Analysis of 2nd Order Differential Equations: Applications to Chaos Synchronisation and Control

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Analysis of 2nd Order Differential Equations:

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Abstract

In this thesis a number of open problems in the theory of ordinary differential equations (ODEs) and dynamical systems are considered. The intention being to address current problems in the theory of systems control and synchronisation as well as enhance the understanding of the dynamics of those systems treated herein. More specifically, we address three central problems; the determination of exact analytical solutions of (non)linear (in)homogeneous ODEs of order 1 and 2, the determination of upper/lower bounds on solutions of nonlinear ODEs and finally, the synchronisation of dynamical systems for the purposes of secure communication.

With regard to the first of these problems we identify a new solvable class of Riccati equations and show that the solution may be written in closed-form. Following this we show how the Riccati equation solution leads us quite naturally to the identification of a new solvable class of 2nd order linear ODEs, as well as a yet more general class of Riccati equations. In addition, we demonstrate a new alternative method to Lagrange's variation of parameters for the solution of 2nd order linear inhomogeneous ODEs. The advantage of our approach being that a choice of solution methods is offered thereby allowing the solver to pick the simplest option. Furthermore, we solve, by means of variable transforms and identification of the first integral, an example of the Duffing-van der Pol oscillator and an associated ODE that connects the equations of Liénard and Riccati. These fundamental results are subsequently applied to the problem of solving the ODE describing a lengthening pendulum and the matter of bounded controller design for linear time-varying systems.

In addressing the second of the above problems we generalise an existing Grönwall-like integral inequality to yield several new such inequalities. Using one of the new inequalities we show that a certain class of nonlinear ODEs will always have bounded solutions and subsequently demonstrate how one can numerically evaluate the upper limits on the square of the solution of any given ODE in this class. Finally, we apply our results to an academic example and verify our conclusions with numerical simulation.

The third and final open problem we consider herein is concerned with the synchronisation of chaotic dynamical systems with the express intention of exploiting that synchronisation for the purposes of secure transmission of information. The particular issue that we concern ourselves with is the matter of limiting the amount of distortion present in the message arriving at the receiver. Since the distortion encountered is primarily a due to the presence of noise and the message itself we meet our ends by employing an observer-based synchronisation technique incorporating a proportional-integral observer. We show how the PI observer used gives us the freedom to reduce message distortion without compromising on synchronisation quality and rate. We verify our results by applying the method to synchronise two parameter-matched Duffing oscillators operating in a chaotic régime. Simulations clearly show the enhanced performance of the proposed method over the more traditional proportional observer-based approach under the same conditions.

The structure of thesis is as follows: first of all we describe the motivation behind object of study before going on to give a general introduction to the theory of ODEs and dynamical systems. This lead-in also includes a brief history of the theory ODEs and dynamical systems, a general overview of the subject (as wholly as is possible without getting into the mathematical detail that is left to the appendices) and concludes with a statement of the scope of the thesis as well as the contributions to knowledge contained herein. We then go on to state and prove our main results and contributions to the solution of those problems detailed above starting with the solution of ODEs and ending with the synchronisation of dynamical systems. Finally, we draw conclusions and offer our ideas on the possible extensions of the work expounded upon in this thesis.

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Author's Publications

1. Johnson, P., Busawon, K. & Barbot, J.P. (2008) "Alternative Solution of the Inhomogeneous Linear Differential Equation of Order Two", *J. Math. Anal. App.*, 1(1) pp. 582-589.

2. Johnson, P. & Busawon, K. (2006) "Chaotic Synchronisation for Secure Communication using PI observers", *Chaos '06: 1st IFAC Conference on Analysis and Control of Chaotic Systems.* Reims, France 28-30 June. Oxford: Elsevier Ltd., pp. 205-210.

3. Busawon, K. & Johnson, P. (2005) "Analytical Solution of a Class of Linear Differential Equations", *WSEAS Trans. Math.,* 4(4), pp. 464-469.

4. Busawon, K. & Johnson, P. (2005) "Solution of a Class of Riccati Equations", *8th WSEAS International Conference on Applied Mathematics.* Tenerife, Spain 16-18 Dec. pp. 334-338.

5. Busawon, K. & Johnson, P. (2005) "Solution of Certain Classes of Linear Time Varying Second Order Systems", *Med. J. Measurement & Control,* 1(3), pp. 109-117.

6. Johnson, P., Busawon, K. & Danaher, S. (2004) "Refining Existing Numerical Integration Methods", *International Conference of Computational Methods in Sciences and Engineering.* Athens, Greece 19-23 Nov. Netherlands: VSP, pp. 234-238.

Notation

$\mathbb{R}$ - field of real numbers

$\mathbb{C}$ - field of complex numbers

$\mathbb{Z}$ - ring of integers

$\mathbb{R}^n$ - Euclidean n-space

$\in$ - is an element of

$\forall$ - for all

$\exists$ - there exists

$\nexists$ - there does not exist

$\blacksquare$ - end of proof

$\hat{\mathbf{x}}$ - fixed point

$|\cdot|$ - absolute value or determinant

$\|\cdot\|$ - l^2-norm on $\mathbb{R}^n$

$[a, b]$ - closed interval with endpoints a and b

(a, b) - open interval with endpoints a and b

$\mathbf{I}$ - identity matrix

z^* - complex conjugate of z

$\triangleq$ - is equal to by definition

i - imaginary unit

∇ - vector operator $\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T$

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Declaration

The author hereby declares that the original work given in this thesis has not been submitted for any other award and further, that the work is that of the author's alone. In addition, on those occasions where use is made of work that is not of the author's hand, every effort has been made to provide the relevant references.

Name: Patrick Johnson

Signature:

Date:

Chapter 1
Introduction

'Dynamical system' is a term which has such enormity of scope that it may refer to almost any element of the physical Universe or indeed the Universe itself. As a consequence, the problems and results in the theory of dynamical systems form a core element of all scientific disciplines. So just as the chemist may be concerned with the dynamics of reacting chemical substances and the biologist may study the population dynamics of feeding biomass, the physicist might query the motion of an iron pendulum suspended over a pair of magnets. However, dynamical systems theory is not concerned with the study of these kinds of problems in isolation since they have a common feature. That is, they all involve an inquiry into the time evolution of a system's global state. The theory therefore brings all these different systems under the scope of a single abstracted mathematical theory, the language of which may then be used to describe any one of these situations and all other conceivable systems besides. As it happens, the language of this theory is split into two 'dialects', each having been in place and studied on its own merits for hundreds of years. The first of these dialects is that of discrete mathematics and is reserved for those systems in which the variable representing time, and by consequence all time-dependant variables, assume discrete values. An example of a system commonly modelled in this dialect might be the population of an herd of Wildebeest. The second 'dialect' is that of Newton's and Leibniz' differential equations which is equipped to deal with systems wherein time, rather than being discretised, occupies a continuum. It is the latter of these two that will form the subject matter of this thesis.

In the continuous problem domain, the dynamics of a system's state variables may be captured by relationships between those variables and their derivatives with respect to certain other variables, in essence, a differential equation (DE). Ordinary differential equations (ODEs) and partial differential equations (PDEs) are as old as the calculus itself and as such have been the subject of a great deal of pure and applied mathematical research. However, the theory of DEs is generally considered to be an element of the latter as the problems considered are more often than not motivated by some physical problem i.e. a continuous dynamical system.

Though it could be argued that research in the field of DEs had its heyday in the 18th and 19th centuries it has recently become an enormously active research area once again. This phenomenon is in part due to Edward Lorenz' observation of what is known in the modern vernacular as 'chaotic' behaviour in a continuous dynamical system [57]. With this seminal event the subject of chaos theory was inaugurated into the field of applied mathematics (although, due to the obscurity of the journal in which Lorenz published his results this would not happen until nearly a decade later).

It is in keeping with this recent activity in the theory of DEs and continuous dynamical systems that this research project has been undertaken. The current open problems and breadth of applicability of theoretical results to physical problems makes this a very attractive area of study indeed. To be more specific, the research presented in this thesis is focused on, but not be restricted to, ODEs of order 2, namely

$$\ddot{x} = f(x, \dot{x}, t) \tag{1.1}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}$ is a function of t (time) and the overdots denote differentiation with respect to t. Examples of open problems related to equation (1.1) include the determination of stability, controllability, observability, general analytical solutions, efficient/reliable numerical solutions, periodicity, capacity for chaotic behaviour and more besides. Given the enormous scope for application of equation (1.1) and related DEs as a model of physical problems, it is easy to see that filling even a modest knowledge gap in one of the abovementioned problems can have far reaching and often profound ramifications for the sciences and engineering. Indeed, it is exactly this issue that is the motivation behind this research.

It ought to be pointed out that, since the manners in which one may tackle the problems of ODEs are manifold, this research is not concerned with one means alone, but a level of variety that will yield the most promise for applications. As far as particular applications are concerned there is a certain focus on matters pertaining to mechanical systems, control theory and chaos synchronisation though again, these considerations are not exclusive.

1.1 Models of Mechanical Systems as DEs

In order to help clarify, and indeed justify what has been said regarding the application of the theory of ODEs, and in particular ODEs of order 2, to the solution of physical problems, we shall now consider the manner in which one would model a mechanical system from two different (but related) standpoints. Indeed, the following discussions should convey some sense of the importance and ubiquity of ODEs of the form (1.1) when modelling physical systems.

1.1.1 Newtonian Mechanics

The modelling of classical mechanical systems in the Newtonian scheme is based on Newton's three laws of motion and centred in particular on the second. This second law constitutes one of the most famous equations in the history of physics and states that the net force $\mathbf{F}^T = \begin{pmatrix} F_x & F_y & F_z \end{pmatrix}$ acting on a body is exactly equal to the rate of change of momentum $\mathbf{p}^T = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}$ brought about by that force. More succinctly we have

$$\mathbf{F} = \dot{\mathbf{p}}$$

where the overdot denotes differentiation with respect to time t. Furthermore, allowing the mass m of the body to be static for the duration of the force's action we obtain the more familiar equation

$$\mathbf{F} = m\ddot{\mathbf{x}} \tag{1.2}$$

where $\mathbf{x}^T = \begin{pmatrix} x & y & z \end{pmatrix}$ represents the position of the body. Additionally, for conservative systems (systems in which the total energy is constant for all time) we know that

$$\mathbf{F} = -\nabla V\left(x, y, z\right)$$

where x, y and z represent the familiar Cartesian coordinates and V is the scalar potential energy function. Substituting this expression into (1.2) and considering each component of the acceleration separately yields the equations

$$\begin{aligned} m\ddot{x} &= -\frac{\partial V}{\partial x} \\ m\ddot{y} &= -\frac{\partial V}{\partial y} \\ m\ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}$$

and thus we have represented a classical mechanical system with the properties of conservative forces and constant mass in terms of three 2nd order ODEs. Consequently we can study this physical scenario using all the myriad techniques and theorems pertaining to the theory of ODEs.

Rather then dwell on the equations we have generated here we shall now proceed to consider another means of expressing mechanical problems in the language of ODEs.

1.1.2 Lagrangian Mechanics

Lagrangian mechanics is essentially a reformulation of Newtonian mechanics based on *Hamilton's principle* (or *the principle of least action*). The fact that the mechanics of Lagrange is recognised as having merits entirely of its own is due to the fact it allows us to find the equations of motion for any system in terms of an arbitrary set of generalised coordinates. However, our interest here is not relative ease of methods but the manner in which ODEs can arise as models of dynamical systems. The Lagrangian analogue of equation (1.2) for conservative systems reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{1.3}$$

where q_i is the i-th generalised coordinate of the system and L is the the *Lagrangian* as defined by relation

$$L = T - V$$

where T is the kinetic energy associated with each coordinate and V again represents the potential. For a body of constant mass m located (see remark at end of section) at the arbitrary coordinates $(q_1, q_2, \ldots, q_n)$ we would therefore write

$$L = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \ldots + \dot{q}_n^2\right) - V\left(q_1, q_2, \ldots, q_n\right)$$

Substituting this Lagrangian into (1.3) yields

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i} \tag{1.4}$$

for $i = 1, 2, \ldots, n$ and so again we have arrived at a set of ODEs that describe the dynamics of a classical mechanical system. In fact, one will notice (1.4) reduces to the equations of Newtonian mechanics derived earlier when the arbitrary coordinates are chosen to be the those of the Cartesian system.

Remark It should be noted than since the Lagrangian approach to dynamics is formulated from considerations of energy, the variables q_i need not necessarily represent location but could just as easily represent, for example, charge in which case the $\dot{q}_i$ would represent current. One would then be in a position to employ the Lagrangian formalism to model the flow of charge in electrical circuits.

1.2 Historical Notes

As previously noted, the theory of continuous dynamical systems, by which one of course means the theory of DEs, is as old as Newton's masterpiece, "Philosiphæ Naturalis Principia Mathematica", and as such has been a subject upon which the eye of scrutiny has been focused by many of history's greatest mathematicians. It is also a subject that has, during the last century, been experiencing a second golden age. Disciplines such as control theory, quantum mechanics, laser dynamics and the more recent advent of chaos theory has thrust nonlinear dynamics, and with it DEs, back into the limelight of applied mathematical research. In what follows we shall discuss how this history applies in the context of our particular focus, namely, dynamical systems of order 2 or 3 (see Appendix A.1 for definition of order) described by the ODE (1.1).

1.2.1 Simple Harmonic Oscillator

To begin with let us consider one of the most simple examples of a dynamical system resulting in an ODE of order 2; the idealised spring-mass system. In formulating the equations of motion for this problem we shall make use of the following assumptions:

1. Hooke's law for springs is universally observed by the system.
2. The system is conservative insofar as there are no damping effects such as friction at play.
3. The mass may be considered to be a point mass.

4. The mass is free to move along the x-axis only.

Bearing these assumptions in mind and denoting the rest position of the system by the coordinate $x = 0$ we write

$$F_x = -kx \tag{1.5}$$

where x denotes the position of the point mass, k is the spring constant and F_x is the net force parallel to the horizontal x-axis. Considering only the x component of equation (1.2) and substituting in (1.5) we get

$$\ddot{x} + \omega_0^2 x = 0 \tag{1.6}$$

where $\omega_0^2 = \dfrac{k}{m}$ and is related to the natural frequency of the system. The ODE (1.6) is known as the equation of the *simple harmonic oscillator* (SHO) and is one of the simplest elements of the family of ODEs defined by (1.1). Without going into the details at this stage we simply state here that there exist two linearly independent solutions of (1.6) and these are

$$\begin{aligned} x_1 &= \sin \omega_0 t \\ x_2 &= \cos \omega_0 t \end{aligned}$$

Let us now suppose that we know $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ i.e. we know the initial state of the system. We are now able to construct the general solution of (1.6). This solution can be verified to be

$$x = \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t + x_0 \cos \omega_0 t \tag{1.7}$$

Our equation of motion derived from Newtonian mechanics is now solved and the solution tells us that the position x of the mass m oscillates (hence SHO) over time with constant amplitude and a period of $\frac{2\pi}{\omega_0}$.

Though the SHO is both a good historical starting point and a nice illustrative example of the utility of the theory surrounding equation (1.1), the solution of this problem shows the behaviour of this simple linear system to be relatively uninteresting. In order to see more exotic behaviours we must move forward through history and consider some of the more complex examples of equation (1.1) that have been studied.

1.2.2 Nonlinear Oscillator

Classifying and analysing the sub-class of equation (1.1) in which f is a nonlinear function of x and/or $\dot{x}$ forms an enormous part of the theory of dynamical systems, namely; *nonlinear dynamics*. A matter of particular interest in nonlinear dynamics is the problem of determining the conditions under which a system will exhibit oscillatory behaviour. The term 'nonlinear oscillator' (NLO) is reserved for just such systems and they were common as far as the physics of radar, lasers and vacuum tubes were concerned. As compared to the SHO, NLOs are not restricted to mere constant amplitude periodic oscillation but may exhibit oscillations of far greater complexity. The NLOs that belong to the class of ODEs described by (1.1) generally assume one of the two following forms:

1. The *unforced, time-invariant* or *autonomous* system

$$\ddot{x} = f(x, \dot{x}) \tag{1.8}$$

which is a 2nd order dynamical system (see Appendix A.1 for a definition of the order of a dynamical system).

2. The *forced, time-variant* or *nonautonomous* system, which is defined by (1.1) itself and is a dynamical system of order 3.

A sub-class of (1.8) which is of particular historical significance is *Liénard's equation*

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0$$

which was classified as being an NLO under the conditions of *Liénard's theorem* [92]. More precisely, given the model of a system with a particular form of $f(x)$ and $g(x)$, the theorem states sufficient conditions that must be satisfied by $f(x)$ and $g(x)$ for the system to admit a *limit cycle*, a feature of dynamical systems that, when stable, allows for asymptotically periodic behaviour.

Dynamical systems described by equation (1.1) on the other hand are free to behave in even more interesting fashions as shall be made apparent later. Two particularly famous examples of forced nonlinear oscillators are the Duffing and van der Pol oscillators, each of which have been studied to exhaustion.

Rather than plunge into too much detail here however it will suffice for the moment to simply highlight importance of these two particular subclasses of equation (1.1) in the context of the whole theory of dynamics.

1.2.3 Stability and the n Body Problem

One of the most significant figures in the history of nonlinear dynamics is the French mathematician and mathematical physicist Henri Poincaré. Poincaré was a man who was equally at home within any mathematical discipline and it is to the great benefit and fortune of all subsequent dynamicists that Poincaré saw fit to turn his attention, upon more than one occasion, to some of the outstanding problems of late 19th c. nonlinear dynamics. One of the more historically important problems upon which he pondered was the n body *problem* *($n \geq 3$)* re-posed by King Oscar II of Sweden in 1887. The n body problem is of interest to us as it is in part concerned with determining the *stability* of planetary motion in the solar system. Practically all dynamical systems of interest that, like equation (1.8) do not depend explicitly on t, contain points which have the property that if the system's initial state is exactly that point then the state of the system will not change for all time. Such points are called the *fixed points* or *equilibrium solutions* of the system and if we were to perturb the system from this equilibrium state by a very small amount the state may either return to the fixed point or rush away from it. Roughly speaking, if the former occurs we would call the point *stable* and if the latter occurs we call the point *unstable*. As it turned out, a solution to the problem was not found (and only recently has one finally been found) and the prize fell to Poincaré who had offered in partial fulfillment of the problem some very important discoveries pertaining to the central question.

This matter of stability is central to practically all dynamical systems and is often the first thing one would attempt to determine when trying to classify and analyse a new system. The above account is a particularly interesting stability problem as it led Poincaré to foreshadow the advent of chaos theory some seventy years ahead of time. We shall return with more rigor to the matter of fixed points and stability later, but for now we shall continue our historical account with a brief discussion of the phenomenon of chaos.

1.2.4 Chaos

We have so far described quite a variety of behaviours accessible to the solution of equation (1.1). This accessibility is intrinsically linked to the particular sub-class of (1.1) that one considers. The more general the example one studies, the greater the variety of possible behaviours exist for that system. If we take equation (1.1) in its most general form therefore we should expect this to give us access to some relatively exotic solutions. In fact we have already briefly mentioned equation (1.1) in its most general form as the equation of a forced NLO. The forcing is provided by the explicit dependence of f on t and in these cases it commonly takes the form

$$f(x, \dot{x}, t) = g(x, \dot{x}) + h(t)$$

The explicit inclusion of this forcing term effects an increase in the order of the dynamical system from 2 to 3. This change is important since it is only in systems of order 3 and higher that chaos may be observed. To put this in perspective we note that the possible behaviours of the solutions of systems of order 2 include:

1. Remaining within a neighbourhood of a stable fixed point.

2. Diverging to infinity due to the repulsive action of an unstable fixed point.

3. Sitting on a periodic orbit for all time.

4. Remaining within a neighbourhood of a limit cycle or cycle graph (a variant of a limit cycle).

5. Diverging to infinity due to the repulsive action of an unstable limit cycle or cycle graph.

Of these five possibilities only 1, 3 and 4 constitute bounded solutions and in each of these instances the solution eventually settles down to small neighbourhood of a some point or periodic cycle. With chaotic behaviour however one has a bounded solution that settles down to none of these but instead wanders around aperiodically with no long term asymptotic solution. In short, chaotic motion is aperiodic, bounded and does not converge on a single state or orbit.

What we have described here is but one feature of chaos. Another prominent feature is the sensitivity with which the long term behaviour of a chaotic system depends on the system's initial state. This is generally termed *sensitive dependence on initial conditions* (SDIC). In fact this sensitivity is so high that initially close states diverge exponentially (up to a certain point as the states are bounded as time increases). The exact definition of chaos is not in fact completely agreed upon, though SDIC can generally be used to classify a behavioural pattern as chaotic.

This description of chaos, brief though it is, should convey how remarkably ground breaking the discovery of chaos was and why, as a consequence, dynamical systems theory has since flourished. Furthermore, the historical account here given should also make clear the importance of equation (1.1), sitting as it does on the border between classical and modern dynamics.

1.3 ODEs

To postpone plunging into the rigorous mathematics that underpin the theory of dynamical systems we shall first give a relatively non-mathematical account of some of the common approaches that exist for solving ODEs. This will help what follows fall into context and also make plain the relevance of the research presented in this thesis.

1.3.1 Analytical Solutions

When one is faced with a problem that has taken the form of an ODE the most desirable outcome of an analysis of that ODE would be an exact analytical solution. By an exact analytical solution of the n-th order ODE

$$\frac{d^n x}{dt^n} = f\left(\frac{d^{n-1}x}{dt^{n-1}}, \ldots, x, t\right) \tag{1.9}$$

we do of course mean finding the form of the function $x = g\,(t)$ where g is a combination of elementary or higher transcendental functions that satisfies the equation (1.9) as well as any additional constraints such as the boundary values or initial conditions (ICs). One will notice that it was an analytical solution that we were able to determine for the initial value problem (IVP) of the SHO. The analytical solution's great value derives from the fact that it expresses, all at once, the manner in which the dependent variable x depends on the independent variable t. An analysis of the function g that defines the solution allows one to determine with little difficulty, the presence or absence of all of those features of a dynamical system that are of central interest e.g. periodicity, stability, boundedness, etc.

The problem is however that an exact analytical solution for an arbitrary ODE of order n is not always possible to find (especially when $n > 1$ and f is nonlinear in its arguments). In fact, those classes of ODEs that are impossible to solve analytically outweigh those that can be solved by far (the reasons for this situation will be discussed later). Assuming that the ODE with which one is faced cannot be solved analytically one must resort to other methods.

1.3.2 Alternative Analytical Methods

It is possible in many examples to determine valuable information about the solution of a given ODE without having to determine the analytical solution itself. These methods obviously yield less information about the system at hand than the full solution would, but they are however more generally applicable than those methods that would seek to determine a general solution. Of these alternative methods there are two approaches; qualitative methods and approximate analytical solutions.

The qualitative methods attack directly those central questions pertaining to the dynamical system. These are generally questions regarding the stability of the system's fixed points, the existence of limit cycles and chaos, solution bounds, etc. These techniques are invaluable in obtaining a general picture of how one can expect a dynamical system to behave.

In contrast to these qualitative approaches, the approximate analytical solution method attempts to find a function, say $h(t)$, that can be known to approximate the actual unknown solution $g(t)$ for some interval of the domain. These approaches are equally popular in the literature as those of the qualitative methods and include the well known method of Frobenius as well as the multitude of perturbation techniques that have been developed over the centuries (see [52],[94],[33],[50],[88],[37],[42]).

Whilst these methods have proven to have a value all of their own they are commonly used in tandem with the final approach to attacking ODEs that we will discuss; the numerical solution.

1.3.3 Numerical Methods

In terms of general applicability to an arbitrary problem the numerical methods for ODEs are second to none. Though numerical solutions of ODEs have been around since the time of Newton, it was only in the last century with the advent of the computer that their true value was realised. In basic terms, a numerical solution is determined by transforming the continuous time problem into a discrete time problem i.e. the time variable t becomes the finite sequence of instants $t_0, t_1, \ldots, t_n$. This allows one to make numerical approximations of the variables, functions of variables and derivatives thereof at each point in the sequence $t_0, t_1, \ldots, t_n$. What this computation results in is a list of dependent variable coordinates, each of which is associated with a point in the time sequence. Plotting these values allows one to get a complete picture of the solution and thereby determine which dynamical properties are present in the system.

The power and generality of this general numerical approach does however come at a price and that is the matter of determining the accuracy of one's numerical solution and its consequent reliability. The most common means of determining reliability is to check the results of a numerical solution with those of a qualitative analysis as described above. If the numerical solution displays all the requisite behaviour of one's qualitative analysis, then it can be taken as reliable. In this way one will always obtain both a qualitative and quantitative view of the solution. In fact, it can be argued compellingly that this two-pronged attack, combining both the qualitative and the quantitative, is the most general and powerful approach to ODE analysis.

1.4 Research Aims, Scope of Thesis and Main Contributions

The discussions in this chapter serve several purposes. Firstly, to highlight the relevance of the theory of ODEs, and in particular equation (1.1), to the physical world. Secondly, to draw attention to some of the important unresolved problems associated with determining the properties of solutions of equation (1.1) and finally, to expose the details (as well as shortcomings) of the common analytical tools that are brought to bear upon these problems (see Appendix B for more on this issue).

What the forthcoming chapters constitute is the culmination of a research effort to realise the aim of this project, a statement of which will now be given.

Aim: It is the primary aim of this research to add to the existing knowledge relating to those classes of ODE subsumed by equation (1.1) by way of determining new analytically solvable classes and carrying out analyses that, in the absence of a solution, yield qualitative information about the solution. In addition, it is desired to, by those means developed herein or otherwise, contribute to those areas in which the classes of ODE considered find contemporary application; in particular, the control and synchronisation of dynamical systems.

1.4.1 Main Contributions

To begin with we identify a new class of solvable 1st order nonlinear ODEs by finding new conditions under which one can solve the Riccati equation

$$\dot{x} + p(t)\,x - x^2 = q(t) \tag{1.10}$$

when considered as an IVP where p and q are real continuous functions.

Since it is well known that (1.10) can be transformed into a 2nd order homogeneous linear ODE, we demonstrate how one can use the solutions of (1.10) to identify a new solvable class linear ODEs of the form

$$\ddot{x} + p(t)\,\dot{x} + q(t)\,x = 0 \tag{1.11}$$

where again, the context of the problem is that of an IVP and where p and q are real continuous functions.

Furthermore, since (1.10) is but a special case of the general Riccati equation, we use our new solutions of (1.10) to identify those conditions under which the Riccati equation

$$\dot{x} + a(t)\,x + b(t)\,x^2 = c(t)$$

may be solved when considered as an IVP where a, b and c are real continuous functions.

Having had success with solving (1.11) it was decided to divert attention to the matter of solving the inhomogeneous counterpart of (1.11). As a result, an alternative method to solving the linear ODE defined by

$$\ddot{x} + p(t)\dot{x} + q(t)x = r(t)$$

was developed using the solutions of (1.11) where p, q and r are real continuous functions.

A further extension of (1.11) was then investigated by introducing nonlinearities into the problem. More specifically, attention was focussed on those Liénard equations that assume the form

$$\ddot{x} + \left(C + \frac{d}{dx}\left(\frac{f(x)}{C}\right)\right)\dot{x} + f(x) = 0 \tag{1.12}$$

where f is real and continuous and C is a real constant. Of these ODEs, we concentrated on two in particular:

Firstly, we consider the homogeneous Duffing-van der Pol equation

$$\ddot{x} + \left(\alpha + \beta x^2\right)\dot{x} + \gamma x + \delta x^3 = 0 \tag{1.13}$$

with α, β, γ and δ being real constants, for which we extended Chandrasekar's solution [19]. This particular system was studied because, when forced, it can behave as a chaotic oscillator and as such may be used in various chaos-based approaches to secure communication.

The second such equation considered is the IVP

$$\ddot{x} + (\alpha + \beta x)\dot{x} + \gamma x + \delta x^2 = 0$$

where α, β, γ and δ are again real constants. The reason for considering this form of (1.12) is that it may be related to the problem of solving a Riccati equation and as such ties in with our earlier investigations.

Applications of the results referred to above were then investigated. As a consequence, an analysis and solution of a simple lengthening pendulum model was carried out. In addition to this, control applications were also considered with the linear ODE solutions being used to design bounded controllers for certain classes of linear time-varying systems.

Again with control applications in mind, we have investigated the stability of some general classes of 2nd order nonlinear systems. The upshot of this is several extensions of Grönwall-type integral inequalities with applications to the determination of solution bounds of ODEs of the form

$$\ddot{x} + \sum_{i=1}^{n} p_i(t) f_i(x) = 0$$

where the functions p and f are continuous and real. Finally, with more direct applications of potentially chaotic oscillators such as (1.13) in mind, we develop an observer-based chaotic masking synchronisation scheme for the express purpose of facilitating low noise secure communication. We use simulations to demonstrate the ideas presented on this topic using a subclass of (1.13) with a forcing term added.

1.4.2 Thesis Structure

The structure of the thesis is designed such that the two chapters which follow immediately deal with the central aim of this thesis, namely; to develop new results for the determination of properties of solutions of ODEs belonging to the class defined by (1.1). The direct application of the ideas contained in this work to scientific problems, though most prominent in Chapter 4 and parts of Chapter 2, will be a theme that is maintained throughout the thesis. This is because the applied aspect of the work given in Chapters 2 and 3 is implicit by virtue of the fact that equation (1.1) is commonly encountered in many areas of scientific research.

Chapter 2
Exact Analytical Solutions of ODEs

The search for exact analytical solutions of ODEs is one that started when Newton first used equation (1.2) to model the behaviour of bodies under the influence of forces. As soon as Newton and his contemporaries started generating ODEs as models of physical systems, functions were being sought that could be shown to satisfy them. However, as more and more complex systems yielded to the power of the new philosophy, more and more complex ODEs were encountered. DEs were soon classified by those methods that could be applied to solve them in terms of an *elementary function* (a formal definition of an elementary function will be given later). As it transpired, many broad classes of ODEs were left outside of the scope of these methods. What was clear to all however was that exact analytical solutions, once found, were extremely valuable as they described completely and at once, the dynamics of the system under study in terms of functions that one could analyse. Indeed, it is this analysis that allows one to determine all of the important dynamical properties of the system such as stability, boundedness, periodicity et cetera. As a consequence, an enormous effort on the part of 18th/19th c. mathematicians led to a great number of different solution schemes that served to broaden the scope of the classes of solvable ODEs and PDEs. Furthermore, when solutions could not be found, mathematicians and physicists developed means of finding solutions in terms of infinite series of functions (e.g. power series) which could subsequently be studied for convergence. If these series were convergent then the limiting function of the series could then be analysed to determine the properties of the system under consideration.

Moving forward to the present day, it would be fair to say that numerical schemes for solving DEs implemented on digital computers have superseded in popularity the often arduous task of finding an exact solution (in fact, there now even exist numerous computer programs that draw on a library of solved equations to provide an investigator with an exact solution). However, the fact remains that the breadth of solvable ODE/PDE classes will always benefit from expansion for it is the exact analytical solution alone that provides more information, in more detail and with greater accuracy than a solution obtained by any other means. Having said this, it should always be borne in mind that, an exact solution of an ODE provides an understanding of the dynamics of the modelled system rather than the system itself. This distinction between one's mathematical model and one's real system is an important one since it is the discrepancies between the two that can cast doubt upon the assertions of the solution and as such we would be wise to exercise a certain degree of incredulity when considering the conclusions of even an analytically derived solution.

In this chapter we will enter a brief discussion on the importance of the ODEs herein considered with an emphasis on applications followed by a summary of some of the central theorems and results pertaining to the solutions of various subclasses of equation (1.1). This will be succeeded by an exposition of the instances in which one can write an expression, either in terms of an elementary function or an integral of an elementary function, for the solution of (1.1) along with brief details of the methods used. We will then proceed to outline some of the more modern methods and recent results appearing in the literature on the determination of exact analytical solutions of ODEs. Finally, details of the author's results on solving the subclasses of (1.1) defined by the IVPs

$$\ddot{x}+p(t)\,\dot{x}+q(t)\,x = 0\,;\quad x(t_0)=x_0,\, \dot{x}(t_0)=\dot{x}_0 \tag{2.1}$$

$$\ddot{x}+p(t)\,\dot{x}+q(t)\,x = r(t)\,;\quad x(t_0)=x_0,\, \dot{x}(t_0)=\dot{x}_0 \tag{2.2}$$

and

$$\ddot{x}+f(x)\,\dot{x}+g(x)=0\,;\quad x(t_0)=x_0,\, \dot{x}(t_0)=\dot{x}_0 \tag{2.3}$$

where $x \in \mathbb{R}$ and $p, q, r, f, g : \mathbb{R} \to \mathbb{R}$, will be detailed along with a discussion of the importance and ubiquity of these equations.

2.1 Relevance of ODEs Considered

It is the intrinsic difficulty in treating ODEs defined in general terms that forces one's analysis and solution techniques to be concentrated upon a small subclass of the more general problem. As such, when starting out on such an endeavour as the one must pick which particular subclass is worthy of treatment. Such a choice is based almost entirely upon one's philosophical motivation, but for the applied mathematician it is the relevance of an ODE to science in general that is the driving factor; is the ODE commonly encountered in problems arising out of purely applied studies? For the ODEs (2.1)-(2.3) the answer is a resounding yes.

The linear ODEs (2.1) and (2.2) in particular are among the most common equations of mathematical physics arising as they do as in the study of mechanical oscillators, circuit analysis, ballistics, astronomy, quantum mechanics, cosmology and so on. However, the equation also appears in the modelling of chemical and biological phenomena. Indeed, the many reference textbooks (see [46],[12],[102],[40]) on the applications of ODEs are testament to the ubiquity of the (in)homogeneous linear ODE of order 2.

As to the importance of equation (2.3), mention has already been made in §1.2.2 to its admission of various nonlinear oscillators that are of particular importance in the study of radar, lasers and vacuum tubes. One particular subclass we consider herein, namely the Duffing-van der Pol equation, is most commonly encountered as the model of an electrical circuit (modified from that conceived by van der Pol [97] in 1927) but as is pointed out in [65], the equation also raises its head in the fields of physics, engineering, biology and neurology.

With these numerous and diverse applications relying upon the analysis and solution of the (2.1)-(2.3) to improve the understanding of these systems one could, without difficulty, argue for the sagacity of focussing one's efforts upon these particular ODEs.

2.2 Fundamental Results on Exact Solutions of ODEs

We will now detail some of the necessary mathematical preliminaries required for an examination of the methods of analytically solving ODEs. Here we will focus on the form of the solution as opposed to the actual solution or solution method. By proceeding in this way one can acquire a better appreciation of why the solution methods that we'll look at later actually work.

N.B. It will be assumed throughout this chapter that the functions p, q, r, f and g satisfy the hypotheses of Theorem A6.1 (see Appendix A.6) $\forall t \geq t_0$ and as such guarantee the existence and uniqueness of the solutions of (2.1), (2.2) and (2.3) over the interval $[t_0, \infty)$.

2.2.1 Elementary and Higher Transcendental Function Solutions

It was clear from as early as the 18th c. that there existed numerous DEs that refused to yield their solution to even the most tenacious and brilliant mathematical minds of the period. With efforts at solution in terms of elementary functions often going nowhere, a fundamental question persisted: *"Under what conditions can it be said that a given DE may be solved in terms of elementary functions?"*. A significant step towards answering this question came in the period 1833-41 with Joseph Liouville's numerous publications on the existence of an *elementary integral* of an elementary function. Before we proceed any further it is necessary to define (in simple terms) what exactly we mean by an elementary function/integral [61].

