On the central quadric ansatz: integrable models and Painlevé reductions

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Abstract

It was observed by Tod [14] and later by Dunajski and Tod [1] that the Boyer-Finley (BF) and the dispersionless Kadomtsev-Petviashvili (dKP) equations possess solutions whose level surfaces are central quadrics in the space of independent variables (the so-called central quadric ansatz). It was demonstrated that generic solutions of this type are described by Painlevé equations P_{III} and P_{II}, respectively. The aim of our paper is threefold:
– Based on the method of hydrodynamic reductions, we classify integrable models possessing the central quadric ansatz. This leads to the five canonical forms (including BF and dKP).
– Applying the central quadric ansatz to each of the five canonical forms, we obtain all Painlevé equations $P_I - P_{VI}$, with P_{VI} corresponding to the generic case of our classification.
– We argue that solutions coming from the central quadric ansatz constitute a subclass of two-phase solutions provided by the method of hydrodynamic reductions.

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1 Introduction

This paper grew from an attempt to understand the construction of [14, 1] reducing the Boyer-Finley (BF) and the dispersionless Kadomtsev-Petviashvili (dKP) equations to Painlevé transcendents via the so-called 'central quadric ansatz'. This procedure applies to PDEs of the form

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0, \tag{1}$$

and consists of seeking solutions $u(x, y, t)$ in implicit form,

$$(x, y, t)M(u)(x, y, t)^T = 1, \tag{2}$$

where $M(u)$ is a 3×3 symmetric matrix of u. The level surfaces of such solutions, $u = const$, are central quadrics in the space of independent variables x, y, t. Remarkably, for both BF and dKP equations considered in [14, 1], differential equations for the matrix $M(u)$ reduce to Painlevé transcendents. Although the occurrence of Painlevé transcendents via similarity reductions of soliton equations is a well-known fact, their appearance in the context of multidimensional *dispersionless* integrable PDEs is, to the best of our knowledge, an entirely new phenomenon. In this paper we address the following questions:

- Classify integrable PDEs of the form (1). Our approach to this problem is based on the method of hydrodynamic reductions which, as demonstrated in [4], provides an efficient criterion for the classification of multidimensional dispersionless integrable equations. In particular, both BF and dKP equations are known to possess an infinity of hydrodynamic reductions. The classification is to be performed modulo (complex) linear changes of the independent variables x, y, t, as well as transformations $u \to \varphi(u)$, which constitute the equivalence group of our problem.

- Describe Painlevé reductions of integrable PDEs resulting from the above classification by applying the central quadric ansatz.

- Relate the central quadric ansatz to the method of hydrodynamic reductions.

To formulate our first result we introduce the symmetric matrix

$$V(u) = \begin{pmatrix} a' & p' & q' \\ p' & b' & r' \\ q' & r' & c' \end{pmatrix},$$

where prime denotes differentiation by u.

Theorem 1 *A PDE of the form (1) is integrable by the method of hydrodynamic reductions if and only if the matrix $V(u)$ satisfies the constraint*

$$V'' = (\ln \det V)'V' + kV, \tag{3}$$

for some scalar function k. Modulo equivalence transformations, this leads to the five canonical forms of nonlinear integrable models possessing the central quadric ansatz:

$$u_{xx} + u_{yy} - g(u)_{yy} - g(u)_{tt} = 0,$$

$$
\begin{aligned}
u_{xx}+u_{yy}+(e^u)_{tt}&=0,\\
(e^u-u)_{xx}+2u_{xy}+(e^u)_{tt}&=0,\\
u_{xt}-(uu_x)_x-u_{yy}&=0,\\
(u^2)_{xy}+u_{yy}+2u_{xt}&=0,
\end{aligned}
$$

here $g(u)=\ln(1-e^u)$. *Examples 2 and 4 are the familiar BF and dKP equations.*

We point out that the constraint (3), which implies $V''\in span\{V,V'\}$, means that the 'curve' $V(u)$ lies in a two-dimensional linear subspace of the space of 3×3 symmetric matrices. The classification of normal forms of such linear subspaces leads to the five canonical forms of Theorem 1. Equations listed in Theorem 1 are not new: in different guises, they have appeared in the classification of multidimensional integrable systems (see Sect. 2 for the proof of Theorem 1 and further discussion).

