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Reappraisal of Whitham's 1967 theory for wave-meanflow interaction in shallow water

Thomas J. Bridges [1] *& Daniel J. Ratliff* [2]

[1]*Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK*

[2]*Department of Mathematics, Physics and Electrical Engineering, Northumbria University, Newcastle upon Tyne, NE1 8ST, UK*

Abstract. The modulation equations for Stokes waves in shallow water coupled to wave-generated meanflow, derived in Whitham (1967), based on an averaged Lagrangian are revisited. Firstly, it is shown that they can be recast into two coupled classical shallow water equations, with modified gravity having the sign of the Whitham index: $\text{sign}(\omega_0'' \omega_2)$. Secondly, it is shown that the amplitude of the meanflow and amplitude of the wave are, in general, independent. Thirdly, the implications of the coalescing characteristics, whose unfolding is associated with the Benjamin-Feir instability, are studied.

1 Introduction

One of the first applications of Whitham modulation theory, based on an averaged Lagrangian, was to Stokes waves in shallow water coupled to meanflow WHITHAM [19]. Four coupled modulation equations were derived with two for the amplitude and mean velocity of the mean flow, coupled to a pair of equations for the amplitude and wavenumber of the wave component. The characteristics of the coupled problem and the unfolding of a double characteristic, and change of characteristic type, was shown to be associated with the onset of the Benjamin-Feir instability. Going forward, the approach outlined by Whitham has been widely used throughout the study of nonlinear waves to identify not only instabilities but to also understand the long-time evolution of such waves. The original paper, however, stops short of discussing this transition as the amplitude of these Stokes waves increases which has been shown to move to lower depth for larger waves(c.f. [12, 13, 8, 1]). Moreover, there are a number of simplifying assumptions within the analysis that only hold in specific cases and may not hold in general.

We revisit this problem, starting with a brief review of the formulation in [19]. Our main new observation is that systems consisting of a mean flow and a wave modulation can be recast into the form of coupled shallow water equations:

$$\begin{aligned}
h_t + (hu)_x &= \mathcal{F}_1(H, V)_x \\
u_t + uu_x + gh_x &= \mathcal{F}_2(H, V)_x \\
H_t + (VH)_x &= \mathcal{F}_3(h, u)_x \\
V_t + VV_x + g'H_x &= \mathcal{F}_4(h, u)_x,
\end{aligned} \tag{1.1}$$

where h, u are the mean depth and mean horizontal velocity, and H, V are the amplitude and group velocity of the wave component. g is the usual gravitational constant and g' is associated

with the Whitham stability index

$$\text{sign}(g') = \text{sign}(\omega''\omega_2)\,, \tag{1.2}$$

which governs the stability of the uncoupled wave component. Explicit expressions for the coupling terms, $\mathcal{F}_j$, $j = 1, \ldots, 4$, are given in §4 below.

While the appearance of a shallow water equation for the modulation of mean flow is expected, it is surprising that the wave component modulation also has the form of the classical shallow water equations. With the coupling, the characteristic type is no longer determined by the sign of g'. Our second observation is to clarify the independent role of the amplitude of the mean flow. In many analyses the bulk variation of the flow and the wave amplitude are directly related, as in the original work of Whitham (e.g. [19] and Section 16.9 of [20]), the Hasimoto-Ono equation [10], and the Davey-Stewartson equation [7], but other models, such as the Benney-Roskes system [3], retain this independence. Our analysis here justifies why they should be treated separately, lending additional insight into why such models may be more effective. Our third observation is to study further the implications of the coalescing characteristics within the water wave problem.

An outline of the paper is as follows. First the derivation of the averaged Lagrangian in [19] is reviewed in §2. Then, in sequence, the SWEs for pure mean flow (§3.1), pure wave (§3.2), and then coupled SWEs (§4) are derived. The coupled modulation equations are studied in §4.1 and §6, with particular attention to the emergence of coalescing characteristics. In §7 and in the Concluding Remarks (§8) some of the nonlinear implications associated with coalescing characteristics are discussed.

2   The averaged Lagrangian and reduction

Luke's variational principle [14], in two space dimensions and time, is

$$\delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L(\eta, \phi)\, \mathrm{d}x \mathrm{d}t = 0\,,$$

for the unknown free surface $\eta(x,t)$ and velocity potential $\phi(x,y,t)$, in an inviscid, irrotational, constant density fluid, with

$$L = \int_{-h_0}^{\eta} \left(\phi_t + \tfrac{1}{2}\left(\phi_x^2 + \phi_y^2 \right) + gy \right)\, \mathrm{d}y\,, \tag{2.1}$$

where g is the gravitational constant and h_0 the still water depth. That this variational principle delivers the governing equations and boundary conditions for irrotational and inviscid water waves is proved in LUKE [14] and in §13.2 of WHITHAM [20].[1]

The free surface is expressed in a Fourier series as

$$\eta(x,t) = N(\theta) = b + \sum_{m=1}^{\infty} a_m \cos(m\theta)\,, \tag{2.2}$$

with $a_1 := a$, and

$$\theta = kx - \omega t\,. \tag{2.3}$$

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[1]Historical note: the paper of Luke immediately preceded W67 in the same issue of JFM, and both papers were submitted on the same date.

The parameters a and b will play important roles as the representative amplitudes of the wave and mean flow respectively. The Fourier expansion of $\phi(x, y, t)$ is

$$\phi(x, y, t) = ux - \gamma t + \widehat{\phi}(\theta, y), \qquad \text{with} \qquad \widehat{\phi}(\theta, y) = \sum_{m=1}^{\infty} \frac{A_m}{m} \cosh\Big(mk(h_0 + y)\Big) \sin(m\theta). \quad (2.4)$$

Substituting the form of N and $\widehat{\phi}$ into L gives

$$L = \left(\tfrac{1}{2}u^2 - \gamma\right)(h_0 + N) + \tfrac{1}{2}gN^2 - (\omega - uk)\int_{-h_0}^{N} \widehat{\phi}_\theta \, dy + \int_{-h_0}^{N} \left(\tfrac{1}{2}k^2\widehat{\phi}_\theta^2 + \tfrac{1}{2}\widehat{\phi}_y^2\right) dy. \quad (2.5)$$

In [19], the Fourier expansion is carried to third order,

$$N(\theta) = b + a\cos(\theta) + a_2\cos(2\theta) + a_3\cos(3\theta) + \cdots,$$

and

$$\begin{aligned}
\widehat{\phi}(\theta, y) &= A_1\cosh\left(k(h_0 + y)\right)\sin(\theta) + \tfrac{1}{2}A_2\cosh\left(2k(h_0 + y)\right)\sin(2\theta) \\
&\quad + \tfrac{1}{3}A_3\cosh\left(3k(h_0 + y)\right)\sin(3\theta) + \cdots.
\end{aligned}$$