Definition 2.1 *An elementary function is one which can be constructed by means of any finite combination of the operations of addition, subtraction, multiplication, division, raising to powers, taking roots, forming trigonometric functions and their inverses and taking exponentials and logarithms.*

Definition 2.2 *An elementary integral is an integral of elementary function that yields an elementary function as its solution.*

As we shall see when we come to look at some of the more common ODE solution régimes, many solutions of ODEs and PDEs can be written in terms of an integral of some function $f(t)$. Expressing such a solution in terms of an elementary function clearly relies on $f(t)$ possessing an elementary integral. At the time of Liouville's work it was well known that, given the set S of all elementary functions and the differential operator $D = \frac{d}{dt}$, then $\forall f \in S$, $D(f) \in S$. The indefinite integral operator D^{-1} on the other hand has the property that, $\exists f \in S$ such that $D^{-1}(f) \notin S$. One of the better known examples of a function of this sort is the equation of the Gaussian probability distribution curve $f(t) = e^{-t^2}$. Here we have that

$$D\left(e^{-t^2}\right) = \frac{d}{dt}e^{-t^2} = -2te^{-t^2}$$

where clearly, $-2te^{-t^2} \in S$. On the other hand

$$D^{-1}\left(e^{-t^2}\right) = \int e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(t) + C$$

where C is a constant and erf (t) is the *error function* evaluated at t. The error function one should note, is an *higher transcendental function* rather than an elementary function and hence, is not an element of the set S. Liouville's Theorem of 1835 [54] provides conditions under which an elementary function possesses an elementary integral and thereby lends itself to determining when a given ODE/PDE possesses an elementary function solution.

With this theorem Liouville dramatically changed the theory of DEs. Before 1835 a mathematician struggling to find a solution of his DE had no good reason to believe that it was anything more than the difficulty of his problem that made the solution so elusive. Though he might eventually resort to solving his problem using an infinite series of elementary functions he could seldom say for certain that he had been forced to do so because the elementary function solution did not exist. For all they knew, the problem may have already been solved and was awaiting publication. After 1835 however, work could begin on polarising DEs into two distinct classes; those that admitted solution in terms of an elementary function and those that did not. Difficulty could now be separated from impossibility and the whole subject took a big step forward. Infinite series solutions of certain problems became accepted as the only workable solutions once a problem was proved to have no elementary function solution. As a result, non-elementary solutions of DEs became named functions and joined the list of higher transcendental functions which today includes Bessel functions, Legendre polynomials, Airy functions, (confluent) hypergeometric functions, the elliptic functions of Jacobi and Weierstraß, the Whittaker function and many more besides. Rather interestingly, finding conditions under which DEs admit elementary function solutions was a topic that received a great deal of attention once again after 1930 as a result of Joseph Fels Ritt's algebraic approach to the problem [79]. More recently, the likes of Rosenlicht [80], [81], [82], Singer [90] and van der Put [98] have continued to use Ritt's algebraic approach in their study of ODEs. One paper of foremost interest in this area is the 1983 work of Prelle and Singer [76]. In this paper the authors provide a semi-decision procedure for determining the elementary function solutions of the 1st order ODE

$$\dot{x} = \frac{P(x,t)}{Q(x,t)}$$

where the ratio of P and Q form a rational function in x and t. More recently however, the *Prelle-Singer method* was extended by Duarte *et al.* [32] to provide a similar procedure for the 2nd order ODE

$$\ddot{x} = \frac{P(x, \dot{x}, t)}{Q(x, \dot{x}, t)}$$

where the ratio of P and Q again form a rational function in x and t.

These two solution methods are extremely powerful and are guaranteed to find the solution of a given ODE provided it is expressible in terms of an elementary function. However, since the semi-decision procedures (which would be run on a computer) are, by definition, not self-terminating and there is no universal means of knowing whether or not an elementary solution exists for any given problem, they can prove less potent practically.

A similar question to that which Liouville considered was: *"Under what conditions can it be said that a 2nd order DE may be solved in terms of either elementary or higher transcendental functions?"*. The higher transcendental functions being defined as the elliptic functions and those non-elementary solutions of (2.1).

This question was most effectively addressed in the late 19th c. by three French mathematicians; E. Picard, P. Painlevé and B. Gambier. In particular they studied the general nonlinear 2nd order ODE [31]

$$A(x,t)\,\ddot{x} + B(x,t)\,\dot{x} + C(x,t)\,\dot{x}^2 + D(x,t) = 0 \tag{2.4}$$

where

$$\begin{aligned} A(x,t) &= \sum_{i=0}^{a} A_i(t)\, x^i \\ B(x,t) &= \sum_{i=0}^{b} B_i(t)\, x^i \\ C(x,t) &= \sum_{i=0}^{c} C_i(t)\, x^i \\ D(x,t) &= \sum_{i=0}^{d} D_i(t)\, x^i \end{aligned}$$

in which $a, b, c, d \in \mathbb{Z}$.

The generality of (2.4) is pointed out rather clearly by Davis when he notes that *"of (the) 249 examples of nonlinear differential equations of second order given by E. Kamke in the extensive list of such equations in the first volume of his Differentialgleichungen* [48] *shows that 132, or somewhat more than half, are subsumed under equation (2.4)"* [31].

Painlevé was led to classify the various subclasses of (2.4) by the nature of the singular points of their solutions. What he discovered was that, if an ODE of the form (2.4) was such that the solution's only movable singular points were poles, then the equation could be solved in terms of elementary or higher transcendental functions (details are deliberately omitted here as they would lead us too far astray). A 2nd order ODE satisfying this condition is said to possess the Painlevé property. Indeed, it was shown that exactly 50 distinct canonical forms of equation (2.4) possessed the Painlevé property and of these 44 could be solved in terms of elementary and higher transcendental functions. As a result, there soon followed the discovery of 6 previously unknown transcendents that served solve the remaining 6 canonical forms. These 6 functions are now known as the 1st, 2nd, 3rd, 4th, 5th and 6th Painlevé Transcendents (the 6th contains the other 5 as special cases). Of these, Painlevé discovered the first 3 while Gambier found the remainder. So, just as Liouville had done before them, Picard, Painlevé and Gambier provided a condition on the solvability of a class of ODEs in given terms.

2.2.2 Linear ODEs and the Wronskian

The current chapter is concerned in part with finding a new exactly solvable subclass of the homogeneous linear ODE of order 2 (2.1) and its equivalent inhomogeneous problem (2.2).

When attempting to solve a given ODE it is always worth bearing in mind any universal features of the solutions of the problem under consideration. For the two equations we are interested in there are some extremely helpful results of this sort to draw on. Before we give these results however we shall proceed to define some common terms.

Definition 2.3 *The Wronskian of a set of n functions $f_1, f_2, \ldots, f_n$ of t, denoted*

$$W\left(f_1, f_2, \ldots, f_n\right)$$

is defined by

$$W\left(f_1, f_2, \ldots, f_n\right) = \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ \frac{d^{n-1} f_1}{dt^{n-1}} & \cdots & \frac{d^{n-1} f_n}{dt^{n-1}} \end{vmatrix}$$

Definition 2.4 *The set of functions $f_1, f_2, \ldots, f_n$ of t are linearly independent on the interval $[t_0, t_1]$ if $\exists \tau \in [t_0, t_1]$ such that*

$$W\left(f_1, f_2, \ldots, f_n\right)_{t=\tau} \neq 0$$

Definition 2.5 *A particular solution of an ODE is one that satisfies the ODE for a particular set of ICs whereas the general solution satisfies the ODE for all ICs and as such will contain arbitrary constants.*

The following theorems are adapted from those appearing in [13] and [107].

Theorem 2.1 *The homogeneous 2nd order linear ODE (2.1) always has 2 linearly independent solutions. If $x_1(t)$ and $x_2(t)$ represent these solutions then the general solution of (2.1) is*

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

where c_1 and c_2 are arbitrary constants.

Theorem 2.2 *If f_1 and f_2 are a pair of continuous, linearly independent functions then there exists an unique homogeneous ODE of order 2 given by*

$$\frac{W(x, f_1, f_2)}{W(f_1, f_2)} = 0$$

whose general solution is

$$x(t) = c_1 f_1(t) + c_2 f_2(t)$$

where c_1 and c_2 are arbitrary constants.

Theorem 2.3 *The general solution of (2.2) is of the form*

$$x(t) = x_h(t) + x_p(t) \tag{2.5}$$

where $x_h(t)$ is the general solution of (2.1) while $x_p(t)$ is a particular solution of (2.2).

The above results tells us a great deal about the common properties of the solutions of 2nd order linear ODEs and each will be of significant utility later.

2.3 Analytical Methods for Solving ODEs

In the introduction to the present chapter it was stated that there exist a great many methods for analytically solving various classes of ODEs. Devising solution methods and identifying those ODEs to which the method could be successfully applied is today considered the classical approach. This way of thinking eventually gave way to a more general philosophy with the development of the Lie theory of ODEs which now stands alongside Prelle, Singer and Duarte's algorithmic approach as the modern methods for ODEs.

In what follows we shall take a look at some of these methods, both classical and modern, with an emphasis on those techniques pertinent to the author's contributions. In doing this we will both lay out the technical preliminaries of the novel work to follow whilst simultaneously putting it into context.

2.3.1 Classical Methods

Exact 1st Order ODEs

The first order DE

$$\dot{x} + \frac{f(x,t)}{g(x,t)} = 0$$

is said to be exact if

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t} \tag{2.6}$$

Provided that (2.6) holds, the solution may be determined from the relation

$$\int_{t_0}^{t} f(x,t)\,dt + \int_{x_0}^{x} g(x,t_0)\,dx = c$$

where x_0 and t_0 are arbitrary while c is governed by the value of $x(t_0)$ [42].

Variables Separable

The method of variables separable applies to nonlinear ODEs of order 1 that are of a separable (and consequently exact) form, that is

$$\dot{x} = f(x)\,g(t) \tag{2.7}$$

The method works by dividing through by $f(x)$ and integrating the result from t_0 to t to get

$$\begin{aligned}\int_{t_0}^{t} g(\tau)\,d\tau &= \int_{t_0}^{t} \frac{\dot{x}(\tau)}{f(x(\tau))}d\tau \\ &= \int_{x(t_0)}^{x(t)} \frac{dx}{f(x)}\end{aligned}$$

Denoting the indefinite integral $\int \frac{dx}{f(x)} = I(x)$ we write the solution as

$$x(t) = I^{-1}\left(I(x_0) + \int_{t_0}^{t} g(\tau)\,d\tau\right)$$

Integrating Factor

An integrating factor (IF) is a function throughout which one can multiply a given DE to transform it into an exact ODE. However, finding an IF for an arbitrary DE is often very difficult or impossible (see notes on Prelle-Singer-Duarte in §2). An example of an ODE for which we know the form of the IF is

$$\dot{x} + p(t)\,x = q(t) \tag{2.8}$$

The IF for this ODE is

$$h(t) = e^{\int_{t_0}^{t} p(\tau)d\tau} \tag{2.9}$$

for if we multiply equation (2.8) throughout by (2.9) we obtain the relation

$$\begin{aligned} e^{\int_{t_0}^{t} p(\tau)d\tau} q(t) &= e^{\int_{t_0}^{t} p(\tau)d\tau} \dot{x} + p(t)\, e^{\int_{t_0}^{t} p(\tau)d\tau} x \\ &= \frac{d}{dt}\left(e^{\int_{t_0}^{t} p(\tau)d\tau} x\right) \end{aligned}$$

Integrating both sides from t_0 to t we have the solution

$$x(t) = e^{-\int_{t_0}^{t} p(\tau)d\tau}\left(x(t_0) + \int_{t_0}^{t} e^{\int_{t_0}^{\lambda} p(\tau)d\tau} q(\lambda)\, d\lambda\right)$$

Transformation I: the Bernoulli Equation

Sometimes a variable transformation can turn a seemingly unsolvable ODE into a form for which a solution is always known. An example of just such an equation is the Bernoulli DE

$$\dot{y} + f(t)\,y = g(t)\,y^n \tag{2.10}$$

where $n \neq 1$ is a real constant. To solve this problem we introduce the variable transform

$$x = y^{1-n} \tag{2.11}$$

which when differentiated becomes

$$\dot{x} = (1-n)\,y^{-n}\dot{y} \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.10) yields the linear ODE

$$\dot{x} + (1-n) f(t) x = (1-n) g(t)$$

which may be solved using the IF $e^{(1-n)\int_{t_0}^{t} f(\tau)d\tau}$ to give

$$x(t) = e^{(n-1)\int_{t_0}^{t} f(\tau)d\tau} \left(x(t_0) + (1-n) \int_{t_0}^{t} e^{(1-n)\int_{t_0}^{\lambda} f(\tau)d\tau} g(\lambda)\, d\lambda \right)$$

The solution of the original problem is therefore

$$y(t) = e^{-\int_{t_0}^{t} f(\tau)d\tau} \left(y_0^{1-n} + (1-n) \int_{t_0}^{t} e^{(1-n)\int_{t_0}^{\lambda} f(\tau)d\tau} g(\lambda)\, d\lambda \right)^{\frac{1}{1-n}}$$

Transformation II: the Riccati Equation

Another example of an ODE that may yield its solution following a variable transform is the Riccati DE

$$\dot{y} + a(t) y + b(t) y^2 = c(t) \tag{2.13}$$

which reduces to the Bernoulli equation for $n = 2$ when $c(t) \equiv 0$. Here we make the variable transform

$$y = \frac{\dot{x}}{b(t) x} \tag{2.14}$$

which when differentiated becomes

$$\dot{y} = \frac{\ddot{x}}{b(t) x} - \frac{\dot{b}(t) \dot{x}}{b^2(t) x} - \frac{\dot{x}^2}{b(t) x^2} \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13) we see that

$$\ddot{x} + \left(a(t) - \frac{\dot{b}(t)}{b(t)} \right) \dot{x} - b(t) c(t) x = 0 \tag{2.16}$$

which is of the form (2.1); an equation for which there exist numerous solved classes.

The direct correspondence between (2.13) and (2.1) shall in fact be relied upon in the novel work presented later in this chapter.

As a final remark on Riccati equations we point out that, as Leonhard Euler was able to prove, if one knows a non-trivial particular solution of (2.13), call it $x_p(t)$, then the general solution is given by

$$x(t) = x_p(t) + \frac{1}{v(t)}$$

where $v(t)$ satisfies the ODE

$$\dot{v}(t) + (2r(t)\,x_p(t) - p(t))\,v(t) + r(t) = 0$$

which is of the form (2.8) and hence may be solved using an IF.

Quadrature

Solving ODEs by a simple quadrature can be performed when the ODE assumes the form

$$\ddot{x} = f(x) \tag{2.17}$$

To solve this equation we multiply through by $2\dot{x}$ thereby obtaining the relation

$$\begin{aligned} 2\dot{x}f(x) &= 2\dot{x}\ddot{x} \\ &= \frac{d}{dt}\left(\dot{x}^2\right) \end{aligned}$$

which can now be integrated with respect to t to give

$$\dot{x} = \sqrt{\dot{x}_0 + 2\int_{t_0}^{t} \dot{x}(\tau)\,f(x(\tau))\,d\tau}$$

or rather

$$\dot{x} = \sqrt{\dot{x}_0 + 2\int_{x(t_0)}^{x(t)} f(x)\,dx} \tag{2.18}$$

Equation (2.18), being an ODE of order one lower than (2.17), is known as the *first integral* of (2.17). Furthermore, equation (2.18) is of the form (2.7) and hence may be solved by the method of variables separable. The solution is

$$x(t) = I^{-1}\left(I(x_0) + (t - t_0)\right)$$

where the function I is defined by the indefinite integral

$$I(x) = \int \frac{dx}{\sqrt{2\int f(x)\,dx + \dot{x}_0}}$$

Undetermined Coefficients for Homogeneous ODEs

The method of undetermined coefficients can be used to solve equation (2.1) when $p(t) \equiv P$ and $q(t) \equiv Q$ where P and Q are constants (i.e. constant coefficients problem). If m_1 and m_2 are the roots of the equation

$$m^2 + Pm + Q = 0$$

then the solution is:

1.

$$x(t) = Ae^{m_1(t-t_0)} + Be^{m_2(t-t_0)}$$

where

$$\begin{aligned} A &= \frac{\dot{x}_0 - m_2 x_0}{m_1 - m_2} \\ B &= \frac{\dot{x}_0 - m_1 x_0}{m_2 - m_1} \end{aligned}$$

when $m_{1,2} \in \mathbb{R}$ and $m_1 \neq m_2$.

2.

$$x(t) = (A(t - t_0) + B)\, e^{m_1(t-t_0)}$$

where

$$\begin{aligned} A &= \dot{x}_0 - m x_0 \\ B &= x_0 \end{aligned}$$

when $m_{1,2} \in \mathbb{R}$ and $m_1 = m_2$.

3.

$$x(t) = e^{\alpha(t-t_0)} \left(A \sin(\beta(t - t_0)) + B \cos(\beta(t - t_0))\right)$$

where

$$\begin{aligned} A &= \frac{\dot{x}_0 - \alpha x_0}{\beta} \\ B &= x_0 \end{aligned}$$

when $m_{1,2} \in \mathbb{C}$ and $m_1 = m_2^* = \alpha + i\beta$.

Variation of Parameters

It was the great 18th c. French mathematician Joseph-Louis Lagrange who developed the method of variation of parameters to solve inhomogeneous linear ODEs. The method applies to linear ODEs of any order but because our interest lies with 2nd order equations we shall restrict our discussion to these alone.

Consider the inhomogeneous ODE (2.2), that is

$$\ddot{x} + p(t)\,\dot{x} + q(t)\,x = r(t)$$

Let us assume that we know the solution of the equivalent homogeneous problem defined by (2.1) and denote that solution

$$x_h(t) = Ax_1(t) + Bx_2(t)$$

where $x_1(t)$ and $x_2(t)$ are the linearly independent solutions of (2.1) (see Theorem 2.1) and constants A and B will be defined later. From Theorem 2.3 we know that the general solution of (2.2) is

$$x(t) = x_h(t) + x_p(t)$$

We now assume that $x_p(t)$ may be written in the the form

$$x_p(t) = u(t)\, x_1(t) + v(t)\, x_2(t)$$

Substituting this ansatz into (2.2) yields the equations

$$\begin{aligned} \dot{u}(t)\, x_1(t) + \dot{v}(t)\, x_2(t) &= 0 \\ \dot{u}(t)\, \dot{x}_1(t) + \dot{v}(t)\, \dot{x}_2(t) &= r(t) \end{aligned}$$

which we can solve for $\dot{u}(t)$ and $\dot{v}(t)$. Integrating the resulting equations from t_0 to t and assuming $u(t_0) = v(t_0) = 0$ we get

$$u(t) = -\int_{t_0}^{t} \frac{r(\tau)\, x_2(\tau)}{W(x_1, x_2)} d\tau \tag{2.19}$$

$$v(t) = \int_{t_0}^{t} \frac{r(\tau)\, x_1(\tau)}{W(x_1, x_2)} d\tau \tag{2.20}$$

hence the solution of (2.2) is

$$x(t) = \left(A - \int_{t_0}^{t} \frac{r(\tau)\, x_2(\tau)}{W(x_1, x_2)} d\tau\right) x_1(t) + \left(B + \int_{t_0}^{t} \frac{r(\tau)\, x_1(\tau)}{W(x_1, x_2)} d\tau\right) x_2(t)$$

where

$$
\begin{aligned}
A &= \frac{W(x, x_2)_{t=t_0} + x_0 x_2(t_0)}{W(x_1, x_2)_{t=t_0}} \\
B &= -\frac{W(x, x_1)_{t=t_0} + x_0 x_1(t_0)}{W(x_1, x_2)_{t=t_0}}
\end{aligned}
$$

2.3.2 Modern Methods

Lie Theory

In the latter half of the 19th century Sophus Lie, using the ideas of symmetry that Galois has introduced into the study of algebraic equations, was able to show why it was that all of the above classical methods and techniques were able to find solutions for particular classes of ODE only. Lie showed that each of the solvable classes thus far identified had a common feature; that they all were invariant under a certain group (this is the symmetry referred to). Furthermore, it followed from his results that if an ODE could be shown to demonstrate this invariant property then it could be solved. Indeed, in some cases the proof of solvability could also yield the solution itself.

Lie's pioneering work was born out of a consideration of a special group of variable transforms. Let us consider the variable transforms

$$
\begin{aligned}
t_1 &= \varphi(t, x; a) \\
x_1 &= \psi(t, x; a)
\end{aligned}
$$

which move the point (t, x) to the new point (t_1, x_1) and where a is the parameter associated with the transform (we assume that φ and ψ are continuous in t, x and a). Let us now suppose that these transforms possess the property that

$$\begin{aligned} \varphi(t, x; a_3) &= \varphi(\varphi(t, x; a_1), \psi(t, x; a_1); a_2) \\ \psi(t, x; a_3) &= \psi(\varphi(t, x; a_1), \psi(t, x; a_1); a_2) \end{aligned}$$

that is to say, for all values of t and x a single transformation with parameter a_3 is equivalent to two successive transformations with parameters a_1 and a_2. All variable transforms of the form considered that possess this property are said to be G_1. Having now defined G_1 we can now define the infinitesimal transformation associated with it (all transformation are hereafter assumed to be G_1).

Let a_0 be such that

$$\begin{aligned} t &= \varphi(t, x; a_0) \\ x &= \psi(t, x; a_0) \end{aligned}$$

If we now think about applying the same transform but with the parameter $a_0 + \delta s$ for small δs then we can expect this to result in a shift of the point (t, x) to a point (t_1, x_1) that is very close to (t, x). It can be shown [42] that this difference can be written

$$t_1 = t + \xi(t, x)\,\delta s \tag{2.22}$$

$$x_1 = x + \eta(t, x)\,\delta s \tag{2.23}$$

Furthermore, the functions ξ and η can be determined by neglecting all second and higher orders of δs.

For example, take the G_1 defined by a translation along the t-axis

$$\begin{aligned} t_1 &= t+a \\ x_1 &= x \end{aligned}$$

Clearly $a_0 = 0$ so let us write $a = a_0 + \delta s = \delta s$ such that

$$\begin{aligned} t_1 &= t+\delta s \\ x_1 &= x \end{aligned}$$

Comparing these with (2.22) and (2.23) gives $\xi(t,x) = 1$ and $\eta(t,x) = 0$.

Finally, it is possible to show that if a function $f(t,x)$ undergoes an infinitesimal transformation then the change in $f(t,x)$ is given to first order by

$$\delta f(t,x) = Uf\delta s$$

where

$$Uf \equiv \xi(t,x)\frac{\partial f}{\partial t} + \eta(t,x)\frac{\partial f}{\partial x}$$

Now, suppose one has the first order ODE (Lie's method applies to higher order ODEs of course but we will stick to order 1 equations for simplicity)

$$F(t,x,p) = 0$$

where $p = \dot{x}$ and F is continuous in its arguments. Suppose a G_1 has been found such that

$$\xi(t,x)\frac{\partial F}{\partial t} + \eta(t,x)\frac{\partial F}{\partial x} + \zeta(t,x)\frac{\partial F}{\partial p} = 0 \tag{2.24}$$

where

$$\zeta(t,x) = \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right)p - \frac{\partial \xi}{\partial y}p^2$$

then we say that the ODE is invariant under that group. This is helpful because the PDE (2.24) may be rewritten as the total differential

$$\frac{dt}{\xi} = \frac{dx}{\eta} = \frac{dp}{\zeta}$$

which may very often be solved to get a relation of the form $p = g(t, x)$ or rather

$$\dot{x} = g(t, x)$$

which may be easier to solve than the original ODE.

Conversely, one can determine all classes ODEs that are invariant under a given group. For example, choosing the G_1 of translations along the t-axis one will find that the class of ODEs invariant under this group are solvable by the method of variables separable. Indeed, it is the underlying symmetry of the separable ODEs that is evident in its invariance under this transformation group that allows this class of ODEs to be solved. Of course, this idea has been fully exploited to expose the symmetries of all the known classes of solvable ODEs and as such Lie unified the disparate tools, methods and tricks for solving ODEs by showing them all to be consequent of an intrinsic symmetry of the ODEs to which they could be applied.

Today, Lie analysis of ODEs forms a significant portion of the total research output on ODEs [59],[18],[91],[105]. The search for symmetries drives the analysis since it is this symmetry which suggests solvability and often aids in the solution process itself. However, it can very often be the case that trying to determine the symmetry transformation is as difficult as guessing the solution itself. Consequently, the most effective means of employing

Lie theory for the solution of ODEs is algorithmic [21],[22] though these search algorithms cannot guarantee that if symmetries exist, they will be found.

Prelle-Singer-Duarte

As mentioned earlier in §2.2.1, one of the most remarkable recent developments in the hunt for the solutions of differential equations is the Prelle-Singer semi-decision procedure for finding, if they exist, elementary function solutions of first order ODEs of the form

$$\dot{x} = \frac{P(x,t)}{Q(x,t)} \tag{2.25}$$

where P and Q are polynomials with coefficients in the field of complex numbers $\mathbb{C}$.

The crucial proof of the 1983 paper [76] was to show that if an elementary first integral of (2.25) exists then an IF $R(x,t)$ can be found with $R^n \in \mathbb{C}$ for some integer n. Once the IF is known the ODE (2.25) can be solved in the manner detailed in §2.3.1.

To understand how the method works we first note that if R is indeed an IF for the ODE in question then from (2.6) we can write

$$\frac{\partial(RP)}{\partial x} + \frac{\partial(RQ)}{\partial t} = 0$$

which can be rewritten in the form

$$\frac{1}{R}\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial t}\right)R = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial t}\right) \tag{2.26}$$

Now let $R = \Pi_i f_i^{n_i}$ with each f_i being an irreducible polynomial and each n_i an integer (i.e. R is a product of powers of polynomials) and substitute into (2.26) then after a good deal of manipulation one is lead to the conclusion that

$$\frac{1}{R}\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial t}\right)R = \sum_i \frac{n_i}{f_i}\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial t}\right)f_i \tag{2.27}$$

However, since the right-hand side (RHS) of (2.26) is a polynomial by virtue of the fact that both P and Q are polynomial then the RHS of (2.27) must be polynomial too. This can only be so however if f_i divides the term

$$\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial t}\right) f_i \tag{2.28}$$

We now have a criterion. To determine R one needst find all the f_i (having decided upon some upper bound for the degree of the f_i) that divide (2.28) and subsequently equate (2.26) and (2.28) to get

$$\sum_i \frac{n_i}{f_i}\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial t}\right) f_i = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial t}\right)$$

Successfully solving this equation will yield the values of the n_i and consequently the IF R which allows for reducing the solution to a quadrature.

As one might imagine, this technique is extremely powerful and general though it is unfortunately not without its limitations. Firstly, the implementation of the Prelle-Singer method is lengthy and as such requires use of symbolic computation. However, the method yields a semi-decision procedure rather than an algorithm and as such it is not self-terminating. That is, the procedure may take an infinite amount of time to generate the IF (by virtue of the fact that there is no known way of determining an upper bound on the degree of the f_i). This non-terminating possibility throws up a second problem, namely that, since it is looking for elementary solutions one cannot always know whether or not the procedure refuses to terminate because there exists no elementary solution (this cannot always be determined in advance) or because the procedure is having difficulty despite there being a solution to be found. Finally, the method is limited by the fact that P and Q must be polynomial in their arguments.

In 2000 a remarkable extension of the Prelle-Singer method was published by Duarte *et al.* [32] which demonstrated the possibility of generalising the method to include 2nd order ODEs of the form

$$\ddot{x} = \frac{P(x, \dot{x}, t)}{Q(x, \dot{x}, t)} \tag{2.29}$$

where P and Q are polynomials with coefficients in the field of complex numbers $\mathbb{C}$.

The approach essentially aims to find the first integral of the 2nd order problem and then to solve that first integral using the Prelle-Singer method. The argument presented in [32] asserts that the supposition of the existence functions $S(x, \dot{x}, t)$ and $R(x, \dot{x}, t)$ which are rational in their arguments and satisfy the PDEs

$$\begin{aligned}
\frac{\partial S}{\partial t} + \dot{x}\frac{\partial S}{\partial x} + \frac{P}{Q}\frac{\partial S}{\partial \dot{x}} &= -\frac{\partial}{\partial x}\left(\frac{P}{Q}\right) + S\frac{\partial}{\partial \dot{x}}\left(\frac{P}{Q}\right) && (2.30)\\
\frac{\partial R}{\partial t} + \dot{x}\frac{\partial R}{\partial x} + \frac{P}{Q}\frac{\partial R}{\partial \dot{x}} &= -R\left(S + \frac{\partial}{\partial \dot{x}}\left(\frac{P}{Q}\right)\right) && (2.31)\\
\frac{\partial R}{\partial x} &= S\frac{\partial R}{\partial \dot{x}} + R\frac{\partial S}{\partial \dot{x}} && (2.32)
\end{aligned}$$

implies the existence of a first integral of (2.29), namely

$$I(x, \dot{x}, t) = C \tag{2.33}$$

where I is rational in it's arguments and C is a constant. Furthermore, if one can determine S and R then I may be found from the relations

$$\frac{\partial I}{\partial t} = R\left(\frac{P}{Q} + S\dot{x}\right) \tag{2.34}$$

$$\frac{\partial I}{\partial x} = -RS \tag{2.35}$$

$$\frac{\partial I}{\partial \dot{x}} = -R \tag{2.36}$$

The solution method generally proceeds as follows:

1. Decide on an upper bound for the polynomials appearing in the rational functions R and S.

2. Set
$$S = \frac{\sum_{i,j,k} a_{ijk} x^i \dot{x}^j t^k}{\sum_{i,j,k} b_{ijk} x^i \dot{x}^j t^k}$$
and enter all possible forms as candidate solutions of (2.30) till the solution S is found.

3. Substitute S into (2.31) and (2.32) and repeat step 2, this time for finding the solution R.

4. Substitute S and R into (2.34), (2.35) and (2.36) to determine I.

5. Substitute I into (2.33) and solve for $\dot{x}$ to arrive at a first order ODE of the form
$$\dot{x} = g(x, t, C)$$
and apply the Prelle-Singer method to find the solution x of the original problem.

Much like Prelle-Singer, Duarte's method is extremely powerful and general but since it too draws on no foreknowledge of the degree of polynomials that should appear in S and

R it suffers from the same problems as Prelle-Singer in terms of the procedure's terminating time. In addition, we're this not the case and an upper bound on degree could be determined, the method still relies on Prelle-Singer which suffers from the same problem. Finally, though the success of the method has been demonstrated [32] through examination of several problems arising from problems in general relativity and the method has provided previously unobtained solutions, the method is based on a conjecture (relating to the assumption of the existence of an elementary solution) that has been proven in special instances only and as such stands on less firm ground than its first order counterpart, Prelle-Singer.

2.3.3 Recent Results

The author's results on the linear ODEs (2.1) and (2.2) which will follow in the next section are somewhat different from some of the other recent works as they offer not just alternative approaches, but also the solution in terms of elementary functions of an as yet unsolved class of ODEs. By contrast however, the novel results presented herein regarding equations of the form (2.3) bear similarity to the cited work of Chandrasekar *et al* (see below), being as they are, akin to the classical methods.

In the above we have examined the details of some of the different methods, both classical and modern, for solving ODEs and the classes to which they are of particular use. It should be noted that the above list is by no means an exhaustive one, though an extensive exposition of these and the myriad alternative methods can be readily found in numerous reference textbooks [66], [75], [107].

With the obvious exceptions of Prelle and Singer's 1983 work and Duarte's extension thereof, the methods we've discussed above and many of those that one will find in the literature are generally more than a century old. Of the more recent results (of which there are few due to the popularity of numerical solvers) in this area there are few examples worth highlighting since they are, like the author's work, classical in approach. For example, in [99] Verde-Star details an alternative method for solving (in)homogeneous linear ODEs with constant coefficients based on the method of *divided differences*. Meanwhile, Taylor expansion methods for the approximate solution of certain subclasses of (2.1) are investigated by Sezer and others in [49], [85], [86] and [87]. A good deal of work has also been produced that is primarily concerned with solving/determining solvability of (2.1) and (2.2) with an emphasis on the applications to symbolic computation, see [51], [96] and [100] and references therein.

On the other hand, Chandrasekar, Senthilvelan and Lakshmanan have been making admirable progress in the study of nonlinear ODEs and in particular, those of the form (2.3). For example, in [19] they provide the solution of the Duffing-van der Pol (DVP) oscillator for certain parametric choices while in [84] and [20] they have made an extensive study of the solvability of subclasses of (2.3) using Painlevé style analyses and Lie algebraic methods.

2.4 Main Results

In this section we shall detail our novel results pertaining to the exact solution of equations (2.1), (2.2) and (2.3) when considered as IVPs. The results of this section can be split into six distinct subsections. These are:

1. New exact analytical solutions of the Riccati equation.
2. New exact analytical solutions of linear ODE (2.1).
3. An alternative to the method of variation of parameters for determining the solution of (2.2).
4. Analytical solution of a DVP oscillator equation.
5. Analytical solution of a Riccati-related Liénard equation.
6. Application of new results to the problem of the lengthening pendulum.
7. Application of the new solutions to the problem of control design for linear systems.

Starting with the first in the above list we shall now detail the specifics of each contribution.

2.4.1 New Solutions of the Riccati Equation and its Associated Linear ODE

In what follows we present new analytical solutions of (2.1) as developed in the series of papers [15], [16] and [17]. The solutions are obtained by first solving a class of Riccati equations and then using that solution to derive the solution of a class of linear ODEs of order 2. This is possible due to the connection that exists between the Riccati equation and the 2nd order linear ODE. In addition, we proceed demonstrate how our Riccati equation solutions can also be used to derive the solutions of more general Riccati problems.

The results of this section are summarised by the following theorem and its corollaries.

Theorem 2.4 *Consider the class of Riccati equations described by*

$$\dot{y} + p(t)\, y - y^2 = \sigma q(t) \ ; \quad y(t_0) = y_0 \tag{2.37}$$

where $\sigma = \pm 1$ *and* $q(t) \geq 0 \; \forall t \geq t_0$. *If the real functions* q *and* p *satisfy the identity*

$$q(t) \equiv \frac{q_0 e^{-2\int_{t_0}^{t} p(\tau)d\tau}}{\left(1 + K\sqrt{q_0}\int_{t_0}^{t} e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} d\lambda\right)^2} \tag{2.38}$$

where $q(t_0) = q_0 \neq 0$ *and* K *is a real constant, then the general solution of (2.37) is*

$$y(t) = f(t)\sqrt{q(t)} \tag{2.39}$$

where $f(t)$ *is given by*

$$f(t) = \begin{cases} -\dfrac{K}{2} + \dfrac{\alpha\left(1 + c_1 v^{\frac{\alpha}{K}}\right)}{2\left(1 - c_1 v^{\frac{\alpha}{K}}\right)} & \text{if } K^2 > 4, \sigma = 1 \\ -\dfrac{K}{2} + \dfrac{\beta}{2}\tan\left(c_2 + \frac{\beta}{2K}\ln v\right) & \text{if } K^2 < 4, \sigma = 1 \\ -\dfrac{K}{2} - \dfrac{K}{Kc_3 + \ln v} & \text{if } K^2 = 4, \sigma = 1 \\ -\dfrac{K}{2} + \dfrac{\gamma\left(1 + c_4 v^{\frac{\gamma}{K}}\right)}{2\left(1 - c_4 v^{\frac{\gamma}{K}}\right)} & \text{if } \sigma = -1 \end{cases} \tag{2.40}$$

in which

$$v = e^{K\int_{t_0}^{t}\sqrt{q(\tau)}d\tau} \tag{2.41}$$

and where

$$
\begin{aligned}
c_1 &= \frac{2y_0 + \sqrt{q_0}\,(K-\alpha)}{2y_0 + \sqrt{q_0}\,(K+\alpha)} && (2.42)\\
c_2 &= \tan^{-1}\left(\frac{2y_0 + K\sqrt{q_0}}{\beta\sqrt{q_0}}\right) && (2.43)\\
c_3 &= -\frac{2\sqrt{q_0}}{2y_0 + K\sqrt{q_0}} && (2.44)\\
c_4 &= \frac{2y_0 + \sqrt{q_0}\,(K-\gamma)}{2y_0 + \sqrt{q_0}\,(K+\gamma)} && (2.45)
\end{aligned}
$$

while

$$
\begin{aligned}
\alpha &= \sqrt{K^2-4}\\
\beta &= \sqrt{4-K^2}\\
\gamma &= \sqrt{K^2+4}
\end{aligned}
$$

Before giving the proof of the above theorem, we shall make a few remarks on the functions p, q and f.

