z}_\theta &= 0 \quad\text{and}\quad \mathbf{L}\widehat{Z}_\omega = \mathbf{M}\widehat{Z}_\theta\,.
\end{aligned}$$

There is currently no theory about multiple Jordan chains, however this paper does not require any to proceed. Instead, we mostly treat each chain as a separate instance and determine their length individually. There is one instance where the $\mathbf{J}$ and $\mathbf{K}$ chains are combined in a nontrivial way, which will be discussed in the next section. The second and third Jordan chain will be assumed to be fixed at length two, and we look at conditions for the first Jordan chain to extend to higher lengths. The main result of this paper assumes that this length is six, which in turn generates higher order dispersive terms in the modulation theory.

Solvability conditions will be needed throughout the analysis to ensure that we can proceed to the desired order. With the assumption (12) and the self-adjointness property of $\mathbf{L}$, the solvability condition for inhomogeneous equations is

$$\mathbf{L}F = G \quad \text{is solvable if and only if} \quad \langle\!\langle \widehat{Z}_\theta, G\rangle\!\rangle = 0\,.$$

3.1 Jordan chain theory

To facilitate the dispersive part of the modulational asymptotics, we discuss the details of the Jordan chain emerging within this formulation. Of interest in this paper is the chain involving $\mathbf{J}$, although similar discussions can be made in relation to the $\mathbf{M}$ and $\mathbf{K}$ chains.

We can see from (11) that we have a Jordan chain of the form

$$\mathbf{L}\xi_1 = \mathbf{0}, \quad \mathbf{L}\xi_{i+1} = \mathbf{J}\xi_i, \; i > 0, \tag{13}$$

with $\xi_1 = \widehat{Z}_\theta, \; \xi_2 = \widehat{Z}_k$. A formality of such a chain existing is that none of its elements lie within the kernel of $\mathbf{J}$, which is satisfied by the example presented later. The chain is always of even length owing to the skew symmetry of $\mathbf{J}$. This can be seen by considering the existence requirement of an even element of the chain:

$$\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\xi_{2n-1} \rangle\!\rangle = -\langle\!\langle \widehat{Z}_k, \mathbf{J}\xi_{2n-2} \rangle\!\rangle = \ldots = \langle\!\langle \xi_n, \mathbf{J}\xi_n \rangle\!\rangle = -\langle\!\langle \mathbf{J}\xi_n, \xi_n \rangle\!\rangle \;\Rightarrow\; \langle\!\langle \widehat{Z}_\theta, \mathbf{J}\xi_{2n-1} \rangle\!\rangle = 0.$$

This corresponds to only dispersive terms arising from the modulational approach in this setting, something that is expected from the conservative structure. Of importance is the point at which the chain terminates and the nonzero constant that will arise at this point, since this is exactly the coefficient of dispersion in the final model we derive. Therefore define

$$\mathscr{K}_i = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\xi_i \rangle\!\rangle, \tag{14}$$

as this nonzero constant. The index i can also be seen to be related to the order of dispersion found within the emergent models, with its order being precisely $i-1$. This allows one to appropriately scale the slow time scale such that the time evolution and dispersion of the reduction are in balance.

The chain has a third element when

$$\mathscr{K}_2 = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\widehat{Z}_k \rangle\!\rangle = \mathscr{B}_k = 0.$$

For the purposes of the working that will follow, we assume this is met for some values of k, m and ω. This condition appears in the majority of the modulational studies (for example, [1, 10, 12]) and is the primary condition for nonlinear terms to appear in the final equation. The theory of this paper focusses on the case of higher order dispersive effects, which in terms of the length of this Jordan chain requires additionally that

$$\mathscr{K}_4 = 0.$$

When this condition is imposed, the third order dispersive terms also vanish and so fifth order dispersion becomes dominant. The chain terminates at exactly length six if

$$\mathscr{K}_6 \neq 0\,,$$

and in the context of this paper we assume that this holds and even higher dispersive terms are not relevant.

Finally, it is worth noting that although these conditions require that a chain of length six exists, in practise one need only compute the first four elements to obtain the desired coefficient. This is since

$$\mathscr{K}_6 = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\xi_6 \rangle\!\rangle = \langle\!\langle \widehat{Z}_k, \mathbf{J}\xi_5 \rangle\!\rangle = -\langle\!\langle \xi_3, \mathbf{J}\xi_4 \rangle\!\rangle\,,$$

by use of the skew-symmetry in $\mathbf{J}$ and the definitions of each element of the chain.