They are substituted into (2.5) and averaged using the standard averaging operator

$$\overline{f} := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta, \quad (2.6)$$

giving

$$\begin{aligned}
\mathscr{L} := \overline{L} &= \left(\tfrac{1}{2}u^2 - \gamma\right)(h_0 + b) + \tfrac{1}{2}gb^2 + \tfrac{1}{4}g\left(a^2 + a_2^2 + a_3^2\right) \\
&\quad - (\omega - uk)\overline{\int_{-h_0}^{N} \widehat{\phi}_\theta \, dy} + \overline{\int_{-h_0}^{N} \left(\tfrac{1}{2}k^2\widehat{\phi}_\theta^2 + \tfrac{1}{2}\widehat{\phi}_y^2\right) dy}.
\end{aligned} \quad (2.7)$$

Whitham [19] gives a detailed account of solving for A_1, A_2, A_3, a_2 and a_3, all in terms of a and b. Back substitution gives the reduced Lagrangian

$$\mathscr{L} = \left(\tfrac{1}{2}u^2 - \gamma\right)h + \tfrac{1}{2}gb^2 + \tfrac{1}{2}E\left\{1 - \frac{(\omega - uk)^2}{gk\tanh(kh)}\right\} + \tfrac{1}{2}E^2\frac{k^2 D_0}{g\tanh(kh_0)} + \cdots, \quad (2.8)$$

where $h = h_0 + b$ with h_0 the still water level,

$$E = \tfrac{1}{2}ga_1^2 \equiv \tfrac{1}{2}ga^2, \quad (2.9)$$

is the energy density and

$$D_0 = \frac{(9T_1^4 - 10T_1^2 + 9)}{9T_1^3}, \quad \text{with } T_1 = \tanh(kh_0). \quad (2.10)$$

The reduced Lagrangian (2.8) is equation (25) in [19]. Expand $h = h_0 + b$, under the assumption that $b \ll 1$ in the first term and the denominator in the third term, thereby generating the averaged Lagrangian to leading order in a and b

$$\begin{aligned}
\mathscr{L}(a, b, \omega, k, \gamma, u) &= \left(\tfrac{1}{2}u^2 - \gamma\right)(h_0 + b) + \tfrac{1}{2}gb^2 + +\tfrac{1}{2}kE\frac{(1-T_1^2)}{T_1}b \\
&\quad + \tfrac{1}{2}E\left\{1 - \frac{(\omega - uk)^2}{gkT_1}\right\} + \tfrac{1}{2}E^2\frac{k^2 D_0}{gT_1} + \cdots,
\end{aligned} \quad (2.11)$$

This is the averaged Lagrangian from which the modulation equations are derived, based on $\mathscr{L}_\omega$, $\mathscr{L}_k$, $\mathscr{L}_\gamma$, and $\mathscr{L}_u$, along with the constraints $\mathscr{L}_b = 0$ and $\mathscr{L}_a = 0$. This Lagrangian is valid up to quadratic order in E and b. An interesting analysis of the Lagrangian (2.7), valid for general finite-amplitude waves, is given in Whitham [21].

3 Limiting cases of the averaged Lagrangian

In order to best demonstrate how the averaged Lagrangian results in a set of coupled set of shallow water equations, it is first illuminating to consider the limiting cases in which the mean flow and wave action are considered separately. In particular, this section will demonstrate the process in which the classical modulation system of waves in the absence of mean flow can be recast into the desired shallow water with a suitable remapping of the modulation variables into one representing the group velocity and the other the energy density.

3.1 Mean Flow without the wave

In this section we look at what the Whitham modulation equations generate when the basic state is just pure mean flow; that is

$$\eta(x,t) = b \quad \text{and} \quad \phi(x,y,t) = ux - \gamma t.$$

In shallow water hydrodynamics this state is called a *uniform flow* with depth $h = h_0 + b$ and mean horizontal velocity u. Whitham called the parameters in basic states of this form pseudo-frequencies (γ) and pseudo-wavenumbers (u). However, they are mathematically of the same form as ordinary frequencies, producing modulation equations of the same form (a discussion is in §3.1 of [6]). The only difference is that averaging is not required in this case. The basic state can be substituted directly into (2.1) giving

$$\mathscr{L}(b, \gamma, u) = \left(\tfrac{1}{2}u^2 - \gamma\right)(h_0 + b) + \tfrac{1}{2}g(h_0 + b)^2. \tag{3.1}$$

The Whitham modulation equations are

$$\begin{aligned}
&\mathscr{L}_b = 0 \\
&\frac{\partial}{\partial x}\mathscr{L}_u - \frac{\partial}{\partial t}\mathscr{L}_\gamma = 0 \\
&\frac{\partial u}{\partial t} + \frac{\partial \gamma}{\partial x} = 0,
\end{aligned} \tag{3.2}$$

Substituting gives

$$\begin{aligned}
&\tfrac{1}{2}u^2 + g(h_0 + b) - \gamma = 0 \\
&\frac{\partial b}{\partial t} + \frac{\partial}{\partial x}(u(h_0 + b)) = 0 \\
&\frac{\partial u}{\partial t} + \frac{\partial \gamma}{\partial x} = 0.
\end{aligned} \tag{3.3}$$

Substitute the first equation into the third, thereby eliminating γ, and replace $h_0 + b$ by h noting that h_0 is constant,

$$h_t + (uh)_x = 0 \quad \text{and} \quad u_t + uu_x + gh_x = 0. \tag{3.4}$$

These equations are exactly the classical shallow water equations for the mean depth h and mean velocity u, delivered by modulation of the uniform flow. The first equation of (3.3) shows that γ can be interpreted as the total head, whereas $\mathscr{L}_u$ can be interpreted as the mass flux.

3.2 Wave without the mean flow

Now consider the averaged Lagrangian for the wave component only,

$$\mathscr{L}(a, \omega, k) = G(\omega, k)a^2 + \tfrac{1}{2}\Gamma a^4 + \cdots. \tag{3.5}$$

We will develop the modulation theory for the abstract Lagrangian (3.5) following §15.1 in [20] where (3.5) is used as a starting point, and then transform to the shallow water equations. For reference, G and Γ for (2.11) are

$$G(\omega, k) = \tfrac{1}{4}g\left(1 - \frac{\omega^2}{gkT_1}\right) \quad \text{and} \quad \Gamma = \frac{gkD_0}{4T_1},$$

although we will not need these explicit expressions in the theory.