Remarks

1. First of all, note that if q and p satisfy the condition (2.38) with $q_0 = 0$ then $q(t) \equiv 0$. In such an instance (2.37) would assume the form of a Bernoulli equation and the solution could therefore be obtained by the transform given in the preceding section.

2. Though the constant $f(t_0)$ is always a finite, real constant, it is possible (depending on the IC y_0) for $f(t)$ to possess a pole for some $t_1 \in [t_0, \infty)$ and as such, the solution $y(t)$ would become singular at $t = t_1$. An example of just such an instance is given by

the conditions: $K^2 > 4$, $\sigma = 1$ and

$$y_0 > \frac{\sqrt{q_0}}{2}(\alpha - K)$$

When these conditions are satisfied there is exactly one element of $[t_0, \infty)$, which we call t_1, such that $y(t_1)$ is undefined.

3. Though the solution (2.39) of the Riccati equation (2.37) does not converge on the solution of the associated Bernoulli equation in the limit as $q_0 \longrightarrow 0$, it does converge on the equation's trivial solution $y(t) \equiv 0$.

4. Because, strictly speaking, $\sqrt{q(t)} = \pm\left|\sqrt{q(t)}\right|$ the solution (2.39) can take two forms; a positive and a negative. However, because the functions p and q are assumed to be continuous then the solution (2.39) must be unique. One may therefore conclude that provided one is consistent in their choice of either $\sqrt{q(t)} = \left|\sqrt{q(t)}\right|$ or $\sqrt{q(t)} = -\left|\sqrt{q(t)}\right|$ then only a single solution shall prevail. Indeed, this is true and for simplicity's sake we will always use the positive solution.

5. Next, it can be shown that $\sqrt{q(t)}$, defined by

$$\sqrt{q(t)} \equiv \frac{\sqrt{q_0}e^{-\int_{t_0}^{t} p(\tau)d\tau}}{1 + K\sqrt{q_0}\int_{t_0}^{t} e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} d\lambda}$$

is the solution of the Bernoulli equation

$$\frac{d}{dt}\sqrt{q(t)} + p(t)\sqrt{q(t)} + Kq(t) = 0 \tag{2.46}$$

6. Finally, it can also be shown that (2.40) is obtained by solving

$$\dot{f}(t) = \left(f^2(t) + Kf(t) + \sigma\right)\sqrt{q(t)} \tag{2.47}$$

In other words, (2.40) is the set of solutions of (2.47) with some specifically defined ICs $f(t_0)$ (see Appendix C for details).

We shall now give the proof of Theorem 2.4.

Proof. Let us first demonstrate that

$$y_p(t) = f(t)\sqrt{q(t)} \tag{2.48}$$

is at least a particular solution of (2.37). Differentiating (2.48) we have

$$\dot{y}_p(t) = \dot{f}(t)\sqrt{q(t)} + f(t)\frac{d}{dt}\sqrt{q(t)} \tag{2.49}$$

Next, by considering the above remarks and substituting (2.47) and (2.46) into (2.49) we obtain

$$\begin{aligned}\dot{y}_p(t) &= f^2(t)\,q(t) + \sigma q(t) - p(t)\,f(t)\sqrt{q(t)} \\ &= y_p^2(t) + \sigma q(t) - p(t)\,y_p(t)\end{aligned}$$

Hence, it is clear that (2.48) is indeed a particular solution of (2.37).

In order to turn our particular solution of (2.37) in to the general solution we must simply set the arbitrary constant $f(t_0)$ in such a way that the relation $y_0 = f(t_0)\sqrt{q_0}$ is always satisfied. Given that we are not interested in the instance where $q_0 = 0$ this may always be done. Indeed, it can be shown that c_1, c_2, c_3 and c_4 are all defined to ensure that $y_0 = f(t_0)\sqrt{q_0}$ holds whenever $q_0 \neq 0$. ■

It has already been demonstrated that to every Riccati equation there corresponds a linear second order homogeneous ODE. In particular, to the Riccati equation (2.37) there corresponds the homogeneous linear ODE (2.1), that is

$$\ddot{x} + p(t)\,\dot{x} + \sigma q(t)\,x = 0$$

where the two are connected by the variable transform

$$y = -\frac{\dot{x}}{x} \tag{2.50}$$

We can therefore use the Riccati equation solution given above to derive the solution of its corresponding second order ODE. This is given in the following corollary:

Corollary 2.1a *Consider the IVP*

$$\ddot{x} + p(t)\,\dot{x} + \sigma q(t)\,x = 0 \tag{2.51}$$

where $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$. If $q_0 \neq 0$, $q(t) \geq 0\ \forall t \geq t_0$ and q and p satisfy the identity (2.38), then the general solution of (2.51) is given by:

$$x(t) = \begin{cases} \dfrac{x_0}{1-c_1}\left(v^{\frac{K-\alpha}{2K}} - c_1 v^{\frac{K+\alpha}{2K}}\right) & \textit{if } K^2 > 4, \sigma = 1 \\ \dfrac{x_0}{\cos c_2}\left(\sqrt{v}\cos\left(c_2 + \ln v^{\frac{\beta}{2K}}\right)\right) & \textit{if } K^2 < 4, \sigma = 1 \\ x_0\left(1 + \frac{1}{Kc_3}\ln v\right)\sqrt{v} & \textit{if } K^2 = 4, \sigma = 1 \\ \dfrac{x_0}{1-c_4}\left(v^{\frac{K-\gamma}{2K}} - c_4 v^{\frac{K+\gamma}{2K}}\right) & \textit{if } \sigma = -1 \end{cases} \tag{2.52}$$

where c_1, c_2, c_3 and c_4 are given by (2.42-2.45) with $y_0 = -\dfrac{\dot{x}_0}{x_0}$.

Proof. Employing the transform (2.50) in (2.51) yields (2.37). By Theorem 2.4 we know the solution of (2.37) is given by (2.39) since we are assuming that the condition (2.46) holds. Solving (2.50) for x it is clear that

$$x(t) = x_0 e^{-\int_{t_0}^{t} f(\tau)\sqrt{q(\tau)}d\tau} \tag{2.53}$$

and hence the solution of (2.51) is (2.53) where $f(t)$ is given by (2.40) and where $y_0 = -\dfrac{\dot{x}_0}{x_0}$. To determine the explicit form of the solution of (2.51) it is necessary to evaluate the integral

$$I(t) = \int_{t_0}^{t} f(\tau)\sqrt{q(\tau)}d\tau$$

To do this we use (2.41) to write $f(\tau)$ in terms of v alone whilst noting that

$$\sqrt{q(\tau)} = \frac{1}{Kv}\frac{dv}{d\tau}$$

We therefore write

$$I(t) = \frac{1}{K}\int_{v(t_0)}^{v(t)} \frac{f(v)}{v}dv$$

and from this simplified form it is possible to show that

$$I(t) = \begin{cases} \left(\frac{\alpha-K}{2K}\right)\ln v + \ln\left(\frac{1-c_1}{1-c_1 v^{\frac{\alpha}{K}}}\right) & \text{if } K^2 > 4, \sigma = 1 \\ -\frac{1}{2}\ln v + \ln\left(\frac{\cos(c_2)}{\cos\left(c_2+\frac{\beta}{2K}\ln v\right)}\right) & \text{if } K^2 < 4, \sigma = 1 \\ -\frac{1}{2}\ln v + \ln\frac{c_3}{c_3+\frac{1}{K}\ln v} & \text{if } K^2 = 4, \sigma = 1 \\ \left(\frac{\gamma-K}{2K}\right)\ln v + \ln\left(\frac{1-c_4}{1-c_4 v^{\frac{\gamma}{K}}}\right) & \text{if } \sigma = -1 \end{cases}$$

which, when substituted into (2.53), gives (2.52). ■

Remarks

1. In solving the IVP (2.51) we have made use of the variable transform $y = -\frac{\dot{x}}{x}$. It is not surprising therefore that when $x_0 = 0$ we start to generate undefined quantities in our solution. The approach to circumventing this problem is given after these remarks.

2. One will notice that equation (2.51), for which we have given the solution, includes all constant coefficient linear ODEs in which the coefficient of x is non-zero. This proviso however does not represent any significant shortcoming of the statements of Corollary 2.1a since the case where the coefficient of x is zero can be easily solved using the substitution $z = \dot{x}$.

In order to address the problem that arises when $x_0 = 0$ we note that the solution of (2.51) (when satisfying the conditions of Corollary 2.1a) is

$$x(t) = \frac{x_0}{1 - c_1}\left(v^{\frac{K-\alpha}{2K}} - c_1 v^{\frac{K+\alpha}{2K}}\right) \tag{2.54}$$

when $K^2 > 4$ and $\sigma = 1$. Before we proceed any further we note that the two indices of v in (2.54) are the two real, distinct roots of the quadratic equation

$$K\lambda^2 - K\lambda + \frac{1}{K} = 0 \tag{2.55}$$

that is to say

$$\lambda_{1,2} = \frac{K \pm \alpha}{2K}$$

From hereon in, we will write all relevant constants in terms of solutions of (2.55).

Returning to the matter in hand, it is possible to discard those constants that become undefined when $x_0 = 0$ by rewriting the solution (2.54) in the form

$$x(t) = Av^{\lambda_1} + Bv^{\lambda_2} \tag{2.56}$$

where A and B aren't computed from the relations

$$\begin{aligned} A &= \frac{x_0}{1 - c_1} \\ B &= \frac{x_0 c_1}{c_1 - 1} \end{aligned}$$

but from the relations

$$\begin{aligned} x_0 &= A + B \\ \dot{x}_0 &= (A\lambda_1 + B\lambda_2) K\sqrt{q_0} \end{aligned}$$

These latter two come from the fact that $v(t_0) = 1$ and $\dot{v}(t_0) = K\sqrt{q_0}$. More explicitly we have that

$$\begin{aligned} A &= -\frac{x_0\lambda_2 K\sqrt{q_0} - \dot{x}_0}{K\sqrt{q_0}(\lambda_1 - \lambda_2)} \\ B &= \frac{x_0\lambda_1 K\sqrt{q_0} - \dot{x}_0}{K\sqrt{q_0}(\lambda_1 - \lambda_2)} \end{aligned}$$

This solves the problem which appears when $x_0 = 0$ because neither of the above can ever be undefined since $K \neq 0$, $\sqrt{q_0} \neq 0$ and $\lambda_1 \neq \lambda_2$ by definition.

Using a same relabelling of constants technique we can rewrite Corollary 2.1a as follows:

Corollary 2.1b *If the ODE (2.51) satisfies all of the hypotheses of Corollary 2.1a then the solution of (2.51) is*

$$x(t) = \begin{cases} A_1 v^{\lambda_1} + B_1 v^{\lambda_2} & \text{if } K^2 > 4, \sigma = 1 \\ v^{m_1}\left(A_2 \cos\left(\ln v^{m_2}\right) + B_2 \sin\left(\ln v^{m_2}\right)\right) & \text{if } K^2 < 4, \sigma = 1 \\ v^{\lambda_3}\left(A_3 + B_3 \ln v\right) & \text{if } K^2 = 4, \sigma = 1 \\ A_4 v^{\lambda_1} + B_4 v^{\lambda_2} & \text{if } \sigma = -1 \end{cases} \tag{2.57}$$

where $A_i, B_i \in \mathbb{R}$ $i = 1, 2, 3$ *are arbitrary constants,* λ_1, λ_2 *and* λ_3 *are the roots of the quadratic equation*

$$K\lambda^2 - K\lambda + \frac{\sigma}{K} = 0 \tag{2.58}$$

for the given value of K *while*

$$\begin{aligned} m_1 &= \operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2) \\ m_2 &= \operatorname{Im}(\lambda_1) = \operatorname{Im}(\lambda_2) \end{aligned}$$

Deriving the solution (2.57) from (2.52) is for the most part trivial since we have simply collected together and renamed groups of constants. The qualifier "for the most part" is inserted however since it may not be immediately obvious how one derives

$$x(t) = v^{m_1}\left(A_2 \cos\left(\ln v^{m_2}\right) + B_2 \sin\left(\ln v^{m_2}\right)\right) \tag{2.59}$$

from

$$x(t) = \frac{x_0}{\cos c_2}\left(\sqrt{v} \cos\left(c_2 + \ln v^{\frac{\beta}{2K}}\right)\right) \tag{2.60}$$

and so we will now prove the equality of (2.59) and (2.60).

Proof. It can be shown that for some real constants $r_{1,2,3}$ and φ their exists the identity

$$r_1 \cos\left(g(t) + \varphi\right) \equiv r_2 \cos g(t) + r_3 \sin g(t) \tag{2.61}$$

where $g : t \longrightarrow \mathbb{R}$ under condition that

$$\begin{aligned} r_1^2 &= r_2^2 + r_3^2 \\ \varphi &= \tan^{-1}\left(-\frac{r_3}{r_2}\right) \end{aligned}$$

Rewriting (2.60) in the form

$$x(t) = \sqrt{v}\left(\frac{x_0}{\cos c_2}\cos\left(c_2 + \ln v^{\frac{\beta}{2K}}\right)\right)$$

and applying the identity (2.61) we have

$$x(t) = \sqrt{v}\left(A_2 \cos\left(\ln v^{\frac{\beta}{2K}}\right) + B_2 \sin\left(\ln v^{\frac{\beta}{2K}}\right)\right) \tag{2.62}$$

where

$$\begin{aligned} A_2^2 + B_2^2 &= \frac{x_0^2}{\cos^2 c_2} \\ \frac{B_2}{A_2} &= -\tan c_2 \end{aligned}$$

Furthermore, the solution (2.60) holds for $K^2 < 4$ and $\sigma = 1$. The roots of (2.58) in such an instance are given by

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2} \pm i\frac{\sqrt{4-K^2}}{2K} \\ &= \frac{1}{2} \pm i\frac{\beta}{2K} \end{aligned}$$

and since

$$\begin{aligned} \mathrm{Re}(\lambda_1) &= \mathrm{Re}(\lambda_2) = \frac{1}{2} \\ \mathrm{Im}(\lambda_1) &= \mathrm{Im}(\lambda_2) = \frac{\beta}{2K} \end{aligned}$$

we can rewrite (2.62) in the form

$$x(t) = v^{m_1}\left(A_2 \cos\left(\ln v^{m_2}\right) + B_2 \sin\left(\ln v^{m_2}\right)\right)$$

■

As a final extension of this work we note at this stage that what Theorem 2.4 provides is a solution of the Riccati equation (2.13) with $b(t) = -1$. However, we can use the solution of (2.37) to determine the solution of (2.13) as shown below.

Corollary 2.2 *If in (2.13) the product $b(t)\,c(t)$ does not change sign and*

$$\frac{d}{dt}\sqrt{|b(t)\,c(t)|} + \left(a(t) - \frac{\dot{b}(t)}{b(t)}\right)\sqrt{|b(t)\,c(t)|} + K\,|b(t)\,c(t)| = 0 \tag{2.63}$$

then the solution of (2.13) with the IC $y(t_0) = y_0$ is

$$y = -\frac{z}{b(t)}$$

where z is the solution of the IVP

$$\dot{z} + \left(a(t) - \frac{\dot{b}(t)}{b(t)}\right)z - z^2 = -b(t)\,c(t)\ ; \quad z(t_0) = -b(t_0)\,y_0 \tag{2.64}$$

and can be obtained by applying Theorem 2.4.

Proof. Starting with the Riccati equation (2.13) we introduce the variable transform

$$y = -\frac{z}{b(t)}$$

and so arrive at the Riccati equation (2.64). Comparing equations (2.37) and (2.64) we write

$$\begin{aligned} p(t) &\equiv a(t) - \frac{\dot{b}(t)}{b(t)} \\ \sigma q(t) &\equiv -b(t)\,c(t) \end{aligned}$$

We can therefore find z by applying Theorem 2.4 if the functions a, b and c are such that

$$\frac{d}{dt}\sqrt{q(t)} + p(t)\sqrt{q(t)} + Kq(t) = 0$$

and the product $b(t)\,c(t)$ does not change sign. In order to rewrite this condition in terms of a, b and c we first note that since the product $b(t)\,c(t)$ does not change sign we may write $b(t)\,c(t) = \bar{\sigma}\,|b(t)\,c(t)|$ where $\bar{\sigma} = \pm 1$. Consequently we have

$$\sigma q(t) = -\bar{\sigma}\,|b(t)\,c(t)|$$

and since $q(t) \geq 0\ \forall t \geq t_0$ we must have that

$$\bar{\sigma} = -\sigma$$

hence

$$q(t) = |b(t)\,c(t)| \tag{2.65}$$

Finally, substituting (2.65) into (2.46) we arrive at (2.63). ■

Theorem 2.4 and Corollaries 2.1a/b and 2.2 constitute the successful determination of a new class of exactly solvable Riccati equations and linear ODEs of order 2. The solutions hold under the single condition (2.46). Furthermore, the solutions we have given are elementary (in the Liouvillian sense) provided that $\int_{t_0}^{t}\sqrt{q(\tau)}d\tau$ and $\int_{t_0}^{t}p(\tau)\,d\tau$ are elementary integrals.

Having established these new solutions of (2.13) and (2.51) we will move on to consider the problem of solving the inhomogeneous equivalent of (2.51).

2.4.2 Alternative Solution of the 2nd Order Inhomogeneous Linear ODE

Earlier in this chapter we indulged in a brief discussion of Lagrange's method of variation of parameters and its applicability to finding the solution of (2.2). There are of course however a number of other methods that may be brought to bear upon such problems as this, a common alternative being the method of undetermined coefficients [13]. As might well be expected, each of the different approaches have their own advantages and disadvantages since some are simpler to implement than others in certain situations. In this section we present another alternative method for solving (2.2), the aim being to extend the ideas of the above linear ODE solutions and to provide a method that is simpler than each of the other methods we have mentioned for certain problems.

We shall now state and prove our main results on the alternative solution of (2.2) as they appear in [45].

Theorem 2.5 *Consider the IVP (2.2), that is*

$$\ddot{x} + p(t)\dot{x} + q(t)x = r(t) \ ; \quad x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0$$

The solution of (2.2) is

$$x(t) = Ax_1 + Bx_2 + x_1 \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_1 e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_1^2} d\lambda \tag{2.66}$$

or

$$x(t) = Ax_1 + Bx_2 + x_2 \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_2 e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_2^2} d\lambda \tag{2.67}$$

where

$$\ddot{x}_i + p(t)\dot{x}_i + q(t)x_i = 0$$

for $i = 1, 2$ while A and B are arbitrary constants.

Proof. It is clear that the solutions (2.66) and (2.67) are by definition, of the form (2.5). Consequently it simply remains to prove that

$$x_p = x_i \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_i e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_i^2} d\lambda \tag{2.68}$$

where x_p is, as usual, a particular solution of (2.2). In order to prove this we differentiate (2.68) to get

$$\dot{x}_p = \dot{x}_i \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_i e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_i^2} d\lambda + \frac{e^{-\int_{t_0}^{t} p(\tau)d\tau} \int_{t_0}^{t} r(\tau)\, x_i e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_i} \tag{2.69}$$

Repeating this step we have

$$\ddot{x}_p = \ddot{x}_i \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_i e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_i^2} d\lambda - \frac{p(t)\, e^{-\int_{t_0}^{t} p(\tau)d\tau} \int_{t_0}^{t} r(\tau)\, x_i e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_i} + r(t) \tag{2.70}$$

Substituting (2.68)-(2.70) into (2.2) gives

$$(\ddot{x}_i + p(t)\,\dot{x}_i + q(t)\,x_i) \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_i e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_i^2} d\lambda = 0$$

which is always satisfied since by definition we have that x_i for $i = 1, 2$ is a solution of

$$\ddot{x}_i + p(t)\,\dot{x}_i + q(t)\,x_i = 0$$

■

Remarks

1. A quick inspection of Theorem 2.5 allows one to see that the particular solution of (2.2) i.e. $x_p(t)$, may be arrived at in two different ways. One may decide to evaluate either (2.66)

$$x_p(t) = x_1 \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_1 e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_1^2} d\lambda \tag{2.71}$$

or (2.67)

$$x_p(t) = x_2 \int_{t_0}^{t} \frac{e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} \int_{t_0}^{\lambda} r(\tau)\, x_2 e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau}{x_2^2} d\lambda \tag{2.72}$$

This gives our method an unique feature, namely a choice with regard to how one is to evaluate $x_p(t)$. Such a choice appears in no other approach for solving equations of the form considered here. We have already seen in §2.3.1 that the method of variation of parameters give the form of the particular solution as ([13])

$$x_p(t) = x_2 \int_{t_0}^{t} \frac{r(\tau)\, x_1}{W(x_1, x_2)} d\tau - x_1 \int_{t_0}^{t} \frac{r(\tau)\, x_2}{W(x_1, x_2)} d\tau \tag{2.73}$$

Clearly here, there is no choice and one must simply attempt to evaluate this form no matter how awkward it may prove to be. The main advantage of our alternative solution is that, for certain problems the simplest of either (2.66) or (2.67) may be easier to evaluate than (2.73) or indeed, any other form of $x_p(t)$ generated by some other method. An example of such an instance would be

$$e^{-\int_{t_0}^{t} p(\tau)d\tau} \int_{t_0}^{t} r(\tau)\, x_1 e^{\int_{t_0}^{\tau} p(\mu)d\mu} d\tau = K \frac{d}{dt}\left(x_1^2\right)$$

for some constant K, since here (2.66) would simply be

$$x_p(t) = 2K x_1 \ln x_1$$

whereas attempts to evaluate (2.67) or (2.73) would not be as straightforward (for an example see [45]).

2. Finally, as one would expect the solutions (2.66) and (2.67) of (2.2) reduce to the solution of (2.51) when $r(t) \equiv 0$.

2.4.3 The DVP Oscillator

One of the most famous and well-studied nonlinear ODEs is Duffing's equation [93]

$$\ddot{x} + a_0\dot{x} + a_1 x + a_2 x^3 = F(t) \tag{2.74}$$

where the a_j for $j = 0, 1, 2$ are constants and $F(t)$ is the forcing function. When $a_0, a_2 > 0$ and $a_1 < 0$ the equation describes the one-dimensional (1-D) position x of a particle moving in a double potential well such as that shown in Figure 2.1 whilst being subjected to damping. In this 'particle in a 1-D well' interpretation, the forcing function (generally periodic) in (2.74) represents an external driving force applied to the particle and has a significant impact on the dynamics of the trajectories generated by the oscillator. In fact, it is when there is a periodic driving force present that the dynamics of equation (2.74) may make the transition to chaos.

This potential for chaotic behaviour is in fact one of the reasons why Duffing's equation is so well studied. Indeed, the Duffing oscillator is not only one of the earliest cited examples of a deterministic system producing chaotic motion but it is also one of the simplest systems to do so.

Another significant equation appearing in the history of nonlinear ODEs is the equation of van der Pol [47]

$$\ddot{x} - a_0\left(1 - x^2\right)\dot{x} + a_1 x = 0 \tag{2.75}$$

This equation arises in models of electronic circuits containing valves and numerous other systems besides. For $a_{0,1} > 0$ the system is such that when $|x| > 1$ energy is removed from the system whilst on the other hand, when $|x| < 1$, energy is fed into the

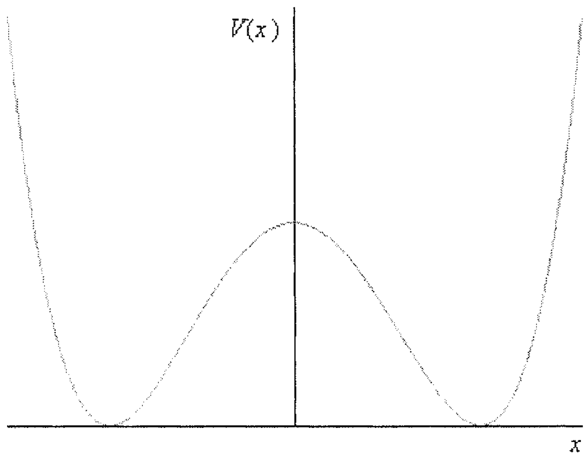

Fig. 2.1. The double potential well of the Duffing oscillator.

system. This property generates limit cycles in the state space of the oscillator. However, like the Duffing oscillator, this system may also exhibit chaos when periodically forced.

Clearly, the nonlinearity in the Duffing equation appears in the restoring force term whilst, for the van der Pol equation, it appears in the damping term. One can therefore generate a new nonlinear ODE that has the properties of both of the above by combining the two equations to into a single ODE. The resulting equation is characterised by both van der Pol-like nonlinear damping and a Duffing-like nonlinear restoring force. The new equation is the DVP oscillator and it takes the form (1.13) i.e.

$$\ddot{x} + \left(\alpha + \beta x^2\right)\dot{x} + \gamma x + \delta x^3 = 0$$

The equation is called an oscillator since one can expect oscillatory behaviour when $\delta, \beta > 0$ and $\alpha < 0$. Furthermore, by setting $\gamma < 0$ one demands the potential field associated with the restoring force to be that of the double potential well. It is interesting note at this stage that equation (1.13) arises in neurological mathematical models, the study of plasmas subjected to RF discharges [77] as well as many other dynamical systems. In addition, like the two equations from which the DVP oscillator is built, it too has the capacity to generate chaotic trajectories. As such it is of particular interest to us as a system since it readily lends itself to applications such as chaos-based message encryption, an application which we will look at in Chapter 4.

It can be plainly seen by inspection that equations (2.74), (2.75) and (1.13) all belong to the family of nonlinear ODEs defined by the Liénard equation (2.3) which we have already discussed briefly in §1.2.2. Of these however, the first two have been far more extensively analysed in the literature than DVP equation. In particular, the instances where equations (2.74) and (2.75) are integrable in the Painlevé sense are far better known than those for equation (1.13). As a result of this, exact analytical solutions of (2.74) and (2.75) are more readily found than for (1.13). Quite recently however, Chandrasekar, Lakshmanan and Senthilvelan published an implicit solution of the DVP oscillator (1.13) with the parametric choices

$$\begin{aligned}
\alpha &= \frac{4}{\beta} \\
\gamma &= \frac{3}{\beta^2} \\
\delta &= 1
\end{aligned}$$

What we now give is a generalisation of Chandrasekar's solution using the methods laid out in [19]. The generalisation we provide is manifested by a greater freedom in the parametric choices that are permissible.

2.4.4 Analytical Solution of a DVP Oscillator

In this section we consider the problem of solving (1.13) under the following parametric constraints

$$\begin{aligned} \alpha &= \frac{4\delta}{\beta} \\ \gamma &= \frac{3\delta^2}{\beta^2} \end{aligned}$$

That is, we intend to solve the DVP oscillator

$$\ddot{x} + \left(\frac{4\delta}{\beta} + \beta x^2\right)\dot{x} + \frac{3\delta^2}{\beta^2}x + \delta x^3 = 0 \tag{2.76}$$

Before we proceed to the solution however we must first establish a result that will be relied upon in our solution's derivation.

Theorem 2.6 *If in the Liénard equation (2.3), the functions f and g satisfy the relation*

$$f(x) = \frac{1}{C}g'(x) + C \tag{2.77}$$

($g'(x)$ being the first derivative of $g(x)$ with respect to x) for some real-valued constant $C \neq 0$, then the first integral of (2.3) is

$$\dot{x} + \frac{1}{C}g(x) = \zeta e^{-C(t-t_0)} \tag{2.78}$$

where

$$\zeta = \dot{x}_0 + \frac{1}{C}g(x_0)$$

Note: equation (1.12) of Chapter 1 satisfies Theorem 2.6 by definition.

Proof. Assuming that (2.77) holds we may write (2.3) in the form

$$\ddot{x} + \left(\frac{1}{C}g'(x) + C\right)\dot{x} + g(x) = 0 \tag{2.79}$$

From the form of the first integral (2.78) we know that

$$\dot{x} = \zeta e^{-C(t-t_0)} - \frac{1}{C} g(x) \tag{2.80}$$

such that

$$\ddot{x} = -\zeta C e^{-C(t-t_0)} - \frac{1}{C} g'(x) \left(\zeta e^{-C(t-t_0)} - \frac{g(x)}{C} \right) \tag{2.81}$$

where the prime denotes differentiation with respect to x. Substituting (2.80) and (2.81) into (2.79) we find that the equality

$$\left(\frac{1}{C} g'(x) + C\right) \left(\zeta e^{-C(t-t_0)} - \frac{1}{C} g(x)\right) + g(x) = \zeta C e^{-C(t-t_0)} + \frac{1}{C} g'(x) \left(\zeta e^{-C(t-t_0)} - g(x)\right)$$

holds for all t. ■

We now intend to show that (2.76) has the property (2.77). Comparing (2.3) and (2.76) we have that

$$\begin{aligned} f(x) &= \frac{4\delta}{\beta} + \beta x^2 \\ g(x) &= \frac{3\delta^2}{\beta^2} x + \delta x^3 \end{aligned}$$

In order for (2.76) to satisfy the hypotheses of Theorem 2.6 we must have that

$$\begin{aligned} \frac{4\delta}{\beta} + \beta x^2 &= \frac{1}{C} \frac{d}{dx} \left(\frac{3\delta^2}{\beta^2} x + \delta x^3 \right) + C \\ &= \frac{3\delta^2}{C\beta^2} + C + \frac{3\delta}{C} x^2 \end{aligned} \tag{2.82}$$

from which it is clear that

$$\beta = \frac{3\delta}{C}$$

and

$$\frac{4\delta}{\beta} = \frac{3\delta^2}{C\beta^2} + C$$

both of which hold for $C = \dfrac{3\delta}{\beta}$ and thus δ is clearly permitted to assume values other than 1 in this case. Clearly therefore, equation (2.76) does indeed satisfy the condition (2.77). Theorem 2.6 leads us to inevitably conclude that the first integral of (2.76) is

$$\dot{x} + \frac{\delta}{\beta}x + \frac{\beta}{3}x^3 = \zeta e^{-\frac{3\delta}{\beta}(t-t_0)} \tag{2.83}$$

Following in the manner of Chandrasekar [19] we rewrite (2.83) in the form

$$\begin{aligned}
\zeta &= e^{\frac{2\delta}{\beta}(t-t_0)}\left(e^{\frac{\delta}{\beta}(t-t_0)}\dot{x} + \frac{\delta}{\beta}e^{\frac{\delta}{\beta}(t-t_0)}x\right) + \frac{\beta}{3}\left(e^{\frac{\delta}{\beta}(t-t_0)}x\right)^3 \\
&= e^{\frac{2\delta}{\beta}(t-t_0)}\frac{d}{dt}\left(e^{\frac{\delta}{\beta}(t-t_0)}x\right) + \frac{\beta}{3}\left(e^{\frac{\delta}{\beta}(t-t_0)}x\right)^3
\end{aligned}$$

Introducing the variable transform $y = e^{\frac{\delta}{\beta}(t-t_0)}x$ we obtain the separable first order ODE

$$\dot{y} = \left(\zeta - \frac{\beta}{3}y^3\right)e^{-\frac{2\delta}{\beta}(t-t_0)}$$

Separating variables and integrating both sides from t_0 to t we have

$$\int_{y(t_0)}^{y(t)} \frac{dy}{\zeta - \frac{\beta}{3}y^3} = \int_{t_0}^{t} e^{-\frac{2\delta}{\beta}(\tau-t_0)}d\tau$$

or equivalently

$$\int_{v(t_0)}^{v(t)} \frac{dv}{1+v^3} = \left(\frac{\zeta^2\beta}{3}\right)^{\frac{1}{3}}\frac{\beta}{2\delta}\left(e^{-\frac{2\delta}{\beta}(t-t_0)} - 1\right)$$

where $v = -\left(\dfrac{\beta}{3\zeta}\right)^{\frac{1}{3}}y$. Evaluating the left-hand side (LHS) we have

$$\int_{v(t_0)}^{v(t)} \frac{dv}{1+v^3} = \frac{1}{6}\ln\left(\frac{(v+1)^2}{v^2-v+1}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2v-1}{\sqrt{3}}\right)$$

from which it can be determined that

$$t = t_0 - \frac{\beta}{2\delta}\left[\ln\left(\left(\frac{3}{\zeta^2\beta}\right)^{\frac{1}{3}}\frac{\delta}{3\beta}\left(\ln\left(\frac{(\omega y+1)^2}{\omega y^2 - \omega y + 1}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2\omega y - 1}{\sqrt{3}}\right)\right)\right) + 1\right]$$

which is the implicit solution of the DVP oscillator (2.76) with $\omega = -\left(\frac{\beta}{3\zeta}\right)^{\frac{1}{3}}$ (N.B. this does not denote frequency).

2.4.5 Analytical Solution of a Liénard Equation

In the above we have considered, amongst other things, general Riccati equations and Liénard equations of the form (2.13). It is therefore of interest to us to look into the equation that connects the ODEs (2.13) and (2.79). The equation to which we refer is the Liénard ODE

$$\ddot{x} + (\alpha + \beta x)\,\dot{x} + \gamma x + \delta x^2 = 0 \tag{2.84}$$

Physically, the ODE (2.84) can be thought of as modelling the one dimensional position x of a particle that moves in a potential field defined by

$$V(x) = \frac{\gamma}{2}x^2 + \frac{\delta}{3}x^3$$

whilst being subjected to a damping force, the size of which varies linearly with x.

Since we are interested in equations of the form (2.79) we must define the parametric constraints that force equation (2.84) into the class defined by (2.79). Comparing (2.84) and (2.79) and recalling (2.77) it can be shown that (2.84) satisfies the hypotheses of Theorem 2.6 iff

$$\alpha = \frac{\gamma}{C} + C$$

where $C = \dfrac{2\delta}{\beta}$.

Assuming that this holds we may apply Theorem 2.6 and so identify the first integral of (2.84) as

$$\dot{x} + \frac{1}{C}\left(\gamma x + \delta x^2\right) = \zeta e^{-C(t-t_0)} \tag{2.85}$$

where

$$\zeta = \dot{x}_0 + \frac{1}{C}\left(\gamma x_0 + \delta x_0^2\right)$$

Looking at equation (2.85) it is clear how the ODE (2.84) connects the equations (2.13) and (2.79) since, when put in the form of (2.79), the ODE (2.84) has a Riccati-type first integral.

Getting back to the matter of obtaining the solution of (2.84), we introduce the dependant variable transform

$$x = \frac{1}{\delta}\left(C\frac{\dot{y}}{y} - \frac{\gamma}{2}\right) \tag{2.86}$$

and substitute into (2.85) to obtain the equation

$$\ddot{y} - \left(\frac{\delta\zeta}{C}e^{-C(t-t_0)} + \frac{\gamma^2}{4C^2}\right) y = 0 \tag{2.87}$$

Though it may not be immediately obvious, we have successfully transformed (2.84) into a form for which the analytical solution is known. That is to say, the solution of equation (2.87) is readily obtained in terms of known transcendents by the use of an independent variable transform. The transform we use is

$$z = \frac{2}{C}\sqrt{-\frac{\delta\zeta}{C}}e^{-\frac{C}{2}(t-t_0)} \tag{2.88}$$

where it is assumed that $\frac{\delta\zeta}{C} < 0$. The transform defined by (2.88) was arrived at by adapting a transform similar to (2.88) appearing in [75].