3.2 Entwining of chains

A consequence of the scalings chosen in the ansatz (1) is that it suggests that a q_{XXXY} term can be balanced in the final PDE. The asymptotics confirm this, highlighting that it does so from a Jordan chain of mixed form. This structure occurs due to the **K**-chain "entwining" with the one above to form an additional chain-like structure. Within the setting of this paper, this takes the form

$$\mathbf{L}\zeta_1 = \mathbf{J}\widehat{Z}_m + \mathbf{K}\xi_2, \quad \mathbf{L}\zeta_{i+1} = \mathbf{J}\zeta_i + \mathbf{K}\xi_{i+3}. \tag{15}$$

We can see that the ζ_1 exists when

$$\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\widehat{Z}_m + \mathbf{K}\widehat{Z}_k \rangle\!\rangle = -\mathscr{B}_m - \mathscr{C}_k = 0 \tag{16}$$

and must have an even number of elements, which can be argued through noting the zero eigenvalue of the linear operator is even. We assume this ζ chain contains at least two elements, the minimum for the theory, so that the above is satisfied. It is then convenient to define

$$\mathscr{M} = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\zeta_2 + \mathbf{K}\xi_4 \rangle\!\rangle = -2\langle\!\langle \xi_3, \mathbf{J}\widehat{Z}_m + \mathbf{K}\widehat{Z}_k \rangle\!\rangle. \tag{17}$$

This may or may not vanish depending on the context. The theory will highlight that this will be precisely the coefficient of the q_{XXXY} term, and as such one expects it to vanish whenever the original system in invariant under the transformation $y \mapsto -y$. In the case that it zero we do not consider the derived equation degenerate, since the highest order of dispersion from the main Jordan chain is still present. Instead, the vanishing of this term corresponds to just the non-presence of mixed dispersion within the final model equation.

These types of chain have been seen in previous modulational studies of this approach [12, 10] although no theory has been constructed for them. Their importance in this paper is that they lead to an additional mixed third order dispersive term in the resulting fifth order KP equation which, as far as the author is aware, does not appear in the literature.

4 The modulation ansatz

The ansatz used to derive the fifth order KP equation is taken to be (1) using the scales (2). The scalings on each of the functions inside of $\widehat{Z}$ are related through the phase consistency conditions, given by

$$\theta_x = k,\ \theta_y = m \quad \text{and} \quad \theta_t = \omega, \quad \Rightarrow \quad \phi_X = q,\ \phi_Y = r \quad \text{and} \quad \phi_T = \Omega\,. \tag{18}$$

This also enforces the relations

$$q_T = \Omega_X, \quad q_Y = r_X \quad \text{and} \quad r_T = \Omega_Y. \tag{19}$$

It is also useful to expand the remainder term W in (1) as a simple asymptotic series,

$$W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \dots\,,$$

so that elements of it appear at each order. The idea is to substitute this ansatz into (5), undertake a Taylor expansion around the $\varepsilon = 0$ state before solving for each order of ε.

The analysis below proceeds under the assumption that

$$\mathscr{B}_k = \mathscr{K}_4 = \mathscr{B}_m = 0\,, \quad \mathscr{K}_6 \neq 0\,,$$

leading to the main result of this paper, that a fifth order KP equation emerges. Below is a summary of the results of this approach at each order of ε.

4.1 $\mathcal{O}(1)$ through $\mathcal{O}(\varepsilon^3)$

The basic state is recovered at order 1 and no other orders exist until ε^3 due to the nature of the scalings chosen. The third order terms in ε give

$$\phi \mathbf{L} \widehat{Z}_\theta = 0,$$

which holds automatically.

4.2 $\mathcal{O}(\varepsilon^4)$

The results at this order give

$$q\mathbf{L}\widehat{Z}_k - \phi_X \mathbf{J}\widehat{Z}_\theta = 0\,,$$

which is satisfied when $q = \phi_X$ by using (11), which has to be the case through the phase consistency condition (18).

4.3 $\mathcal{O}(\varepsilon^5)$

This order gives

$$\mathbf{L}W_0 = q_X \mathbf{J}\widehat{Z}_k\,,$$

which is solvable when $\mathscr{B}_k = 0$ as discussed in §3.1, giving

$$W_0 = q_X \xi_3.$$

There is the potential to add terms proportional to the kernel element $\widehat{Z}_\theta$ to this solution, however they can always be shown not to contribute at subsequent orders of the analysis. In light of this, we do not include these additional terms.

4.4 $\mathcal{O}(\varepsilon^6)$

This order gives

$$\mathbf{L}W_1 + \frac{1}{2}\phi^2\left(\mathbf{L}\widehat{Z}_{\theta\theta} + \mathrm{D}^3 S(\widehat{Z})(\widehat{Z}_\theta, \widehat{Z}_\theta)\right) + r\mathbf{L}\widehat{Z}_m - \phi_Y \mathbf{K}\widehat{Z}_\theta - q_{XX}\mathbf{J}\xi_3 = 0.$$

The r term cancels with the ϕ_Y term due to both the phase consistency condition (19) and (11). The ϕ^2 term is identically zero since by differentiating (6) twice with respect to θ one finds

$$\mathbf{L}\widehat{Z}_{\theta\theta} = -\mathrm{D}^3 S(\widehat{Z})(\widehat{Z}_\theta, \widehat{Z}_\theta).$$