The Whitham modulation equations, analogous to (3.2) are

$$\begin{aligned} \mathscr{L}_a &= 0 \\ \frac{\partial}{\partial x}\mathscr{L}_k - \frac{\partial}{\partial t}\mathscr{L}_\omega &= 0 \\ \frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} &= 0. \end{aligned} \tag{3.6}$$

Our main observation is that Whitham modulation equations (3.6), for the weakly nonlinear Lagrangian (3.5), are exactly the shallow water equations with modified gravity,

$$H_t + (HV)_x = 0 \quad \text{and} \quad V_t + VV_x + g'H_x = 0, \tag{3.7}$$

with V and H defined by

$$V = -\frac{G_k(\omega_0(k), k)}{G_\omega(\omega_0(k), k)} = c_g \quad \text{and} \quad H = |G_\omega|a^2, \tag{3.8}$$

where c_g is the group velocity of the linear waves, and the modified gravity

$$g' = \frac{1}{|G_\omega|}\omega_0''\omega_2 \quad \Rightarrow \quad \text{sign}(g') = \text{sign}(\omega_0''\omega_2), \tag{3.9}$$

with ω_2 the weakly nonlinear correction to the frequency (defined below in terms of Γ).

The frequency $\omega_0(k)$ is deduced from the dispersion relation of the linear problem defined by $G(\omega_0(k), k) = 0$, with the assumption $G_\omega(\omega_0(k), k) \neq 0$, required for the definition of H.

The familiar Whitham instability criterion (e.g. §15.1 in [20]) shows up in the shallow water equations (3.7) in the modified gravity. The shallow water equations (3.7) are ill posed ($g' < 0$) precisely when the Whitham modulation equations are elliptic.

There are a number of subtleties in going from (3.6) to the shallow water equations (3.7), and the details are now given. The nonlinear dispersion relation is obtained from the first equation, $\mathscr{L}_a = 0$, in (3.6),

$$G(\omega, k) + \Gamma a^2 = 0. \tag{3.10}$$

Writing the nonlinear dispersion relation in the conventional form

$$\omega = \omega_0(k) + \omega_2(k)a^2 + \cdots,$$

and substituting into (3.10) gives

$$G(\omega_0, k) = 0 \quad \text{and} \quad \omega_2 = -\frac{\Gamma}{G_\omega} \,. \tag{3.11}$$

The second equation in (3.6) gives conservation of wave action

$$0 = \frac{\partial}{\partial x}\mathscr{L}_k - \frac{\partial}{\partial t}\mathscr{L}_\omega = \frac{\partial}{\partial x}\left(G_k a^2\right) - \frac{\partial}{\partial t}\left(G_\omega a^2\right), \tag{3.12}$$

to leading order. Now substitute $a^2 = H/|G_\omega|$ and multiply by -1

$$\frac{\partial}{\partial t}\left(\frac{G_\omega}{|G_\omega|}H\right) + \frac{\partial}{\partial x}\left(-\frac{G_k}{G_\omega}\frac{G_\omega}{|G_\omega|}H\right) = 0\,. \tag{3.13}$$

With the assumption $G_\omega(\omega_0(k), k) \neq 0$ the ratio $G_\omega/|G_\omega|$ is constant and so with $V = -G_k/G_\omega$ this equation reduces to the first of (3.7).

Now multiply the third equation in (3.6) by G_ω

$$G_\omega k_t + G_\omega \omega_x = 0\,. \tag{3.14}$$

To transform the second term, differentiate the nonlinear dispersion relation (3.10) with respect to x

$$G_\omega \omega_x + G_k k_x + (\Gamma a^2)_x = 0\,.$$

Substitute into (3.14), and using the relation between ω_2 and Γ in (3.11), and so

$$k_t + V k_x + \frac{1}{G_\omega}\left(\frac{G_\omega}{|G_\omega|}\omega_2 H\right)_x = 0\,.$$

Now multiply by dV/dk

$$\frac{dV}{dk}k_t + V\frac{dV}{dk}k_x + \frac{1}{G_\omega}\frac{dV}{dk}\left(\frac{G_\omega}{|G_\omega|}\omega_2 H\right)_x = 0\,. \tag{3.15}$$

The first two terms simplify to $V_t + V V_x$. To simplify the third term requires some calculation. First, the expression for dV/dk is

$$\frac{dV}{dk} = \left(-\frac{G_k(\omega_0(k), k)}{G_\omega(\omega_0(k), k)}\right)_k = \frac{1}{G_\omega^3}\det\begin{bmatrix} G_{\omega\omega} & G_{\omega k} & G_\omega \\ G_{k\omega} & G_{kk} & G_k \\ G_\omega & G_k & 0 \end{bmatrix}. \tag{3.16}$$

But we need to relate this to $\omega_0''(k)$ and that relationship is obtained by differentiating the linear dispersion relation $G(\omega_0(k), k) = 0$ with respect to k twice gives

$$G_\omega^3 \omega_0''(k) = \det\begin{bmatrix} G_{\omega\omega} & G_{\omega k} & G_\omega \\ G_{k\omega} & G_{kk} & G_k \\ G_\omega & G_k & 0 \end{bmatrix}. \tag{3.17}$$

Comparing (3.16) with (3.17) gives

$$\frac{dV}{dk} = \omega_0''(k)\,,$$

6

and so

$$V_t + VV_x + \frac{1}{G_\omega}\omega_0''(k)\left(\frac{G_\omega}{|G_\omega|}\omega_2 H\right)_x = 0\,. \tag{3.18}$$

Noting that $G_\omega/|G_\omega|$ is constant and that ω_2 is treated as a constant at this order, gives the second of the shallow water equations in (3.7).

The above result is general, showing that Whitham modulation theory applied to any weakly nonlinear wave, with averaged Lagrangian in the form (3.5), leads to the shallow water equations. Ironically, one can conclude from this result that the WMEs show that the modulation of weakly nonlinear deep water waves is governed by the shallow water equations albeit with $g' < 0$ and so ill-posed. However, with the addition of surface tension there are parameter values where $\omega_2\omega_0''(k) > 0$ (see Figure 1 in [9]). In this case $g' > 0$ and so the WMEs are indeed a shallow water model for modulation of deep water capillary-gravity Stokes waves.

Unfortunately, the shallow water form of the WMEs does not appear to carry over to fully finite amplitude waves. In §7 of WHITHAM [21] (see also §15.2 in [20]), it is shown, by taking a Legendre transform of the averaged Lagrangian, that the Whitham modulation equations can be cast in the form

$$I_t + \frac{\partial}{\partial x}J(k,I) = 0 \quad \text{and} \quad k_t + \frac{\partial}{\partial x}\omega(k,I) = 0\,,$$

where $J = H_k$ and $H(k,I)$ is the Hamiltonian function (Legendre transform of $\mathscr{L}$). These equations are close but not exactly of the form of shallow water equations.