Substituting (2.88) into (2.87) we obtain the ODE

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{\gamma^2}{C^4z^2}\right)y=0 \tag{2.89}$$

which one will notice is the Bessel differential equation. We can therefore write the two linearly independent solutions of (2.89) as

$$y_1 = J_\eta(z) \tag{2.90}$$

$$y_2 = Y_\eta(z) \tag{2.91}$$

where J_η and Y_η are the Bessel functions of the first and second kind respectively with each one being of order $\eta=\frac{\gamma}{C^2}=\frac{\beta^2\gamma}{4\delta^2}$. In order to return to our original variables we substitute (2.90) and (2.91) into (2.86) and so obtain

$$\begin{aligned} x_1 &= \frac{1}{\delta}\left(\frac{C\dot{z}}{J_\eta(z)}\frac{d}{dz}J_\eta(z)-\frac{\gamma}{2}\right)\\ x_2 &= \frac{1}{\delta}\left(\frac{C\dot{z}}{Y_\eta(z)}\frac{d}{dz}Y_\eta(z)-\frac{\gamma}{2}\right)\end{aligned}$$

Since it can be shown [103] that

$$\begin{aligned} \frac{d}{dz}J_\eta(z) &= J_{\eta-1}(z)-\frac{\eta}{z}J_\eta(z)\\ \frac{d}{dz}Y_\eta(z) &= Y_{\eta-1}(z)-\frac{\eta}{z}Y_\eta(z)\end{aligned}$$

we may express the solutions $x_{1,2}$ thus

$$\begin{aligned} x_1 &= -\frac{C^2zJ_{\eta-1}(z)}{2\delta J_\eta(z)}\\ x_2 &= -\frac{C^2zY_{\eta-1}(z)}{2\delta Y_\eta(z)}\end{aligned}$$

or in terms of t

$$
\begin{aligned}
x_1 &= -\sqrt{-\frac{C\zeta}{\delta}}\frac{J_{\eta-1}(z)}{J_\eta(z)}e^{-\frac{C}{2}(t-t_0)} \\
x_2 &= -\sqrt{-\frac{C\zeta}{\delta}}\frac{Y_{\eta-1}(z)}{Y_\eta(z)}e^{-\frac{C}{2}(t-t_0)}
\end{aligned}
$$

These solutions are however only real when $\frac{\delta\zeta}{C} < 0$. In order to find the real-valued solutions when $\frac{\delta\zeta}{C} > 0$ we must introduce the alternative variable transform

$$w = \frac{2}{C}\sqrt{\frac{\delta\zeta}{C}}e^{-\frac{C}{2}(t-t_0)}$$

in (2.87) and so obtain the modified Bessel differential equation

$$\frac{d^2y}{dw^2} + \frac{1}{w}\frac{dy}{dw} - \left(1 + \frac{\gamma^2}{C^4w^2}\right)y = 0$$

for which the two linearly independent particular solutions are

$$
\begin{aligned}
y_3 &= I_\eta(w) \\
y_4 &= K_\eta(w)
\end{aligned}
$$

where I_η and K_η are the η order modified Bessel functions of the first and second kind respectively. Just as before we can return to our original variables by recalling the variable transforms and using the recursion relations given in [103] i.e.

$$
\begin{aligned}
\frac{d}{dw}I_\eta(w) &= I_{\eta-1}(w) - \frac{\eta}{w}I_\eta(w) \\
\frac{d}{dw}K_\eta(w) &= -K_{\eta-1}(w) - \frac{\eta}{w}K_\eta(w)
\end{aligned}
$$

the result being that for $\frac{\delta\zeta}{C} > 0$ we have that

$$
\begin{aligned}
x_3 &= -\sqrt{\frac{C\zeta}{\delta}}\frac{I_{\eta-1}(w)}{I_\eta(w)}e^{-\frac{\zeta}{2}(t-t_0)} \\
x_4 &= \sqrt{\frac{C\zeta}{\delta}}\frac{K_{\eta-1}(w)}{K_\eta(w)}e^{-\frac{\zeta}{2}(t-t_0)}
\end{aligned}
$$

The solution of (2.84) when $\alpha = \frac{\gamma}{C} + C$ is therefore

$$
x(t) = \begin{cases} A_1 x_1(t) + A_2 x_2(t) & \text{if } \frac{\delta\zeta}{C} < 0 \\ A_3 x_3(t) + A_4 x_4(t) & \text{if } \frac{\delta\zeta}{C} > 0 \end{cases}
$$

where the A_i for $i = 1, 2, 3, 4$ are constants whose values are governed by the ICs of the system.

2.5 Applications

In order to demonstrate the utility of the main results of this chapter we shall now present two distinct applications. The first of these is concerned with finding the solution of the model of a physical system while the second constitutes an assault upon a significant problem in control theory.

2.5.1 The Lengthening Pendulum

As a first application of the foregoing results we shall here present an example of a physical system, the model of which is an ODE of the form (2.51) which also satisfies the condition (2.46). This being the case we are hence able to solve the ODE that models the system by recalling the results of Corollary 2.1. The system that we consider is that of the lengthening pendulum and is motivated by the consideration of simple models of mechanical cranes.

Furthermore, in order to fully appreciate the application, we will derive the model of the system at hand before solving it and subsequently analyse the solution in the context of the physics of the problem.

The Model

In constructing our model of the lengthening pendulum shown in Figure 2.2 we shall make use of the following assumptions:

1. The pendulum arm is constrained to move in a vertical plane.

2. The resistance of the air to the motion of the pendulum is negligible.

3. The range of angular displacement of the pendulum arm is small.

4. The pendulum arm is massless.

5. The bob can be considered as a point mass located at a distance l from the pivot.

6. The pivot point is fixed and the length l changes in accordance with the relation

$$l(t) = \left(\sqrt{l_0} + at\right)^2 \tag{2.92}$$

where $l_0 = l(0) \geq 0$ is the initial length of the arm and $a \neq 0$ is a real constant.

Before we proceed to the model the system we will take a little time to interpret the physical meaning of setting the time dependence of l as we have above. Firstly, one will notice that since $l(t) \geq 0 \; \forall t \geq 0$ irrespective of the value of a, the pendulum arm may not retract through the pivot point. Secondly, one can see from (2.92) that if a is positive then

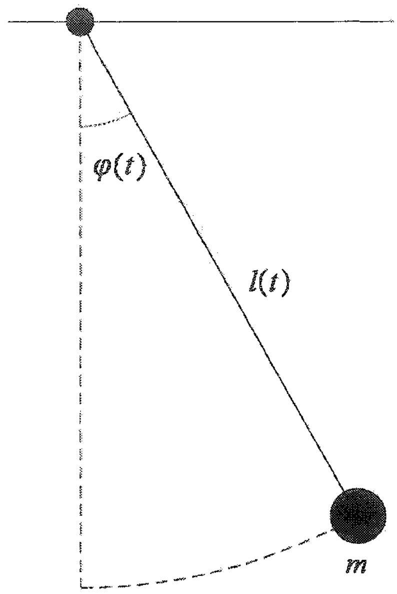

Fig. 2.2. Schematic of the simple lengthening pendulum.

the arm starts off at some positive length l_0 and proceeds to grow indefinitely. If however a is negative then the arm again begins at some positive length l_0 and proceeds to shrink until $l = 0$ before growing back to its original length and beyond.

The parameter a also has another effect on the growth of l. This effect can be seen from differentiating (2.92) twice to obtain the expression

$$\ddot{l} = 2a^2$$

Clearly a also governs the acceleration of the arm growth and this acceleration is both constant and positive irrespective of the sign of a.

Having established the nature of the variable length pendulum we are dealing with here we will now model the system using the Lagrangian formalism. Bearing the above assumptions in mind we write the kinetic energy of the bob at time t as

$$T = \frac{ml^2\dot{\varphi}}{2}$$

where m is the mass of the bob and φ is the angular displacement of the arm from the vertical. Furthermore, the potential energy function for the system can be shown to be

$$V = mg\left(H - l\cos\varphi\right)$$

where $g = 9.80665\,\mathrm{m\,s^{-2}}$ and H is the height of the pivot above ground (where the potential of the ground is set to 0). The Lagrangian for the system is therefore

$$L = \frac{ml^2\dot{\varphi}}{2} - mg\left(H - l\cos\varphi\right) \tag{2.93}$$

Substituting (2.93) into the Euler-Lagrange equation (1.3) we get

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\varphi}}\left(\frac{ml^2\dot{\varphi}^2}{2} - mg\left(H - l\cos\varphi\right)\right)\right) = \frac{\partial}{\partial\varphi}\left(\frac{ml^2\dot{\varphi}^2}{2} - mg\left(H - l\cos\varphi\right)\right)$$

which when simplified takes the form

$$\ddot{\varphi} + \frac{2\dot{l}}{l}\dot{\varphi} + \frac{g}{l}\sin\varphi = 0$$

Given assumption 2 the small angle approximation may be validly applied and so we write

$$\ddot{\varphi} + \frac{2\dot{l}}{l}\dot{\varphi} + \frac{g}{l}\varphi = 0$$

or more specifically

$$\ddot{\varphi} + \frac{4a}{\sqrt{l_0} + at}\dot{\varphi} + \frac{g}{\left(\sqrt{l_0} + at\right)^2}\varphi = 0 \tag{2.94}$$

which is of the form (2.51) with $\sigma = 1$ and

$$\begin{aligned} p(t) &= \frac{4a}{\sqrt{l_0} + at} \\ q(t) &= \frac{g}{\left(\sqrt{l_0} + at\right)^2} \end{aligned} \tag{2.95}$$

and represents an equation similar to that of the damped SHO.

The Solution

It is clear from the above that

$$\sqrt{q(t)} = \frac{\sqrt{g}}{\sqrt{l_0} + at} \tag{2.96}$$

while

$$\frac{d}{dt}\sqrt{q(t)} = -\frac{a\sqrt{g}}{\left(\sqrt{l_0} + at\right)^2} \tag{2.97}$$

Substituting (2.96) and (2.97) into (2.46) and solving for $p(t)$ we get

$$p(t) = \frac{a - K\sqrt{g}}{\sqrt{l_0} + at} \tag{2.98}$$

Comparing (2.95) with (2.98) leads us to conclude that condition (2.46) of Corollary 2.1 is satisfied for $K = -\frac{3a}{\sqrt{g}}$. Additionally, since $\sqrt{q(0)} \neq 0$ the ODE (2.94) meets all of the conditions of Corollary 2.1 and therefore a direct substitution of this problem's particulars into (2.57) will yield the exact analytical solution of (2.94) i.e.

$$\varphi(t) = \begin{cases} A_1 v^{\frac{1}{2} + \frac{\sqrt{9a^2 - 4g}}{6a}} + B_1 v^{\frac{1}{2} - \frac{\sqrt{9a^2 - 4g}}{6a}} & \text{if } a^2 > \frac{4g}{9} \\ \sqrt{v}\left(A_2 \cos\left(\ln v^{\frac{\sqrt{4g - 9a^2}}{6a}}\right) + B_2 \sin\left(\ln v^{\frac{\sqrt{4g - 9a^2}}{6a}}\right)\right) & \text{if } a^2 < \frac{4g}{9} \\ \sqrt{v}\left(A_3 + B_3 \ln v\right) & \text{if } a^2 = \frac{4g}{9} \end{cases} \tag{2.99}$$

where

$$v = \frac{l_0^{\frac{3}{2}}}{\left(\sqrt{l_0} + at\right)^3}$$

and where the constants $A_i, B_i \in \mathbb{R}$ $i = 1, 2, 3$ may be selected to satisfy the ICs of the system.

The solution (2.99) describes, in an explicit way, how the value of the state variable φ varies over time. Analysing the functions that define the solution of (2.94) allows one to determine the answers to the questions of system stability and periodicity as well as easing the problems associated with controller design.

Solution Analysis

An immediately clear feature of the solution (2.99) is that it is split into three distinct forms. Each of these forms is defined by the value of the parameter a, a fact that tells us that the behaviour of the system is heavily dependent on a. Bearing this in mind we recall the equation of the damped SHO

$$\ddot{\varphi} + \beta\dot{\varphi} + \omega_0^2\varphi = 0 \tag{2.100}$$

which can be solved by the method of undetermined coefficients to yield the solution

$$\varphi = \begin{cases} \bar{A}_1 e^{\left(-\frac{\beta}{2}+\sqrt{\beta^2-4\omega_0^2}\right)t} + \bar{B}_1 e^{\left(-\frac{\beta}{2}+\sqrt{\beta^2-4\omega_0^2}\right)t} & \text{if } \beta^2 > 4\omega_0^2 \\ e^{-\frac{\beta}{2}t}\left(\bar{A}_2 \sin\left(\sqrt{4\omega_0^2-\beta^2 t}\right) + \bar{B}_2 \cos\left(\sqrt{4\omega_0^2-\beta^2 t}\right)\right) & \text{if } \beta^2 < 4\omega_0^2 \\ \left(\bar{A}_3 t + \bar{B}_3\right) e^{-\frac{\beta}{2}t} & \text{if } \beta^2 = 4\omega_0^2 \end{cases} \tag{2.101}$$

where $\bar{A}_i$ and $\bar{B}_i$ $i = 1, 2, 3$ are real constants which serve to satisfy the ICs of (2.100).

The solution (2.101) one will mark, bears a striking resemblance to (2.99). Indeed, this is not terribly surprising if one considers that (2.94) is essentially a damped SHO with a time variable restoring force parameter, namely $\frac{g}{l}$ and a time variable damping parameter, namely $\frac{2\dot{l}}{l}$. Furthermore, given equation (2.92), the values of $\frac{g}{l}$ and $\frac{2\dot{l}}{l}$ are both always positive after some finite period and so could conceivably produce a behaviour similar to that which is observed in (2.100) since β and ω_0^2 are always positive.

The three different solution régimes present in (2.101) correspond to three distinct modes of operation of the oscillator, namely overdamped, underdamped and critically damped. In the first case, when the system is overdamped, one would see no oscillations at all since the damping is too high to allow any. In the second, underdamped case, the damping is sufficiently small to allow oscillations. However, since there is still damping present even in the underdamped case these oscillations will decay over time. The critically damped case represents the border (separatrix) that separates the two other behavioural modes. As a matter of interest we note that in the critically damped scenario, just as in the overdamped scenario, there can be no oscillations.

In fact, these same three distinct behavioural modes also correspond to the three different solution régimes of equation (2.99). To illustrate this point we plot the solution (2.99) for three different values of a, namely;

$$\begin{aligned} a^2 &= \frac{9}{2} > \frac{4g}{9} \\ a^2 &= 1 < \frac{4g}{9} \\ a^2 &= \frac{4g}{9} \end{aligned}$$

where we impose $a > 0$ to avoid singularities in the solution. In plotting the solutions associated with each a we keep all the other parameters fixed. Figure 2.3 demonstrates the similarity between the solutions of (2.94) and (2.100) admirably, showing as it does the overdamped, underdamped and critically damped solution modes. The figure also shows

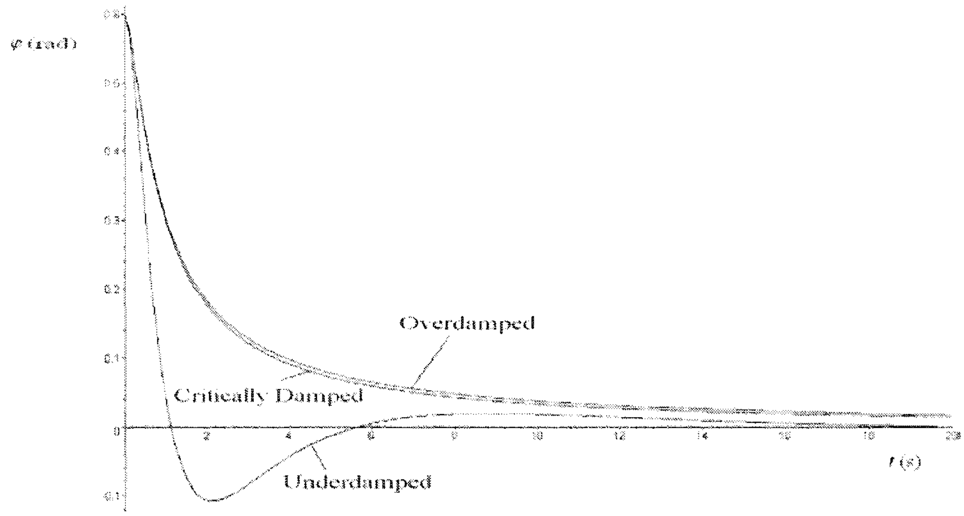

Fig. 2.3. Three distinct behavioural modes of the variable length pendulum with $l_0 = 1\,\mathrm{m}$, $\varphi_0 = \frac{3}{5}\,\mathrm{rad}$ and $\dot{\varphi}_0 = 0\,\mathrm{rad\,s^{-1}}$.

how each behavioural mode decays exponentially (though admittedly, at different rates) to the stable equilibrium position $\varphi = 0$.

N.B. The solutions plotted in Figure 2.3 were corroborated using numerical simulations implemented in MATLAB Simulink® version 5.2 and employing the Dormand-Prince RKF45 solver.

2.5.2 Linear Systems Control

The second application we consider addresses an outstanding problem in control theory. To appreciate the importance of this application we must divert ourselves for a moment and indulge in a brief introduction to the motivations and objectives of control theory.

Objectives of Control Theory

Control theory is a significant branch of applied mathematics that has immediate practical implications for physics and engineering. Those who practise and study control are very often attempting to address one essential research question, that is: *"Given a dynamical system, how can we, by the application of some external input, force the trajectory of the system to behave in a prescribed manner?"*. This *"prescribed manner"* is commonly a single state i.e. the control engineer wishes to drive the state of a system to a single point (regulation point) in the state space and have it remain there for all time. This is generally achieved by designing an external input, or controller, in such a way that the state to which one wishes to drive the system's trajectory becomes a stable attractor for that system. Consequently, the kinds of stability analyses described in Appendix B.1 are invaluable to the control designer as they allow identification of the conditions that must be satisfied before the ends of the control strategy are met.

For example, consider the linear time-invariant (LTI) system

$$\dot{\mathrm{x}} = \mathrm{A}\mathrm{x} \tag{2.102}$$

where $\mathrm{x} \in \mathbb{R}^n$ represents the state of the system at time t. If we were to apply some external drive $\mathbf{u}$, in order to control this system it would adopt the new form

$$\dot{\mathrm{x}} = \mathrm{A}\mathrm{x} + \mathrm{B}\mathbf{u}$$

where the matrix B tells us which elements of x may be influenced by $\mathbf{u}$. The methods of designing the controller $\mathbf{u}$ for such systems as these are well-established [23]. In addition, the stability of the fixed point (which we denote $\hat{\mathrm{x}}$ - see Appendix A §A.4) of (2.102) can be determined by proving that the eigenvalues of $\mathbf{A}$ all have negative real parts. For this autonomous linear system, the fixed point is simply $\hat{\mathbf{x}} = \mathbf{0}$.

On the other hand, there is no universal method for designing a controller for the general linear time-varying (LTV) system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} + \mathbf{B}(t)\,\mathbf{u}$$

N.B. This is not to suggest that there does exist such a method for linear time-invariant (LTI) systems, for this too is an open problem.

This stems mainly from the fact that the stability of such systems cannot be determined from mere eigenvalue inspection. Indeed, there are LTV systems for which the eigenvalues of $\mathbf{A}(t)$ have negative real parts for each fixed time and yet the system is unstable. One such (hypothetical) example is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{2.103}$$

Here, the eigenvalues of $\mathbf{A}(t)$ are found by solving the *characteristic equation* $0 = |\mathbf{A}(t) - \lambda I|$. This yields the two time-invariant eigenvalues $\lambda_{1,2} = -0.25 \pm i0.25\sqrt{7}$, both of which have negative real parts. Nevertheless, it can be proven (see [101]) that $\lim_{t\to+\infty} \mathbf{x} \to +\infty$.

This last example highlights one significant open problem in control theory; namely, the determination of the stability of LTV systems. With this issue in mind we will, in what follows, make use of the novel results established earlier in this section to design new controllers for certain classes of LTV systems.

The Problem

Consider the 2nd order LTV system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{2.104}$$

where $u, x_{1,2} \in \mathbb{R}$ and the real-valued functions a and b are assumed to be continuous and possessing of continuous first derivatives $\forall t \geq t_0$. Denoting $x = x_1$ and $\dot{x} = x_2$ we may rewrite the system (2.104) as the 2nd order linear ODE

$$\ddot{x} + a(t)\,\dot{x} + b(t)\,x = u \tag{2.105}$$

We will assume throughout that the ICs $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$ are given.

The problem we will address is the development of a strategy for designing a controller that will drive the solution $x(t)$ of (2.105) to some regulation point using a bounded controller. The latter of these stipulations is included such that the controller may be implemented practically (it should be pointed out here that other considerations besides boundedness must be taken into account before a controller can be said to be entirely practicable). Finally, we will also assume that we must deal with the additional difficulty of not being able to monitor the state variable $\dot{x}$.

Controller Design

Here we consider the problem of driving the solution $x(t)$ of (2.105) to some real constant value, which we denote $\tilde{x}$, under the restriction that the controller may be a function of x and t only. That is, we assume that only x is measured by some sensor. Consequently, our controlled system takes the form

$$\ddot{x} + a(t)\,\dot{x} + b(t)\,x = u(x,t) \tag{2.106}$$

In designing the controller $u(x,t)$ we first note that for any given $a(t)$ we can find a find a function, say $c(t)$, such that the functions a and c satisfy the hypotheses of Corollary 2.1. Bearing this in mind we set

$$u(x,t) = (b(t) - \sigma c(t))\,x \tag{2.107}$$

with $\sigma = \pm 1$ and

$$c(t) = \frac{c_0 e^{-2\int_{t_0}^{t} a(\tau)d\tau}}{\left(1 + K\sqrt{c_0}\int_{t_0}^{t} e^{-\int_{t_0}^{\lambda} a(\tau)d\tau}d\lambda\right)^2} \tag{2.108}$$

where $c(t) \geq 0\ \forall t \geq t_0$, the constant $c_0 = c(t_0) > 0$ but is otherwise a free parameter while K may be chosen without restriction.

N.B. What we have just done here can be termed linear cancellation since we have used our controller to cancel an unwanted linear term and replace it with another, more desirable one. However, there is nothing stopping us from extending this approach to problems where the cancellation of a nonlinear term is required.

Notice that since we want $c(t)$ to be continuous $\forall t \geq t_0$ it is vital to ensure that $\nexists t \geq t_0$ such that

$$1 + K\sqrt{c_0}\int_{t_0}^{t} e^{-\int_{t_0}^{\lambda} a(\tau)d\tau}d\lambda = 0$$

This may not always possible (since we have no control of the form of the function a), but one can often satisfy this requirement by judicious selection of the parameters K and c_0.

Substituting (2.107) into (2.106) we obtain the closed-loop system

$$\ddot{x} + a(t)\dot{x} + c(t)x = 0 \tag{2.109}$$

and from Corollary 2.1b we know that

$$x(t) = \begin{cases} A_1 v^{\lambda_1} + B_1 v^{\lambda_2} & \text{if } K^2 > 4, \sigma = 1 \\ v^a \left(A_2 \cos\left(\ln v^b\right) + B_2 \sin\left(\ln v^b\right)\right) & \text{if } K^2 < 4, \sigma = 1 \\ v^\lambda \left(A_3 + B_3 \ln v\right) & \text{if } K^2 = 4, \sigma = 1 \\ A_4 v^{\lambda_1} + B_4 v^{\lambda_2} & \text{if } \sigma = -1 \end{cases}$$

where of course

$$\begin{aligned} v &= 1 + K\sqrt{c_0}\int_{t_0}^{t} e^{-\int_{t_0}^{\lambda} a(\tau)d\tau} d\lambda \\ &= 1 + K\sqrt{c_0}\left(\frac{1}{a(t)}e^{-\int_{t_0}^{t} a(\tau)d\tau} - \frac{1}{a_0}\right) \end{aligned}$$

So that we might isolate an individual solution form we set $K^2 > 4$ (or equivalently $|K| > 2$) and $\sigma = 1$ such that

$$\begin{aligned} x(t) &= A_1 v^{\lambda_1} + B_1 v^{\lambda_2} \\ &= A_1 e^{K\lambda_1 \int_{t_0}^{t}\sqrt{c(\tau)}d\tau} + B_1 e^{K\lambda_2 \int_{t_0}^{t}\sqrt{c(\tau)}d\tau} \end{aligned}$$

If $c(t)$, as defined by (2.108) is continuous, then $\int_{t_0}^{t}\sqrt{c(\tau)}d\tau$ is a non-decreasing function of t. For arbitrary $a(t)$ however, we cannot say anything about the boundedness of $\int_{t_0}^{t}\sqrt{c(\tau)}d\tau$ and so we must examine the bounded and unbounded cases individually.

1. If $\lim_{t\to+\infty}\int_{t_0}^{t}\sqrt{c(\tau)}d\tau = \infty$ we can say that $\lim_{t\to+\infty} x(t) = 0$ if $K\lambda_1 < 0$ and $K\lambda_2 < 0$.

   Recalling that

$$\begin{aligned} \lambda_1 &= \frac{1}{2} + \frac{\sqrt{K^2-4}}{2K} \\ \lambda_2 &= \frac{1}{2} - \frac{\sqrt{K^2-4}}{2K} \end{aligned}$$

it is clear that, irrespective of the sign of K, at least one of λ_1 or λ_2 will be positive. Consequently, the only way to ensure that $K\lambda_1 < 0$ and $K\lambda_2 < 0$ is to set $K < -2$ whilst ensuring that $\lambda_{1,2} > 0$. To satisfy this requirement we set $K = -(2+\varepsilon)$ for some $\varepsilon > 0$. For this choice of K, $\lambda_2 > 0$ irrespective of the size of ε. It therefore remains only to find those values of ε that satisfy the inequality

$$\lambda_1(\varepsilon) = \frac{1}{2} - \frac{\sqrt{(2+\varepsilon)^2 - 4}}{2(2+\varepsilon)} > 0 \tag{2.110}$$

Since $\lambda_1(0) = \frac{1}{2}$ and $\lim_{\varepsilon \to +\infty} \lambda_1 = 0$ we can prove that (2.110) holds for all $\varepsilon > 0$ if we can show that $\lambda_1(\varepsilon)$ has no turning points. To this end we differentiate $\lambda_1(\varepsilon)$ to obtain the expression

$$\frac{d\lambda_1}{d\varepsilon} = -\frac{4}{2(2+\varepsilon)^2\sqrt{4\varepsilon+\varepsilon^2}}$$

from which we conclude that $\nexists \varepsilon > 0$ such that $\frac{d\lambda_1}{d\varepsilon} = 0$ hence there are no turning points and (2.110) holds $\forall \varepsilon > 0$.

To summarise, using the controller defined by (2.107) and (2.108) with $K < 2$ and $\sigma = 1$ the solution of the closed-loop system (2.106) will always be driven to 0 provided that (2.108) contains no singularities and $\lim_{t \to +\infty} \int_{t_0}^{t} \sqrt{c(\tau)}d\tau = \infty$.

2. If $\lim_{t \to +\infty} \int_{t_0}^{t} \sqrt{c(\tau)}d\tau = L$ a constant then

$$\lim_{t \to +\infty} x(t) = A_1 e^{K\lambda_1 L} + B_1 e^{K\lambda_2 L}$$

irrespective of the values of K, λ_1 and λ_2. Supposing therefore that the object of the controller is to drive x to some constant value M, then one would select c_0 or K in such a way as to satisfy the equality

$$M = A_1 e^{K\lambda_1 L} + B_1 e^{K\lambda_2 L}$$

whilst still guaranteeing the continuity of $c(t)$. Such a selection would then achieve the aims of the control scheme.

Controller Bounds

It now remains to find those conditions which, when satisfied, guarantee the boundedness of the controller. To this end we recall that our controller is

$$u(x,t) = (b(t) - c(t))\,x$$

or more specifically

$$u(x,t) = \left(b(t) - \frac{c_0 e^{-2\int_{t_0}^{t} a(\tau)d\tau}}{\left(1 + K\sqrt{c_0}\left(\frac{1}{a(t)} e^{-\int_{t_0}^{t} a(\tau)d\tau} - \frac{1}{a_0}\right)\right)^2} \right) x \tag{2.111}$$

wherein only K and c_0 are free parameters (the rest of the terms being governed by the nature of the system being modelled).

It would be a significant analytical task in itself to place sufficient and/or necessary conditions on the boundedness of (2.111). This being the case, we shall simply state one instance for which the controller is bounded.

N.B. Since our controller ensures that $\lim_{t\to+\infty} x(t)$ bounded and we select K and c_0 in such a way as to ensure that $c(t)$ possesses no singularities. As a result, establishing the boundedness of the first term in the product appearing in (2.111) is sufficient to prove the boundedness of u.

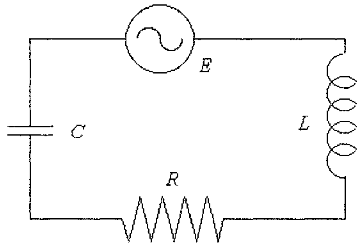

Fig. 2.4. The LCR series circuit.

If $b(t)$ is bounded and $a(t) \equiv a \neq 0$ a constant then

$$u(x,t) = \left(b(t) - \frac{c_0 e^{-2a(t-t_0)}}{\left(1 + \frac{K\sqrt{c_0}}{a}\left(e^{-a(t-t_0)} - 1\right)\right)^2} \right) x \tag{2.112}$$

and $u(x,t)$ is quite obviously bounded.

Example

For the purposes of clarifying the above exposition on controller design we will now present the control design technique within the context of a series LCR circuit problem.

Modelling the series LCR circuit given in Figure (2.4) one obtains the ODE

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}(q - q_0) = \frac{E(t)}{L}$$

where $q(t)$ is the charge on the capacitor at time t, while L, R and C represent the values of the series inductance, resistance and capacitance of the circuit respectively and $E(t)$ is the voltage applied to the circuit at time t. The ICs of the system are $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$.

In this problem we suppose that, though L and R are constant, the value of C varies in some fashion and consequently impacts upon the normal dynamics of q. Let us also suppose that we want the the charge on the capacitor to be maintained at some constant value Q for all time, notwithstanding the inherent dynamics of the system and the variations in series capacitance. In order to do this we propose to measure the values of C and q as they vary and use those values to determine the voltage E that should be applied to the circuit in order to force $q(t) \to Q$.

First however, we will make the change of variable $x = q - q_0$ to obtain the modified equation

$$\ddot{x} + \frac{R}{L}\dot{x} + \frac{1}{LC(t)}x = \frac{E(t)}{L} \tag{2.113}$$

In order to meet the aims of our control system we design E as follows:

$$E(t) = \left(\frac{1}{C} - Lc(t)\right)x$$

where $c(t) \geq 0 \; \forall t \geq t_0$. Our closed-loop system is therefore

$$\ddot{x} + \frac{R}{L}\dot{x} + c(t)\,x = 0 \tag{2.114}$$

with $x(0) = 0$ and $\dot{x}(0) = \dot{q}_0$.

The function $c(t)$ appearing in our controller is defined by the relation

$$c(t) = \frac{c_0 e^{-\frac{2R}{L}t}}{\left(1 + \frac{KL\sqrt{c_0}}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right)^2} \tag{2.115}$$

for some $\sqrt{c_0} > 0$ and $|K| > 2$. Note that it is possible to ensure the continuity of $c(t)$ by setting either

$$\begin{aligned} \sqrt{c_0} &> -\frac{R}{(2+\varepsilon)L} \quad \text{for } K = 2 + \varepsilon \\ \sqrt{c_0} &< \frac{R}{(2+\varepsilon)L} \quad \text{for } K = -(2 + \varepsilon) \end{aligned}$$

where $\varepsilon > 0$.

The first of these inequalities is true by definition since $\sqrt{c_0} > 0$, while the latter may be satisfied by setting

$$\sqrt{c_0} = \frac{R}{\kappa L(2+\varepsilon)}$$

for some $\kappa > 1$. Indeed, for this problem we will choose $K = -(2+\varepsilon)$ and $\sqrt{c_0} = \frac{R}{\kappa L(2+\varepsilon)}$. By consequence, $c(t)$ now assumes the form

$$c(t) = \frac{R^2 e^{-\frac{2R}{L}t}}{L^2(2+\varepsilon)^2\left((\kappa-1)+e^{-\frac{R}{L}t}\right)^2}$$

It can be shown that the coefficients of $\dot{x}$ and x in (2.114) satisfy the hypotheses of Corollary 2.1b and so we write the solution of (2.114) as

$$x(t) = A_1 v^{\lambda_1} + B_1 v^{\lambda_2}$$

where

$$\begin{aligned} v &= 1 - \frac{1}{\kappa}\left(1 - e^{-\frac{R}{L}t}\right) \\ \lambda_1 &= \frac{1}{2} - \frac{\sqrt{4\varepsilon+\varepsilon^2}}{2(2+\varepsilon)} \\ \lambda_2 &= \frac{1}{2} + \frac{\sqrt{4\varepsilon+\varepsilon^2}}{2(2+\varepsilon)} \\ A_1 &= \frac{\kappa L \dot{q}_0(2+\varepsilon)}{R\sqrt{4\varepsilon+\varepsilon^2}} \\ B_1 &= -A_1 \end{aligned}$$

Taking the limit of the solution as $t \to +\infty$ we find that

$$\lim_{t\to+\infty} x(t) = A_1\left(\left(1-\frac{1}{\kappa}\right)^{\lambda_1} - \left(1-\frac{1}{\kappa}\right)^{\lambda_2}\right)$$

or in terms of $q(t)$

$$\lim_{t\to+\infty} q(t) = A_1\left(\left(1-\frac{1}{\kappa}\right)^{\lambda_1} - \left(1-\frac{1}{\kappa}\right)^{\lambda_2}\right) + q_0$$

Recalling that it is our intention to control this system in such a way that $\lim_{t \to +\infty} q(t) = Q$ we write

$$Q = A_1 \left(\left(1 - \frac{1}{\kappa}\right)^{\lambda_1} - \left(1 - \frac{1}{\kappa}\right)^{\lambda_2} \right) + q_0 \tag{2.116}$$

which is an equation in the variables $\varepsilon > 0$ and $\kappa > 1$. We can therefore drive the capacitor charge $q(t)$ to the desired value Q provided we can find an ε and κ that solve (2.116) for a given set of parameters q_0, $\dot{q}_0$, Q, R and L.

This control problem was simulated in MATLAB Simulink® for the following parametric choices

$$\begin{aligned} q_0 &= 0\,\mathrm{C} \\ \dot{q}_0 &= 1\,\mathrm{A} \\ Q &= 0.01\,\mathrm{C} \\ R &= 180\,\Omega \\ L &= 20\,\mathrm{H} \\ C(t) &= \frac{1}{1 + 0.01\sin(2t)}\,\mu\mathrm{F} \end{aligned}$$

and was performed under the assumption that the cancellation of the variable capacitance by the controller $E(t)$ was perfect. The choice of the variable capacitance follows in the vein of [43].

For these values (2.116) becomes

$$0.01 = \frac{\kappa(2+\varepsilon)}{9\sqrt{4\varepsilon + \varepsilon^2}} \left(\left(1 - \frac{1}{\kappa}\right)^{\lambda_1} - \left(1 - \frac{1}{\kappa}\right)^{\lambda_2} \right)$$

To determine appropriate values of ε and κ we selected $\varepsilon = 0.01$ and used a graphical method to find the corresponding κ that would solve (2.116). It was consequently found that

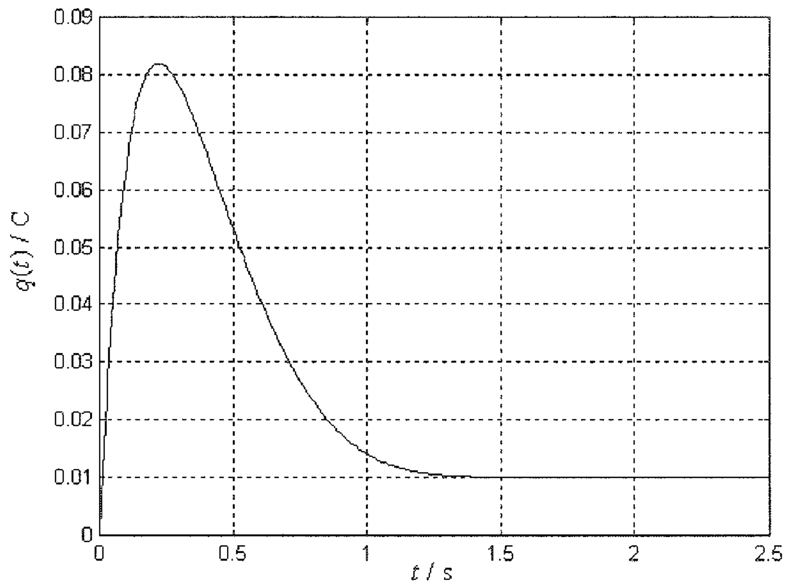

Fig. 2.5. Charge on capactior $q(t)$ plotted against time t. The convergence of solution on the desired value of $Q = 0.01$ C is clearly acheived.