It can similarly be shown that all terms of the form $\phi^a q^b$ vanish by considering a suitable combination of θ and k derivatives of (6) and so these terms will be omitted from the discussion. The q_{XX} term is automatically solvable using the fact that the Jordan chain is of even length. Therefore the result of this order is that

$$W_1 = q_{XX}\xi_4.$$

4.5 $\mathcal{O}(\varepsilon^7)$

Here we have

$$\mathbf{L}W_2 - q_Y(\mathbf{K}\widehat{Z}_k + \mathbf{J}\widehat{Z}_m) - q_{XXX}\mathbf{J}\xi_4 = 0,$$

where (19) has been used to replace the r_X term. The solvability of this first term, by (8), requires that

$$\mathscr{C}_k + \mathscr{B}_m = 0. \tag{20}$$

This gives the second condition for the fifth order KP to emerge, and by using (9) it may be seen that this may just be reduced to the condition that $\mathscr{B}_m = 0$. One can however show that when the system has a transverse symmetry (i.e. invariance under the transformation $y \to -y$) that this condition is satisfied when $m = 0$. Full details of this symmetry argument can be found in [10]. The last term is solvable when

$$-\mathscr{K}_4 = \langle\!\langle \widehat{Z}_\theta, \mathbf{J}\xi_4 \rangle\!\rangle = 0.$$

When both of the above conditions are met, we are able to solve for W_2 as

$$W_2 = q_Y\zeta_1 + q_{XXX}\xi_5, \quad \mathbf{L}\zeta_1 = \mathbf{K}\widehat{Z}_k + \mathbf{J}\widehat{Z}_m.$$

The case in which $\mathscr{K}_4 \neq 0$ has been discussed previously, revealing that a different set of scales provides a more suitable balance [1]. However in this case we assume that this constant is indeed zero and that higher order derivatives appear instead.

4.6 $\mathcal{O}(\varepsilon^8)$

At the penultimate order we have

$$\mathbf{L}W_3 - q_{XY}(\mathbf{J}\zeta_1 + \mathbf{K}\xi_3) - q_{XXXX}\mathbf{J}\xi_5 - \phi q_X\big(\mathbf{J}\widehat{Z}_{\theta k} - \mathrm{D}^3 S(\widehat{Z})(\widehat{Z}_\theta, \xi_3)\big) + \Omega\mathbf{L}\widehat{Z}_\omega - \phi_T\mathbf{M}\widehat{Z}_\theta = 0.$$

Appealing to solvability, the first term vanishes since

$$\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\zeta_1 + \mathbf{K}\xi_3 \rangle\!\rangle = -\langle\!\langle \widehat{Z}_k, \mathbf{K}\widehat{Z}_k + \mathbf{J}\widehat{Z}_m \rangle\!\rangle + \langle\!\langle \widehat{Z}_\theta, \mathbf{K}\xi_3 \rangle\!\rangle = \langle\!\langle \widehat{Z}_\theta, \mathbf{K}\xi_3 \rangle\!\rangle + \langle\!\langle \xi_3, \mathbf{K}\widehat{Z}_\theta \rangle\!\rangle = 0.$$

The last two terms vanish via the phase consistency condition (18) that $\phi_T = \Omega$ along with (11). The ϕq_X term is solved by $\partial_\theta \xi_3$, which can be shown through differentiation of the defining equation for ξ_3 (13) with respect to θ. The other term is solvable since

$$\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\xi_5 \rangle\!\rangle = -\langle\!\langle \mathbf{L}\widehat{Z}_k, \xi_5 \rangle\!\rangle = -\langle\!\langle \widehat{Z}_k, \mathbf{J}\xi_4 \rangle\!\rangle = \langle\!\langle \mathbf{L}\xi_3, \xi_4 \rangle\!\rangle = \langle\!\langle \xi_3, \mathbf{J}\xi_3 \rangle\!\rangle = 0,$$

as discussed before. Therefore we have

$$W_3 = q_{XY}\zeta_2 + \phi q_X \partial_\theta \xi_3 + q_{XXXX}\xi_6, \quad \mathbf{L}\zeta_2 = \mathbf{J}\zeta_1 + \mathbf{K}\xi_3,$$

emerging at this order.

4.7 $\mathcal{O}(\varepsilon^9)$

At the last order, the non-vanishing terms are

$$\mathbf{L}W_4 - q_T\mathbf{M}\widehat{Z}_k - \Omega_X\mathbf{J}\widehat{Z}_\omega - qq_X\left(\mathbf{J}\widehat{Z}_{kk} + \mathbf{J}\partial_\theta\xi_3 - \mathrm{D}^3S(\widehat{Z})(\widehat{Z}_k,\xi_3)\right) \\ - q_{XXY}(\mathbf{J}\zeta_1 + \mathbf{K}\xi_4) - r_Y\mathbf{K}\widehat{Z}_m - q_{XXXXX}\mathbf{J}\xi_6 = 0.$$

We now impose the solvability condition at this order. From (9) we see that

$$-\mathscr{A}_k = \langle\!\langle \widehat{Z}_\theta, \mathbf{M}\widehat{Z}_k\rangle\!\rangle = \langle\!\langle \widehat{Z}_\theta, \mathbf{J}\widehat{Z}_\omega\rangle\!\rangle = -\mathscr{B}_\omega.$$