In Sect. 3 we apply the central quadric ansatz (2) to the five canonical forms of Theorem 1. Differentiating (2) implicitly and substituting into (1) one obtains equations for the matrix $M(u)$ which can be most naturally written in terms of the inverse matrix $N=-M^{-1}$:

$$
gN'=V \quad \text{where} \quad g^2\det N=\xi=const, \tag{4}
$$

see [14, 1]. For matrices V associated with the five canonical forms of Theorem 1, *generic* solutions of equations (4) reduce to the following Painlevé transcendents:

Case 1: General case of P_{VI};
Case 2: Special case of P_V reducible to P_{III};
Case 3: General case of P_V;
Case 4: General case of P_{II} with a reduction to P_I;
Case 5: General case of P_{IV}.

It is quite remarkable that the whole variety of Painlevé equations is contained in the pair of matrix equations (3), (4): setting $\xi=1$ and eliminating the scalar g we can rewrite them in the equivalent form,

$$
N'=V\sqrt{\det N}, \quad (V'/\det V)'=\kappa V.
$$

In Sect. 4 we show that, at least for the dKP equation, solutions coming from the central quadric ansatz constitute a particular subclass of two-phase solutions provided by the method of hydrodynamic reductions.

2 Classification of integrable PDEs of the form (1): proof of Theorem 1

Following the method of hydrodynamic reductions [7, 8, 4], we seek multi-phase solutions of equation (1) in the form

$$
u=u(R^1,\dots,R^N), \tag{5}
$$

where the phases $R^i(x,y,t)$ are required to satisfy a pair of compatible systems of hydrodynamic type,

$$
R^i_t=\lambda^i(R)R^i_x, \quad R^i_y=\mu^i(R)R^i_x, \tag{6}
$$

$R = (R^1, \ldots, R^N)$. We say that equations (6) provide an N-component hydrodynamic reduction of the equation (1). Note that the number of phases, N, is allowed to be arbitrary. The compatibility of equations (6) requires the following conditions [15],

$$\frac{\lambda^i_j}{\lambda^j - \lambda^i} = \frac{\mu^i_j}{\mu^j - \mu^i}, \tag{7}$$

$i \neq j$, here $\lambda^i_j = \partial_{R^j}\lambda^i$, $\mu^i_j = \partial_{R^j}\mu^i$. It was observed in [4] that the requirement of the existence of N-phase solutions parametrized by N arbitrary functions of one variable imposes strong constraints on the original system (1), and provides an efficient approach to the classification of multidimensional dispersionless integrable systems.

In the present context the integrability conditions can be derived as follows. Upon substitution of (5) and (6) into equation (1) we obtain that the characteristic speeds λ^i and μ^i obey the dispersion relation,

$$\Delta^i = a' + b'{\mu^i}^2 + c'{\lambda^i}^2 + 2\left(p'\mu^i + q'\lambda^i + r'\lambda^i\mu^i\right) = 0. \tag{8}$$

Differentiating this relation with respect to R^j, $i \neq j$, and taking into account the equations (7) and (8), we obtain

$$\lambda^i_j = \frac{1}{2}(\lambda^i - \lambda^j)\frac{\dot{\Delta}^i}{\Delta^{ij}}u_j, \quad \mu^i_j = \frac{1}{2}(\mu^i - \mu^j)\frac{\dot{\Delta}^i}{\Delta^{ij}}u_j, \tag{9}$$

where we introduced the notation

$$\dot{\Delta}^i = a'' + b''{\mu^i}^2 + c''{\lambda^i}^2 + 2\left(p''\mu^i + q''\lambda^i + r''\lambda^i\mu^i\right),$$
$$\Delta^{ij} = a' + b'\mu^i\mu^j + c'\lambda^i\lambda^j + p'(\mu^i + \mu^j) + q'(\lambda^i + \lambda^j) + r'(\lambda^i\mu^i + \lambda^j\mu^i).$$

Moreover, we obtain the following expressions for the second order derivatives of u,

$$u_{ij} = -\frac{\dot{\Delta}^{ij}}{\Delta^{ij}}u_i u_j, \tag{10}$$

where

$$\dot{\Delta}^{ij} = a'' + b''\mu^i\mu^j + c''\lambda^i\lambda^j + p''(\mu^i + \mu^j) + q''(\lambda^i + \lambda^j) + r''(\lambda^i\mu^i + \lambda^j\mu^i).$$