(2.116) is satisfied provided that $\varepsilon = 0.01$ and $\kappa = 1.000086$. The result of the simulation is shown in Figure 2.5 and demonstrates the success of our control scheme in driving $q(t)$ to Q. Furthermore, Figure 2.6 demonstrates, as one would expect, the boundedness of the control input function $E(t)$.

Remark One will recall that in this simulation, a perfect cancellation of the coefficient of x in (2.113) was assumed. As a consequence, it could be argued that the results here presented appear to be better than what may be practically attainable. Furthermore, it should be borne in mind that the proposed control scheme is based on the theory of continuous functions, however, in practice the control system would most likely be implemented using microprocessors which make use of discrete time steps. As such,

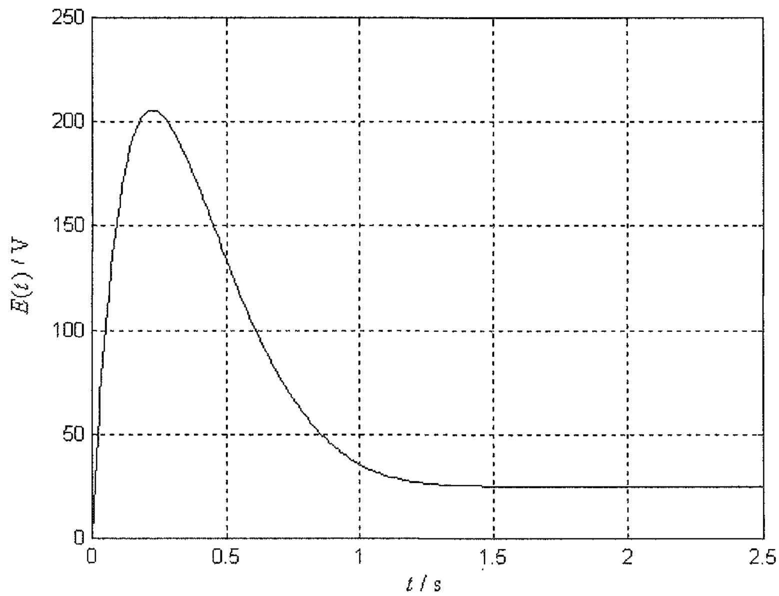

Fig. 2.6. The control input function $E(t)$ plotted against time t. The control function is, by consequence of the design scheme, bounded.

the practicability of the scheme here given could be justifiably regarded with a certain amount of scepticism.

2.6 Summary

In this chapter we have seen how important and powerful exact analytical solutions of ODEs are. We have also seen however that these solutions may only be obtained for certain classes of problems and that these classes are rather restricted. In particular, we have looked at these matters in the context of the Riccati equation and the (in)homogeneous linear ODE of order 2 i.e. equations (2.13), (2.1) and (2.2) and Liénard's equation (2.3). We have also looked in some detail at some of the well-established methods for analytically solving (non)linear ODEs of order 1 and 2. In addition, the current direction of research in this field has also been reviewed with particular attention being focussed on the current trend of using/developing numerical solutions of ODEs. More importantly however, we have stated and proven our main results on a new exactly solvable class of Riccati equations and homogeneous linear ODEs of order 2 (the class being defined by the condition (2.46)). More specifically, we obtained a solution of a certain class of Riccati equations and applied the result to the solution of the associated 2nd order linear ODE with variable coefficients. This work was subsequently extended yet further by identifying a distinct and familiar solution structure to the class of linear ODEs considered. As a final addendum to this work, we used our original Riccati equation solution in the identification of a more general solvable class of Riccati ODEs.

In addition, we have stated and validated our alternative method for the solution of ODEs of the form (2.2) wherein the exact solution of the corresponding homogeneous problem was known. The method's advantages over Lagrange's variation of parameters in certain circumstances were also discussed with regard to the choice offered in the determination of the solution.

Motivated by an interest in systems that have a capacity for chaotic behaviour and connecting Liénard and Riccati-type problems we followed our treatment of linear problems by considering the solution of two equations belonging to the class of equations defined by (2.79), which is itself a subclass of the more general Liénard equation (2.3). The results obtained included an extension of Chandrasekar's analytical solution [19] of the DVP oscillator as well as the analytical solution of the nonlinear ODE defined by (2.84) which connects the equations of Liénard and Riccati. Each of these were solved by finding appropriate variable transforms that modified the original problems into known solvable forms.

Finally the new results have been vindicated as a piece of applied mathematics by using the newly derived solutions in two applications. Firstly, we used our results to determine the exact analytical solution of the ODE (2.94); which models the dynamics of a 1-D variable length pendulum, the results obtained being verified by numerical simulation. Following this, we showed how our solutions can aid in the design of bounded controllers for LTV systems. Again, we vindicated our claims by applying the ideas presented to the problem of driving the charge on a variable capacitor in a series LCR circuit to some desired constant using a bounded controller. However, it should be noted that the LCR simulation was an idealised one and a more thorough investigation into the limitations of the proposed scheme is required before any concrete conclusions may be drawn regarding its practicability.

In the following chapter we shall move away from the problem of obtaining exact analytical solutions and instead assume that the systems under consideration are such that there is no means by which an exact solution may be obtained. In particular, we will look at how one might determine one of the most important features of the solution without ever having to solve the governing equation. The feature referred to is the boundedness of the solution.

Chapter 3
Boundedness of Solutions of ODEs

In the previous chapter we stressed the point that not all ODEs that one might encounter may possess an elementary solution. In these instances one might attempt to determine a solution in the form of an infinite series of elementary functions. These sorts of solutions are, like elementary solutions, still very helpful when trying to classify solution properties. However, there exist ODEs that not only refuse to yield an elementary solution but also admit no solution in terms of an infinite series either. In such instances as this one is left with but one resort; employ a numerical scheme to find an approximation of the solution. Indeed, there exist many such schemes as this; all of which work admirably within the scope of problems for which they were designed to tackle. One could ask however how it is that we know these schemes work well when the ODE at hand cannot be solved, since one has no actual solution to employ as a yardstick by which one could measure the quality of one's approximate solution. One possible, yet unsatisfactory answer, would be to argue that if a given numerical method, when used to approximate the solution to a problem where the actual solution is known, makes a superb approximation of the solution, then it must necessarily perform equally well on problems for which the solution is not known. One needn't however make this leap of faith if one had some means of determining certain testable properties of the solution without requiring any access to the solution itself. Such means as these constitute the qualitative methods of ODEs, an example of which we give in Appendix B.1 in the guise of the linearised stability analysis. The linearised analysis was developed under the assumption that the ODEs to which it would be applied possessed solutions that were intractable. With this in mind, the method sets about determining the behaviour of particular solutions within a small neighbourhood of the system's fixed points. Checking the nature of a numerically derived solution against what one expects the solution to do in the vicinity of the fixed points allows one to determine the validity of that approximate solution.

As helpful as the qualitative methods are in the manner just described, it would be unfair to suggest that such methods exist for the exclusive application of verifying numerical solutions. Indeed, important questions regarding a given dynamical system's controllability, observability, stability, boundedness and periodicity can be answered in no completely satisfactory way by any numerical solver. Now obviously these questions could be answered immediately if one had access to the system's analytical solution. However, assuming that such a solution cannot be found, one is left with the task of answering these questions using qualitative analytical methods and results, examples of which include; the linearised analysis, Lyapunov's direct method (see Theorem B1.1 in Appendix B.1) and the Poincaré-Bendixson Theorem (see Theorem B.1.3 in Appendix B.1) to name but a few. In fact, the primary use of the qualitative methods for ODEs is to determine the various properties of an ODE's solution without actually finding that solution.

The particular solution property of interest to us in this chapter is boundedness. Determining the conditions under which the solutions of a given ODE are bounded is extremely important if we are interested in operating the dynamical system described by that ODE in such a way that one or more of its state variables should remain confined to a prescribed region of state space. For example, if the solution of an ODE represented the variation in temperature of some electronic device over time then in order to avoid damaging the device one would want to ensure that the solution was confined to some pre-defined range of values. This would be achieved by adhering to the conditions laid out by the qualitative analysis for such bounds to be seen as impassable by the solution.

The primary approach to determining conditions for the boundedness of the solutions of an ODE makes great use of integral inequalities i.e. an inequality relating a function to its antiderivative. In the following sections we shall show how one can use integral inequalities to determine boundedness conditions for ODEs as well as look at some of the different integral inequalities that have been developed with this application in mind. Finally, we will present several novel integral inequalities and show how one in particular may be applied to determine the boundedness of solutions of class of nonlinear ODEs of order 2.

3.1 Integral Inequalities and the Boundedness of Solutions of ODEs

It has long been established that the integral inequality known as Grönwall's Lemma, first reported in 1919 by T.H. Grönwall [36], and its subsequent generalisation by R. Bellman in [4] is a tool of the utmost utility in both the qualitative and quantitative study of linear ODEs and integro-differential equations (IDEs). Indeed, there exists no small number of textbooks [2],[3],[26],[60],[70],[83] that either deal with, or are dedicated to, the theory and application of Grönwall-Bellman type inequalities and as such, are testament to their value.

In the subsections below we will chart the history of the integral inequality starting with Bellman's generalisation of Grönwall's Lemma and going all the way up to the present day. Since our focus here is boundedness, we will also show by way of example, how the Grönwall-Bellman inequality may be employed to determine the bounds of solutions of certain ODEs.

Throughout this chapter we shall denote the closed intervals $[t_0, t_1] \subseteq \mathbb{R}$ and $[0, \infty) \subset \mathbb{R}$ by I and $\mathbb{R}_+$ respectively.

3.1.1 The Grönwall-Bellman Inequality

Here we state and prove the Grönwall-Bellman inequality [83].

Lemma 3.1 *Let $u, q : I \to \mathbb{R}_+$ (where I is the closed interval $[t_0, t_1] \subseteq \mathbb{R}$) be continuous functions while $m \geq 0$ and $\alpha > 0$ are real constants. If*

$$u(t) \leq \alpha + m \int_{t_0}^{t} q(\tau)\, u(\tau)\, d\tau \tag{3.1}$$

$\forall t \in I$, then

$$u(t) \leq \alpha e^{m \int_{t_0}^{t} q(\tau) d\tau}$$

$\forall t \in I$.

Proof. Let us denote the RHS of (3.1) by $v(t)$ and differentiate to get

$$\dot{v}(t) = mq(t)\, u(t)$$

By definition $v(t) \geq u(t)\ \forall t \in I$ so

$$\dot{v}(t) \leq mq(t)\, v(t)$$

Since the sign of $v(t)$ does not change over I we may divide throughout by $v(t)$ and subsequently integrate from t_0 to $t \in I$ to get

$$\ln\left(\frac{v(t)}{\alpha}\right) \leq m \int_{t_0}^{t} q(\tau)\, d\tau$$

or alternatively

$$v(t) \leq \alpha e^{m \int_{t_0}^{t} q(\tau) d\tau}$$

hence

$$u(t) \leq \alpha e^{m \int_{t_0}^{t} q(\tau) d\tau}$$

■

Lemma 3.1 is an useful result because, even if u is unknown, then one may still say that u is bounded $\forall t \in I$ provided

$$\int_{t_0}^{t} q(\tau)\, d\tau < M$$

$\forall t \in I$ where M is some positive constant. To test for this provision one would only require the knowledge of q. In fact, we will show in the proof of the next theorem how this kind of idea can be applied to the determination of solutions bounds of ODEs [83].

Theorem 3.1 *If q is a continuously differentiable function satisfying the conditions*

$$\begin{aligned} \lim_{t \to +\infty} q(t) &= 0 \\ \int_{0}^{\infty} |q(\tau)|\, d\tau &< \infty \end{aligned}$$

then all solutions of the equation

$$\ddot{x}(t) + (1 + q(t))\, x(t) = 0 \tag{3.2}$$

are bounded.

Proof. Multiplying (3.2) throughout by $\dot{x}(t)$ and integrating from 0 to t we get

$$\dot{x}^2(t) - \dot{x}_0^2 + x^2(t) - x_0^2 + 2\int_0^t q(\tau)\, x(\tau)\, \dot{x}(\tau)\, d\tau = 0$$

where $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ are known. Using integration by parts it can be verified that

$$\int_0^t q(\tau)\, x(\tau)\, \dot{x}(\tau)\, d\tau = \frac{q(t)\, x^2(t)}{2} - \frac{q_0 x_0^2}{2} - \frac{1}{2}\int_0^t \dot{q}(\tau)\, x^2(\tau)\, d\tau$$

hence

$$\begin{aligned} x^2(t)(1 + q(t)) &= K - \dot{x}^2(t) + \int_0^t \dot{q}(\tau)\, x^2(\tau)\, d\tau \\ &\le K + \int_0^t \dot{q}(\tau)\, x^2(\tau)\, d\tau \end{aligned}$$

where $K = x_0^2(1 + q_0) + \dot{x}_0^2$.

Since $\lim_{t\to+\infty} q(t) = 0$ $\exists t_0 \ge 0$ such that $1 + q(t) \ge \frac{1}{2}$ $\forall t \ge t_0$. We may therefore infer that

$$x^2(t) \le 2K + 2\int_0^t \dot{q}(\tau)\, x^2(\tau)\, d\tau$$

$\forall t \ge t_0$. Taking absolute values we have

$$\begin{aligned} \left|x^2(t)\right| &\le \left|2K + 2\int_0^t \dot{q}(\tau)\, x^2(\tau)\, d\tau\right| \\ &\le 2|K| + 2\int_0^t |\dot{q}(\tau)|\left|x^2(\tau)\right| d\tau \end{aligned}$$

Applying Lemma 3.1 we see that

$$\begin{aligned} \left|x^2(t)\right| &\leq 2|K| e^{2\int_0^t |\dot{q}(\tau)| d\tau} \\ &\leq 2|K| e^{2\int_0^\infty |\dot{q}(\tau)| d\tau} = M < \infty \end{aligned}$$

$\forall t \geq t_0$. Furthermore, since $q(t)$ is continuously differentiable, $x(t)$ is continuous throughout $t \in [0, t_0]$ and hence $x(t)$ is bounded $\forall t \in \mathbb{R}_+$. ■

It is worth remarking here that since the bounds derived in the above apply to the value of $|x^2(t)|$ we can state that the actual solution $x(t)$ is bounded both above and below.

3.1.2 Bihari's Lemma

The application of the Grönwall-Bellman inequality to the resolution of various stability and existence problems associated with linear ODEs inspired further research into extending and generalising the above result. The aim of this research was simply this: to establish yet more powerful tools for probing the properties of dynamical systems. An example of one of the more well-cited early contributions is Ou-Iang's paper of 1957 [68]. However, the biggest stride forward in this period was made in 1956 when Bihari published his paper [5] entitled "A Generalization of a Lemma of Bellman and its Application to Uniqueness Problems of Differential Equations" in which he stated and proved a nonlinear Grönwall-Bellman type inequality. The significance of Bihari's result was that it could be brought to bear upon the same stability and uniqueness problems that the Grönwall-Bellman inequality had been applied to, but now with respect nonlinear ODEs. Indeed, Bihari himself was quick to demonstrate this in both his paper of 1956 and another paper published in the following year [6]. We summarise Bihari's Lemma below.

Lemma 3.2 *Let $u, q : I \to \mathbb{R}_+$ be continuous functions while $m, \alpha \geq 0$ are real constants. Furthermore, define a function $g(u)$ that is non-negative, non-decreasing and continuous $\forall u \geq 0$. If*

$$u(t) \leq \alpha + m \int_{t_0}^{t} q(\tau) g(u(\tau)) d\tau$$

$\forall t \in I$, then

$$u(t) \leq G^{-1}\left(G(\alpha) + m \int_{t_0}^{t} q(\tau) d\tau\right)$$

$\forall t \in [t_0, t_1']$ where

$$G(z) = \int \frac{dz}{g(z)}$$

and where $t_1' \leq t_1$ is chosen such that $G(\alpha) + m \int_{t_0}^{t} q(\tau) d\tau$ is always within the domain of definition of $G^{-1}(z)$.

We shall not prove Bihari's Lemma as the exercise is not sufficiently fruitful to justify the effort of doing so. We shall simply remark that the above lemma reduces to that of Grönwall and Bellman's when $g(u) = u$ and as such may be applied to give conditions for the solutions of nonlinear ODEs to be bounded in a similar way as we saw in the proof of Theorem 3.1.

3.1.3 Recent Results

Over the last fifty years the investigations of authors such as Dafermos [30], Lipovan [55],[56], Pachpatte [69],[71],[72] and Cheung [25] have resulted in further generalisations of Grönwall-Bellman and Bihari type inequalities and with each generalisation there is associated a more general ODE, IDE or PDE, upon the solutions of which one can establish new conditions for boundedness.

Of the recent generalisations mentioned above, there is one of particular interest to us that is due to Pachpatte. In [72] Pachpatte stated and proved the following theorem.

Theorem 3.2 *Let $u, a, b : I \to \mathbb{R}_+$ and $\beta : I \to I$ with $\beta(t) \leq t$ be continuous functions and $\alpha \geq 0$ be some real constant. Furthermore, for $i = 1, 2$, let $g_i(u)$ be a non-negative, non-decreasing continuous function of u. If*

$$u(t) \leq \alpha + \int_{t_0}^{t} a(\tau)\, g_1(u(\tau))\, d\tau + \int_{\beta(t_0)}^{\beta(t)} b(\tau)\, g_2(u(\tau))\, d\tau$$

$\forall t \in I$ then

$$u(t) \leq G_1^{-1}(G_1(\alpha) + A(t) + B(t))$$

$\forall t \in [t_0, t_2]$ when $g_2(u) \leq g_1(u)$ or

$$u(t) \leq G_2^{-1}(G_2(\alpha) + A(t) + B(t))$$

$\forall t \in [t_0, t_2]$ when $g_1(u) \leq g_2(u)$ where

$$\begin{aligned} A(t) &= \int_{t_0}^{t} a(\tau)\, d\tau \\ B(t) &= \int_{\beta(t_0)}^{\beta(t)} b(\tau)\, d\tau \end{aligned}$$

and for $i = 1, 2$; G_i^{-1} are inverse functions of

$$G_i(r) = \int_{r_0}^{r} \frac{du}{g_i(u)}, \qquad r > 0,\ r_0 > 0$$

and $t_2 \in I$ is chosen such that the value of $G_i(\alpha) + A(t) + B(t)$ remains within the domain of G_i^{-1} respectively $\forall t \in [t_0, t_2]$.

N.B. when $g_1(u) = g_2(u)$ the two alternative bounds on $u(t)$ given by Theorem 3.2 are equal and as such there are no consistency issues relating to the choice offered.

This result lends itself immediately to the determination of solution bounds of the nonlinear ODE (1.1) under certain conditions by the method used in the proof of Theorem 3.1. Indeed, in the main results which follow we offer a generalisation of Theorem 3.2. This new result is then applied to obtain conditions under which certain classes of nonlinear ODEs of order 2 have bounded solutions.

3.2 Main Results

Before stating our main results we must first formally define the notation we will be using throughout the remainder of this chapter. First note that we denote by $[1, n]$ and $[1, K]$ the integer sequences $1, 2, \ldots, n$ and $1, 2, \ldots, K$ respectively (where n and K are of course integers themselves).

Definition 3.1 *For $k \in [1, K]$ let $\cup_{k=1}^{K} J_k = \mathbb{R}_+$ (where $K \geq 1$ is otherwise undefined) and let $g_i(u)$ be both non-negative and non-decreasing $\forall u \in \mathbb{R}_+$ and $\forall i \in [1, n]$ where $n \geq 1$. We define $J_k \subseteq \mathbb{R}_+$ by*

$$J_k = [U_{k-1}, U_k]$$

where $U_0 = 0$ and $U_k > U_{k-1}$ is the smallest value for which $\exists i, j \in [1, n]$ with $i \neq j$ such that

$$g_i(U_k) = g_j(U_k)$$

Theorem 3.3 *$\forall u \in J_k$, $\exists m(J_k) \in [1,n]$ such that $g_{m(J_k)}(u) \geq g_i(u)$, $\forall i \in [1,n]$ with $i \neq m(J_k)$.*

The proof of Theorem 3.3 is elementary.

Remarks:

1. It is possible to get a better grasp on the meaning of Definition 3.1 if one considers the family of curves $g_i(u)$ plotted against u. The sub-intervals J_k (J-intervals hereafter) of the u-axis are those intervals, between the boundaries of which, no two curves belonging to the family $g_i(u)$ intersect (see Figure 3.1).

2. Theorem 3.3 simply states the obvious point that the J-intervals are so defined that, between (and at) the boundaries of any given interval, there exists one and only one function in the family $g_i(u)$ which is uniquely defined by the fact that it is greater than or equal to any other function in the family within that interval.

3.2.1 A Generalisation of an Inequality of Pachpatte

The following lemma is a generalisation of Theorem 3.2 and will be used to prove the boundedness of solutions of certain nonlinear ODEs.

Lemma 3.3 *Let $u, a_i : I \rightarrow \mathbb{R}_+$ $\forall i \in [1,n]$ and let $g_i(u)$ be both non-negative and non-decreasing $\forall u \in \mathbb{R}_+$ and $\forall i \in [1,n]$. If $\alpha \geq 0$ is a real constant and*

$$u(t) \leq \alpha + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, g_i(u(\tau))\, d\tau \tag{3.3}$$

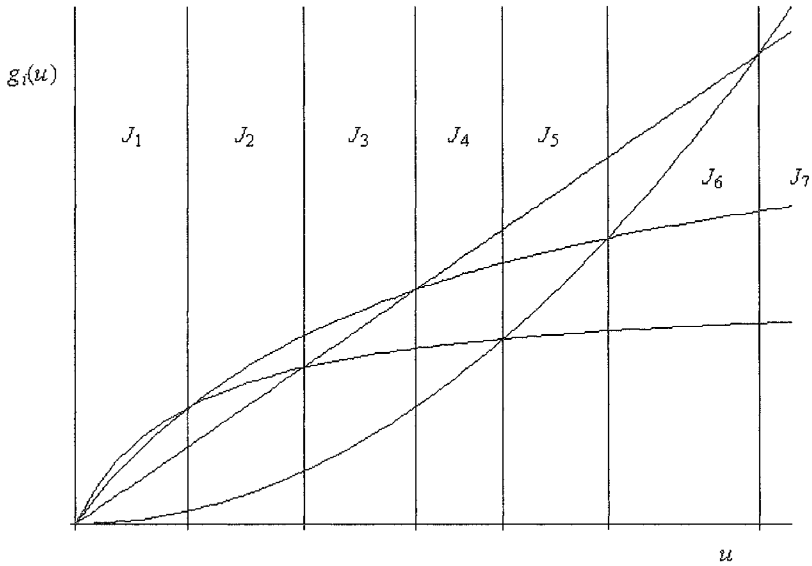

Fig. 3.1. J-interval boundaries for a family of $g_i(u)$ curves. Each intersection of two distinct curves defines a new boundary and hence J-interval.

$\forall t \in I$, then $\forall u \in J_k$

$$G_{m(J_k)}(u(t)) \leq G_{m(J_k)}(\alpha) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau$$

$\forall t \in I$ where the function $G_{m(J_k)}$ is defined by the relation

$$G_{m(J_k)}(z) = \int \frac{dz}{g_{m(J_k)}(z)} \tag{3.4}$$

N.B. For ease of writing we shall hereafter refer to those functions defined by (3.4) as G-functions.

Proof. Let $\alpha > 0$ and denote the RHS of (3.3) by the variable $v(t)$, that is

$$v(t) = \alpha + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, g_i(u(\tau))\, d\tau$$

Differentiating yields

$$\dot{v}(t) = \sum_{i=1}^{n} a_i(t)\, g_i(u(t)) \tag{3.5}$$

Since by definition $u(t) \leq v(t)$ and $g_i(u)$ is non-decreasing $\forall u \in \mathbb{R}_+$ and $\forall i \in [1, n]$ we may conclude that

$$\dot{v}(t) \leq \sum_{i=1}^{n} a_i(t)\, g_i(v(t)) \tag{3.6}$$

$\forall t \in I$. Bearing Theorem 3.3 in mind, if we consider any one sub-interval of the u-axis J_k we can isolate a single function in the family $g_i(v(t))$, which we denote $g_{m(J_k)}(v(t))$, that is greater than each of the others in that family over the entirety of J_k. Hence $\forall u \in J_k$

$$\dot{v}(t) \leq g_{m(J_k)}(v(t)) \sum_{i=1}^{n} a_i(t)$$

$\forall t \in I$. Dividing through by $g_{m(J_k)}(v(t))$ and integrating from t_0 to t we have

$$\int_{t_0}^{t} \frac{dv}{g_{m(J_k)}(v(\tau))} \leq \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau$$

Applying the Fundamental Theorem of Calculus and substituting from (3.4) we write

$$G_{m(J_k)}(v(t)) \leq G_{m(J_k)}(v(t_0)) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau$$

but $v(t_0) = \alpha$ so

$$G_{m(J_k)}(v(t)) \leq G_{m(J_k)}(\alpha) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau$$

$\forall u \in J_k$ and $\forall t \in I$. Given the monotonity of $G_i(v(t))$ $\forall i \in [1, n]$ we may take its inverse and so arrive at the expression

$$v(t) \leq G_{m(J_k)}^{-1}\left(G_{m(J_k)}(\alpha) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau\right)$$

for all t in some sub-interval of I in which $G_{m(J_k)}(\alpha) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau$ remains within the domain of $G_{m(J_k)}^{-1}$. Now, by definition $u(t) \leq v(t)$ hence

$$u(t) \leq G_{m(J_k)}^{-1}\left(G_{m(J_k)}(\alpha) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau\right)$$

and consequently

$$G_{m(J_k)}(u(t)) \leq G_{m(J_k)}(\alpha) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau$$

$\forall u \in J_k$ and $\forall t \in I$.

To prove the case where $\alpha \geq 0$ we replace α in the above with $\alpha + \varepsilon$ where $\varepsilon > 0$ is an arbitrarily small constant and examine the limiting value as $\varepsilon \longrightarrow 0$ to obtain the required result. ■

3.2.2 Further Generalisations

Let us consider for a moment the case where α in (3.3) is not a constant but a continuous function of t, that is

$$u(t) \leq \alpha(t) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, g_i(u(\tau))\, d\tau \tag{3.7}$$

Now, given that we are assuming the functions u, a_i and g_i are as defined in Lemma 3.3, $\alpha(t)$ must itself be constrained by an inequality. Indeed, it is easily shown that

$$\alpha(t) \geq -\sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, g_i(u(\tau))\, d\tau \tag{3.8}$$

$\forall t \in I$. One will notice that the RHS of (3.8) is 0 when $t = t_0$ and negative and monotonic decreasing $\forall t > t_0$. This inequality therefore imposes a limit on the negativity of $\alpha(t)$ throughout the interval I whilst also requiring that $\alpha(t_0) \geq 0$.

An obvious question now arises: "*Under what conditions on $\alpha(t)$ may Lemma 3.3 be applied to integral inequalities of the type defined by (3.7)?*". Indeed, it is with this question in mind that we deduce the following theorems.

Theorem 3.4 *Let the functions u, a_i and g_i be as defined in Lemma 3.3 and let $u(t)$ satisfy (3.7) where $\alpha(t)$ satisfies (3.8) and $\alpha(t_0) \geq 0$. If $\dot{\alpha}(t) \geq 0\, \forall t \in I$, then $\forall u \in J_k$*

$$G_{m(J_k)}(u(t)) \leq G_{m(J_k)}(\alpha(t_0)) + \sum_{j=1}^{n+1} \int_{t_0}^{t} a_j(\tau)\, d\tau$$

$\forall t \in I$ where $a_{n+1}(\tau) = \dot{\alpha}(\tau)$ and $g_{n+1}(u(t)) = 1$.

Proof. By the Fundamental Theorem of Calculus we have that

$$\alpha(t) = \alpha(t_0) + \int_{t_0}^{t} \dot{\alpha}(\tau)\, d\tau \tag{3.9}$$

Substituting (3.9) into (3.7) we obtain

$$u(t) \leq \alpha(t_0) + \int_{t_0}^{t} \dot{\alpha}(\tau)\, d\tau + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, g_i(u(\tau))\, d\tau$$

$\forall t \in I$. Denoting $a_{n+1}(\tau) = \dot{\alpha}(\tau)$ and $g_{n+1}(u(\tau)) = 1$ we rewrite the above to read

$$u(t) \leq \alpha(t_0) + \sum_{j=1}^{n+1} \int_{t_0}^{t} a_j(\tau)\, g_j(u(\tau))\, d\tau$$

which is of the form (3.3). Applying Lemma 3.3 therefore yields the required result. ∎

Theorem 3.5 *Let the functions u, a_i and g_i be as defined in Lemma 3.3 and let $u(t)$ satisfy (3.7) where $\alpha(t)$ satisfies (3.8) and $\alpha(t_0) \geq 0$. If $\dot{\alpha}(t) \leq 0\, \forall t \in I$, then $\forall u \in J_k$*

$$G_{m(J_k)}(u(t)) \leq G_{m(J_k)}(\alpha(t_0)) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, d\tau$$

$\forall t \in I$.

Proof. By the Fundamental Theorem of Calculus we have that

$$\alpha(t) = \alpha(t_0) + \int_{t_0}^{t} \dot{\alpha}(\tau)\, d\tau$$

and since $\dot{\alpha}(t) \leq 0 \; \forall t \in I$ we know $\int_{t_0}^{t} \dot{\alpha}(\tau)\, d\tau \leq 0$ over the same interval. Hence

$$\alpha(t) \leq \alpha(t_0)$$

$\forall t \in I$. Consequently, inequality (3.7) becomes

$$u(t) \leq \alpha(t_0) + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, g_i(u(\tau))\, d\tau$$

which is of the form (3.3). Applying Lemma 3.3 therefore yields the required result. ■

Theorem 3.6 *Let the functions u, a_i and g_i be as defined in Lemma 3.3 and let $u(t)$ satisfy (3.7) where $\alpha(t)$ satisfies (3.8) and $\alpha(t_0) \geq 0$. If $\dot{\alpha}(t) \leq A \; \forall t \in I$ where $A > 0$ is a constant, then $\forall u \in J_k$*

$$G_{m(J_k)}(u(t)) \leq G_{m(J_k)}(\alpha(t_0)) + \sum_{j=1}^{n+1} \int_{t_0}^{t} a_i(\tau)\, d\tau$$

$\forall t \in I$ where $a_{n+1}(\tau) = A$ and $g_{n+1}(u(t)) = 1$.

Proof. By the Fundamental Theorem of Calculus we have that

$$\alpha(t) = \alpha(t_0) + \int_{t_0}^{t} \dot{\alpha}(\tau)\, d\tau$$

and since $\dot{\alpha}(t) \leq A \; \forall t \in I$ we know $\int_{t_0}^{t} \dot{\alpha}(\tau)\, d\tau \leq \int_{t_0}^{t} A d\tau$ over the same interval. Hence

$$u(t) \leq \alpha(t_0) + \int_{t_0}^{t} A d\tau + \sum_{i=1}^{n} \int_{t_0}^{t} a_i(\tau)\, g_i(u(\tau))\, d\tau$$

Denoting $a_{n+1}(\tau) = A$ and $g_{n+1}(u(\tau)) = 1$ we rewrite the above to read

$$u(t) \leq \alpha(t_0) + \sum_{j=1}^{n+1} \int_{t_0}^{t} a_j(\tau)\, g_j(u(\tau))\, d\tau$$

which is of the form (3.3). Applying Lemma 3.3 therefore yields the required result. ■

3.2.3 Bounds on the Solutions of a Nonlinear ODE

We shall now make use of Lemma 3.3 in a more direct manner by employing it to obtain a nonlinear version of Theorem 3.1.

Example 3.1 Consider equation (1.1) with

$$f(x,\dot{x},t) = -\sum_{i=1}^{n} p_i(t)\, f_i(x(t))$$

where $p_i(t)$ and $f_i(x(t))$ are continuous and defined over the interval I and

(a) $p_i : I \to \mathbb{R}_+ \; \forall t \in I$ and $\forall i \in [1, n-1]$.

(b) $\exists T \in [t_0, t_1)$ such that $p_n(t) \geq P \; \forall t \in [T, t_1]$ where P is a positive real constant.

(c) $f_n(x(t)) = x(t)$.

(d) $h_i(x^2(t)) \triangleq \int f_i(x(t))\, dx$ (ignoring the constant of integration) is a single valued function of $x^2(t)$ that is non-negative and non-decreasing $\forall t \in I$ and $\forall i \in [1, n-1]$.

(e) $H_i(x^2(t)) \triangleq \int \frac{dx^2}{\int f_i(x(t))dx}$ (ignoring the constant of integration) is such that $H_i^{-1}(z) \geq 0 \; \forall z \in \mathbb{R}_+$ and $\forall i \in [1, n-1]$.

The ODE under investigation is therefore

$$\ddot{x}(t) + \sum_{i=1}^{n} p_i(t)\, f_i(x(t)) = 0 \tag{3.10}$$

with $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$ and where $t \in I$, $i \in [1, n]$.

Multiplying (3.10) throughout by $2\dot{x}(t)$ and integrating from t_0 to t yields the expression

$$2\int_{t_0}^{t}\ddot{x}(\tau)\dot{x}(\tau)\,d\tau+2\sum_{i=1}^{n}\int_{t_0}^{t}p_i(\tau)f_i(x(\tau))\dot{x}(\tau)\,d\tau=0 \tag{3.11}$$

It can be demonstrated quite simply using integration by parts that the following relations hold:

$$2\int_{t_0}^{t}\ddot{x}(\tau)\dot{x}(\tau)\,d\tau \equiv (\dot{x}(t))^2-\dot{x}_0^2 \tag{3.12}$$

$$\begin{aligned}\int_{t_0}^{t}p_i(\tau)f_i(x(\tau))\dot{x}(\tau)\,d\tau \equiv\; & p_i(t)h_i\left(x^2(t)\right)-p_i(t_0)h_i\left(x_0^2\right) \\ & -\int_{t_0}^{t}\dot{p}_i(\tau)h_i\left(x^2(\tau)\right)d\tau\end{aligned} \tag{3.13}$$

where $h_i\left(x^2(t)\right)$ is as defined above. Substituting (3.12) and (3.13) into (3.11) we see that

$$(\dot{x}(t))^2+2\sum_{i=1}^{n}p_i(t)h_i\left(x^2(t)\right)-2\sum_{i=1}^{n}\int_{t_0}^{t}\dot{p}_i(\tau)h_i\left(x^2(\tau)\right)d\tau=C_0 \tag{3.14}$$

where

$$C_0=\dot{x}_0^2+2\sum_{i=1}^{n}p_i(t_0)h_i\left(x_0^2\right)$$

Since it is known that $f_n(x(t))=x(t)$ we may conclude that

$$\begin{aligned}h_n\left(x^2(t)\right) &= \int x(t)\,dx \\ &= \frac{x^2(t)}{2}\end{aligned}$$

Substituting this into (3.14) we obtain the relation

$$(\dot{x}(t))^2+2\sum_{i=1}^{n-1}p_i(t)h_i\left(x^2(t)\right)+p_n(t)x^2(t)-2\sum_{i=1}^{n}\int_{t_0}^{t}\dot{p}_i(\tau)h_i\left(x^2(\tau)\right)d\tau=C_0$$

Rearranging this expression it is possible to show that

$$p_n(t)\,x^2(t) = C_0 - (\dot{x}(t))^2 - 2\sum_{i=1}^{n-1} p_i(t)\,h_i\left(x^2(t)\right) + 2\sum_{i=1}^{n}\int_{t_0}^{t} \dot{p}_i(\tau)\,h_i\left(x^2(\tau)\right)d\tau$$

Removing the non-positive terms from the RHS of the above we arrive at the inequality

$$p_n(t)\,x^2(t) \le C_0 + 2\sum_{i=1}^{n}\int_{t_0}^{t} \dot{p}_i(\tau)\,h_i\left(x^2(\tau)\right)d\tau$$

$\forall t \in I$. Now, it is known that $p_n(t) \ge P$ when $T \le t \le t_1$ hence

$$x^2(t) \le \frac{C_0}{P} + \frac{2}{P}\sum_{i=1}^{n}\int_{t_0}^{t} \dot{p}_i(\tau)\,h_i\left(x^2(\tau)\right)d\tau$$

$\forall t \in [T, t_1]$. Taking absolute values leads us to conclude that

$$\left|x^2(t)\right| \le \frac{|C_0|}{P} + \frac{2}{P}\sum_{i=1}^{n}\int_{t_0}^{t} |\dot{p}_i(\tau)|\left|h_i\left(x^2(\tau)\right)\right|d\tau$$

One will notice that this inequality is of the form (3.3) with $u(t) = |x^2(t)|$, $g_i(x^2(t)) = |h_i(x^2(t))|$ and moreover, satisfies all the conditions associated therewith. Hence, from Lemma 3.3 we have that $\forall x^2(t) \in J_k$

$$G_{m(J_k)}\left(\left|x^2(t)\right|\right) \le G_{m(J_k)}\left(\frac{|C_0|}{P}\right) + \frac{2}{P}\sum_{i=1}^{n}\int_{t_0}^{t} |\dot{p}_i(\tau)|\,d\tau \qquad (3.15)$$

$\forall t \in [T, t_1]$ where $G_{m(J_k)}(|x^2(t)|) = H_{m(J_k)}(|x^2(t)|)$ and is as defined in the details of Lemma 3. The upper bound of $|x^2(t)|$ is therefore given by the relation

$$\left|x^2(t)\right| \le G^{-1}_{m(J_k)}\left(G_{m(J_k)}\left(\frac{|C_0|}{P}\right) + \frac{2}{P}\sum_{i=1}^{n}\int_{t_0}^{t} |\dot{p}_i(\tau)|\,d\tau\right)$$

$\forall\,|x^2(t)| \in J_k$ and $\forall t \in [T, t_1']$ where t_1' is chosen such that the argument

$$G_{m(J_k)}\left(\frac{|C_0|}{P}\right) + \frac{2}{P}\sum_{i=1}^{n}\int_{t_0}^{t} |\dot{p}_i(\tau)|\,d\tau$$

is confined to the domain of $G^{-1}_{m(J_k)}$.