The nonlinearity can be shown to have the coefficient $\mathscr{B}_{kk}$ since

$$\begin{aligned}\mathscr{B}_{kk} =& \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_{kk}\rangle\!\rangle + \langle\!\langle \mathbf{J}\widehat{Z}_{\theta k}, \widehat{Z}_k\rangle\!\rangle\,,\\ =& \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_{kk}\rangle\!\rangle - \langle\!\langle \widehat{Z}_{\theta k}, \mathbf{L}\xi_3\rangle\!\rangle\,,\\ =& \langle\!\langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_{kk}\rangle\!\rangle - \langle\!\langle \mathbf{J}\widehat{Z}_{\theta\theta} - \mathrm{D}^3S(\widehat{Z})(\widehat{Z}_k,\widehat{Z}_\theta),\xi_3\rangle\!\rangle,\\ =& -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\widehat{Z}_{kk}\rangle\!\rangle - \langle\!\langle \mathbf{J}\widehat{Z}_{\theta\theta},\xi_3\rangle\!\rangle + \langle\!\langle \mathrm{D}^3S(\widehat{Z})(\widehat{Z}_k,\widehat{Z}_\theta),\xi_3\rangle\!\rangle,\\ =& -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\widehat{Z}_{kk}\rangle\!\rangle - \langle\!\langle \widehat{Z}_\theta, \mathbf{J}(\xi_3)_\theta\rangle\!\rangle + \langle\!\langle \widehat{Z}_\theta, \mathrm{D}^3S(\widehat{Z})(\widehat{Z}_k,\xi_3)\rangle\!\rangle.\end{aligned}$$

Finally, we can use the definition of the termination constants to see that

$$\mathscr{M} = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\zeta_2 + \mathbf{K}\xi_4\rangle\!\rangle\,, \tag{21}$$

as well as

$$\mathscr{K}_6 = -\langle\!\langle \widehat{Z}_\theta, \mathbf{J}\xi_6\rangle\!\rangle \neq 0\,,$$

by assumption to find that we have

$$(\mathscr{A}_k + \mathscr{B}_\omega)q_T + \mathscr{B}_{kk}qq_X + \mathscr{K}_6 q_{XXXXX} + \mathscr{M}q_{XXY} + \mathscr{C}_m r_Y = 0. \tag{22}$$

Differentiation with respect to X leads to (3).

5 Example 1 - flexural gravity waves

To initally illustrate the theory, we use a example from the literature to validate our approach. As such, we take the shallow water potential case of the biharmonic flexural gravity waves (corresponding to the assumption that the elastic sheet satisfies the Euler-Bernoulli beam condition) with no loading, following the approach of Parau and Vanden-Broeck [8] but taking a shallow water limit. This gives the system

$$\begin{aligned}\eta_t + \nabla\cdot(\eta\nabla\phi) =& 0,\\ \phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta + \frac{D}{\rho}\nabla^4\eta =& R,\end{aligned} \tag{23}$$

for fluid height η, velocity potential ϕ, Bernoulli constant R, fluid density ρ and D the flexural rigidity of the plate above the fluid. This system has the solution

$$\phi = kx + \omega t + \phi_0 = \theta, \quad \eta = \eta_0(k,m,\omega) = \frac{1}{g}\left(R - \omega - \frac{k^2+m^2}{2}\right).$$

The conservation law for this system is simply the first equation and is associated with mass conservation, and so we find

$$\mathscr{A} = \eta_0, \quad \mathscr{B} = k\eta_0, \quad \text{and} \quad \mathscr{C} = m\eta_0. \tag{24}$$

We can now find the desired criticalities $\mathscr{B}_k = \mathscr{B}_m = 0$, which occur when

$$\eta_0 - \frac{k^2}{g} = 0 \;\Rightarrow\; \mathrm{Fr}^2 \equiv \frac{k^2}{g\eta_0} = 1, \quad -\frac{km}{g} = 0 \;\Rightarrow\; m = 0$$

where Fr is the Froude number for this system. The coefficients required for the fifth order KP can then be calculated to find

$$\mathscr{A}_k = \mathscr{B}_\omega = -\frac{k}{g}, \quad \mathscr{C}_m = \eta_0, \quad \mathscr{B}_{kk} = -\frac{3k}{g}.$$

The final step in determining the fifth order KP is to determine the $\mathbf{J}$ Jordan chain and compute its termination constants. There are two ways that this can be done, either through the dispersion relation (as these are linear terms) or from the Jordan chain argument presented in this paper. The latter is chosen here to link better with the theory presented here. This requires one to determine the multisymplectic form for this system. We start from the Lagrangian density for this system, given by

$$\mathcal{L} = \eta\phi_t + \frac{1}{2}\eta(\phi_x^2 + \phi_y^2) + \frac{1}{2}g\eta^2 + \frac{D}{2\rho}(\nabla^2\eta)^2\,,$$

and introducing

$$p = \eta\phi_x,\;\; s = \eta\phi_y,\;\; \sigma = \eta_x,\;\; \gamma = \eta_y\;\; \tau = -\frac{D}{\rho}(\eta_{xx} + \eta_{yy}),\;\; \kappa = \tau_x + \varphi_y,\;\; \chi = \tau_y - \varphi_x,$$

for some dummy variable φ that is introduced to preserve the structure, one finds the multisymplectic form (5) with

$$Z = \left(\phi, \eta, p, s, \sigma, \gamma, \tau, \varphi, \kappa, \chi\right)^T, \quad S(Z) = R\eta - \frac{1}{2}g\eta^2 + \frac{p^2 + s^2}{2\eta} + \frac{\rho\tau^2}{2D} + \sigma\kappa + \gamma\chi.$$