Equations (9) and (10) constitute the so-called generalised Gibbons-Tsarev system governing multi-phase solutions (hydrodynamic reductions) of equation (1). The conditions of compatibility of Eqs (9)-(10), $(\lambda^i_j)_k = (\lambda^i_k)_j$, $(\mu^i_j)_k = (\mu^i_k)_j$ and $(u_{ij})_k = (u_{ik})_j$, lead to polynomial expressions in the characteristic speeds $\lambda^i, \lambda^j, \lambda^k$ and μ^i, μ^j, μ^k which are required to vanish modulo the dispersion relation (8). This leads to the integrability conditions (3).

To solve Eq. (3) let us view $V(u)$ as a curve in the space of 3×3 symmetric matrices. Since the acceleration vector V'' is a linear combination of the position vector V and the velocity vector V', this curve must belong to a fixed two-dimensional linear subspace (the case when this subspace is one-dimensional corresponds to linear equations: in this case the central quadric ansatz was analysed in [2]). The classification of two-dimensional linear subspaces of the space of 3×3 symmetric matrices leads to the five canonical forms which are presented below in the format $span\{A, B\}$ for some particular choices of symmetric matrices A and B. Recall that pairs of symmetric matrices, $\tilde{A}$ and $\tilde{B}$, are classified by Jordan normal forms of the operator $\tilde{B}\tilde{A}^{-1}$, further subcases correspond to coincidences among eigenvalues of this operator. Choosing a suitable basis A, B in the subspace $span\{\tilde{A}, \tilde{B}\}$ leads to the five canonical forms mentioned above. In each of these cases one can set $V = A + f(u)B$ (to be precise, $V = g(u)A + f(u)B$, however, $g(u)$ can be set equal to 1 via an equivalence transformation $u \to \varphi(u)$). The substitution into the integrability condition (3) leads to simple ODEs for $f(u)$ which can be readily solved.

Case 1.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The substitution of $V = A + f(u)B$ into the integrability condition (3) leads to $k = 0$ and $f'' = f'^2\left(\frac{1}{f} + \frac{1}{f+1}\right)$, so that without any loss of generality one can take $f(u) = e^u/(1 - e^u)$. This leads to the first equation of Theorem 1. Setting $u = v_{yy} + v_{tt}$ we obtain the equation $e^{v_{xx}+v_{yy}} + e^{v_{tt}+v_{yy}} = 1$ or, equivalently, $e^{v_{xx}} + e^{v_{tt}} = e^{-v_{yy}}$, which first appeared in [5] in the classification of integrable PDEs of the form $F(v_{xx}, v_{yy}, v_{tt}) = 0$. It can be viewed as an integrable generalisation of the BF equation.

Case 2.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The substitution of $V = A + f(u)B$ into the integrability condition (3) leads to $k = 0$ and $f'' = f'^2/f$, so that without any loss of generality one can take $f(u) = e^u$. This leads to the BF equation.

Case 3.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The substitution of $V = A + f(u)B$ into the integrability condition (3) leads to $k = 0$ and $f'' = f'^2/(f+1)$, so that without any loss of generality one can take $f(u) = e^u - 1$. This leads to the third equation of Theorem 1 which is yet another integrable deformation of the BF equation: its equivalent forms, $u_{xy} = (e^u)_{tt} - c(e^u)_{xx}$ and $u_{xy} = (e^u)_{tt} + (e^u)_{ty}$, have appeared in [4, 3]. Both can be reduced to case 3 via appropriate (complex) linear transformations of the independent variables x, y, t. An alternative first order form of this equation is provided by the system $v_t = \frac{v_y + w_y}{v+w}$, $w_x = \frac{v_y + w_y}{v+w}$ which appeared in [6] in the classification of integrable Hamiltonian systems of hydrodynamic type in 2+1 dimensions. Setting $v = s_{xy}, w = s_{ty}$ we obtain the second order PDE $s_{xy} + s_{ty} = e^{s_{xt}}$. For $u = s_{xt}$ this gives the equation $u_{xy} + u_{ty} = (e^u)_{xt}$ which is also equivalent to case 3.