4   Wave-meanflow coupling

We have shown that modulation of mean flow on its own satisfies a shallow water equation with conventional gravity (3.4) and modulation of the wave component on its own also satisfies a shallow water equation with a modified gravity (3.7). We now consider the coupled problem treated in [19] and show that the modulation equations there are coupled shallow water equations.

The coupled Whitham modulation equations are obtained from

$$\mathscr{L}_b = 0$$
$$\frac{\partial}{\partial x}\mathscr{L}_u - \frac{\partial}{\partial t}\mathscr{L}_\gamma = 0 \tag{4.1}$$
$$\frac{\partial u}{\partial t} + \frac{\partial \gamma}{\partial x} = 0\,,$$

and

$$\mathscr{L}_a = 0$$
$$\frac{\partial}{\partial x}\mathscr{L}_k - \frac{\partial}{\partial t}\mathscr{L}_\omega = 0 \tag{4.2}$$
$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0\,,$$

with the averaged Lagrangian given in (2.11). The derivatives needed for the first three equations,

to leading order, are

$$\begin{aligned}
\mathscr{L}_b &= \tfrac{1}{2}u^2 - \gamma + gb + \tfrac{1}{2}kE\frac{(1-T_1^2)}{T_1} \\
\mathscr{L}_\gamma &= -(h_0 + b) \\
\mathscr{L}_u &= u(h_0 + b) - \frac{(\omega + uk)}{gT_1}E \, .
\end{aligned}$$

(4.3)

Generalising the interpretation of the first equation in (3.3), setting $\mathscr{L}_b = 0$ gives a generalisation of total head for the coupled flow

$$\gamma = gb + \tfrac{1}{2}u^2 + \tfrac{1}{2}kE\frac{(1-T_1^2)}{T_1} \, .$$

Substitute this expression for γ into the third equation in (4.1). The second and third equation in (4.3) are inserted into conservation of wave action. Combining these two equations gives the mean flow equations forced by the wave, to leading order,

$$\begin{aligned}
b_t + (u(h_0 + b))_x &= -(E/c_0)_x \\
u_t + uu_x + gb_x &= -\big(B_0 E/(h_0 c_0)\big)_x \, ,
\end{aligned}$$

(4.4)

where

$$B_0 = c_g - \tfrac{1}{2}c_0 = \frac{\omega_0 h_0}{2}\left(\frac{1-T_1^2}{T_1}\right) \, , \quad c_0 = \frac{\omega_0}{k} \, .$$

(4.5)

We now consider the second triad of equations in (4.2). Firstly, solve $\mathscr{L}_a = 0$ for ω, taking the positive square root,

$$\omega = \omega_0 - uk + \frac{kB_0}{h_0}b + k^2\frac{D_0}{c_0}E + \cdots , \quad \omega_0(k) = \sqrt{gk\tanh(kh_0)} \, .$$

(4.6)

This then allows one to write, to leading order,

$$\begin{aligned}
\mathscr{L}_\omega &= -\frac{(\omega - uk)}{gkT_1}E = -\frac{E}{\omega_0} + \mathcal{O}(Eb, E^2) \\
\mathscr{L}_k &= \frac{(\omega - uk)}{gkT_1}\left\{u + \frac{\omega_0'(\omega - uk)}{\omega_0}\right\}E + \mathcal{O}(Eb, E^2) \approx c_g\frac{E}{\omega_0} + \mathcal{O}(Eb, E^2) \, .
\end{aligned}$$

(4.7)

and substitute into the third equation, which gives to leading order,

$$k_t + c_g k_x + \left(k^2\frac{D_0}{c_0}E\right)_x = -\frac{\partial}{\partial x}\left(\frac{kB_0}{h_0}b\right) - ku_x \, .$$

(4.8)

The second and third equations in (4.7) are inserted into conservation of wave action,

$$0 = \frac{\partial}{\partial x}\mathscr{L}_k - \frac{\partial}{\partial t}\mathscr{L}_\omega = \frac{\partial}{\partial t}\left(\frac{E}{\omega_0}\right) + \frac{\partial}{\partial x}\left(c_g\frac{E}{\omega_0}\right) = 0 \, .$$

(4.9)

The two equations (4.8) and (4.9) can be put into the form of shallow water equations by taking

$$H = \frac{E}{\omega_0} \quad \text{and} \quad V = c_g := u + \omega_0'(k) \, .$$

(4.10)

Conservation of wave action (4.9) then becomes

$$H_t + (VH)_x = 0 \, .$$

(4.11)

To transform the equation (4.8) into shallow water form, multiply (4.8) by $V_k = \omega_0''(k)$ and take ω_2 to be the coefficient of a^2 in (4.6). This sequence of computations transforms (4.8) into

$$V_t + VV_x + g'H_x = -kV_k u_x - V_k \frac{kB_0}{h_0} b_x - \left(\frac{kB_0}{h_0}\right)_k bV_x \,, \tag{4.12}$$

to leading order. Here terms of the form $(\cdot)V_x$ on the right hand side are neglected in [19] as it is argued that they are convective terms (this argument is discussed in [19] as "neglect of the $F\kappa_x$ term" in equation (51)). However, as we will show later in §6, this term contributes to the characteristics and influences the Benjamin-Feir stability boundary as the amplitude varies. The modified gravity is

$$g' = \frac{2\omega_0}{g}\, \omega_0'' \omega_2 \quad \Rightarrow \quad \text{sign}(g') = \text{sign}(\omega_0'' \omega_2)\,. \tag{4.13}$$

The sign of g' no longer controls the characteristic type since the equations are coupled. Analysis of the characteristics of the fully coupled problem are given in §6 below.

To summarise, the Whitham modulation equations (4.4), (4.11) and (4.12) are coupled shallow water equations for H, V, b, u,

$$\begin{aligned}
b_t + (u(h_0 + b))_x &= -(kH)_x \\
u_t + uu_x + gb_x &= -\left(kB_0 H/h_0\right)_x \\
H_t + (VH)_x &= 0 \\
V_t + VV_x + g'H_x &= -kV_k u_x - V_k \frac{kB_0}{h_0} b_x - \left(\frac{kB_0}{h_0}\right)_k bV_x\,.
\end{aligned} \tag{4.14}$$

The first pair are mean-flow equations forced by the wave component, and the second pair are wave component equations forced by the mean-flow.