Due to their size, $\mathbf{M}, \mathbf{J}$ and $\mathbf{K}$ are given in the appendix. To determine the chain, we notice its first two elements are θ independent, and so its sensible to assume all elements are as well. Therefore we solve $\mathrm{D}^2 S(\widehat{Z})\xi_{i+1} = \mathbf{J}\xi_i$, and so the Jordan chain can be found to be

$$\begin{aligned}
\xi_1 &= \left(1,0,0,0,0,0,0,0,0,0\right)^T, & \xi_2 &= \left(0, -\tfrac{k}{g}, \eta_0 - \tfrac{k^2}{g}, 0,0,0,0,0,0,0\right)^T,\\
\xi_3 &= \left(0,0,0,0,-\tfrac{k}{g},0,0,0,0,0\right)^T, & \xi_4 &= \left(0,0,0,0,0,0,\tfrac{Dk}{\rho g},0,0,0\right)^T,\\
\xi_5 &= \left(0,0,0,0,0,0,0,0,\tfrac{Dk}{\rho g},0\right)^T, & \xi_6 &= \left(0,\tfrac{D\eta_0}{\rho g k},\tfrac{D\eta_0}{\rho g},0,0,0,0,0,0,0\right)^T.
\end{aligned}$$

From these and calculating $\widehat{Z}_m$ at m=0, we find that

$$\mathscr{M} = -2\langle\!\langle \xi_3, \mathbf{J}\widehat{Z}_m + \mathbf{K}\widehat{Z}_k \rangle\!\rangle = 0,$$

so there is no mixed dispersion. One also finds

$$\mathscr{K}_6 = \frac{D\eta_0}{\rho g}, \tag{25}$$

and so the fifth order 2+1 KP in this case is given by

$$\begin{aligned} &\left(-\frac{2k}{g} q_T - \frac{3k}{g} q q_X + \frac{D\eta_0}{\rho g} q_{XXXXX} \right)_X + \eta_0 q_{YY} = 0, \\ \Rightarrow &\left(q_T + \frac{3}{2} q q_X \mp \frac{D}{2\rho} \sqrt{\frac{\eta_0}{g}} q_{XXXXX} \right)_X \pm \frac{\sqrt{g\eta_0}}{2} q_{YY} = 0, \end{aligned} \tag{26}$$

where we have used the criticality $k = \pm\sqrt{g\eta_0}$. We note that both signs of k are admissible since the velocity may be oriented to the right or left. A comparision with [4] shows agreement between coefficients, accounting for differences in the model used. We can transform this into free surface variables through the transformation $H = -\frac{k}{g} q$ to obtain

$$\left(H_T - \frac{3g}{2k} H H_X - \frac{Dk}{2\rho g} H_{XXXXX} \right)_X \pm \frac{\sqrt{g\eta_0}}{2} H_{YY} = 0. \tag{27}$$

6 Example 2 - fourth order NLS model

The novel application we can apply our theory to is a defocussing Nonlinear Schrödinger model with fourth order dispersion, given by

$$iA_t + i\beta A_y + \lambda \nabla^4 A + \nabla^2 A + A - |A|^2 A = 0, \tag{28}$$

for $A(x, y, t)$ a complex valued function, β a real costant and λ characterises the higher order dispersive effects. It will be this parameter along with the wave number that will be used to eliminate the third order dispersion terms and lead to the fifth order KdV. When $\beta = 0$, such a system has been introduced by Karpman et al. to investigate highly dispersive phenomenon in condensed matter and plasmas [6, 7]. The introduction of the extra y derivative can be argued, for instance by considering a frame of reference moving in the y direction at speed β, and is done here to demonstrate how the breaking of the $y \mapsto -y$ symmetry leads to the mixed dispersive term in (3). It is however noted that the $\beta \neq 0$ has yet to be rigorously derived in any context.