It may not be obvious why it was necessary to introduce condition (e) in the above example as its effect is subtle and is only ever seen plainly when one considers a less general problem. It suffices however to point out that if this condition were omitted we would not always be able to rearrange (3.15) into a form where we would have $x^2(t)$ being less than or equal to some function of t.

So as to justify the conclusions of the forgoing we will proceed now to apply the method of Example 3.1 to a more specific case and then compare our results with those of a numerical solution.

Example 3.2 Consider the 2nd order nonlinear ODE

$$\ddot{x}(t) + t^2 e^{-t} \frac{\left(2x^2(t)+1\right)^2+1}{\left(2x^2(t)+1\right)^2} x(t) + \left(\frac{t}{t^2+1}+1\right) x(t) = 0 \tag{3.16}$$

where $t \in \mathbb{R}_+$, $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$. As one might expect, we begin by multiplying (3.16) throughout by $2\dot{x}(t)$ and then integrate the result from 0 to t to get

$$\begin{aligned} 0 &= 2\int_0^t \ddot{x}(\tau)\dot{x}(\tau)\,d\tau + 2\int_0^t \tau^2 e^{-\tau} \frac{\left(2x^2(\tau)+1\right)^2+1}{\left(2x^2(\tau)+1\right)^2} x(\tau)\dot{x}(\tau)\,d\tau \\ &\quad +2\int_0^t \left(\frac{\tau}{\tau^2+1}+1\right) x(\tau)\dot{x}(\tau)\,d\tau \end{aligned} \tag{3.17}$$

Now, from (3.13) we have that

$$\begin{aligned} \int_0^t \tau^2 e^{-\tau} \frac{\left(2x^2(\tau)+1\right)^2+1}{\left(2x^2(\tau)+1\right)^2} x(\tau)\dot{x}(\tau)\,d\tau &= t^2 e^{-t} \frac{x^4(t)+x^2(t)}{2x^2(t)+1} \\ &\quad - \int_0^t (2-\tau)\tau e^{-\tau} \frac{x^4(\tau)+x^2(\tau)}{2x^2(\tau)+1} d\tau \end{aligned}$$

and

$$\int_0^t \left(\frac{\tau}{\tau^2+1}+1\right) x(\tau)\dot{x}(\tau)\,d\tau = \left(\frac{t}{t^2+1}+1\right)\frac{x^2(t)}{2} - \frac{x_0^2}{2} - \frac{1}{2}\int_0^t \frac{1-\tau^2}{\left(1+\tau^2\right)^2} x^2(\tau)\,d\tau$$

Substituting these last two expressions and (3.12) into (3.17) yields

$$\begin{aligned}\left(\frac{t}{t^2+1}+1\right)x^2(t) &= x_0^2+(\dot{x}_0)^2-(\dot{x}(t))^2-2t^2e^{-t}\frac{x^4(t)+x^2(t)}{2x^2(t)+1}\\ &+2\int_0^t(2-\tau)\tau e^{-\tau}\frac{x^4(\tau)+x^2(\tau)}{2x^2(\tau)+1}d\tau+\int_0^t\frac{1-\tau^2}{(1+\tau^2)^2}x^2(\tau)\,d\tau\end{aligned}$$

from which we may naturally conclude that

$$\begin{aligned}\left(\frac{t}{t^2+1}+1\right)x^2(t) &\le x_0^2+(\dot{x}_0)^2+2\int_0^t(2-\tau)\tau e^{-\tau}\frac{x^4(\tau)+x^2(\tau)}{2x^2(\tau)+1}d\tau\\ &+\int_0^t\frac{1-\tau^2}{(1+\tau^2)^2}x^2(\tau)\,d\tau\end{aligned}$$

$\forall t\in\mathbb{R}_+$. Furthermore, since $\frac{t}{t^2+1}+1\ge 1\ \forall t\in\mathbb{R}_+$ we may infer that

$$x^2(t)\le x_0^2+(\dot{x}_0)^2+2\int_0^t(2-\tau)\tau e^{-\tau}\frac{x^4(\tau)+x^2(\tau)}{2x^2(\tau)+1}d\tau+\int_0^t\frac{1-\tau^2}{(1+\tau^2)^2}x^2(\tau)\,d\tau$$

$\forall t\in\mathbb{R}_+$. Taking absolute values we have

$$\left|x^2(t)\right|\le x_0^2+(\dot{x}_0)^2+2\int_0^t|2-\tau|\,\tau e^{-\tau}\frac{|x^4(\tau)|+|x^2(\tau)|}{2\,|x^2(\tau)|+1}d\tau+\int_0^t\frac{|1-\tau^2|}{(1+\tau^2)^2}\left|x^2(\tau)\right|d\tau \tag{3.18}$$

$\forall t\in\mathbb{R}_+$. We are now in a position to apply Lemma 3 since (3.18) is of the form (3.3) with $u(t)=|x^2(t)|$. However, in order to make proper use of the lemma it is first necessary to determine the J-intervals and G-functions associated with inequality (3.18). In this example we denote

$$\begin{aligned}g_1\left(\left|x^2(t)\right|\right) &= \frac{|x^4(t)|+|x^2(t)|}{2\,|x^2(t)|+1}\\ g_2\left(\left|x^2(t)\right|\right) &= \left|x^2(t)\right|\end{aligned}$$

Given that $g_2\left(\left|x^2(t)\right|\right) \geq g_1\left(\left|x^2(t)\right|\right)\ \forall x \in \mathbb{R}$ we have but one J-interval i.e. $J_1 = \mathbb{R}_+$. Consequently

$$\begin{aligned} g_{m(J_1)}\left(\left|x^2(t)\right|\right) &= g_2\left(\left|x^2(t)\right|\right) \\ &= \left|x^2(t)\right| \end{aligned}$$

Furthermore

$$\begin{aligned} G_{m(J_1)}\left(\left|x^2(t)\right|\right) &= \int \frac{dx^2}{g_{m(J_1)}\left(\left|x^2(t)\right|\right)} \\ &= \int \frac{dx^2}{\left|x^2(t)\right|} \\ &= \ln\left|x^2(t)\right| \end{aligned}$$

and thus, applying Lemma 3 to the inequality (3.18) leads us to conclude that $\forall\left|x^2(t)\right| \in \mathbb{R}_+$

$$\ln\left|x^2(t)\right| \leq \ln\left(x_0^2 + (\dot{x}_0)^2\right) + 2\int_0^t |2-\tau|\,\tau e^{-\tau}d\tau + \int_0^t \frac{\left|1-\tau^2\right|}{\left(1+\tau^2\right)^2}d\tau$$

$\forall t \in \mathbb{R}_+$. Splitting and solving the integrals and rearranging for $\left|x^2(t)\right|$ we conclude that

$$\left|x^2(t)\right| \leq \begin{cases} \left(x_0^2 + (\dot{x}_0)^2\right)\exp\left(2t^2e^{-t} + \frac{t}{t^2+1}\right) & \forall t \in [0,1] \\ \left(x_0^2 + (\dot{x}_0)^2\right)\exp\left(\frac{1}{2} + 2t^2e^{-t} + \frac{t^2-2t+1}{2(t^2+1)}\right) & \forall t \in [1,2] \\ \left(x_0^2 + (\dot{x}_0)^2\right)\exp\left(16e^{-2} + \frac{1}{2} - 2t^2e^{-t} + \frac{t^2-2t+1}{2(t^2+1)}\right) & \forall t \geq 2 \end{cases} \tag{3.19}$$

and since

$$\lim_{t\to+\infty}\left(16e^{-2} + \frac{1}{2} - 2t^2e^{-t} + \frac{t^2-2t+1}{2\left(t^2+1\right)}\right) = 16e^{-2} + 1$$

the inequality (3.19) demonstrates that the value of $\left|x^2(t)\right|$ and hence the solution of (3.16) is globally bounded. In fact, by plotting a numerical solution of (3.16) along with

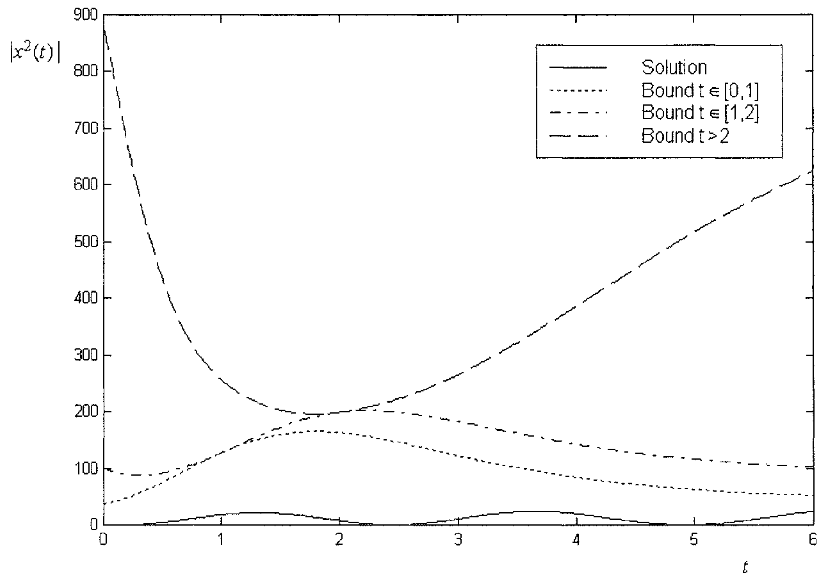

Fig. 3.2. Numerically derived plot of $|x^2(t)|$ against t. The bounds defined by (3.19) for $t \in [0,1]$, $t \in [1,2]$ and $t > 2$ are also plotted and are seen to exceed $|x^2(t)|$ within the plot range.

the bounds of that solution defined by (3.19) we may positively verify this conclusion (see Figure 3.2).

N.B. The numerical solution of (3.16) was computed in MATLAB Simulink® version 5.2 using a variable step ode113(Adams) solver with default settings.

3.3 Summary

In this chapter we have demonstrated the important role played by the integral inequality when engaging in qualitative analyses of ODEs. In particular, we have charted the history of these inequalities through their history from their origination in the guise of Grönwall's

Lemma, to the more modern works of Pachpatte and others. We have also shown explicitly how one can make use of these integral inequalities to investigate the boundedness of solutions of (non-)linear ODEs without requiring any knowledge of the particular form of the solution. Following this literature review, we stated and proved a generalisation of an inequality of Pachpatte and subsequently extended the result yet further to cope with the case where the constant α in Lemma 3.3 is permitted to vary as a function of time t. Then, using the same approach as used in the proof of Theorem 3.1, we applied our result to classify those nonlinear ODEs for which we can explicitly determine solution bounds. We then applied the lemma to a specific, purely academic, problem and in order to demonstrate the validity of the results obtained, we graphed the derived solution bounds along with the numerically derived solution. The theoretical results and those of the numerical experiment proved to be in perfect agreement and thus corroborated our assertions convincingly.

In the following chapter we shall maintain the above interests of qualitative analysis, however in the context of a much more applied problem. More specifically, we shall consider those methods that seek to synchronise dynamical systems for the purposes of secure communication and offer refinements thereon.

Chapter 4
Synchronisation of Dynamical Systems

So far we have looked at dynamical systems and ODEs with a single question in mind; *"What can we do to learn as much as possible about the solution of that system/ODE?"*. This question has so far led us to develop a method for obtaining exact analytical solutions and a method for determining whether or not solutions are bounded. This chapter however is concerned with answering a quite different question; *"What can we do to ensure that the trajectories of two parameter-matched dynamical systems converge on one another without requiring the systems to have matched ICs?"*. This is in essence a consideration of the problem of synchronising parameter-matched, but initially different dynamical systems.

As is pointed out in [8], one of the first people to document his considerations of synchronised motion is the Dutch natural philosopher Christiaan Huygens, who investigated the ultimately synchronised motion of identical pendula hanging from a common beam. Huygens' research lead him primarily to muse over the reasons for which the phases of these coupled pendula coincided. In more recent times, a great deal of interest has been aroused in synchronised dynamical systems as a result of a proliferation in the study of naturally occurring synchronised biological processes. However, researches such as those just mentioned are tackling the exact inverse of the question that we consider here. That is to say, the above examples are concerned with studying naturally occurring synchronous behaviour whereas we wish to force synchronisation to appear where it would not normally do so. In particular, it is the aim of our work to synchronise those systems that, until

Pecora and Carroll's seminal paper [74], one would have thought to be totally beyond the capacities of synchronisation. We are of course talking about chaotic systems.

In what follows we shall introduce the theory and methods developed in the early 1990's that lead to the first synchronisation of chaotic systems. The relationship between the early approaches and observer theory shall be subsequently expounded upon, before proceeding to the utility of synchronised chaotic systems in the secure communication of information. Following this discussion there will be presented the author's contributions to the use of observers in the synchronisation of chaotic systems as well as an exposition of the applicability of the work to secure communication.

4.1 Synchronisation: the general approach

Suppose one had two (almost) identical dynamical systems, the behaviour of which one wished to synchronise. There are two ways that one could go about achieving this; 1) match the ICs of the two systems to such a degree that the difference in the long term behaviour of the systems is small or; 2) send dynamical state information from one system (the master) to the second system (the slave) in such a way as to bring its motion into synchronicity with that of the master. Both of these approaches can work quite admirably in certain circumstances but may also be practically impossible to implement in others. For example, the first method can be used to achieve good trajectory matching in systems of low dimensionality since this eases the difficulty of matching ICs. However, practical IC matching is never exact and any delay between the trajectories will be extremely pronounced in systems with rapidly changing states. Furthermore, in chaotic systems any slight difference in the ICs will grow exponentially, causing the systems to evolve into wildly different behaviours.

Of the two techniques just alluded to, the latter is by far the more robust, for much the same reason that closed loop control systems are more robust than open loop systems. That is to say, the system to be synchronised is constantly receiving information about the state of the master and consequently may automatically react to any changes. Only in this régime may one hope to synchronise two parameter matched chaotic systems. Indeed, it is precisely this method we intend to focus on in the following.

4.1.1 Synchronisation of Chaotic Systems

Chaos synchronisation is an extremely active research area and has been since the publication Pecora and Carroll's 1990 paper [74] in which was given the first indication that such systems could be made to synchronise. However, rather than rely on the details of this particular paper to discuss the methodology introduced, the account given below will mirror the one given in [92] which is based on the simplified arguments outlined by Cuomo and Oppenheim in [27] and [28]. Central to this account is the Lorenz system which constitutes a simplified model of a 2D fluid constrained above and below by two parallel plates at different temperatures [92]. The system takes the form of three coupled ODEs

$$\begin{aligned}
\dot{x} &= \sigma\,(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{aligned}$$

where $x, y, z \in \mathbb{R}$ describe the state of the fluid while σ (Prandtl number), r (Rayleigh number) and b are positive constants. This system will act as the master in the following chaos synchronisation scheme.

Given the above Lorenz master system, the parameter matched slave system is simply given by

$$\begin{aligned}
\dot{x}' &= \sigma \left(y' - x' \right) \\
\dot{y}' &= r x' - y' - x' z' \\
\dot{z}' &= x' y' - b z'
\end{aligned}$$

where for the two systems we have the ICs

$$\begin{aligned}
x\left(t_0\right) - x'\left(t_0\right) &= c_x \\
y\left(t_0\right) - y'\left(t_0\right) &= c_y \\
z\left(t_0\right) - z'\left(t_0\right) &= c_z
\end{aligned}$$

where $c_{x,y,z}$ are constants.

Our objective is to drive the trajectory of the slave system into synchronicity with that of the master system. In order to achieve this we replace one of the state variables of the slave with the corresponding master state variable in all equations (except the one that contains the derivative of the replaced variable). For example, replacing x' in the slave with x from the master we have the new slave system

$$\begin{aligned}\dot{x}' &= \sigma\left(y' - x'\right)\\ \dot{y}' &= rx - y' - xz'\\ \dot{z}' &= xy' - bz'\end{aligned}$$

and since $\dot{x}'$ is the time derivative of the variable being replaced we have left the first equation unchanged. By defining the error in each variable as follows:

$$\begin{aligned}e_x &= x - x'\\ e_y &= y - y'\\ e_z &= z - z'\end{aligned}$$

and substituting we have that

$$\begin{aligned}\dot{e}_x &= \sigma\left(e_y - e_x\right)\\ \dot{e}_y &= -e_y - xe_z\\ \dot{e}_z &= xe_y - be_z\end{aligned}$$

These three equations form a third order linear dynamical system with time variable coefficients and may be written using vector notation as

$$\dot{\mathrm{e}} = \begin{pmatrix} -\sigma & \sigma & 0\\ 0 & -1 & -x\\ 0 & x & -b \end{pmatrix} \mathrm{e} \tag{4.1}$$

where $\mathrm{e} = \begin{pmatrix} e_x & e_y & e_z \end{pmatrix}^T$. If the trajectories of the master and slave systems synchronise then one would observe that $\lim\limits_{t\to+\infty} \mathrm{e} = \mathbf{0}$. In order to prove that this is limit holds it is sufficient to prove the stability of the only fixed point of (4.1), namely $\hat{\mathrm{e}} = \mathbf{0}$. In order to do this we will employ Lyapunov's direct method. To this end we select a Lyapunov function of the form

$$V(\mathrm{e}) = \frac{1}{2}\left(\frac{1}{\sigma}e_x^2 + e_y^2 + e_z^2\right)$$

From this definition it can be verified that

$$\dot{V} = -\left(e_x - \frac{1}{2}e_y\right)^2 - \frac{3}{4}e_y^2 - be_z^2$$

therefore $\dot{V} \leq 0 \ \forall \mathrm{e} \in \mathbb{R}^3$ and $V = 0$ iff $\mathrm{e} = \mathbf{0}$ hence proving the global and asymptotic stability of the fixed point $\hat{\mathrm{e}} = \mathbf{0}$, the upshot of which is the conclusion that the two chaotic systems synchronise irrespective of the values assumed by $c_{x,y,z}$. Furthermore, as is shown in [29], not only is it true that $\lim_{t \to +\infty} \mathrm{e} = \mathbf{0}$ but the convergence of e on $\mathbf{0}$ takes place at an exponential rate!

As we shall see in the next section, despite the seeming novelty of Pecora and Carroll's work on chaos synchronisation, their approach as given above is not at all far removed from an idea that first appeared in 1964 in relation to state variable estimators for control systems. Indeed, the ideas on estimators, or observers, have been successfully applied to precisely the same problems of chaos synchronisation as those discussed above.

4.1.2 Observer Theory and Synchronisation

In 1964 D.G. Luenberger published a paper entitled "Observing the State of a Linear System" [58] and in doing so he inaugurated the field of observer theory, a subject which has proliferated ever since. Luenberger's observers and their subsequent generalisations are of great utility in applications involving control systems. In particular, they address a significant problem that can arise when one wishes to control the trajectory of some dynamical system to ensure that it behaves in some predetermined fashion. To clarify, consider the linear dynamical system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ while $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are constant matrices of dimensions $n \times n$, $n \times m$ and $p \times n$ respectively. Here the vector $\mathbf{x}$ represents all of the state variables of the system, the vector $\mathbf{y}$ represents the outputs (or measurables) of the system while the vector $\mathbf{u}$ is the control input and the matrix $\mathbf{B}$ that premultiplies it defines which state variables $\mathbf{u}$ may influence.

Let us suppose that we want the trajectory $\mathbf{x}(t)$ of the above system to behave in such a way that $\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{0}$, hence we must design the controller $\mathbf{u}$ such that this always holds. One of the simplest forms that $\mathbf{u}$ could take is that of a linear state feedback controller i.e. $\mathbf{u} = \mathbf{L}\mathbf{x}$, where $\mathbf{L}$ is a constant $m \times n$ matrix. Substituting this into the system we have

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{A} + \mathbf{B}\mathbf{L})\,\mathbf{x} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

It is known [39] that if all of the eigenvalues of the matrix $\mathbf{A} + \mathbf{B}\mathbf{L}$ have negative real parts (in such an event we say the matrix $\mathbf{A} + \mathbf{B}\mathbf{L}$ is *stable*) then the system's only fixed point fixed point, $\hat{\mathbf{x}} = \mathbf{0}$, is an attractor hence $\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{0}$ which is what we initially set out to do. To achieve this one would therefore simply select the elements of $\mathbf{L}$ so that real parts of the solutions, λ, of $|\mathbf{A} + \mathbf{B}\mathbf{L} - \lambda\mathbf{I}| = 0$ are all negative.

Theoretically sound as all of the this is, there is a problem in the controller design given above. In particular, we have implicitly assumed that we have access to all of the elements of the state vector $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$ and that we may subsequently feed back that information (amplified to some degree defined by $\mathbf{L}$) into our system using the control input $\mathbf{u}$. However, in any given dynamical system one cannot necessarily have access to information about each individual state. State feedback controllers must be designed such that they only make use of the available measurables and it is the matrix $\mathbf{C}$ that defines this availability (each zero element of $\mathbf{C}$ sets one state as inaccessible). Consequently, if one's state feedback controller design relies on inaccessible state information then one is left with an impractical controller.

One possible way to remedy this problem is to turn to the information offered by observers. An observer is essentially a software model of the real world system at hand that relies on information about one or more of the states of the real world system to estimate the values of all the other state variables. To see how the idea works we will consider the same linear system as before, namely

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

Our state observer is simply a copy of the above system with an additional scaled residual term [23] and appears in the form

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\left(\mathbf{y} - \tilde{\mathbf{y}}\right) \\ \tilde{\mathbf{y}} &= \mathbf{C}\tilde{\mathbf{x}} \end{aligned}$$

where $\mathbf{L}$ is a $n \times p$ constant matrix and the tilde denotes the estimate made by the observer of a state variable viz. $\tilde{\mathbf{x}}$ is an estimate of $\mathbf{x}$ and $\tilde{\mathbf{y}}$ is an estimate of $\mathbf{y}$. Defining the error between the states of the two systems as $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$ we have

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\,\mathbf{e}$$

which is of course an homogeneous linear system. Assuming the system satisfies the hypotheses of the observability criterion, the matrix $\mathbf{L}$ can be chosen such that all the eigenvalues of $\mathbf{A} - \mathbf{LC}$ have negative real parts. The reason for this choice is to ensure that $\lim_{t \to +\infty} \mathbf{e}(t) = \mathbf{0}$ (irrespective of IC differences) which in turn ensures that the state of the observer $\tilde{\mathbf{x}}$ converges upon the state of the system $\mathbf{x}$ exponentially in t. Furthermore, because the observer is implemented in a software domain we have access to all the elements of the estimated state vector $\tilde{\mathbf{x}}$, which being almost identical to the real state vector $\mathbf{x}$, can be output from the computer as though they were measurements taken from the real system. The matrix $\tilde{\mathbf{x}}$ may subsequently be used for the kind of state feedback control discussed above. Notice also that in the design of the observer we have only used the inputs and outputs of the original system and we have not had to rely on anything that may not be available to us.

It is important to see that what we have done here is send incomplete information about the state of a system to a copy of that system in order to match the behaviour of the latter to the former. Indeed, the observer designed above is a solution to a synchronisation problem and as Nijmeijer points out in [67], chaos synchronisation problems may easily be recast as observer problems. As shall be seen later, the main results appearing in this chapter exploit this equivalence in order to design a synchronisation scheme for the chaotic Duffing oscillator.

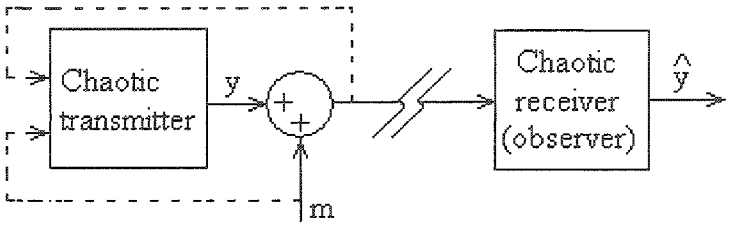

Fig. 4.1. Classical observer-based synchronisation scheme.

4.1.3 Synchronisation for Secure Communications

In the closing remarks of first paper published on the subject of chaos synchronisation, Pecora and Carroll alluded to the possibility of applying their new ideas to the secure transmission of information. As it turned out, Cuomo and Oppenheim [27] transformed this conjecture into a reality only two years later when they used a chaotic masking technique to transmit information securely with the direct aid of chaos synchronisation. Since that time, several generations of secure transmission techniques have been developed [106], all of which exploit the susceptibility of chaotic systems to synchronisation, with a good deal of these relying upon an observer-based approach [1], [11], [24], [34], [35], [53], [63], [64], [65].

The author's own results, presented in the following section and also in [44], constitute a contribution to the chaos synchronisation approach to secure communication. In particular, consideration is made of the classical observer-based chaotic masking scheme [29] illustrated in Figure 4.1.

In this scheme the chaotic transmitter generates a chaotic signal $y(t)$ to which a message $m(t)$ is added. The output $y(t)$ of the transmitter therefore acts as a message carrier. At the receiver end, the transmitted signal $y'(t) = y(t) + m(t)$ is processed by an observer in order to produce an estimate $\tilde{y}(t)$ of the carrier $y(t)$. This implies that a certain degree of robustness must be exhibited by the observer in generating the estimated output $\tilde{y}(t)$ since it is excited by the transmitted signal $y'(t)$ which of course differs from the carrier signal $y(t)$. This also suggests that the message should not be of a too high an amplitude compared to that of the carrier. The received message $m_r(t)$ is generally obtained by performing the following subtraction

$$m_r(t) = y'(t) - \tilde{y}(t) = y(t) - \tilde{y}(t) + m(t)$$

The observer is generally designed such that $\lim_{t \to +\infty} |y(t) - \tilde{y}(t)| = 0$. As a result, the received message $m_r(t)$ will asymptotically converge to the transmitted message $m(t)$. Obviously, if the estimated output $\tilde{y}(t)$ converges exponentially to the output $y(t)$, then we will have a better convergence between $m_r(t)$ and $m(t)$. Unfortunately, in many cases it is not possible to obtain an exponential convergence of the output error to zero. This is mainly due to the fact that, in many instances, the error dynamics of the observer contain some residual terms that are dependent on the transmitted message and so give rise to a distortion in the received message. As has been shown in [63], an appropriate filter might be employed in some cases to filter out the message from the residual terms. In other cases, either the message $m(t)$ or the transmitted signal $y'(t)$ is fed back into the chaotic drive system in order to compensate for these residual terms. These are symbolised by the dotted lines in Figure 4.1.

The main difficulties in the design and implementation of the above scheme are:

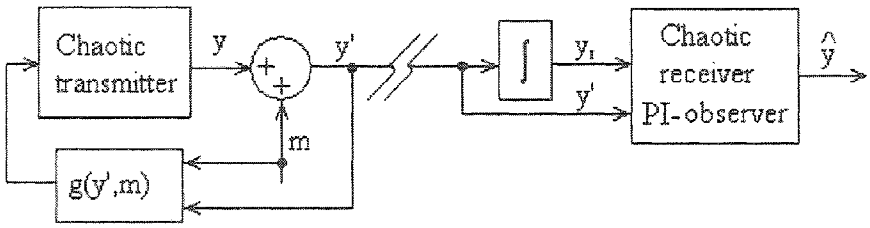

Fig. 4.2. PI observer-based synchronisation scheme.

1. Nonlinear observer design is, in general, not an easy task. Some hypotheses on the nonlinearities are generally required (Lipschitz condition [95], triangular structure [41], persistency of inputs [9] etc.).

2. The injection of either the message or the transmitted signal cannot be done in an arbitrary fashion. This has to be done via an input function. As a result, full cancellation of nonlinearities and full compensation of any residual terms appearing in the error dynamics might not be possible.

3. Noise in the output or message could hamper the recovery of the message at the receiving end. In effect, if $\eta(t)$ is an additive noise affecting the message $m(t)$ or the output $y(t)$ then the received message will be affected by the noise as follows: $m_r(t) = y(t) - \tilde{y}(t) + m(t) + \eta(t)$. This implies that even if $\tilde{y}(t)$ tracks $y(t)$ exponentially the received message will not be exempt from the noise; that is $m_r(t) \approx m(t) + \eta(t)$. Consequently, some mechanism for attenuating the effect of the noise influence in the received message must be found.

Bearing in mind this 3rd problem, it is shown in [14] that proportional integral observers (PI observers) have the capability of attenuating disturbances. Taking into account the above remarks, we propose in the following section a PI observer-based chaotic synchronisation scheme for secure communication. This scheme is illustrated in Figure 4.2.

In the proposed scheme, the transmitted signal $y'(t)$ is integrated at the receiver end yielding $x_0 = \int_0^t y'(\tau)\, d\tau$. This integrated signal, together with the transmitted signal, is used to design a PI observer. It is shown that the proposed PI observer has the capability of reducing the message or sensor noise up to some acceptable level. In addition, an output feedback $g(y', u)$ is derived and fed back to the chaotic drive system via an input function. This function allows the cancellation of the nonlinearities appearing in the error dynamics of the observer and also clarifies how a nonlinear output injection is actually performed in practice.

In the next section, the proposed synchronisation scheme is described and an example using the Duffing oscillator is given in order to show the design procedure. It is shown that the proposed scheme has a better message recovery performance than the classical proportional observer-based (or P observer-based) synchronisation scheme in the presence of measurement and/or message noise. We also present simulation results to demonstrate the performance of the proposed scheme compared to the P observer-based synchronisation scheme.

4.2 Main results

In this section, the proposed PI observer-based chaotic synchronisation scheme for secure communication (illustrated by Figure 4.2) is described. We shall assume that the transmitter system is composed of a chaotic system described by

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}f(y) + \mathbf{h}(t) + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} \end{cases} \tag{4.2}$$

where $\mathbf{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\mathbf{h}$ is some forcing function. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ assume the form

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ & & & 1 \\ a_1 & & \cdots & a_n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \\ \mathbf{C} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \end{aligned}$$

The function u represents an input function through which an output feedback can be applied. For simplicity, we have considered only one nonlinearity to clarify the design procedure. However, the procedure can be extended to the case where we have several nonlinearities provided each nonlinearity can be influenced by some input function.

We also note at this stage that many chaotic systems, if they are not already of the above form, can be brought into the above form by a change of variable. In particular the Duffing oscillator is subsumed by the above form.

4.2.1 The Masking Scheme

The transmitted signal is described as

$$y' = y + m = \mathbf{C}\mathbf{x} + m.$$

where $m(t)$ denotes a message. Note that the difference defined by $f(y) - f(y')$ can always be expressed as a function of the message and the transmitted signal only; that is to say

$$f(y) - f(y') = f(y' - m) - f(y') = g(y', m).$$

Consequently, to create the masking system we apply the following feedback via the input function $u(t)$:

$$u = -g(y', m).$$

As a result the masking scheme will be defined as follows:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}(t) + \mathbf{B}(f(y) - g(y', m)) \\ y' = \mathbf{C}\mathbf{x} + m \end{cases}$$

which means that the closed-loop the masking scheme is given by

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}f(y') + \mathbf{h}(t) \\ y' = \mathbf{C}\mathbf{x} + m \end{cases} \tag{4.3}$$

Note that it is because the nonlinearity $f(y)$ is directly affected by the input function that the above output injection is possible.

4.2.2 The PI Observer-Based Receiver Design

By placing an integrator at the receiving end we obtain an integrated transmitted signal defined by

$$x_0 = \int_0^t y'(\tau)\, d\tau = \int_0^t (\mathbf{C}\mathbf{x}+m)\, d\tau = y_I.$$

or alternatively

$$\dot{x}_0 = \mathbf{C}\mathbf{x} + m.$$

By combining the above equation with the transmitter system, we can create the augmented system

$$\begin{cases} \dot{x}_0 = \mathbf{C}\mathbf{x} + m \\ \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}f(y') + \mathbf{h}(t) \\ y_I = x_0 \\ y' = \mathbf{C}\mathbf{x} + m \end{cases}$$

which can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}_{aug} = \bar{\mathbf{A}}\mathbf{x}_{aug} + \bar{\mathbf{B}}f(y') + \bar{\mathbf{h}}(t) + \mathbf{C}_0^T m \\ y_I = \mathbf{C}_0\mathbf{x}_{aug} \\ y' = \bar{\mathbf{C}}\mathbf{x}_{aug} + m \end{cases}$$

where

$$\begin{aligned} \mathbf{x}_{aug} &= \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix}, \bar{\mathbf{A}} = \begin{pmatrix} 0 & \mathbf{C} \\ \mathbf{0}_{n\times 1} & \mathbf{A} \end{pmatrix}, \\ \bar{\mathbf{B}} &= \begin{pmatrix} 0 \\ \mathbf{B} \end{pmatrix}, \bar{\mathbf{h}}(t) = \begin{pmatrix} 0 \\ \mathbf{h}(t) \end{pmatrix}, \\ \mathbf{C}_0 &= \begin{pmatrix} \mathbf{C} & 0 \end{pmatrix} \text{ and } \bar{\mathbf{C}} = \begin{pmatrix} 0 & \mathbf{C} \end{pmatrix} \end{aligned}$$

A PI observer, based that appearing in [14], for this system is given by

$$\dot{\tilde{\mathbf{x}}}_{aug} = \bar{\mathbf{A}}\tilde{\mathbf{x}}_{aug} + \bar{\mathbf{B}}f(y') + \bar{\mathbf{h}}(t) + \mathbf{K}_I(y_I - \mathbf{C}_0\tilde{\mathbf{x}}_{aug}) + \mathbf{K}_p\left(y' - \bar{\mathbf{C}}\tilde{\mathbf{x}}_{aug}\right) \tag{4.4}$$

By defining $\mathbf{e} = \mathbf{x}_{aug} - \tilde{\mathbf{x}}_{aug}$, the error dynamics can be shown to be

$$\begin{aligned} \dot{\mathbf{e}} &= \left(\bar{\mathbf{A}} - \mathbf{K}_p\bar{\mathbf{C}} - \mathbf{K}_I\mathbf{C}_0\right)\mathbf{e} + \left(\mathbf{C}_0^T - \mathbf{K}_p\right)m \\ &= \mathbf{F}\mathbf{e} + \left(\mathbf{C}_0^T - \mathbf{K}_p\right)m \end{aligned}$$

It simply remains now to choose $\mathbf{K}_p$ such that the pair $\left(\bar{\mathbf{A}} - \mathbf{K}_p\bar{\mathbf{C}},\ \mathbf{C}_0\right)$ is observable while at the same time ensuring that $\mathbf{C}_0^T - \mathbf{K}_p \approx \mathbf{0}$. Once $\mathbf{K}_p$ is chosen, we can then choose $\mathbf{K}_I$ such that the matrix $\mathbf{F}$ is stable. As a result, one can expect the error to converge exponentially on $\mathbf{0}$ as $t \to +\infty$.