Case 4.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The substitution of $V = A + f(u)B$ into the integrability condition (3) leads to $k = 0$ and $f'' = 0$, so that without any loss of generality one can take $f(u) = u$. This leads to the equation $(uu_x)_x + 2u_{xy} + u_{tt} = 0$, which is equivalent to the dKP equation.

Case 5.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The substitution of $V = A + f(u)B$ into the integrability condition (3) leads to $k = 0$ and $f'' = 0$, so that without any loss of generality one can take $f(u) = u$. This leads to the last equation in Theorem 1. Setting $u = v_{xy}$ we obtain the equation $v_{xy}^2 + v_{yy} + 2v_{xt} = 0$. In this form, it appeared in [13] as the simplest case in the classification of integrable hydrodynamic chains satisfying the so-called 'Egorov' property.

This finishes the proof of Theorem 1.

3 Reduction to Painlevé equations

In this section we investigate systems (4),

$$gN' = V, \quad g^2 \det N = \xi = const,$$

for all canonical forms of Theorem 1, and reduce them to Painlevé transcendents. For the BF and dKP equations this was done in [14, 1]: we include these results for the sake of completeness. We find it more convenient to work with σ-forms of Painlevé equations as listed in [11], Appendix C.

Case 1: $u_{xx} + u_{yy} - \ln(1 - e^u)_{yy} - \ln(1 - e^u)_{tt} = 0$. We have

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-e^u} & 0 \\ 0 & 0 & \frac{e^u}{1-e^u} \end{pmatrix}, \quad N = \begin{pmatrix} F & \alpha & \beta \\ \alpha & G & \gamma \\ \beta & \gamma & H \end{pmatrix},$$

so that the first equation (4) implies that α, β, γ are constants, while $F' = \frac{1}{g}$, $G' = \frac{1}{(1-e^u)g}$, $H' = \frac{e^u}{(1-e^u)g}$. Thus, $G' = \frac{1}{1-e^u}F'$, $H' = \frac{e^u}{1-e^u}F'$. Setting $s = \frac{1}{1-e^u}$ one obtains $G_s = sF_s$, $H_s = (s-1)F_s$. Parametrizing these relations in the form $F = \sigma_s$, $G = s\sigma_s - \sigma$, $H = s\sigma_s - \sigma + \mu - \sigma_s$, and substituting into the second equation (4),

$$FGN - \alpha^2 H - \beta^2 G - \gamma^2 F + 2\alpha\beta\gamma = \xi F'^2,$$

(note that $F' = s(s-1)F_s$), we obtain an ODE for $\sigma(s)$,

$$\xi s^2(s-1)^2\sigma_{ss}^2 = \sigma_s(s\sigma_s - \sigma)(s\sigma_s - \sigma + \mu - \sigma_s) - \gamma^2\sigma_s - \beta^2(s\sigma_s - \sigma) - \alpha^2(s\sigma_s - \sigma + \mu - \sigma_s) + 2\alpha\beta\gamma.$$

This ODE belongs to the class

$$s^2(s-1)^2\sigma_{ss}^2 = a\sigma_s(s\sigma_s - \sigma)^2 + (s\sigma_s - \sigma)P_2(\sigma_s) + P_1(\sigma_s),$$

where P_2 and P_1 are polynomials in σ_s of the order two and one, respectively, such that the leading term of P_2 is $-a\sigma_s^2$. Modulo transformations $\sigma \to a_1\sigma + a_2$ this form is equivalent to the general case of P_{VI},

$$\sigma_s s^2(s-1)^2\sigma_{ss}^2 + 2[\sigma_s(s\sigma_s - \sigma) - \sigma_s^2 - \nu_1\nu_2\nu_3\nu_4]^2 = (\sigma_s + \nu_1^2)(\sigma_s + \nu_2^2)(\sigma_s + \nu_3^2)(\sigma_s + \nu_4^2).$$

see [11].