Whitham simplifies these equations further by assuming that all the modulation parameters can be expressed in terms of a^2 as

$$b = \mathcal{O}(a^2)\,, \quad \gamma = \mathcal{O}(a^2)\,, \quad \text{and} \quad u = \mathcal{O}(a^2)\,. \tag{4.15}$$

The argument for the approximation on b is discussed in §4.1 below. With the approximations (4.15) the modulation equations (4.14) simplify to

$$\begin{aligned}
b_t + h_0 u_x &= -kH_x \\
u_t + gb_x &= -\frac{kB_0}{h_0}H_x - \frac{(kB_0)_k}{h_0 V_k}HV_x \\
H_t + c_g H_x + HV_x &= 0 \\
V_t + c_g V_x + g'H_x &= -kV_k u_x - V_k \frac{kB_0}{h_0} b_x\,.
\end{aligned} \tag{4.16}$$

The last term in the second equation can be neglected at this order as it does not affect the characteristic calculations in §6 but it is retained for comparison as Whitham includes it. The c_g on the left-hand side of the second pair is the constant c_g of the basic state. The equations (4.16) are equivalent, to equations (51)-(54) in [19].

4.1 Independence of the amplitude parameters b and a

The amplitude parameters a and b are independent small parameters. This can be seen by re-arranging the equations $\mathscr{L}_a = 0$ and $\mathscr{L}_b = 0$. Re-arranging $\mathscr{L}_b = 0$ in (4.3) gives

$$gb + \frac{gk}{4}\left(\frac{1}{T_1} - T_1\right)a^2 = \gamma - \tfrac{1}{2}u^2.$$

Similarly, re-arranging $\mathscr{L}_a = 0$ and taking the square root gives, to leading order

$$\frac{\omega}{\omega_0} = 1 + \frac{D_0}{2T_1}(ka)^2 + \frac{1}{2}\left(\frac{1}{T_1} - T_1\right)kb + \frac{ku}{\omega_0}.$$

Combining into one equation

$$\begin{bmatrix} \frac{D_0}{2T_1} & \frac{1}{2}\left(\frac{1}{T_1} - T_1\right) \\ \frac{1}{2}\left(\frac{1}{T_1} - T_1\right) & 2 \end{bmatrix}\begin{pmatrix} (ka)^2 \\ kb \end{pmatrix} = \begin{pmatrix} \frac{\omega}{\omega_0} - 1 - \frac{u}{c_0} \\ \frac{2\gamma k}{g} - \frac{ku^2}{g} \end{pmatrix}. \tag{4.17}$$

(kb) and $(ka)^2$ are independent since the determinant is

$$\det\begin{bmatrix} \frac{D_0}{2T_1} & \frac{1}{2}\left(\frac{1}{T_1} - T_1\right) \\ \frac{1}{2}\left(\frac{1}{T_1} - T_1\right) & 2 \end{bmatrix} = \frac{1}{8T_1^4}\left(9 - 12T_1^2 + 13T_1^4 - 2T_1^6\right).$$

and it is nonzero for $0 \leq T_1^2 \leq 1$. The equation (4.17) can be solved uniquely for a and b as functions of the modulation parameters

$$a := a(\omega, k, \gamma, u) \quad \text{and} \quad b := b(\omega, k, \gamma, u). \tag{4.18}$$

These expressions for a and b can then be substituted into the reduced Lagrangian (3.5), giving a Lagrangian dependent only on the modulation parameters

$$\mathcal{L}(\omega, k, \gamma, u) = \mathscr{L}(a(\omega, k, \gamma, u), b(\omega, k, \gamma, u), \omega, k, \gamma, u).$$

The Whitham modulation equations then reduce to the closed system

$$k_t + \omega_x = 0, \quad (\mathcal{L}_k)_x - (\mathcal{L}_\omega) = 0, \quad u_t + \gamma_x = 0, \quad (\mathcal{L}_u)_x - (\mathcal{L}_\gamma) = 0.$$

These equations can be much more complicated and more difficult to work with, than the equivalent equations that include the amplitudes. However, this approach – elimination of the amplitudes – is the more useful choice in some examples (e.g. [16, 5, 6]).

The above result highlights definitively that the amplitude parameters a and b are in general independent. However, implicitly in [19] and explicitly in §16.9 of [20] the two parameters are taken to be related. The relationship between b and a^2 is not true in general as shown above, but in equation (16.99) of [20], with additional assumptions, a relationship between b and a of the following form is derived

$$kb = -\frac{1}{2kh_0(1 - c_g^2/gh_0)}\left(\frac{2c_g}{c_0} - \frac{1}{2}\right)k^2a^2. \tag{4.19}$$

This relationship induces a constraint on modulation space. To see this, substitute (4.19) into (4.18) and eliminate b and a. The result is a constraint between the parameters,

$$F(\omega, k, \gamma, u) = 0.$$

This constraint defines a hypersurface in parameter space, and restricts the equations (4.14) to be valid only on the hypersurface. This restriction may be valid in the weakly nonlinear limit, but it does not carry over to higher order. More importantly it is not necessary.

5 Comparison with coupled NLS equations

We digress briefly to give another point of view on the "independence of amplitudes" question, using the modulation of the two-phase wavetrains of coupled nonlinear Schrödinger (CNLS) equations. Take the CNLS equations in the following form

$$
\begin{aligned}
2\mathrm{i}\frac{\partial \Psi_1}{\partial t} + \alpha_1 \frac{\partial^2 \Psi_1}{\partial x^2} + (\beta_{11}|\Psi_1|^2 + \beta_{12}|\Psi_2|^2)\Psi_1 &= 0 \\
2\mathrm{i}\frac{\partial \Psi_2}{\partial t} + \alpha_2 \frac{\partial^2 \Psi_2}{\partial x^2} + (\beta_{12}|\Psi_1|^2 + \beta_{22}|\Psi_2|^2)\Psi_2 &= 0,
\end{aligned}
\tag{5.1}
$$

where $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are complex-valued, and α_1, α_2 and $\beta_{11}, \beta_{12}, \beta_{22}$ are nonzero real parameters with $\beta_{11}\beta_{22} - \beta_{12}^2 \neq 0$. More general versions of CNLS exist, but this model will be sufficient to clarify our argument.