In general, if we wish to minimise the influence of the message on the error dynamics, we can choose $\mathbf{K}_p$ such that $\mathbf{K}_p = \alpha \mathbf{C}_0^T$ with $0 < \alpha < 1$. As a result of this selection, the error dynamics reduce to

$$\dot{\mathbf{e}} = \mathbf{F}\mathbf{e} + (1 - \alpha)\, \mathbf{C}_0^T m$$

In order to totally eliminate the residual term in the error dynamics above, one would naturally select $\alpha = 1$ thereby achieving ideal synchronisation. However, if one does that we have $\mathbf{K}_p = \mathbf{C}_0^T$ and the pair $\left(\bar{\mathbf{A}} - \mathbf{K}_p \bar{\mathbf{C}},\ \mathbf{C}_0\right) = \left(\bar{\mathbf{A}} - \mathbf{C}_0^T \bar{\mathbf{C}},\ \mathbf{C}_0\right)$ becomes unobservable. Consequently, one cannot totally negate the message in the error dynamics but instead must choose α such that the difference $1 - \alpha$ is small but non-zero.

4.2.3 Performance with Respect to Noise

If the message or the output was corrupted by an additive noise $\eta(t)$, then the error dynamics would become

$$\dot{\mathbf{e}} = \mathbf{F}\mathbf{e} + \left(\mathbf{C}_0^T - \mathbf{K}_p\right)(m + \eta)$$

Just as before, one can choose $\mathbf{K}_p$ appropriately such that the amplitude of noise and the message is attenuated up to some reasonable level. Having chosen the proportional gain $\mathbf{K}_p$, the integral gain $\mathbf{K}_I$ can then be selected to stabilise the matrix $\mathbf{F}$.

4.2.4 Comparison with the P Observer

Here we compare the performance of the above design procedure with the classical method which utilises a P observer as a receiver. In this case, the masking system will be given as before by the system (4.3). At the receiving end, the observer will be given by

$$\dot{\tilde{\mathbf{x}}}= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}f\left(y'\right) + \mathbf{h}\left(t\right) + \mathbf{K}_p\left(y' - \mathbf{C}\tilde{\mathbf{x}}\right)$$

By defining $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$, the error dynamics are given by

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{A}\mathbf{e} - \mathbf{K}_p\left(y' - \mathbf{C}\tilde{\mathbf{x}}\right) \\ &= \left(\mathbf{A} - \mathbf{K}_p\mathbf{C}\right)\mathbf{e} - \mathbf{K}_p m \end{aligned}$$

In this case, it can be seen that the proportional gain is multiplied by the message $m\left(t\right)$. If too high a gain is chosen, the message recovery could be compromised. On the other hand, if too low a gain is chosen then the stability of the error dynamics may be compromised. Otherwise, whenever possible, one can use a filter in order to recover the message as suggested in [63]. Note that if the message or the output was corrupted by an additive noise $\eta\left(t\right)$, then the error dynamics would be given by

$$\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{K}_p\mathbf{C}\right)\mathbf{e} - \mathbf{K}_p\left(m + \eta\right)$$

It is clear that in this case, the proportional gain cannot adequately handle both the noise and the stability of the matrix $\left(\mathbf{A} - \mathbf{K}_p\mathbf{C}\right)$ simultaneously, offering as it does a kind of trade-off between the stability of the point $\mathbf{e} = \mathbf{0}$ and noise attenuation. It is therefore clear that this scheme is not suitable for handling noise.

Remark

It is important to note that in the proposed synchronisation scheme, the integrator is placed at the receiving end of the communication channel. There are two main reasons for doing this: firstly, if the integrator was placed at the end of the transmitter, then two signals i.e. the transmitted signal $y'\left(t\right)$ and the integrated transmitted signal y_I, would have to be sent through the communication channel. This is obviously less practical than transmitting only one signal. Secondly, if any intruder were to intercept these two signals, then after some signal processing it would be possible to decode the message. For these reasons it is judicious to place the integrator at the receiving end.

4.3 Application: the Duffing oscillator

In this section, we shall apply the above PI observer-based synchronisation scheme by using the Duffing oscillator as the drive system. For comparison purposes, the classical P observer-based synchronisation scheme will also be presented.

Consider the particular Duffing oscillator system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{y}{4} - y^3 + 11\cos t + u\\ y &= x_1.\end{aligned}$$

which is known to generate chaotic trajectories [64]. The system is of the form (4.2) with

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0\\ 1\end{pmatrix},\\ \mathbf{C} &= \begin{pmatrix} 1 & 0\end{pmatrix},\\ f(y) &= -\frac{y}{4} - y^3,\\ \mathbf{h}(t) &= \begin{pmatrix} 0\\ 11\cos t\end{pmatrix}\end{aligned}$$

By setting the $y' = x_1 + m$ we see that

$$\begin{aligned} f(y) - f(y') &= \left(\frac{y'}{4} - \frac{y}{4}\right) + \left(y'^3 - y^3\right) \\ &= \frac{1}{4}m + 3(y')^2 m - 3y'm^2 + m^3 \\ &= g(y', m) \end{aligned}$$

4.3.1 The Masking System

By applying the output control $u(t) = -g(y', m)$, the masking system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{y}{4} - y^3 + 11\cos t - \left(\frac{1}{4}m + 3(y')^2 m - 3y'm^2 + m^3\right) \end{aligned}$$

or rather

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{y'}{4} - y'^3 + 11\cos t \end{aligned}$$

4.3.2 PI Observer-Based Scheme

By setting $x_0 = \int_0^t y'(\tau)\,d\tau = y_I$, we have $\dot{x}_0 = x_1 + m$ so that we have the following augmented system:

$$\begin{aligned}
\dot{x}_0 &= x_1 + m \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{y'}{4} - y'^3 + 11\cos t \\
y' &= x_1 + m \\
y_I &= x_0
\end{aligned}$$

Recalling (4.4) we have that the PI observer for the above masking scheme is given by

$$\begin{cases}
\dot{\tilde{x}}_0 = \tilde{x}_1 + k_{I0}\left(x_0 - \tilde{x}_0\right) + k_{p0}\left(y' - \tilde{x}_1\right) \\
\dot{\tilde{x}}_1 = \tilde{x}_2 + k_{I1}\left(x_0 - \tilde{x}_0\right) + k_{p1}\left(y' - \tilde{x}_1\right) \\
\dot{\tilde{x}}_2 = -\dfrac{y'}{4} - y'^3 + 11\cos t + k_{I2}\left(x_0 - \tilde{x}_0\right) + k_{p2}\left(y' - \tilde{x}_1\right)
\end{cases}$$

where $\mathbf{K}_p^T = \begin{pmatrix} k_{p0} & k_{p1} & k_{p2} \end{pmatrix}$ is the proportional gain and $\mathbf{K}_I^T = \begin{pmatrix} k_{I0} & k_{I1} & k_{I2} \end{pmatrix}$ is the integral gain.

The error dynamics are therefore represented by the system

$$\begin{cases}
\dot{e}_0 = -k_{I0}e_0 + \beta e_1 + \beta m \\
\dot{e}_1 = -k_{I1}e_0 - k_{p1}e_1 + e_2 - k_{p1}m \\
\dot{e}_2 = -k_{I2}e_0 - k_{p2}e_1 - k_{p2}m
\end{cases}$$

where $\beta = (1 - k_{p0})$. We choose $k_{p1} = k_{p2} = 0$, so that we have

$$\begin{pmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} -k_{I0} & \beta & 0 \\ -k_{I1} & 0 & 1 \\ -k_{I2} & 0 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} m$$

It can be shown that the Laplace transfer function between $e_1(t)$ and $m(t)$ is given by

$$\begin{aligned} G_{PI}(s) &= \frac{E_1(s)}{M(s)} \\ &= -\frac{\beta k_{I1} s + \beta k_{I2}}{s^3 + s^2 k_{I0} + k_{I1}\beta s + k_{I2}\beta} \end{aligned}$$

In order for $G_{PI}(s)$ to be small for all s whilst maintaining stability we choose β very small and $k_{I0} > 1$. The gains represented by k_{I1} and k_{I2} can be chosen to be reasonably large..

4.3.3 P Observer-Based Scheme

In this case the receiver is given by

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 + k_{p1}(y' - \tilde{x}_1) \\ \dot{\tilde{x}}_2 = -\frac{y'}{4} - (y')^3 + 11\cos t + k_{p2}(y' - \tilde{x}_1) \end{cases}$$

and the error dynamics are given by

$$\begin{cases} \dot{e}_1 = -k_{p1}e_1 + e_2 - k_{p1}m \\ \dot{e}_2 = -k_{p2}e_1 - k_{p2}m. \end{cases}$$

In this case it can be shown that the transfer function between $e_1(t)$ and $m(t)$ is given by

$$\begin{aligned} G_P(s) &= \frac{E_1(s)}{M(s)} \\ &= -\frac{k_{p1}s + k_{p2}}{s^2 + k_{p1}s + k_{p2}} \end{aligned}$$

Here again, in order for $G_P(s)$ to be small for all s we should choose k_{p1} and k_{p2} very small. This means however, that the poles of the P observer should be placed very closed to the origin; thereby slowing the convergence of the observer.

Remark 1.

It is difficult to carry out a clear cut comparison between $G_P(s)$ and $G_{PI}(s)$ due to the fact that they are of different order. However, by observing $G_P(s)$ and $G_{PI}(s)$, it can be noticed that the gain k_{I0} provides an additional degree of freedom to improve the stability of the transfer function $G_{PI}(s)$. Indeed, if we set $\beta k_{I1} = k_{p1}$ and $\beta k_{I2} = k_{p2}$, then we have

$$G_{PI}(s) = -\frac{k_{p1}s + k_{p2}}{s^3 + s^2 k_{I0} + k_{p1}s + k_{p2}}.$$

In this case the numerator of both transfer functions $G_P(s)$ and $G_{PI}(s)$ are the same. However, the gain k_{I0} can be used adjust the poles of $G_{PI}(s)$ while the poles of $G_P(s)$ are already fixed by k_{p1} and k_{p2}.

Remark 2.

One will notice in addition that $\lim_{s\to 0} G_P(s) = \lim_{s\to 0} G_{PI}(s) = -1$ and thus it follows that the gain of the transfer functions G_{PI} and G_P are equal, irrespective of the choice of $\mathbf{K}_p$ and $\mathbf{K}_I$.

4.3.4 Simulations Results

Simulation studies of the above PI observer and P observer synchronisation schemes are carried out to show their message recovery performance both in the presence and absence of message and/or measurement noise. The message $m(t) = \cos \pi t$ was used throughout the simulation. We have chosen $k_{p0} = 0.9$ so that $\beta = 0.1$. To make the comparison with the P observer relevant, we shall choose $k_{p1} = k_{p2} = \beta k_{I1} = \beta k_{I2}$ since we are mainly concerned in making the numerators of the transfer functions $G_P(s)$ and $G_{PI}(s)$

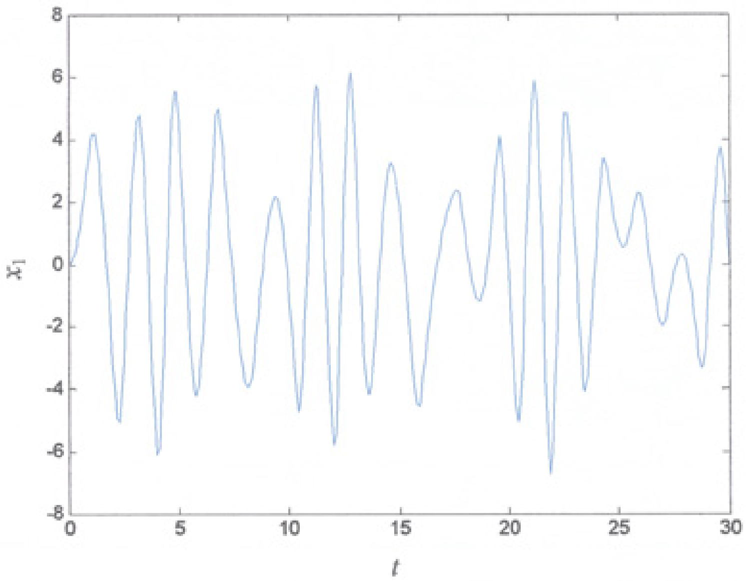

Fig. 4.3. The carrier $x_1(t)$.

(defined previously) as small as possible. All simulations were carried out using MATLAB Simulink® version 5.2 and the numerical solver ode113(Adams).

Figures 4.3 and 4.4 shows the output of the chaotic drive and the transmitted message. It can be seen that both signals look similar and the message is not discernible from the transmitted signal. Figures 4.5 and 4.6 show the original message $m(t)$ and the received message $m_r(t)$ (in dotted lines and denoted as 'mr' in the figure legend) by the PI observer and the P observer respectively when a fairly low gain was applied. This set of simulations were carried out using the following gains: $k_{I0} = k_{I1} = k_{I2} = 10$, $k_{p1} = k_{p2} = 1$. That is, we have the poles of the P observer located at $-\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$ and that of the PI observer at -9.9093 and $-4.5366 \times 10^{-2} \pm 0.31442i$. In this case, the poles of the P observer are close to the imaginary axis. It can be observed that both the PI and the P observer demonstrate

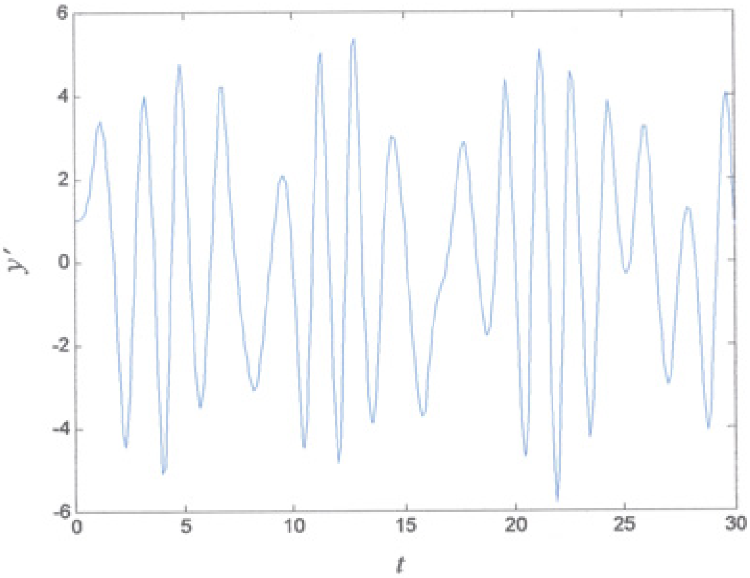

Fig. 4.4. The transmitted signal $y'(t)$.

good message recovery performance. However, there is a clear difference in the transient behaviour of the two observers. Where the P observer demonstrates rapid convergence to the message, the PI observer exhibits significant overshooting until around $t = 8$ after which the signal settles down and converges on the message. This could certainly be viewed as a practical problem associated with the PI observer when operated at low gain.

Figures 4.7 and 4.8 show the original message and the received message for the PI observer and the P observer respectively when an high gain was employed. This set of simulations was carried out using the following gains: $k_{I0} = 10$, $k_{I1} = k_{I2} = 100$, $k_{p1} = k_{p2} = 10$. That is, we have the poles of the P observer located at $-5 \pm \sqrt{15}$ and that of the PI observer at -9.9093 and $-0.49317 \pm .9307i$. It is clear from these figures that the PI observer still performs well while the P observer struggles to effectively

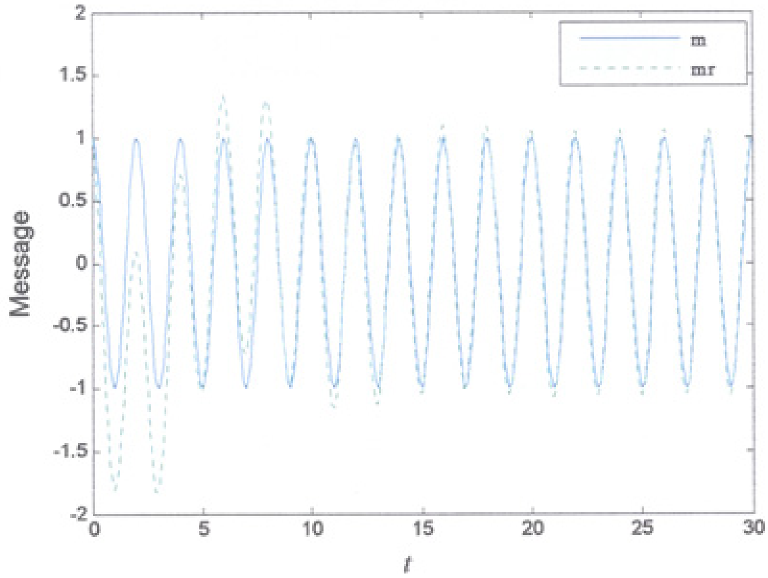

Fig. 4.5. $m(t)$ and $m_r(t)$ with PI observer - low gain.

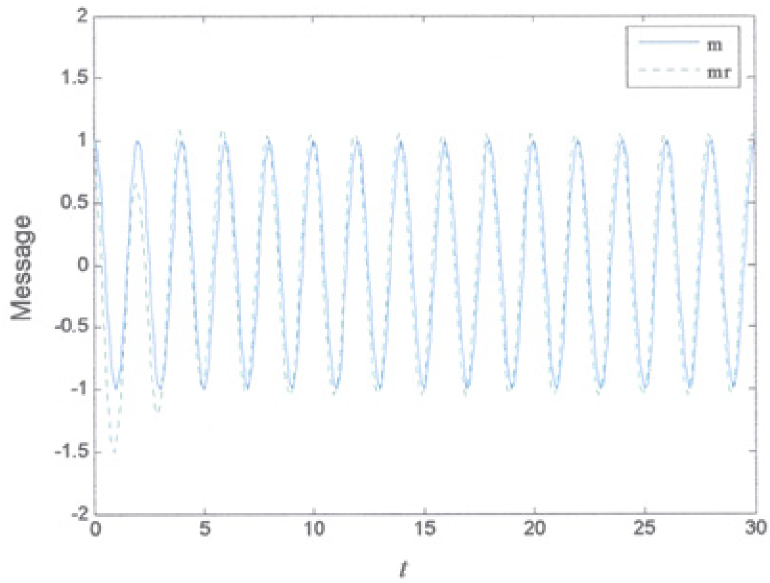

Fig. 4.6. $m(t)$ and $m_r(t)$ with P observer - low gain.

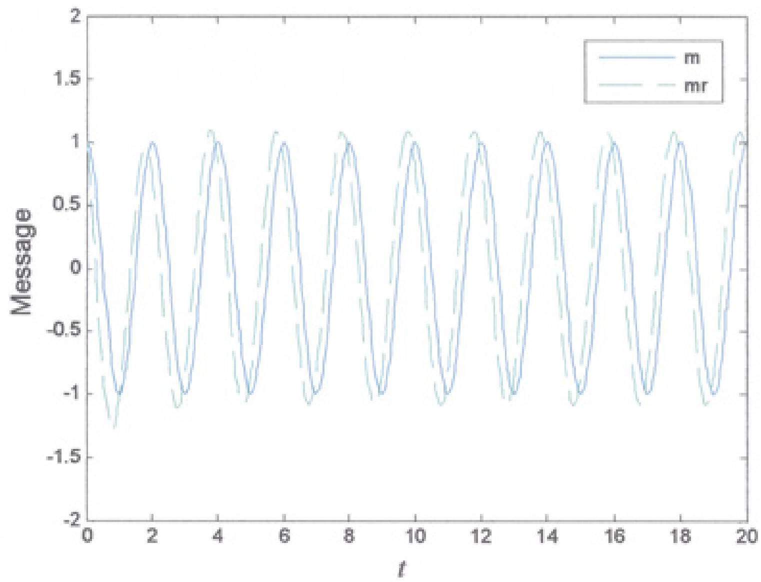

Fig. 4.7. $m(t)$ and $m_r(t)$ with PI observer - high gain.

recover the message, deciphering as it does, a phase shifted and heavily damped version of the intended signal. One will also notice that the transient overshooting seems to have disappeared from PI observer signal.

Figures 4.9 and 4.10 show the original message and the received message as recovered by the PI observer and P observer respectively in the presence of noise. A random, zero-mean Gaussian noise of 10% of the message amplitude was used and the following gains were set as in the high gain case: $k_{I0} = 10$, $k_{I1} = k_{I2} = 100$, $k_{p1} = k_{p2} = 10$. As expected, the message recovery performance of the PI observer is good, the presence of noise notwithstanding.

It should be stressed that the above comparison is not perfect since the PI observer is of order 3 while the P observer is of order 2. The comparison could also have been made

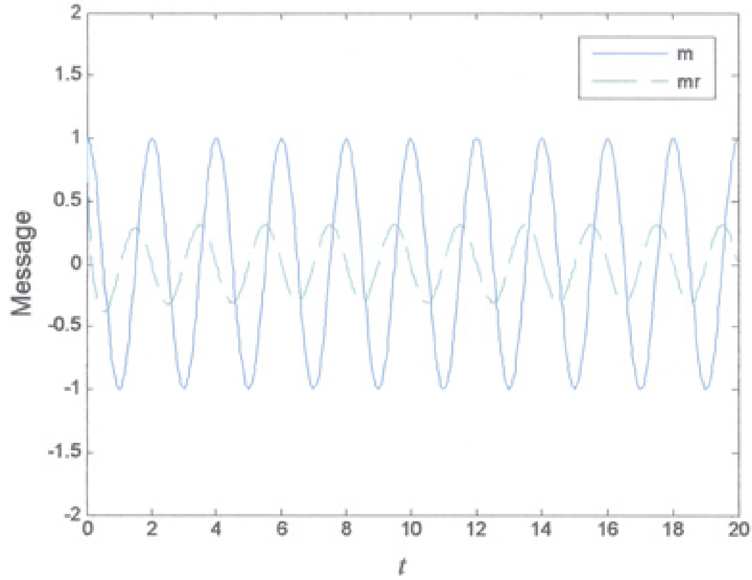

Fig. 4.8. $m(t)$ and $m_r(t)$ with P observer - high gain.

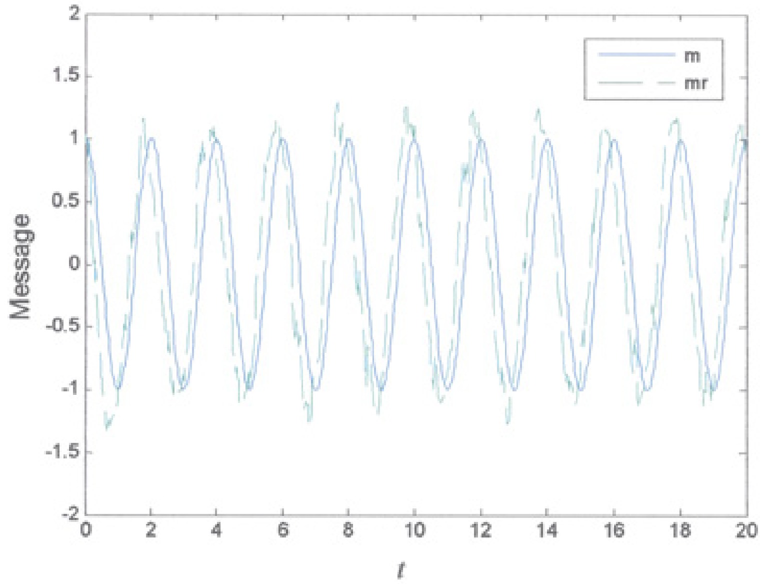

Fig. 4.9. $m(t)$ and $m_r(t)$ with PI observer under noisy conditions.

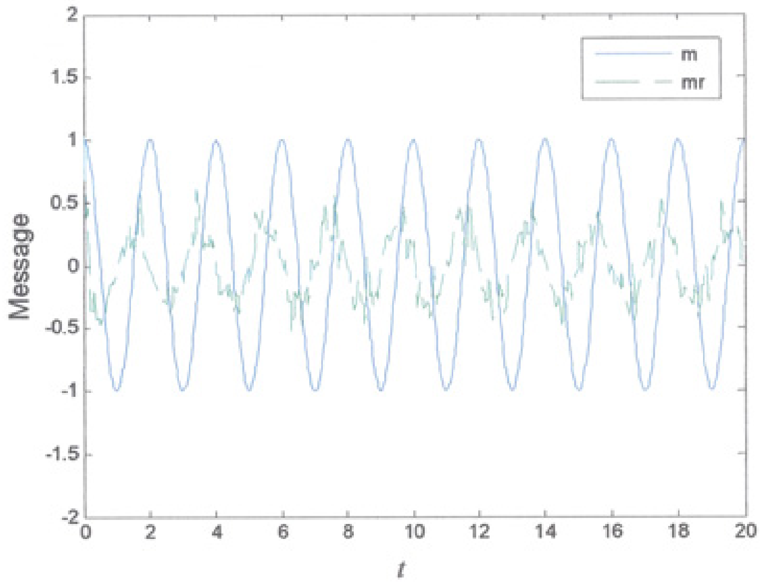

Fig. 4.10. $m(t)$ and $m_r(t)$ with P observer under noisy conditions.

by fixing two of the poles of both observers at similar locations. However, the general conclusions obtained above would still be valid in such a case. It is clear that there are better choice of gains of both observers. For example, when $k_{I0} = 20$, $k_{I1} = k_{I2} = 10$ it can be shown that the PI observer would perform particularly well. Finally, while there may indeed be better choices of gains for both observers than those given above, it can nevertheless be concluded that the performance of the PI observer in the presence of noise is better than that of the P observer, though it should also be borne in mind that the P observer suffered far less from transient effects for low gain than the PI observer.

4.4 Summary

In this chapter we have presented a general approach to designing a chaotic synchronisation scheme for secure communication using PI observers. The particular security scheme considered was that of additive masking with a chaotic signal. We have shown that this proposed synchronisation scheme is capable of effectively addressing the problem of limited message recovery introduced by ineffective handling of measurement and/or message noise. The scheme proposed worked by attenuating the noise inherent to the system without compromising stability of the error dynamics and thereby maintaining exponential convergence of the observer. As a result, the proposed synchronisation scheme generates better message recovery performance in the presence of noise compared to the classical P observer-based chaotic synchronisation scheme, as was demonstrated by a series of software simulations.. On the other hand, the PI observer suffered a tendency towards transient overshooting, something not exhibited by the P observer. The performance of the PI observer did in general offer more desirable than undesirable features and as such could, under noisy conditions at least, be said to perform measurably better than the P observer.

Chapter 5
Conclusions

In this work we have given novel contributions to problems associated with the determination of exact analytical solutions of ODEs, the boundedness of solutions of nonlinear ODEs and the synchronisation of dynamical systems for the purposes of secure communications. In particular we have identified a new solvable class of Riccati equations and used that solution to solve a class of 2nd order homogeneous linear ODEs and a yet more general Riccati equation. We have also offered an alternative method to solve 2nd order linear inhomogeneous ODEs, which in some cases is simpler than Lagrange's variation of parameters. These results were subsequently applied to solve the model of a 1-D lengthening pendulum as well as to design bounded controllers for LTV systems. We also provided the solution of two previously unsolved equations of Liénard type; the first being of the form of the DVP oscillator and the result given was a generalisation of that Chandrasekar's *et al.* [19], while the second was an ODE connecting the equations of Liénard and Riccati. The solution of the latter was obtained by employing a variable transform to modify the ODE into a Bessel equation, thereby allowing us to give the solution in terms of Bessel functions. Following this we moved on to consider the matter of identifying solution bounds of nonlinear ODEs. To this end we generalised a result of Pachpatte to yield essentially four new integral inequalities, one of which we successfully employed in the identification a class of nonlinear ODEs with bounded solutions. Furthermore, we proceeded to verify these results by considering an academic example and comparing the conclusions with a numerical simulation.

The final contribution presented in this thesis was a novel PI observer-based approach to secure communications using chaotic masking. It was shown, both through theoretical investigation and numerical experiment, the capacity of the PI observer to reduce distortion in the message at the receiver end without compromising on trajectory convergence of the two chaotic systems. This was made possible because the observer allowed for independent control of both convergence and noise limitation. The simulations in this section were performed using a chaotic Duffing oscillator and the results were compared with the performance of a P observer under the same conditions. Indeed, this comparison showed the measurable improvement on the P observer offered by the PI observer save certain deficiencies with regards the transient behaviour.

In summary, we have detailed numerous novel contributions in various areas of dynamical systems theory. We have demonstrated our motivation and why it is that these results are important. Moreover, we have shown how the results given herein can be successfully applied to help solve problems in physics and engineering; fields which clearly lie outside the interests of pure mathematical research.

Chapter 6
Future Work

Each of the three distinct problems we have addressed in this thesis have the scope to be extended beyond their current reach. One of the more interesting possible extensions is related to the work given in Chapter 2. In this chapter we gave exact analytical solutions of a class of Riccati equations. However, this problem was only considered for y a scalar. If instead we allow our variable y to represent a number of states, that is, let

$$\mathrm{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

then we transform our scalar Riccati equation into a matrix Riccati equation. It is interesting therefore to suppose the possibility of using our solution to the scalar problem to find the solution of the matrix Riccati equation

$$\dot{\mathrm{y}} + \mathbf{P}^T\mathrm{y} + \mathrm{y}\mathbf{P} - \mathrm{y}\mathbf{S}\mathrm{y} = \pm\mathbf{Q} \tag{6.1}$$

where $\mathbf{Y}$, $\mathbf{P}$, $\mathbf{S}$ and $\mathbf{Q}$ are $n \times n$ matrices with $\mathbf{P}$, $\mathbf{S}$ and $\mathbf{Q}$ being positive symmetric definite. The issue of solving equation (6.1) is a problem of central importance in control theory and particularly optimal control.

It should of course be remembered that in trying to use our solution to solve (6.1) one would expect to generate a solvability condition perhaps not unlike the condition we gave in Corollary 2.2 for the solution of the equivalent scalar problem. We would therefore not expect to solve (6.1) universally but rather, a class of equations taking the form of (6.1).

Furthermore, the application of the analytical solutions to solving problems associated with the design of bounded controllers for LTV systems could be said to lack a certain realism. No considerations were made of the effects of model mismatch, time delays or the fact that the implementation of the control scheme would most likely be performed by a microprocessor that would make discrete measurements as opposed to the continuous functions we relied upon. Investigations into the robustness of the controllers generated by the proposed design scheme to these effects could yield to some welcome refinements. For example, any model mismatch would certainly mean that the assumption of complete cancellation by the controller is false. Furthermore, time delays in the system would lead to a delay differential equation that would most certainly be more taxing to solve. Finally, discretising the problem for practical considerations would lead one out of the realms of real analysis and into problems involving recurrence relations and as such would require an alternative mathematical treatment.

A promising line to pursue with regard to our results on integral inequalities would be to investigate the possible application of the four new inequalities to resolve; not only problems of boundedness, but of IC/parameter dependence too. This particular extension is motivated by the fact that many of the integral inequalities appearing in the literature are applied to *"...investigations of uniqueness and dependence of the solutions of differential equations and systems on the initial conditions and parameters..."* [5].

A final possible extension of the work given in this thesis pertains to the application of PI observers for use in chaos synchronisation-based secure communication. Because our focus in this area was aimed directly at the performance and capabilities of the PI observer, we applied it to the simplest chaos communication régime i.e. chaotic masking. However, chaotic masking has been proven to offer little in the way of security, since techniques exist that will effectively retrieve the masked message without the need for synchronisation [89]. In light of this shortcoming it would be interesting to see how one may exploit the inherent advantages offered by PI observers in the subsequent generations of chaotic communication schemes.

Appendix A
Some Basic Concepts and Definitions

What follows here is an introduction to some of the requisite mathematical concepts, definitions and results that allow one to look at 2nd and 3rd order dynamical systems from an approximate analytical perspective. That is, we can use these concepts and their surrounding theory to carry out a useful analysis of dynamical systems despite the absence of any analytical solution. The reason for proceeding in this fashion is that, having an analytical solution is the exception and not the rule. As such, a general analysis can only be carried out by using approximate analytical methods.

A.1 The Dynamical System

We have already seen the way in which one can use ODEs to model time-varying systems. However, not all continuous systems may be modelled using a single ODE such as (1.9). Consider for example the Lorenz system which takes the form

$$\begin{aligned}\dot{x} &= \sigma\,(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

where σ, r and b are constants and $x, y, z \in \mathbb{R}^1$. It is possible to show that these 3 coupled 1st order ODEs cannot be rewritten as a single ODE that describes the same system. A complete theory of continuous dynamical systems must therefore be built upon a general equation that all systems may be described by. This equation is the matrix DE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{A.1}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

describes the state of the system at time t while

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}$$

and is called the *velocity vector field.* The general autonomous dynamical system of order n is defined by equation (A.1).

We note that since any continuous dynamical system may be modelled by an equation of the form (A.1), then a single ODE that describes a continuous dynamical system may be rewritten in this form. To see how this reformulation is performed we consider the ODE

$$\frac{d^n x}{dt^n} = f\left(\frac{d^{n-1}x}{dt^{n-1}}, \ldots, x\right) \tag{A.2}$$

which is clearly of order n since n is the highest derivative of x that appears in the equality. Notice also that this is an autonomous ODE since the function f does not depend explicitly on t.

Let $x = x_1$ and $\frac{d^i x}{dt^i} = x_{i+1}$ for $i = 1, 2, \ldots, n-1$ and rewrite (A.2) as the system of n coupled 1st order ODEs thus

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f\left(x_1, \ldots, x_{n-1}\right) \end{aligned}$$

Writing this as a matrix DE we have

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right)$$

where $\mathbf{x} = \left(x_1, \ldots, x_n\right)^{\mathrm{T}}$ and

$$\mathbf{f}\left(\mathbf{x}\right) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f\left(x_1, \ldots, x_n\right) \end{pmatrix}$$

which completes the reformulation.

The above transformation demonstrates an interesting point; namely that an autonomous ODE is of the same order as its equivalent autonomous dynamical system, and we are therefore lead to ask if the same is true of nonautonomous ODEs. To answer this question we consider the ODE (1.9), which is clearly nonautonomous. Proceeding as above we may rewrite (1.9) in the form of the matrix DE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, t\right) \tag{A.3}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ and

$$\mathbf{f}(\mathbf{x}, t) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(x_1, \ldots, x_n, t) \end{pmatrix}$$

Equation (A.3) is not yet of an identical in form to (A.1). To rectify this discrepancy we define a new variable $\dot{x}_{n+1} = 1$ with the IC $x_{n+1}(t_0) = 0$. We then have an $n+1$-th order system

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) \tag{A.4}$$

where $\mathbf{z} = (x_1, \ldots, x_{n+1})^{\mathrm{T}}$ and

$$\mathbf{f}(\mathbf{z}) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(x_1, \ldots, x_{n+1}) \\ 1 \end{pmatrix}$$

hence the reformulation is complete.

What we see therefore is that a nonautonomous ODE of order n is equivalent to an autonomous dynamical system of order $n+1$. This result bears particular significance for equation (1.1), since in the case where f does not explicitly depend on t the equation describes a dynamical system of order 2, otherwise equation (1.1) describes a system of order 3.

A.2 Classification of Systems

Now that we have established a single equation with sufficient generality to be able to describe all dynamical systems we are concerned with, we can start to classify some of its important subclasses. What follows therefore is a succinct list of the subclasses of equation (A.1), so that we might make the terminology used throughout this work comprehensible. In each instance $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{B}$ is an $n \times 1$ matrix and $\mathbf{f}$ is a nonlinear vector function.