Case 2: $u_{xx}+u_{yy}+(e^u)_{tt}=0$. We have

$$V=\begin{pmatrix}1&0&0\\0&1&0\\0&0&e^u\end{pmatrix},\quad N=\begin{pmatrix}F&\alpha&\beta\\\alpha&G&\gamma\\\beta&\gamma&H\end{pmatrix},$$

so that the first equation (4) implies that α,β,γ are constants, while $F'=1/g,\ G'=1/g,\ H'=e^u/g$. Thus, $G'=F',\ H'=e^uF'$. Setting $e^u=s$, parametrizing these relations in the form $F=\sigma_s,\ G=\sigma_s+\mu,\ H=s\sigma_s-\sigma$, and substituting into the second equation (4) we obtain an ODE for $\sigma(s)$,

$$\xi s^2\sigma_{ss}^2=\sigma_s(\sigma_s+\mu)(s\sigma_s-\sigma)-\gamma^2\sigma_s-\alpha^2(s\sigma_s-\sigma)-\beta^2(\sigma_s+\mu)+2\alpha\beta\gamma.$$

This ODE belongs to the class

$$s^2\sigma_{ss}^2=(s\sigma_s-\sigma)P_2(\sigma_s)+P_1(\sigma_s),$$

where P_2 and P_1 are polynomials in σ_s of the order two and one, respectively. This is the special case of P_V which is reducible to P_{III}, see Case 3 below.

Case 3: $(e^u-u)_{xx}+2u_{xy}+(e^u)_{tt}=0$. We have

$$V=\begin{pmatrix}e^u-1&1&0\\1&0&0\\0&0&e^u\end{pmatrix},\quad N=\begin{pmatrix}F&G&\alpha\\G&\beta&\gamma\\\alpha&\gamma&H\end{pmatrix},$$

so that the first equation (4) implies that α,β,γ are constants, while $F'=(e^u-1)/g,\ G'=1/g,\ H'=e^u/g$. Thus, $H'=e^uG',\ F'=(e^u-1)G'$. Setting $e^u=s$, parametrizing these relations in the form $G=\sigma_s,\ H=s\sigma_s-\sigma,\ F=s\sigma_s-\sigma+\mu-\sigma_s$, and substituting into the second equation (4) we obtain an ODE for $\sigma(s)$,

$$\xi s^2\sigma_{ss}^2=\beta(s\sigma_s-\sigma+\mu-\sigma_s)(s\sigma_s-\sigma)-\gamma^2(s\sigma_s-\sigma+\mu-\sigma_s)-\sigma_s^2(s\sigma_s-\sigma)+2\alpha\gamma\sigma_s-\alpha^2\beta.$$

This ODE belongs to the class

$$s^2\sigma_{ss}^2=a(s\sigma_s-\sigma)^2+(s\sigma_s-\sigma)P_2(\sigma_s)+P_1(\sigma_s),\tag{11}$$

where P_2 and P_1 are polynomials in σ_s of the order two and one, respectively. In the case $a\neq 0$, transformations $\sigma\to a_1\sigma+a_2s+a_3,\ s\to b_1s$ bring any generic equation of the form (11) to the canonical P_V form,

$$s^2\sigma_{ss}^2=[\sigma-s\sigma_s+2\sigma_s^2+(\nu_1+\nu_2+\nu_3)\sigma_s]^2-4\sigma_s(\nu_1+\sigma_s)(\nu_2+\sigma_s)(\nu_3+\sigma_s).$$

The case $a=0$ leads to the special case of P_V which is reducible to P_{III},

$$s^2\sigma_{ss}^2=(s\sigma_s-\sigma)P_2(\sigma_s)+P_1(\sigma_s).$$

Case 4: $u_{xt}-(uu_x)_x-u_{yy}=0$. We have

$$V=\begin{pmatrix}-u&0&1/2\\0&-1&0\\1/2&0&0\end{pmatrix},\quad N=\begin{pmatrix}F&\alpha&G\\\alpha&H&\beta\\G&\beta&\gamma\end{pmatrix},$$

so that the first equation (4) implies that α, β, γ are constants, while $F' = -u/g,\ G' = 1/(2g),\ H' = -1/g$. Thus, $G' = -H'/2,\ F' = uH'$. Parametrizing these relations in the form $H = \sigma',\ H = \mu - \sigma'/2,\ F = u\sigma' - \sigma$, and substituting into the second equation (4) we obtain an ODE for $\sigma(u)$,

$$\xi(\sigma'')^2 = \gamma\sigma'(u\sigma' - \sigma) - \beta^2(u\sigma' - \sigma) - \sigma'(\mu - \sigma'/2)^2 + 2\alpha\beta(\mu - \sigma'/2) - \alpha^2\gamma.$$