Consider the basic two-phase wavetrain solution,

$$
\Psi_1(x,t) = A_1 \mathrm{e}^{\mathrm{i}(k_1 x - \omega_1 t)} \quad \text{and} \quad \Psi_2(x,t) = A_2 \mathrm{e}^{\mathrm{i}(k_2 x - \omega_2 t)}.
\tag{5.2}
$$

Substitution into (5.1) gives the nonlinear dispersion relations

$$
2\omega_1 = \alpha_1 k_1^2 - \beta_{11}|A_1|^2 - \beta_{12}|A_2|^2 \quad \text{and} \quad 2\omega_2 = \alpha_2 k_2^2 - \beta_{12}|A_1|^2 - \beta_{22}|A_2|^2.
\tag{5.3}
$$

Rewrite as a matrix-vector equation to highlight the independence of the amplitudes

$$
\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{bmatrix} \begin{pmatrix} |A_1|^2 \\ |A_2|^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 k_1^2 - \omega_1 \\ \alpha_2 k_2^2 - \omega_2 \end{pmatrix}.
$$

This equation is the analogue of (4.17). With $\beta_{11}\beta_{22} - \beta_{12}^2 \neq 0$, it is clear that the amplitudes $|A_1|^2$ and $|A_2|^2$ are independent, and if a relationship between the amplitudes, $|A_2| = \sigma|A_1|$, exists then a constraint on wavenumber-frequency space would be introduced,

$$
(\beta_{12} + \beta_{22}\sigma)(\alpha_1 k_1^2 - \omega_1) = (\beta_{11} + \beta_{12}\sigma)(\alpha_2 k_2^2 - \omega_2).
\tag{5.4}
$$

This constraint is a hypersurface in $(\omega_1, \omega_2, k_1, k_2)$ space, which will not be satisfied in general. Moreover, this constraint would restrict the admissable parameter values in the modulation of the two-phase wavetrain (see below for WMEs).

Another reason this CNLS example is interesting is that the WMEs, modulating the two-phase wavetrain (5.2), are in fact equivalent to the coupled SWEs,

$$
\begin{aligned}
\frac{\partial h_1}{\partial t} + \frac{\partial(h_1 u_1)}{\partial x} &= 0 \\
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + g_1 \frac{\partial h_1}{\partial x} &= \alpha_1 \beta_{12} \frac{\partial h_2}{\partial x} \\
\frac{\partial h_2}{\partial t} + \frac{\partial(h_2 u_2)}{\partial x} &= 0 \\
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + g_2 \frac{\partial h_2}{\partial x} &= \alpha_2 \beta_{12} \frac{\partial h_1}{\partial x}.
\end{aligned}
\tag{5.5}
$$

with $g_1 = -\alpha_1 \beta_{11}$ and $g_2 = -\alpha_2 \beta_{22}$. These equations have almost the same form as the modulation equations in W67, with the main difference being the form of the coupling terms. A derivation of the modulation equations (5.5) is given in Appendix A.

6   Coalescing characteristics

The main application in W67 is to show that the system (4.16) has coalescing characteristics whose unfolding captures the Benjamin-Feir instability and its transition. Here we review that theory and discuss the nonlinear implications.

The system (4.16) is a system of first-order PDEs

$$\mathbf{U}_t + \mathbf{M}(\mathbf{U})\mathbf{U}_x = \mathbf{0}\,, \tag{6.1}$$

with

$$\mathbf{U} = \begin{pmatrix} b \\ u \\ H \\ V \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} u & h_0 & k & 0 \\ g & u & \frac{kB_0}{h_0} & \frac{(kB_0)_k}{h_0 V_k}H \\ 0 & 0 & c_g & H \\ \frac{kB_0}{h_0}V_k & kV_k & g' & c_g + \frac{(kB_0)_k}{h_0}b \end{bmatrix}\,. \tag{6.2}$$

The characteristics, denoted by C, are the eigenvalues of $\mathbf{M}(\mathbf{U})$ evaluated at a given state, which we can choose as $\mathbf{U}_0 = (b, 0, H, c_g)$ without loss of generality since the fluid velocity can be removed via a suitable Gallilean shift. These characteristics satisfy the quartic equation

$$
\begin{aligned}
\Delta(C, k, E, b) & := \Delta(C; a, b) := -\left(gh_0 + 2B\right) \\
& = (C^2 - gh_0)\left[(C - c_g)^2 - g'H\right] \\
& \quad - (gh_0 + 2B_0 C + B_0^2)\frac{k^2 V_k}{h_0}H - (C + B_0)(kB_0)_k\frac{k}{h_0}(C - c_g)H \\
& \quad - (C - c_g)\left[(c^2 - gh)\frac{(kB_0)_k}{h_0} + g(C - c_g)\right]b + \mathcal{O}(H^2, b^2, bH) = 0\,.
\end{aligned} \tag{6.3}
$$

This equation is exact to the order of approximation of the averaged Lagrangian (3.5). We are interested in the type (elliptic or hyperbolic) of the characteristics. The amplitude parameter b appears to the same order as the energy density H, as expected. Hence, we expect the mean level plays a role in the stability and characteristic speeds of the wave to some order in the analysis.

A complete analysis of the roots of this quartic requires computation. However, Whitham notes that in the limit as the amplitude of the wave goes to zero (limit $H \to 0$ here) two of the characteristics coalesce. When $H = 0$, the characteristics are

$$C = \pm\sqrt{gh_0} \quad \text{and} \quad C = c_g \text{ (multiplicity 2)}\,.$$

The first two characteristics are the shallow water modes, and will remain real for H small. The second two are coalesced characteristics. These latter characteristics can become complex, thereby making (6.1) elliptic, when H is perturbed away from zero.

To unfold the double characteristic let

$$C = c_g + \Upsilon\sqrt{H} + \mathcal{O}(H)\,, \tag{6.4}$$

and substitute into (6.3)

$$(c_g^2 - gh_0)\left[\Upsilon^2 H - g'H\right] - \left(gh_0 + 2B_0 c_g + B_0^2\right)\frac{k^2 V_k}{h_0}H = \mathcal{O}(H^{3/2})\,. \tag{6.5}$$

Equating the $\mathcal{O}(H)$ terms to zero gives

$$\Upsilon^2 = g' - \frac{(gh_0 + 2B_0c_g + B_0^2)}{gh_0 - c_g^2}\frac{k^2V_k}{h_0} = \frac{k^2\omega_0''(k)}{h_0}\left(kh_0D_0 - \frac{(gh_0 + 2B_0c_g + B_0^2)}{gh_0 - c_g^2}\right) \quad (6.6)$$

which agrees with equation (57) in [19] noting that $g' = k^3D_0V_k$ from (4.13). Whitham computes the sign of the right-hand side of (6.6) and shows that it is negative when kh_0 is greater than ≈ 1.363, which is the familiar Benjamin-Feir instability threshold [2, 4]. Hence for $H \ll 1$ the double characteristic at $C = c_g$ splits into two

$$C = c_g \pm \sqrt{\omega_0''(k)\omega_2^{eff}(k)H} + \mathcal{O}(H)\,, \quad \text{where} \quad \omega_2^{eff} = \frac{k^2}{h_0}\left(kh_0D_0 - \frac{(gh_0 + 2B_0c_g + B_0^2)}{gh_0 - c_g^2}\right) \quad (6.7)$$

The link of the elliptic characteristic type with instability can be seen by looking at the eigenvalues of the linearised time dependent problem associated with (6.1),

$$\mathbf{U}_t + \mathbf{M}(\mathbf{U}_0)\mathbf{U}_x = 0\,,$$

with a spectral ansatz

$$\mathbf{U}(x,t) = \widehat{\mathbf{U}}e^{\lambda t + \mathrm{i}\alpha x}\,,$$

where α the wavenumber of the perturbation, the eigenvalues are

$$\lambda_j = -\mathrm{i}C_j\alpha\,, \quad j = 1,2,3,4\,, \quad (6.8)$$

where C_j, $j = 1,2,3,4$, are the four characteristics. Let

$$C_1 = -\sqrt{gh_0} + \mathcal{O}(H)\,, \quad C_2 = +\sqrt{gh_0} + \mathcal{O}(H)\,,$$

and

$$C_3 = c_g + \Upsilon\sqrt{H} + \mathcal{O}(H)\,, \quad C_4 = c_g - \Upsilon\sqrt{H} + \mathcal{O}(H)\,,$$

with Υ the positive square root of (6.6).