The periodic wave considered in this scenario is simply the plane wave solution $A = A_0 e^{i\theta}$, where the complex amplitude A_0 satisfies

$$|A_0|^2 = 1 - (k^2 + m^2) + \lambda(k^2 + m^2)^2 - \omega - \beta m.$$

The system (28) admits the following conservation laws:

$$\begin{aligned} A =& \frac{1}{2}|A|^2, \\ B =& \Im\big(A^* A_x + \lambda(2A^* A_{xyy} + A^* A_{xxx} - A_x^* A_{xx})\big), \\ C =& \Im\big(A^* A_y + \lambda(2A^* A_{xxy} + A^* A_{yyy} - A_y^* A_{yy})\big) + \frac{\beta}{2}|A_0|^2, \end{aligned}$$

where $*$ denotes the complex conjugate. These can be evaluated along the plane wave to give the quantities

$$\mathscr{A} = \frac{1}{2}|A_0|^2, \quad \mathscr{B} = k(1-2(k^2+m^2)\lambda)|A_0|^2, \quad \mathscr{C} = m(1-2(k^2+m^2)\lambda)|A_0|^2 + \frac{\beta}{2}|A_0|^2. \tag{29}$$

Therefore, for the theory of this paper to be applicable we require that

$$\begin{aligned} \mathscr{B}_k &= \big(1 - 2\lambda(m^2 + 3k^2)\big)|A_0|^2 - 2k^2(1 - 2(k^2 + m^2)\lambda)^2 = 0\,, \\ \text{and} \quad \mathscr{C}_k &= -4mk\lambda|A_0|^2 - 2mk(1 - 2(k^2 + m^2)\lambda)^2 + \beta k(2\lambda(k^2 + m^2) - 1) = 0. \end{aligned}$$

These form two of the three necessary conditions, and below we will show that the condition for higher order dispersion is

$$\mathscr{K}_4 = \frac{\lambda|A_0|^4 + 8\lambda k^2(2\lambda(k^2 + m^2) - 1)|A_0|^2 + (6\lambda k^2 + 2\lambda m^2 - 1)k^2(2\lambda(k^2 + m^2) - 1)^2}{|A_0|^2}\,,$$

Due to the complexity of the polynomials involved, we use a Newton scheme to find solutions to these three constrains simultaneously for (k, m, λ) whilst keeping ω fixed since λ plays a role in the criticality akin to the Bond number in fluid mechanics. We consider two cases for this, one where $\beta = 0$ so that the original Karpman model is analysed as well as the case where $\beta \neq 0$. For example, when $\omega = \beta = 0$, we find that the criticality occurs when

$$k \approx 0.38387, \quad \lambda \approx 0.92979, \quad m = 0\,, \tag{30}$$

and when $\beta = 1$

$$k \approx 0.42085, \quad \lambda = 0.73780, \quad m = -0.17700\,. \tag{31}$$

The fact that $m \neq 0$ already suggests that the mixed dispersive term should occur. It should also be noted that either sign of k is admissible since each of the criticality conditions is even in k. Additional criticalities may be found by varying $\omega$and β, although not all of them are expected to be physically relevant. At these values, the coefficients related to the conservation laws take the values

$$\mathscr{A}_k \approx -0.27868, \quad \mathscr{B}_{kk} \approx -4.03590, \quad \mathscr{C}_m \approx 0.63367\,,$$

and

$$\mathscr{A}_k \approx -0.29140, \quad \mathscr{B}_{kk} \approx -4.02506, \quad \mathscr{C}_m \approx 0.69285\,,$$

respectively. All that remains is to compute the Jordan chain to obtain the dispersive coefficients.

To accomplish this, we first must obtain the system's multisymplectic form. To do this, we decompose A into its real and imaginary parts so that $A = u + iv$. Then introduce the vectors

$$\mathbf{a} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{b} = \mathbf{a}_x, \quad \mathbf{c} = \mathbf{a}_y, \quad \mathbf{d} = -\lambda(\mathbf{b}_x + \mathbf{c}_y), \quad \mathbf{e} = -\mathbf{d}_x - \boldsymbol{\alpha}_y, \quad \mathbf{f} = -\mathbf{d}_y + \boldsymbol{\alpha}_x,$$

where we have introduced again a dummy variable $\boldsymbol{\alpha}$ to preserve the system's structure. This generates (5) with

$$Z = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \\ \boldsymbol{\alpha} \\ \mathbf{e} \\ \mathbf{f} \end{pmatrix}, \quad S(Z) = \frac{1}{2}\left(\mathbf{a}\cdot\mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{a})^2}{2} + \mathbf{b}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{c} - \frac{\mathbf{d}\cdot\mathbf{d}}{\lambda}\right) + \mathbf{b}\cdot\mathbf{e} + \mathbf{c}\cdot\mathbf{f}\,,$$

and the skew symmetric matrices are given in the appendix. We undertake the Jordan chain analysis using the approach of [1] for the NLS example, and so we can write the state vector and the Jordan chain elements as

$$\widehat{Z} = \begin{pmatrix} R_\theta\hat{\mathbf{u}} \\ kR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \\ mR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \\ (k^2+m^2)\lambda R_\theta\hat{\mathbf{u}} \\ \mathbf{0} \\ -k(k^2+m^2)\lambda R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \\ -m(k^2+m^2)\lambda R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \end{pmatrix}, \quad \xi_j = \begin{pmatrix} R_\theta\mathbf{a}_j \\ R_\theta\mathbf{b}_j \\ R_\theta\mathbf{c}_j \\ R_\theta\mathbf{d}_j \\ \mathbf{0} \\ R_\theta\mathbf{e}_j \\ R_\theta\mathbf{f}_j \end{pmatrix},$$

noting that the $\boldsymbol{\alpha}$ elements will not contribute and so can be chosen to be zero, along with

$$\|\hat{\mathbf{u}}\|^2 = |A_0|^2\,, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \boldsymbol{\sigma} = \left.\frac{dR_\theta}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