This ODE belongs to the class

$$(\sigma'')^2 = a\sigma'(u\sigma' - \sigma) + b(u\sigma' - \sigma) + P_3(\sigma'), \tag{12}$$

where P_3 is a third order polynomial in σ'. In the case $a \neq 0$ one can eliminate b by a transformation $\sigma \to \sigma + su$. Further transformations $\sigma \to a_1\sigma + a_2,\ u \to b_1u + b_2$ bring any equation of the form (12) to the canonical P_{II} form,

$$(\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(u\sigma' - \sigma) + \delta = 0.$$

Similarly, for $a = 0,\ b \neq 0$, transformations $\sigma \to a_1\sigma + su + a_2,\ u \to b_1u + b_2$ bring any equation of the form (12) to the canonical P_I form,

$$(\sigma'')^2 + 4(\sigma')^3 + 2(u\sigma' - \sigma) = 0.$$

Case 5: $(u^2)_{xy} + u_{yy} + 2u_{xt} = 0$. We have

$$V = \begin{pmatrix} 0 & u & 1 \\ u & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} \alpha & F & G \\ F & H & \beta \\ G & \beta & \gamma \end{pmatrix},$$

so that the first equation (4) implies that α, β, γ are constants, while $G' = 1/g,\ H' = 1/g,\ F' = u/g$. Thus, $H' = G',\ F' = uG'$. Parametrizing these relations in the form $G = \sigma',\ H = \sigma' + \mu,\ F = u\sigma' - \sigma$, and substituting into the second equation (4) we obtain an ODE for $\sigma(u)$,

$$\xi(\sigma'')^2 = -\gamma(u\sigma' - \sigma)^2 + 2\beta\sigma'(u\sigma' - \sigma) - \sigma'^2(\sigma' + \mu) + \alpha\gamma(\sigma' + \mu) - \alpha\beta^2.$$

This ODE belongs to the class

$$(\sigma'')^2 = a(u\sigma' - \sigma)^2 + b\sigma'(u\sigma' - \sigma) + P_3(\sigma'), \tag{13}$$

where P_3 is a third order polynomial in σ'. In the case $a \neq 0$ one can eliminate b by a translation of u. Further transformations $\sigma \to a_1\sigma + a_2u,\ u \to b_1u$ bring any equation of the form (13) to the canonical P_{IV} form,

$$(\sigma'')^2 = 4(u\sigma' - \sigma)^2 - 4\sigma'(\sigma' + \nu_1)(\sigma' + \nu_2),$$

Similarly, for $a = 0,\ b \neq 0$, transformations $\sigma \to a_1\sigma + a_3,\ u \to b_1u + b_2$ bring any equation of the form (13) to the canonical P_{II} form,

$$(\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(u\sigma' - \sigma) + \delta = 0.$$

4 Central quadric ansatz versus hydrodynamic reductions

The main observation of this section is that solutions coming from the central quadric ansatz (2) can also be obtained as a particular case of two-phase solutions,

$$u = u(R^1, R^2),$$

where the phases $R^1(x,y,t),\ R^2(x,y,t)$ satisfy a pair of two-component systems of hydrodynamic type,

$$R^i_t = \lambda^i(R)R^i_x, \quad R^i_y = \mu^i(R)R^i_x, \tag{14}$$

$i = 1, 2$. The general solution of equations (14) is given by the so-called generalised hodograph formula [15],

$$\begin{aligned} x + \lambda^1(R^1, R^2)t + \mu^1(R^1, R^2)y &= \nu^1(R^1, R^2), \\ x + \lambda^2(R^1, R^2)t + \mu^2(R^1, R^2)y &= \nu^2(R^1, R^2), \end{aligned} \tag{15}$$

where ν^i are the characteristic speeds of yet another flow,

$$R^i_\tau = \nu^i(R)R^i_x, \tag{16}$$

which commutes with (14). Equations (15) define $R^1(x,y,t)$ and $R^2(x,y,t)$ implicitly. These relations can also be viewed as the equations of a line congruence (two-parameter family of lines) in the space of independent variables x, y, t, parametrised by R^1, R^2. As $u = u(R^1, R^2)$ is constant along the lines of this congruence, the level surfaces $u = const$ will automatically be ruled. Since a quadric carries two (complex) one-parameter families of lines, solutions constant on quadrics can therefore be viewed as two-phase solutions in 'two different ways': they are constant along two distinct congruences of lines.