The change of Υ from real to complex is reflected in the eigenvalues in (6.8) becoming a complex quartet. The position of the four eigenvalues is shown in Figure 1 below. The outer eigenvalues are the simple eigenvalues $\pm\mathrm{i}\sqrt{gh_0}\alpha$. The inner eigenvalues are the complex quartet. They are two double eigenvalues at $H = 0$ and then in the unfolding (kh_0 moved away from 1.363) they become complex for $kh_0 > 1.363$. This figure is consistent with Figure 2 in [4].

7   Nonlinear continuation of coalescing characteristics

In the previous section we looked at the effect of amplitude on the unfolding of the coalesced characteristics with kh_0 fixed. In this section we enforce the coalescence and look for a curve in (kh_0, a) space where the coalesence is continued to finite amplitude. In this case it is necessary to solve the "coalescing characteristic equations"

$$\Delta(C; a, k) = 0 \quad \text{and} \quad \frac{\partial}{\partial C}\Delta(C; a, k) = 0\,.$$
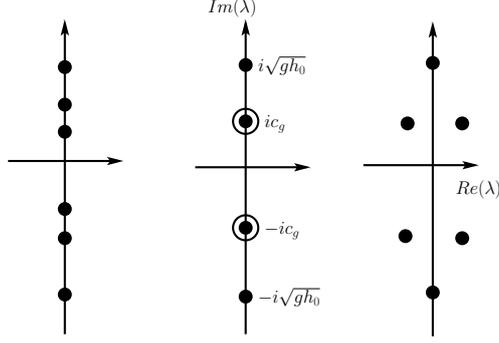
13

Figure 1: Collision of purely imaginary eigenvalues in the Whitham equations, with $\alpha = 1$. The left picture corresponds to $kh_0 < 1.363$ and the right picture corresponds to $kh_0 > 1.363$. The middle picture shows the case $H = 0$.
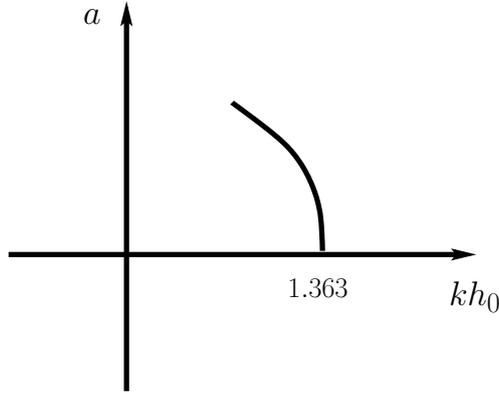

Figure 2: Curve along which $\det[\mathbf{M}(U_0) - C\mathbf{I}]$ has a double characteristic.

These two equations define a curve in the (kh_0, a) plane. At leading order, this curve is a parabola emerging from the point $(kh_0, a) = ((kh_0)^{crit}, 0)$ with $(kh_0)^{crit} = 1.363$. The leading order form of this curve can be obtained by expanding C, k, and E in terms of the amplitude a, resulting in the coalescing characteristic branch

$$C = c_g + \frac{1}{2}\chi g a^2 + \frac{1}{2}\frac{(kB_0)_k}{h_0}b + \mathcal{O}(a^3, b, ab) \;, \tag{7.9}$$

and

$$kh_0 = (kh_0)^{crit} - \frac{\chi^2}{2\omega_0''(\omega_2^{eff})_k} + \xi b + \mathcal{O}(a^3) \;,$$

where

$$\chi = -\frac{k(B_0 + c_g)}{2h_0(gh_0 - c_g^2)}\left((kB_0)_k + \frac{k\omega_0''(B_0c_g + gh_0)}{gh_0 - c_g^2}\right),$$

and

$$\xi = \frac{k}{h_0^2(gh_0 - c_g^2)(\omega_2^{eff})_{h_0}}\left[\frac{(c_g + B_0)(kB_0)_k^2}{2h_0\omega_0''} + \frac{k(B_0c_g + gh_0)}{gh_0 - c_g^2}\left((B_0c_g + gh_0) - (B_0 + c_g)(kB_0)_k)\right)\right] \;.$$

14

It becomes apparent that, to the order of the Lagrangian presented here, that the Benjamin-Feir stability boundary is (incorrectly) increasing with amplitude. This is rectified by including the higher order Stokes frequency correction of order a^4 given by $\omega_4^{eff} > 0$, which includes the effects of mean flow:

$$kh_0 = (kh_0)^{crit} - \frac{1}{2}\left(\omega_4^{eff} + \frac{\chi^2}{\omega_0''(k)}\right)\frac{gh_0 a^2}{(\omega_2^{eff})_k} + \xi b + \mathcal{O}(a^3)$$

which then gives the stability threshold as a decreasing function of wave amplitude, in agreement with the results of other works [11, 18], and is shown schematically in Figure 2. The role of the bulk variations are encoded into this higher order correction using (4.19) and an open question remains as to how the mean variation b contributes to this term, which could be discerned via a higher order analysis of the Lagrangian using the notions of this paper but is outside the scope of the work presented here. At each point on the curve there is a double characteristic. This is just the starting point for the curve. At finite amplitude there is the possibility of bifurcations in the (kh_0, a) plane, producing multiple coalescing characteristics. Even simple problems like the coupled nonlinear Schrödinger equations have more than one set of coalescing characteristics at some parameter values (e.g. Figure 5 in §5 of [5]).

8 Concluding remarks

In this paper we have revisited the classical Whitham approach to the modulation of gravity waves coupled to mean flow and recast these equation in the form of shallow water waves to aid in their interpretation and analysis. The abstraction of the form of the wave action in this setting is of benefit, which will likely allow for the treatment of Stokes waves when capillarity is considered or flexural terms are considered. Furthermore the Benjamin-Feir threshold is formulated using these shallow water principles, allowing for more accurate theories than that presented here (such as with further Stokes expansion terms) to be addressed and studied.