One may then substitute these into the multisymplectic Euler-Lagrange equations, and once simplified lead to the recursion relations for the chain

$$\begin{aligned}
-2(\hat{\mathbf{u}}\cdot\mathbf{a}_i)\hat{\mathbf{u}} &= (\lambda(k^2+m^2)-1)\mathbf{b}_{i-1} + k\boldsymbol{\sigma}\mathbf{d}_{i-1} - \mathbf{e}_{i-1} + k(\lambda(k^2+m^2)-1)\boldsymbol{\sigma}\mathbf{a}_{i-1}\,, \\
\mathbf{b}_i &= k\boldsymbol{\sigma}\mathbf{a}_i + \mathbf{a}_{i-1}\,, \\
\mathbf{c}_i &= m\boldsymbol{\sigma}\mathbf{a}_i\,, \\
\mathbf{d}_i &= \lambda(k^2+m^2)\mathbf{a}_i - \lambda\mathbf{b}_{i-1} - \lambda k\boldsymbol{\sigma}\mathbf{a}_{i-1}\,, \\
\mathbf{e}_i &= k\lambda\boldsymbol{\sigma}\mathbf{b}_{i-1} - \lambda k^2\mathbf{a}_{i-1} - \mathbf{d}_{i-1} - \lambda k(k^2+m^2)\boldsymbol{\sigma}\mathbf{a}_i\,, \\
\mathbf{f}_i &= \lambda m\boldsymbol{\sigma}\mathbf{b}_{i-1} - \lambda km\mathbf{a}_{i-1} - \lambda m(k^2+m^2)\boldsymbol{\sigma}\mathbf{a}_i\,.
\end{aligned}$$

Solving this generates

$$\xi_1 = \begin{pmatrix} R_\theta \boldsymbol{\sigma}\hat{\mathbf{u}} \\ -kR_\theta\hat{\mathbf{u}} \\ -mR_\theta\hat{\mathbf{u}} \\ (k^2+m^2)\lambda R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \\ \mathbf{0} \\ k(k^2+m^2)\lambda R_\theta\hat{\mathbf{u}} \\ m(k^2+m^2)\lambda R_\theta\hat{\mathbf{u}} \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} R_\theta\hat{\mathbf{u}}_k \\ R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} + kR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}}_k \\ mR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}}_k \\ 2\lambda kR_\theta\hat{\mathbf{u}} + \lambda(k^2+m^2)R_\theta\hat{\mathbf{u}}_k \\ \mathbf{0} \\ -\lambda(3k^2+m^2)R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} - k(k^2+m^2)\lambda R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}}_k \\ -2km\lambda R_\theta\hat{\mathbf{u}} - m(k^2+m^2)\lambda R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}}_k \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} \mathbf{0} \\ R_\theta\hat{\mathbf{u}}_k \\ \mathbf{0} \\ -\lambda(R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} + 2kR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}}_k) \\ \mathbf{0} \\ -3\lambda k(R_\theta\hat{\mathbf{u}} + kR_\theta\hat{\mathbf{u}}_k) - \lambda m^2 R_\theta\hat{\mathbf{u}}_k \\ -\lambda m(R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} + 2kR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}}_k) \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} \Gamma R_\theta\hat{\mathbf{u}} \\ k\Gamma R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \\ m\Gamma R_\theta\hat{\mathbf{u}} \\ \lambda\Gamma(k^2+m^2)R_\theta\hat{\mathbf{u}} - \lambda R_\theta\hat{\mathbf{u}}_k \\ \mathbf{0} \\ \lambda\big(1-k\Gamma(k^2+m^2)\big)R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} + 3\lambda kR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \\ \lambda mR_\theta\boldsymbol{\sigma}\hat{\mathbf{u}}_k - \lambda\Gamma m(k^2+m^2)R_\theta\boldsymbol{\sigma}\hat{\mathbf{u}} \end{pmatrix}.$$

where

$$\Gamma = -\frac{k(6\lambda k^2+2\lambda m^2-1)(2\lambda(k^2+m^2)-1)+4\lambda k|A_0|^2}{2|A_0|^4}.$$