Taking dKP equation as an example, we will now provide explicit derivation of the central quadric ansatz from the general two-phase solutions. Let us rewrite dKP as a two-component first order system,

$$u_t = uu_x + w_y, \quad u_y = w_x.$$

It will be more convenient for our purposes to work with variables u, w rather than R^1, R^2.

Proposition 1. *Two-phase solutions of the dKP equation,* $u(x,y,t)$ *and* $w(x,y,t)$*, are given by the implicit formulae*

$$x + (z_u + u)t = m_u, \quad y + z_w t = m_w, \tag{17}$$

where the functions $z(u,w)$ *and* $m(u,w)$ *satisfy the PDEs*

$$\begin{aligned} z_{uu} + z_w z_{uw} - z_u z_{ww} + 1 &= 0, \\ m_{uu} + z_w m_{uw} - z_u m_{ww} &= 0. \end{aligned} \tag{18}$$

Proof:

The analogues of equations (14) read

$$u_t = uu_x + z_x,\ w_t = q_x \qquad \text{and} \qquad u_y = w_x,\ w_y = z_x, \tag{19}$$

where the commutativity conditions, $u_{ty} = u_{yt}$ and $w_{ty} = w_{yt}$, imply that the functions $q(u,w)$ and $z(u,w)$ satisfy the relations $q_u = z_u z_w,\ q_w = z_w^2 + z_u + u$. Notice that the compatibility condition, $q_{uw} = q_{wu}$, implies the first relation (18). This equation for $z(u,w)$ first appeared in

[7] in the context of hydrodynamic reductions of Benney's moment equations. It is known as the Monge-Ampère form of the Gibbons-Tsarev system. The analogue of the third commuting flow (16) is

$$u_\tau = m_x, \; w_\tau = n_x, \tag{20}$$

where the commutativity conditions imply that the functions $m(u, w)$ and $n(u, w)$ satisfy the relations $n_u = z_u m_w, \; n_w = z_w m_w + m_u$. Their compatibility condition, $n_{uw} = n_{wu}$, implies the second relation (18). Rewriting equations (19) and (20) in matrix form we obtain

$$\begin{pmatrix} u \\ w \end{pmatrix}_t = \begin{pmatrix} z_u + u & z_w \\ z_u z_w & z_w^2 + z_u + u \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_x, \qquad \begin{pmatrix} u \\ w \end{pmatrix}_y = \begin{pmatrix} 0 & 1 \\ z_u & z_w \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_x,$$

and

$$\begin{pmatrix} u \\ w \end{pmatrix}_\tau = \begin{pmatrix} m_u & m_w \\ z_u m_w & z_w m_w + m_u \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_x,$$

respectively. In matrix notation, the generalised hodograph formula (15) is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} z_u + u & z_w \\ z_u z_w & z_w^2 + z_u + u \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ z_u & z_w \end{pmatrix} y = \begin{pmatrix} m_u & m_w \\ z_u m_w & z_w m_w + m_u \end{pmatrix}.$$

One can readily see that only two of these relations are independent, and they are equivalent to (17).

Remark. Differentiating (17) implicitly by x, y, t we obtain the relations

$$u_t = (z_u + u)u_x + z_w u_y, \quad w_t = (z_u + u)w_x + z_w w_y, \tag{21}$$

which manifest the invariance of the corresponding solutions under the conditional symmetry

$$X = (z_u + u)\partial_x + z_w \partial_y - \partial_t. \tag{22}$$

The presence of such symmetry is characteristic of two-phase solutions for equations in $2 + 1$ dimensions [9, 10]. Differentiating (21) with respect to x, y, t and eliminating the derivatives of z we obtain second order differential constraints for u and w characterising two-phase solutions,

$$\begin{aligned} \delta_{xy}^2 u_{tt} + \delta_{xt}^2 u_{yy} + \delta_{yt}^2 u_{xx} - 2\left(\delta_{xt}\delta_{xy}u_{ty} + \delta_{tx}\delta_{ty}u_{xy} + \delta_{yt}\delta_{yx}u_{tx}\right) = 0, \\ \delta_{xy}^2 w_{tt} + \delta_{xt}^2 w_{yy} + \delta_{yt}^2 w_{xx} - 2\left(\delta_{xt}\delta_{xy}w_{ty} + \delta_{tx}\delta_{ty}w_{xy} + \delta_{yt}\delta_{yx}w_{tx}\right) = 0, \end{aligned} \tag{23}$$