At coalescing characteristics the Whitham equations are no longer valid, but a remodulation with a slower time scale leads to a new modulation equation. It is shown in [6], that conservation of wave action, on the slower time scale, and after projection, leads to a modulation equation of the form

$$\mu U_{TT} + \kappa U U_X + \mathcal{K} U_{XXXX} = 0 \,, \tag{8.1}$$

where $X = \varepsilon x$ and $T = \varepsilon^2 t$ are slow space and time scales, U is a projection on wavenumber space, and the parameters μ, κ, and $\mathcal{K}$ are determined from the averaged Lagrangian. This modulation equation is a form of the two-way Boussinesq equation which has both nonlinearity and dispersion and has a multitude of interesting solutions. The story turns out to be far richer than this previous stipulation owing to a loss of genuine nonlinearity at the Benjamin-Feir transition for Stokes waves, invoking further nonlinear terms into the two-way Boussinesq:

$$\mu U_{TT} + \alpha_1 (U^3)_{XX} + \alpha_2 (2U U_T + U_X \partial_X^{-1} U_T) + \mathcal{K} U_{XXXX} = 0$$

where ∂_X^{-1} denotes the antiderivative with respect to X. This modulation equation was first derived in RATLIFF [15]. Analysis of this equation is more involved, and more interesting, than the two-way Boussinesq equation, and it will be discussed elsewhere [17].

One outcome of the study of the water wave problem from this Whitham perspective is that it highlights how the approach is unable to capture the high frequency instabilities reported in

other works [8]. These non-modulational stabilities are omitted from these analyses since there are finite-wavelength instabilities as opposed to the long-wavelength nature of modulation (and related) approaches.

— **Appendix** —

A   WMEs for two-phase wavetrains of CNLS

The CNLS equations (5.1) are generated by the Lagrangian variational principle

$$\delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L(\Psi_1, \Psi_2)\, \mathrm{d}x \mathrm{d}t = 0\,,$$

with Lagrangian density

$$
\begin{aligned}
L \;=\; & \mathtt{i}\left(\overline{\Psi}_1 \frac{\partial \Psi_1}{\partial t} - \Psi_1 \overline{\frac{\partial \Psi_1}{\partial t}}\right) - \alpha_1 \left|\frac{\partial \Psi_1}{\partial x}\right|^2 + \tfrac{1}{2}\beta_{11}|\Psi_1|^4 + \beta_{12}|\Psi_1|^2|\Psi_2|^2 \\
& + \mathtt{i}\left(\overline{\Psi}_2 \frac{\partial \Psi_2}{\partial t} - \Psi_2 \overline{\frac{\partial \Psi_2}{\partial t}}\right) - \alpha_2 \left|\frac{\partial \Psi_2}{\partial x}\right|^2 + \tfrac{1}{2}\beta_{22}|\Psi_2|^4\,.
\end{aligned}
$$

Here, a derivation of the WMEs in (5.5) is sketched following the theory in Ratliff [16].

Substitute the two-phase wavetrain (5.2) into the Lagrangian density and average over the phase,

$$
\begin{aligned}
\mathscr{L}\big(\omega_1, k_1, |A_1|^2, \omega_2, k_2, |A_2|^2\big) \;=\; & \big(2\omega_1 - \alpha_1 k_1^2\big)|A_1|^2 + \big(2\omega_2 - \alpha_2 k_2^2\big)|A_2|^2 \\
& + \tfrac{1}{2}\beta_{11}|A_1|^4 + \tfrac{1}{2}\beta_{22}|A_2|^4 + \beta_{12}|A_1|^2|A_2|^2\,.
\end{aligned}
$$

The derivatives are

$$
\begin{aligned}
\mathscr{L}_{\overline{A_1}} &= (2\omega_1 - \alpha_1 k_1^2)A_1 + \beta_{11}|A_1|^2 A_1 + \beta_{12}|A_2|^2 A_1 \\
\mathscr{L}_{\omega} &= 2|A_1|^2 \\
\mathscr{L}_{k} &= -2\alpha_1 k_1 |A_1|^2\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{L}_{\overline{A_2}} &= (2\omega_2 - \alpha_2 k_2^2)A_2 + \beta_{12}|A_1|^2 A_2 + \beta_{22}|A_2|^2 A_2 \\
\mathscr{L}_{\omega} &= 2|A_2|^2 \\
\mathscr{L}_{k} &= -2\alpha_2 k_2 |A_2|^2\,.
\end{aligned}
$$

Solving the first equation of each set gives the nonlinear dispersion relations (5.3). Conservation of wave action for each phase is

$$0 = (\mathscr{L}_{\omega_j})_t - (\mathscr{L}_{k_j})_x = \big(2|A_j|^2\big)_t + \big(2\alpha_j k_j |A_j|^2\big)_x\,,$$

or

$$\left(|A_j|^2\right)_t + \left(\alpha_j k_j |A_j|^2\right)_x = 0\,.$$

Noting that $c_g^{(j)} = \alpha_j k_j$, define

$$h_j = \tfrac{1}{2}|A_j|^2 \quad \text{and} \quad u_j = c_g^{(j)} = \alpha_j k_j\,.$$

Then conservation of wave action becomes

$$\frac{\partial}{\partial t}\left(h_j\right) + \frac{\partial}{\partial x}\left(u_j h_j\right) = 0\,, \quad j = 1, 2\,.$$

This result confirms the first and third equations in (5.5).

Now look at conservation of waves. For the $j = 1$ component,

$$\frac{\partial}{\partial t}\left(k_1\right) + \frac{\partial}{\partial x}\left(\omega_1\right) = 0\,.$$

Multiply by α_1 and substitute for ω_1 from the nonlinear dispersion relation (5.3),

$$\frac{\partial}{\partial t}\left(\alpha_1 k_1\right) + \alpha_1 \frac{\partial}{\partial x}\left(\tfrac{1}{2}\alpha_1 k_1^2 - \tfrac{1}{2}\beta_{11}|A_1|^2 - \tfrac{1}{2}\beta_{12}|A_2|^2\right) = 0\,.$$

Now substitute for $u_1 = \alpha_1 k_1$ and $2h_j = |A_j|^2$,

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + g_1' \frac{\partial h_1}{\partial x} = \alpha_1 \beta_{12} \frac{\partial h_2}{\partial x}\,,$$

with

$$g_1' = -\alpha_1 \beta_{11}\,.$$

This result confirms the second equation in (5.5). A similar argument reduces the second component of the conservation of waves, $(k_2)_t + (\omega_2)_x = 0$, to the fourth equation in (5.5). This completes the derivation of (5.5).

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