These are all the Jordan chain elements we need to compute not only the condition for higher order dispersion to emerge but the relevant coefficients for (3). Firstly, one finds that

$$\mathscr{K}_4 = \frac{\lambda|A_0|^4 + 8\lambda k^2(2\lambda(k^2+m^2)-1)|A_0|^2 + (6\lambda k^2+2\lambda m^2-1)k^2(2\lambda(k^2+m^2)-1)^2}{|A_0|^2},$$

which provides the final criterion for the fifth order KP to emerge. We also find that

$$\begin{aligned} \mathscr{K}_6 &= \Gamma\big(4\lambda k|A_0|^2 + k(6\lambda k^2+2m^2-1)(2\lambda(k^2+m^2)-1)\big) - \frac{\lambda k^2(2\lambda(k^2+m^2)-1)^2}{|A_0|^2}, \\ &= -2\Gamma^2|A_0|^4 - \frac{\lambda k^2(2\lambda(k^2+m^2)-1)^2}{|A_0|^2}. \end{aligned} \tag{32}$$

as well as

$$\begin{aligned} \mathscr{M} &= 16\lambda km(2\lambda(k^2+m^2)-1) - 4\beta k\lambda \\ &\quad + (6\lambda k^2+2\lambda m^2+4km\lambda-1)\frac{2k(2\lambda(k^2+m^2)-1)(4m\lambda(k^2+m^2)-2m-\beta)}{|A_0|^2} \end{aligned} \tag{33}$$

At criticality (30), these coefficients are approximately

$$\mathscr{K}_6 \approx -1.18457, \quad \mathscr{M} = 0,$$

and at (31)

$$\mathscr{K}_6 \approx -0.89651, \quad \mathscr{M} \approx -0.62210.$$

Therefore, the relevant nonlinear phase equation for the original Karpman model ($\beta = 0$) is

$$\left(q_T + a_1 q q_X + a_2 q_{XXXXX}\right)_X + a_3 q_{YY} = 0\,, \tag{34}$$

where along this criticality

$$a_1 \approx 7.24109\,, \quad a_2 \approx 2.12532\,, \quad a_3 \approx -1.13690\,.$$

In the case where the transverse symmetry is broken we instead use the criticality (31) to obtain the relevant phase equation as

$$\left(q_T + b_1 q q_X + b_2 q_{XXY} + b_3 q_{XXXXX}\right)_X + b_4 q_{YY} = 0\,, \tag{35}$$

where the coefficients are approximately

$$b_1 \approx 6.90633\,, \quad b_2 \approx 1.06742\,, \quad b_3 \approx 1.53826\,, \quad b_4 \approx -1.18881\,.$$

7 Concluding remarks

The ideas presented here demonstrate how under the assumptions of the existence of a periodic travelling wave satisfying the conditions

$$\mathscr{B}_k = \mathscr{K}_4 = \mathscr{B}_m = 0\,, \quad \mathscr{K}_6 \neq 0\,,$$

one is able to obtain the equation (3) as a higher order dispersive correction to the linearised Whitham equations when they are degenerate. In the topic so far, it has been demonstrated how third and now fifth order dispersion arise in models. The equation (3) has been shown to admit many interesting solutions [4]. Although unlikely, the methodology provided here can be extended to obtain seventh order dispersion and beyond, however due to the rarity of such systems that would support such dispersion it would be difficult to find an example of where these new models arise.

In typical derivations of fifth order KP models, the third order dispersion is considered weak instead of vanishing, and this can be incorporated into the modulational approach. If it is instead the case that $\mathscr{K}_4 = \varepsilon^2 \mu$ for some $\mu = \mathcal{O}(1)$, the analysis proceeds as normal but the weak unfolding of the third order dispersion instead appears in the final equation, giving

$$\left((\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_{kk} q q_X + \mu q_{XXX} + \mathscr{M} q_{XXY} + \mathscr{K}_6 q_{XXXXX}\right)_X + \mathscr{C}_m q_{YY} = 0,$$

along with the remaining criterion established within the paper.

The present work considers the case of only one conservation law, however there are many interesting problems that possess at least two of them, such as stratified hyrodynamics. It is expected that with the relevant adjustments to the theory of this paper a similar result may hold in such cases, in a similar way to how the KdV has been shown to arise in these scenarios [11].

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Data statement

Details of the data associated with this paper and how to request access are available from the University of Surrey publications repository: http://epubs.surrey.ac.uk/.

Appendix

The multisymplectic operators for the first example are

$$\mathbf{M}=\begin{pmatrix}0&-1&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix},\quad \mathbf{J}=\begin{pmatrix}0&0&-1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&-1&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&-1&0&0\\0&0&0&0&-1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix},$$

$$\mathbf{K}=\begin{pmatrix}0&0&0&-1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&-1&0&0&0&0\\0&0&0&0&-1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\end{pmatrix},$$

and for the second read

$$\mathbf{M}=\begin{pmatrix}-\boldsymbol{\sigma}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix},\quad \mathbf{J}=\begin{pmatrix}\mathbf{0}&-\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&-\mathbf{I}&\mathbf{0}\\\mathbf{I}&\mathbf{0}&\mathbf{0}&-\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&-\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix},$$

$$\mathbf{K}=\begin{pmatrix}-\beta\boldsymbol{\sigma}&\mathbf{0}&-\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&-\mathbf{I}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&-\mathbf{I}&\mathbf{0}&\mathbf{0}\\\mathbf{I}&\mathbf{0}&\mathbf{0}&-\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix},$$

with each bold entry being a 2×2 matrix and

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

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