where the functions $\delta_{yt}, \delta_{tx}, \delta_{xy}$ are 2×2 minors of the Jacobi matrix $J = \partial(u, w)/\partial(x, y, t)$, i.e. $\delta_{yt} = u_y w_t - u_t w_y$, etc. Introducing the vector field $X = (\delta_{yt}, \delta_{tx}, \delta_{xy})$ one can rewrite these constraints in the equivalent form $XUX^T = 0, \; XWX^T = 0$ where U, W are the Hessian matrices of u and w. The constraints (23) completely characterize two-phase solutions of the dKP equation, providing an easy criterion to verify if a given solution can be constructed using this method. We emphasize that differential constraints (23) are *universal*: they govern two-phase solutions of *any* two-component quasilinear system. Indeed, the general solution of (23) is given by implicit relations

$$x + p(u, w)t = q(u, w), \quad y + r(u, w)t = s(u, w), \tag{24}$$

where p, q, r, s are arbitrary functions. In other words, functions u, w solve (23) if and only if they are constant along a two-parameter family of lines. This follows from the following equivalent geometric representation of the constraints (23):

$$L_X u = L_X w = 0, \quad XUX^T = XWX^T = 0.$$

The first two conditions mean that the functions u and w are constant along integral trajectories of the vector field X. This, in particular, implies that $X = (\delta_{yt}, \delta_{tx}, \delta_{xy})$. The last two conditions, which are equivalent to (23), mean that integral trajectories of X are asymptotic curves on the level surfaces $u = const$ and $w = const$. It remains to point out that if two surfaces intersect transversally along a curve which is an asymptotic curve on both of them, this curve must be a straight line (indeed, its osculating plane is tangential to both surfaces, and hence degenerates into a line). Adding the relations (24) to the dKP equation one obtains constraints for p, q, r, s resulting in (17).

Following [12] we will demonstrate that the system (18) possesses solutions expressible in terms of the Painlevé equation P_{II}.

Proposition 2. *The system (18) possesses solutions of the form*

$$z(u, w) = s_1(u) + s_2(u)e^w + s_3(u)e^{-w}, \quad m(u, w) = s_2(u)e^w - s_3(u)e^{-w},$$

where the functions $s_1(u), s_2(u), s_3(u)$ satisfy the ODEs

$$s_1' = 2s_2s_3 - u, \quad s_2'' = (2s_2s_3 - u)s_2, \quad s_3'' = (2s_2s_3 - u)s_3. \tag{25}$$

This expression for z was obtained in [12] based on the method of linear differential constraints. Substituting the above z, m into the equations (19) and eliminating w, one obtains an implicit relation of the form (2) for u,

$$(s_2s_3)'^2(t^2 - 1) - s_2s_3(x + 2s_2s_3t)^2 + (s_3s_2' - s_2s_3')(x + 2s_2s_3t)y + s_2's_3'y^2 = 0, \tag{26}$$

which shows that u is indeed constant on central quadrics. It remains to demonstrate that the system (25) is equivalent to P_{II}. Indeed, equations (25) possess a conservation law, $s_3s_2' - s_2s_3' = c = const$, which can be resolved in the form $s_2' = ps_2 + c/s_3$, $s_3' = ps_3$. This, in particular, implies the relation $(s_2s_3)' = 2ps_2s_3 + c$. Differentiating $s_3' = ps_3$ and using the last equation (25) we obtain $2s_2s_3 = p' + p^2 + u$. Substituting this expression for s_2s_3 into the previous relation we obtain the P_{II} equation,

$$p'' = 2p^3 + 2pu + 2c - 1.$$

Ultimately, all coefficients of the relation (26) are expressed in terms of p,

$$s_2s_3 = \frac{1}{2}(p' + p^2 + u), \quad s_3s_2' - s_2s_3' = c, \quad s_2's_3' = p^2s_2s_3 + cp = \frac{1}{2}p^2(p' + p^2 + u) + cp.$$

This gives a solution equivalent to the one constructed in [1].

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