1. **Autonomous linear system**:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Elements of A are constant.

2. **Nonautonomous linear system**:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} + \mathbf{B}(t)$$

Elements of A and B vary with t.

3. **Autonomous nonlinear system**:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

4. **Forced autonomous nonlinear system**:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(t)$$

5. **Nonautonomous nonlinear system**:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)$$

It was demonstrated above that it is possible to transform n-th order nonautonomous systems into $n+1$-th order autonomous systems. Given that this is the case one might well wonder why we classify nonautonomous and autonomous systems separately. The reason being that the form of a transformed nonautonomous system is quite distinct from a 'naturally' autonomous system. In order to appreciate the nature of this distinction consider for a moment the dynamical system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_n, t) \\ f_2(x_1, \ldots, x_n, t) \\ \vdots \\ f_n(x_1, \ldots, x_n, t) \end{pmatrix}$$

which is nonlinear and nonautonomous. Defining a new variable $\dot{x}_{n+1} = 1$ where $x_{n+1}(t_0) = t_0$ we may rewrite the above in its autonomous form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \\ \dot{x}_{n+1} \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_{n+1}) \\ f_2(x_1, \ldots, x_{n+1}) \\ \vdots \\ f_n(x_1, \ldots, x_{n+1}) \\ 1 \end{pmatrix}$$

Comparing this with the general $n+1$-th order autonomous system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \\ \dot{x}_{n+1} \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_{n+1}) \\ f_2(x_1, \ldots, x_{n+1}) \\ \vdots \\ f_n(x_1, \ldots, x_{n+1}) \\ f_{n+1}(x_1, \ldots, x_{n+1}) \end{pmatrix}$$

we see that the former absolutely guarantees that there is no solution to the identity $\mathbf{f}(\mathbf{x}) \equiv \mathbf{0}$ whereas the latter does not necessarily promise the same. The importance of this distinction will become clear when we encounter fixed points.

A.3 Trajectories and State Space

Dynamical systems theory is concerned with the determination of the nature of solutions of a system with an emphasis on the qualitative rather than quantitative. When we talk about the behaviour of a system it is always the meanderings of its solutions that we are really referring to. The term solution, however, is generally reserved for the study of DEs of scalar functions such as (A.2) and assumes the form of a scalar function of time, $x(t)$. On the other hand, in dynamical systems theory where our problems are of the form (A.1), we tend to refer to the *trajectories* of a system and these assume the form of a column vector of time varying functions $\mathbf{x}(t)$. In the context of the system (A.1) we would define a trajectory as follows:

Definition A3.1 *A vector function* $\mathbf{x}(t)$ *that satisfies the matrix DE (A.1) and the IC* $\mathbf{x}(t_0) = \mathbf{x}_0$ *is a trajectory of that system.*

In the parlance of ODE theory we would call this a particular solution of the matrix DE. The reason this terminological distinction exists is due to the way we represent the solutions of dynamical systems. The scalar function $x(t)$ that solves a DE is generally plotted in the x-t plane. Trajectories however are plotted in *state space*.

In order to convey the nature and utility of state space we consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. At any given time t, the state of the above system may be uniquely specified by the value each of the state variables $x_1(t), x_2(t), \ldots, x_n(t)$. *State space* is the n-D space in which a plot of these n state variables may be made made over a continuous time interval (obviously we may only physically plot these variables for $n \leq 3$ however that does not mean that higher order spaces do not exist, it simply means that we cannot visualise them). An example of a particular state space is the *phase plane* which is the state space of the 2nd order system

$$\dot{x}_1 = f(x_1, x_2)$$
$$\dot{x}_2 = g(x_1, x_2)$$

Plotting the solution of this system for a given set of ICs, x_1 on the x-axis and x_2 on the y-axis for some time interval $t \in [t_0, t_1]$, would give us the trajectory or *phase path* described by that solution. In this sense therefore, the ICs of a solution are simply the starting coordinates of a trajectory in state space and each trajectory for a given system is uniquely defined by its associated ICs. In addition to the trajectory of the system, the state space is also populated by the system's fixed points, limit cycles and various other system features (if present). In fact, it is the nature and location of these features within the state space that govern the behaviour of a trajectory with a certain set of ICs. It should be clear therefore how extremely useful the idea of state space is. Furthermore, theorems relating to a state space can be terribly powerful tools due to their overwhelming generality. We shall in fact note this first hand when we encounter an example of just such a theorem later on.

A.4 Fixed Points

In order to formally define the concept of a fixed point of a system we recall equation (A.1) which takes the form

$$\dot{\mathrm{x}} = \mathbf{f}(\mathrm{x})$$

where $\mathrm{x} \in \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and we denote the initial state of the system $\mathrm{x}(t_0) = \mathrm{x}_0$. A fixed point of (A.1) is defined as a constant vector $\hat{\mathrm{x}}$ such that $\mathbf{f}(\hat{\mathrm{x}}) \equiv \mathbf{0}$. In less formal terms this definition simply states that a solution starting at the state space coordinate $\hat{\mathbf{x}}$ will remain there for all time (furthermore, if a trajectory were to coincide with a fixed point it also would remain there for all time). The fixed points are essentially the trivial solutions of (A.1) and they serve to divide the state space up into to regions of different behavioural traits.

One will notice that the definition given above was stated with reference to an autonomous system. This was done because, generally speaking, the idea of a fixed point is inconsequential for nonautonomous systems because in such systems it is impossible to find a vector $\hat{\mathbf{x}}$ that satisfies the equality $\mathbf{f}\left(\hat{\mathbf{x}}\right) = \mathbf{0}\ \forall t \geq t_0$ (see section on classification of systems above).

A.5 Stability

In the absence of an analytical solution the state space of a system gives us an enormously useful environment in which to analyse that system. Let us consider for a moment the system (A.1). We have associated with (A.1) the fixed points $\hat{\mathbf{x}}$ satisfying the identity $\mathbf{f}(\hat{\mathbf{x}}) \equiv \mathbf{0}$. These points populate the n-D state space and affect the behaviour of trajectories in that space in a manner governed by the stability of each such point. The stability of a fixed point is therefore an extremely important property that can help us analyse a dynamical system. As such the stability of a fixed point must be given a formal definition such as that given in [104] which we detail below.

Definition A5.1 *A fixed point* $\hat{\mathrm{x}}$ *of (A.1) is stable if, given an* $\varepsilon > 0$, *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that, for any other solution* $\mathrm{x}(t)$ *of (A.1) satisfying* $|\hat{\mathrm{x}} - \mathrm{x}(t_0)| < \delta$, *then* $|\hat{\mathrm{x}} - \mathrm{x}(t)| < \varepsilon \, \forall t > t_0$.

In addition to the above, we may define those fixed points whose nearby trajectories approach the fixed point ever closer as $t \to +\infty$. We call such fixed points asymptotically stable and these may be formally defined as follows [104]:

Definition A5.2 *A fixed point* $\hat{\mathrm{x}}$ *of (A.1) is asymptotically stable if it is stable and for any other solution* $\mathrm{x}(t)$ *of (A.1) there exists a constant* $b > 0$ *such that, if* $|\hat{\mathrm{x}} - \mathrm{x}(t_0)| < b$, *then* $\lim_{t \to +\infty} |\hat{\mathrm{x}} - \mathrm{x}(t)| = 0$.

With these definitions in place, one can engage in an effective analysis of dynamical systems and their fixed points.

A.6 Existence and Uniqueness of Solutions

Before we proceed to discuss some of the more interesting facets of trajectories and the solutions they represent, we will look at an important theorem pertaining to these solutions. The theorem gives sufficient conditions for the existence and uniqueness of a solution of a system of 1st order ODEs and is stated below [83].

Theorem A6.1 *Consider equation (A.1), that is*

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x})$$

where $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ *is defined and continuous throughout some domain* $D \subseteq \mathbb{R}^n$. *Furthermore, let* $\dfrac{\partial \mathbf{f}}{\partial x_i}$, $i = 1, 2, \ldots, n$ *be defined and continuous in* D. *Then for every IC* $\mathbf{x}_0 \in D$ *there exists an unique solution* $\mathbf{\Phi}(t)$ *of (A.1) satisfying* $\mathbf{\Phi}(t_0) = \mathbf{x}_0$ *that is defined in some neighbourhood of* $\mathbf{x}_0$.

The above theorem (a proof of which using Picard's method of successive approximations can be found in [95]) provides essential information for anyone who wishes to analyse a dynamical system of the form (A.1). Furthermore, an immediate corollary of the theorem actually provides some terribly helpful analytical information regarding solution trajectories of (A.1) [38].

Corollary A6.1 *If the system (A.1) satisfies the hypotheses of Theorem A6.1 then two distinct trajectories, say* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$, *of that system (i.e.* $\mathbf{x}_2(t)$ *does not start on any point of the path traced out by* $\mathbf{x}_1(t)$ *and vice versa) can never intersect each other or themselves (unless they are periodic) in their* n *dimensional state space.*

This corollary indicates why the behavioural complexity of trajectories of dynamical systems goes up with the order of the system. An high order system will after all have an high dimensional state space and consequently the possible non-intersecting paths one can trace in that state space may have more varied and complex features compared to those of a low order system. This fact can be made immediately obvious by trying to trace out trajectories with chaotic features in a 2-D state space. The 2 degrees of freedom offered by the phase plane are simply not sufficient to house non-intersecting trajectories that display boundedness, aperiodicity and SDIC. 3rd order systems on the other hand have a 3-D state space and trajectories can wrap round and under each other without crossing, thus allowing for the existence of chaotic orbits.

A.7 Periodic Solutions and Limit Cycles

Consider the autonomous system (A.1) where $n = 1$, that is

$$\dot{x}_1 = f(x_1) \tag{A.5}$$

The state space of this system is the x_1-axis since the system is of order 1 and so the fixed points and trajectories of the system are both confined to the line. Trajectories starting between two fixed points are confined to the region separating them since the trajectories cannot pass through a fixed point. Trajectories which have only one adjacent fixed point are either attracted to it so that $\lim_{t\to+\infty} |x_1(t) - \hat{x}| \to 0$ or repelled by it so that $\lim_{t\to+\infty} |x_1(t) - \hat{x}| \to \infty$. It can be shown that trajectories of a first order system can never turn back on themselves when the velocity vector field is invariant in time. One can therefore never observe periodic behaviour in such systems. All in all, the permissible trajectories of (A.5) could be considered as rather dull. However, this quickly changes as one moves up to 2nd order systems and beyond.

The trajectories of a planar (i.e. 2nd order) dynamical system occupy a plane rather than a line and as such are plotted in the phase plane as opposed to an axis of that plane. The phase plane of the 2nd order system, like the phase line of the 1st order system, is pocked with fixed points that affect the behaviour of trajectories that stray near them. However, with the extra dimension afforded by this higher order system comes a greater freedom of trajectory paths. A trajectory in the plane, though influenced by local fixed points, is not constrained to sit on the straight line connecting them.

One of the most significant changes encountered in the transition from dynamical systems of order 1 to 2 is the appearance of periodic trajectories. These periodic trajectories appear in nonlinear systems of order 2 (and higher) and are the result of a new feature of the phase space; the limit cycle. Limit cycles are simple closed loops in state space that contain a fixed point and just like fixed points, they can attract or repel nearby trajectories and so be deemed to be (un)stable. However, where fixed points attract/repel trajectories to/from a point in state space, limit cycles attract/repel trajectories to/from a closed orbit. Indeed, a trajectory that periodically returns to its initial state x_0 must be sitting exactly on a limit cycle.

In nonlinear 2-D problems limit cycles divide the phase space into regions inside and outside the closed loop. If the limit cycle is stable, then nearby trajectories are attracted, such that those trajectories starting outside the cycle spiral ever inwards, asymptotically approaching the trajectory defined by the cycle's shape. On the other hand, trajectories starting inside the cycle it asymptotically from within. Logically, unstable limit cycles repel nearby trajectories so that those starting outside remain outside and move further away with time, while those that start inside move away from the inner edge and converge on a stable fixed point inside the cycle's boundary. It is also possible to have limit cycles that attract on one side while repelling on the other (these particular limit cycles being analogous to the saddle-type fixed points).

It should be pointed out here that linear systems cannot give rise to limit cycles since a single periodic trajectory of a planar linear system implies the existence of an infinite number of concentric periodic trajectories. There is therefore no attraction or repulsion provided by any one orbit that might be named the limit cycle.

Periodic solutions are important features of dynamical systems and methods for determining the existence of limit cycles in the absence of an analytical solution are thereby equally important. Later on we shall encounter later a rather powerful theorem that pertains to this matter exactly.

A.8 Cycle Graphs

Another type of limiting behaviour of orbital trajectories is defined by the cycle graph. The behaviour of trajectories in the presence of cycle graphs is similar to that which is witnessed in the presence of limit cycles; the distinction being that trajectories cannot orbit round a cycle graph, only tend towards or away from it. Cycle graphs are comprised of $N \geq 1$ vertices which are fixed points of the system and at least N edges. An example of a cycle graph can be seen in the solutions of the system [10]

$$\begin{aligned}\dot{x}_1 &= x_1\left(1 - x_1 - \frac{15}{4}x_2 + 2x_1x_2 + x_2^2\right)\\ \dot{x}_2 &= x_2\left(-1 + x_2 + \frac{15}{4}x_1 - x_1x_2 - 2x_1^2\right)\end{aligned}$$

Three of the system's fixed points, namely $(0,0)$, $(0,1)$ and $(1,0)$, form the vertices of the system's cycle graph while its edges are defined by the x_1 and x_2 axes and the line $y = 1 - x$. This triangular cycle graph encloses an unstable fixed point at $\left(\frac{1}{4}, \frac{1}{4}\right)$ and trajectories starting within the triangle are repelled by this point and the shape of the resulting trajectory approaches the shape drawn out by the edges of the cycle graph. Although the trajectories of this system are seen to tend towards a periodic (and evermore triangular) orbit, one can rule out the possibility that this behaviour is due to the presence of a limit cycle. This can be shown by observing that those trajectories beginning on the limiting shape of the orbits do not have periodic behaviour (as one would expect from a limit cycle), but rather converge upon one of the fixed points that make up the cycle graph's vertices.

Interesting though they are, we shall not dwell on cycle graphs in what follows as there are many other areas of system behaviour that will require greater focus.

A.9 Poincaré Sections

Generally used in conjunction with numerical simulations, the Poincaré section is an effective tool for analysing the limit cycles of nonlinear systems. This is because it is both conceptually simple and reduces the dimension of oscillatory trajectory problems by 1. In systems of order 2 (the lowest order systems to exhibit periodic solutions) the Poincaré section constitutes a line (perhaps one of the axes) in the phase plane. This line is positioned such that it is crossed by all (asymptotically) periodic trajectories and on it one plots the positions of each of the crossings. If for a given trajectory these points converge asymptotically on a single point on the line, then we see that we have an attracting limit cycle. Conversely, diverging crossing points would indicate an unstable limit cycle. In 3rd order systems, rather than a line, the Poincaré section consists of a plane in the 3-D phase space.

Crossings are therefore represented by points on the plane and their behaviour would be interpreted with the simplicity of 2-D problem.

As an aside, it is common when dealing with forced systems to shape one's Poincaré section so that the time between crossings is equal to the period of the forcing function as it is the frequency of this forcing that ultimately determines the oscillatory period.

Though the above ideas and their higher dimensional extensions are quite satisfactory for many problems, there are exceptions that require a little more thought. Recall that a nonautonomous system of order 2

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2, t) \\ \dot{x}_2 &= g(x_1, x_2, t)\end{aligned}$$

can be transformed into the autonomous system of order 3

$$\begin{pmatrix}\dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3\end{pmatrix} = \begin{pmatrix} f(y_1, y_2, y_3) \\ g(y_1, y_2, y_3) \\ 1 \end{pmatrix}$$

with $x_3(t_0) = t_0$, so that the z-axis in our 3-D state space is $y_3 = t - t_0$. Clearly all trajectories will propagate along the direction this axis indefinitely and thereby cross the plane of the Poincaré section just once. In these instances it is necessary to wrap the z-axis round on itself so that the period of the oscillation is equal to one round-trip of the z-axis. The resulting space through which the solutions $\mathbf{y}(t) = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^{\mathrm{T}}$ travel is that of a toroid (in the 2-D analogue we would wrap the y-axis on itself and yield a cylindrical phase space). The Poincaré section would then be a plane intersecting the toroid and on this plane one would plot coordinates of each crossing made by a given trajectory.

It is common when plotting the crossings on one's Poincaré section to ignore the crossings of a trajectory until it has settled down close enough to its asymptote (assuming the limit cycle is stable) for each crossing to be indistinguishable from the next. This measure serves to stop the transient behaviour of a trajectory confusing the analysis. As a result, one would interpret a single point on the Poincaré section after the transient period as evidence for a one-period limit cycle that attracts on one side at least. As we shall see in the next section, it is the analysis of the Poincaré sections associated with a dynamical system that can help us identify the different kinds of oscillatory motion.

A.10 Quasiperiodicity

In the same way that 2nd order systems allow for more varied trajectories than those of order 1, systems of order 3 afford yet more variation still. Unlike the trajectories of 2nd order systems, the trajectories of 3rd order systems can exhibit *frequency locked multiperiodic* motion and *quasiperiodic* motion. These types of motion result from systems of the same type, namely, systems in which there are a finite number of distinct frequencies at work. Periodic trajectories in 3-D state space, as we mentioned above, can be considered as motion on the surface of a distorted toroid whose annular radius may vary along the wrapped z-axis. The Poincaré section would intersect this toroid and capture the crossing points of the trajectories travelling on its surface. What the Poincaré section allows us to do in this instance is distinguish between two types of oscillatory motion: frequency locked and quasiperiodic.

Considering a two frequency system, we can identify frequency locked motion by a finite number of Poincaré section crossings. This kind of behaviour appears in systems where two working frequencies are commensurate (their ratio is rational). Trajectories exhibiting quasiperiodic motion on the other hand make an infinite number of different Poincaré section crossings and hence show completely aperiodic behaviour. This motion can be observed when the two working frequencies are incommensurate.

Because aperiodic behaviour is indicative of chaotic motion it is important to be able to distinguish between chaotic and quasiperiodic behaviour. The first important distinction is that, unlike quasiperiodic trajectories, chaotic trajectories cannot be said to be born from any number of finite frequency contributions. Secondly, because a quasiperiodic trajectory can be considered to traverse the surface of a distorted toroid, the points plotted on the Poincaré section will form a closed curve. This is not seen with chaotic motion. The final distinction is that quasiperiodic motion does not display SDIC.

Though these distinctions make a clear separation between chaos and quasiperiodicity, the latter can sometimes lead to the former by the alteration of one or more system parameters. That is, quasiperiodic motion can be indicative of the onset of chaos.

A.11Chaotic Behaviour

Chaotic motion represents something terribly counterintuitive, manifested as it is by apparently random behaviour born from a purely deterministic equation. This curious and only partially understood phenomenon is the focus of a great deal of nonlinear dynamics research. Despite this however, chaos still eludes definition. In fact, the best one can do to effect a definition is to simply list the properties of chaotic trajectories. These are:

1. SDIC - initially nearby trajectories diverge exponentially over time.

2. Bounded trajectory - $\|\mathbf{x}(t)\| < K \; \forall t \geq t_0$.

3. Long-term (i.e. post-transient) aperiodic motion - $\nexists \tau$ a constant such that $\mathbf{x}(t) = \mathbf{x}(t+\tau)$.

There are some qualifiers that need to be added to the above if they are to constitute a full description of the properties of chaotic orbits. Firstly, recall that for nonautonomous systems of order 2 all trajectories will propagate along the z-axis indefinitely and thereby be unbounded. This does not mean, however, that n-th order nonautonomous systems cannot have chaotic trajectories. We simply say that the chaotic trajectories of such systems are bounded in the n-D state space rather than the transformed system's $n+1$-D state space.

Secondly, considering properties 1 and 2 together we have a contradiction since for two trajectories $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to diverge exponentially we must have that $\lim_{t \to +\infty} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| = \infty$ whilst $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ occupy only a finite volume of the state space $\forall t \geq t_0$. Indeed, there is no pair of trajectories that could ever satisfy both of these conditions. The qualifier we must add is: initially nearby trajectories diverge exponentially for a limited time interval.

This last qualifier leads us rather nicely onto our final point on chaotic orbits. Given that one of the central properties of chaotic trajectories is SDIC, one might be led to think that chaos is a phenomenon displayed by pairs of trajectories for only then can one witness

SDIC. However, since this exponential divergence only takes place for a limited period, we notice that for a chaotic trajectory $\mathrm{x}(t)$, $\exists\tau$ such that $\mathrm{x}(t) = \mathrm{x}(t+\tau) + \delta\mathrm{x}$ where $\delta\mathrm{x}$ is 'small'. That is, chaotic trajectories pass near to previously occupied points in state space. We can consider the instant at which this occurs as a new initial time and effectively reset the clock to $t = t_0$ and thereby effectively consider two initially close trajectories, namely; $\mathrm{x}(t)$ and $\mathrm{x}(t+\tau)$. Now, since $\mathrm{x}(t)$ is chaotic, these initially close trajectories must diverge at an exponential rate. SDIC is then a property of both pairs of trajectories and of single trajectories. To summarise, the phase paths of chaotic trajectories pass close to previously visited points at varying intervals, only to exponentially diverge from the course taken upon its previous visit.

This idea of intrinsically chaotic trajectories can also be understood from the perspective of attractors. One can consider the 1-D motion of a trajectory of a 1st order system as being subject to an attraction due to a nearby stable fixed point. Equally, one may consider the meanderings of a chaotic trajectory as being the subject of attraction of some kind of attracting set of points which form the *limit set* of the trajectory. Assuming there is only one chaotic attractor of the system then the limit set of the chaotic trajectories form a *strange attractor*. This strange attractor can be considered to be the feature of the system that forces its trajectories to exhibit chaos.

Appendix B
Analysing 2nd and 3rd Order Dynamical Systems

In the following we will concentrate mainly (but not wholly) on the qualitative dynamical behaviour of 2nd and 3rd order systems using the alternative analytical methods for dynamical systems. This approach will allow us to obtain a good clear picture of the dynamics of these systems without having to worry about the reliability of numerical simulation results.

We shall assume throughout that all systems under consideration satisfy the conditions of Theorem A6.1 for existence and uniqueness of solutions.

B.1 2nd Order Systems

The most general system of order 2 is the autonomous nonlinear system

$$\dot{\mathrm{x}} = \mathbf{f}\,(\mathrm{x}) \tag{B.1}$$

where $\mathrm{x} \in \mathbb{R}^2$ and $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$. This system of course subsumes the nonautonomous system of order 1.

An initial analysis of a system such as this would generally involve determining the stability of any fixed points of the system. This is because the stability of the fixed points of the system will allow us to map the system's state space and get a good qualitative picture of the global behaviour of the trajectories. Stability can be analysed in a number of ways, however, a simple and universally applicable method is the linearised analysis. This method involves making a linear approximation of our system, solving the simpler linear model and drawing conclusions about the nonlinear system from the linear solution. Obviously, since the results we obtain are from an approximation they do not apply globally, but they do give a good sense of the local behaviour about the fixed points. In what follows we will present the linearised analysis in the context of an n-th order problem before looking specifically at the implications for the 2nd order problem.

Consider the system (A.1). In order to determine the stability of a fixed point of (A.1) we must investigate how the separation between a fixed point and a trajectory (that starts close to but not on the fixed point) varies in time. In this analysis stability would be indicated by a decrease in this separation whereas instability would be indicated by an increase.

Proceeding in a similar manner to Wiggins [104] we denote the separation at time t between a fixed point $\hat{\mathrm{x}}$ and a trajectory $\mathrm{x}(t)$ by $\delta\mathrm{x}(t)$ such that

$$\mathrm{x}(t) = \hat{\mathrm{x}} + \delta\mathrm{x}(t)$$

where $\delta\mathrm{x}(t_0) \neq 0$ is assumed to be 'small' i.e. the separation between the trajectory and the fixed point is initially small. Differentiating the above with respect to t yields

$$\dot{\mathrm{x}} = \frac{d}{dt}\delta\mathrm{x}$$

or alternatively

$$\frac{d}{dt}\delta\mathrm{x} = \mathbf{f}(\hat{\mathrm{x}} + \delta\mathrm{x})$$

Since we have assumed that $\delta\mathrm{x}(t_0)$ is very small then we may, for sufficiently small $t - t_0$, replace the RHS of this expression with a curtailed Taylor expansion and so obtain the approximation

$$\frac{d}{dt}\delta\mathrm{x} \approx \mathbf{f}(\hat{\mathrm{x}}) + \begin{pmatrix} \sum_{j=1}^{n} \delta x_j \frac{\partial}{\partial x'_j} f_1(\mathrm{x}')\Big|_{\mathrm{x}'=\hat{\mathrm{x}}} \\ \sum_{j=1}^{n} \delta x_j \frac{\partial}{\partial x'_j} f_2(\mathrm{x}')\Big|_{\mathrm{x}'=\hat{\mathrm{x}}} \\ \vdots \\ \sum_{j=1}^{n} \delta x_j \frac{\partial}{\partial x'_j} f_n(\mathrm{x}')\Big|_{\mathrm{x}'=\hat{\mathrm{x}}} \end{pmatrix}$$

where $\mathrm{x}' = \begin{pmatrix} x'_1 & x'_2 & \cdots & x'_n \end{pmatrix}^{\mathrm{T}}$ is a set of dummy variables introduced for conciseness of notation. Since by definition $\mathbf{f}(\hat{\mathrm{x}}) \equiv \mathbf{0}$ we can reduce the above expression to the linear constant matrix DE

$$\frac{d}{dt}\delta\mathrm{x} \approx J\delta\mathrm{x} \tag{B.2}$$

where

$$\mathbf{J} = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x'_1} & \cdots & \frac{\partial f_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x'_1} & \cdots & \frac{\partial f_n}{\partial x'_n} \end{pmatrix}\right|_{\mathrm{x}'=\hat{\mathrm{x}}}$$

and is the $n \times n$ Jacobian matrix evaluated at $\mathrm{x}' = \hat{\mathrm{x}}$. Since (B.2) is a linear constant matrix DE we can solve it analytically and so determine the time variation of $\delta\mathrm{x}$ and thereby identify the fixed point's properties. The general solution of (B.2) can be shown to be

$$\delta\mathrm{x}(t) \approx \sum_{k=1}^{n} a_k \mathrm{x}^{(k)} e^{\lambda_k t} \tag{B.3}$$

where the a_k values are arbitrary constants, the values of which are governed by the IC $\delta\mathrm{x}(t_0)$, while λ_k and $\mathrm{x}^{(k)}$ are respectively the k-th eigenvalues and eigenvectors of J. We have implicitly assumed here that $\lambda_{m_1} = \lambda_{m_2}$ iff $m_1 = m_2$ where $m_1, m_2 \in [1, n]$. The eigenvalues λ_k are the *Lyapunov exponents* of (A.1) and are defined by the relation

$$|\mathbf{J} - \lambda \mathbf{I}| = 0 \tag{B.4}$$

which assumes the form of an n-th order polynomial in λ. Associated with each of these eigenvalues there is also an eigenvector defined by

$$\mathbf{J}\mathbf{x}^{(k)} = \lambda_k \mathbf{x}^{(k)} \tag{B.5}$$

Returning now to our 2nd order system (B.1) and applying the above analysis, we see that the separation $\delta\mathbf{x}$ at time t between a fixed point $\hat{\mathbf{x}}$ and an initially nearby trajectory $\mathbf{x}$ is given by

$$\delta\mathbf{x}\,(t) \approx a_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + a_2 \mathbf{x}^{(2)} e^{\lambda_2 t} \tag{B.6}$$

where $\delta\mathbf{x} \in \mathbb{R}^2$, $a_{1,2} \in \mathbb{R}$ are arbitrary constants and

$$\mathbf{J} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{B.7}$$

where for brevity's sake we have denoted

$$\begin{aligned} p &= \left.\frac{\partial f_1}{\partial x'_1}\right|_{\mathbf{x}'=\hat{\mathbf{x}}} \\ q &= \left.\frac{\partial f_1}{\partial x'_2}\right|_{\mathbf{x}'=\hat{\mathbf{x}}} \\ r &= \left.\frac{\partial f_2}{\partial x'_1}\right|_{\mathbf{x}'=\hat{\mathbf{x}}} \\ s &= \left.\frac{\partial f_2}{\partial x'_2}\right|_{\mathbf{x}'=\hat{\mathbf{x}}} \end{aligned}$$

Substituting (B.7) into (B.4) and performing a modest amount of calculation we arrive at the expression for the system's two Lyapunov exponents

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

where $\tau = p + s$ and $\Delta = |\mathbf{J}|$.

The linearised solution (B.6) of (B.1) is of such a form that the growth/decay of $||\delta\mathbf{x}||$ is governed entirely by the system's Lyapunov exponents. Therefore the (in)stability of any fixed point to which this analysis is applied is also governed by $\lambda_{1,2}$. In order to classify the fixed points that may occur in a system appearing in the form (B.1) we must consider all the various combinations of $\lambda_{1,2}$ and qualitatively analyse the resulting solution. We shall first consider the non-degenerate case i.e. $\lambda_1 \neq \lambda_2$ [93].

1. $\lambda_1 < \lambda_2 < 0$ are real - a *stable node*. The fixed point attracts nearby trajectories towards it. Trajectories flow almost parallel to $\mathbf{x}^{(2)}$ (the slow eigenvector) near the fixed point whilst flowing almost parallel to $\mathbf{x}^{(1)}$ (the fast eigenvector) further away. The fast eigenvector is $\mathbf{x}^{(1)}$ in this example because $|\lambda_1| > |\lambda_2|$.

2. $\lambda_1 > \lambda_2 > 0$ are real - an *unstable node*. The fixed point repels nearby trajectories from it. Here $\mathbf{x}^{(1)}$ is the fast eigenvector again.

3. $\lambda_1 < 0 < \lambda_2$ are real - an *hyperbolic point*. The fixed point attracts along the line of the vector $\mathbf{x}^{(1)}$ and repels along the line of $\mathbf{x}^{(2)}$. Consequently, trajectories that do not begin on the line $\mathbf{x}^{(1)}$ are eventually repelled away in a direction almost parallel to $\mathbf{x}^{(2)}$.

4. $\lambda_1 = \lambda_2^* = -\alpha + i\beta$ for $\alpha > 0$ - a *stable spiral point*. Trajectories flow in towards the fixed point following a spiral path.

5. $\lambda_1 = \lambda_2^* = \alpha + i\beta$ for $\alpha, \beta > 0$ - an *unstable spiral point*. Trajectories flow away from the fixed point following a spiral path.

6. $\lambda_1 = \lambda_2^* = i\beta$ - an *elliptic point*. Nearby trajectories flow periodically around the fixed points following closed paths. the trajectories are neither repelled from nor attracted to the point but simply remain in orbit about it.

To complete the classification of fixed points we must consider the instance where the Lyapunov exponents are found to be degenerate. In this case the solution (B.6) no longer applies. The solution instead takes the form

$$\delta \mathbf{x}(t) \approx \left(a_1 \mathbf{x}^{(1)} + a_2 \left(\mathbf{x}^{(2)} + \mathbf{x}^{(1)} t\right)\right) e^{\lambda t}$$

where $\lambda = \frac{\mathcal{T}}{2}$ and the vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are defined by the relations

$$\begin{aligned} (\mathbf{J} - \lambda \mathbf{I})\, \mathbf{x}^{(1)} &= \mathbf{0} \\ (\mathbf{J} - \lambda \mathbf{I})\, \mathbf{x}^{(2)} &= \mathbf{x}^{(1)} \end{aligned}$$

In this case we can identify the following fixed points:

6. $\mathbf{x}^{(2)} = \mathbf{0}$ - a *star*. Trajectories flow in straight lines; directly towards the fixed point for a stable star ($\lambda < 0$) and directly away for an unstable star ($\lambda > 0$).

7. $\mathbf{x}^{(2)} \neq \mathbf{0}$ - an *improper node*. Trajectories flow along curved paths. The flow is towards the fixed point for a stable node ($\lambda < 0$) and away for an unstable node ($\lambda > 0$).

Carrying out this kind of linearised analysis for an autonomous 2nd order system allows one to be able to populate the state space of the system with its fixed points. Furthermore, the information yielded by the analysis regarding the local behaviour of trajectories within the vicinity of these fixed points allows one to map out a selection of the numerous trajectories that populate the phase plane. The resulting picture is called a *phase portrait* and provides a qualitative picture of the dynamics of the system solutions. Indeed, an accurate phase portrait is often sufficient to facilitate a comprehensive understanding of the qualitative features of a given dynamical system.

Of course, the linearised analysis here expounded is not the only method that exists to determine whether or not a fixed point is stable. An example of a commonly used alternative is *Lypaunov's direct method* which we will now describe (in a manner similar to that given in [104]) in the context of 2nd order systems.

In attempting to establish the stability of a fixed point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)^{\mathrm{T}}$ of a vector field $\mathbf{f}(\mathbf{x}) = (f(x_1, x_2), g(x_1, x_2))^{\mathrm{T}}$ we would be satisfied if we could determine some neighbourhood U of that point (containing no other fixed points) such that any trajectory starting in U remained there for all $t \geq t_0$. To prove this it is sufficient to demonstrate that the vector field points inwards at all points along the boundary of the neighbourhood U. To infer stability this condition must hold even as we shrink the U down onto the point $(\hat{x}_1, \hat{x}_2)$. In order to test a fixed point for satisfaction of this condition we define a function $V(x_1, x_2)$ with the following properties:

1. $V(x_1, x_2)$ is scalar, that is $V : \mathbb{R}^2 \to \mathbb{R}$ and is differentiable at least once with respect to both x_1 and x_2.

2. The locus of points satisfying $V(x_1, x_2) = C$ form closed curves in the plane for different values of $C > 0$ encircling $(\hat{x}_1, \hat{x}_2)$ with $V(x_1, x_2) > 0$ in a neighbourhood of $(\hat{x}_1, \hat{x}_2)$.

Such a function as this is called a *Lyapunov function*. Recall that $\nabla V(x_1, x_2)$ is a vector that is perpendicular to the curve $V(x_1, x_2) = C$ which points in the direction of increasing $V(x_1, x_2)$ (which, given that C is positive must be outward from the centre of the enclosed area). Hence, if the vector field $(f(x_1, x_2), g(x_1, x_2))$ were always either a tangent or pointing inwards for each of the curves $V(x_1, x_2) = C$ surrounding $(\hat{x}_1, \hat{x}_2)$, then

$$\nabla V(x_1, x_2) \cdot (f(x_1, x_2), g(x_1, x_2)) \leq 0$$

The n-D case is summarised by the following theorem.

Theorem B1.1 *Consider the system*

$$\dot{\mathrm{x}} = \mathbf{f}(\mathrm{x})$$

where $\mathrm{x} \in \mathbb{R}^n$. *Let* $\hat{\mathrm{x}}$ *be a fixed point of the system. If one can find a Lyapunov function* $V : U \to \mathbb{R}$ *defined on some neighbourhood* U *of* $\hat{\mathrm{x}}$ *such that* $V(\hat{\mathrm{x}}) = 0$, $V(\mathrm{x}) > 0$ *if* $\mathrm{x} \neq \hat{\mathrm{x}}$ *and* $\dot{V}(\mathrm{x}) \leq 0$ *in some neighbourhood* U *of* $\hat{\mathrm{x}}$ *then* $\hat{\mathrm{x}}$ *is stable.*

Further, if $\dot{V}(\mathrm{x}) < 0$ *in some neighbourhood* U *of* $\hat{\mathrm{x}}$, *then* $\hat{\mathrm{x}}$ *is asymptotically stable.*

It is simple to see how this method may be adapted to obtain results pertaining to instability too. However, since we are concerned here with 2nd order systems there are more than just fixed points that affect trajectories in the phase plane, there may also be limit cycles.

The possible presence of limit cycles in a planar system presents a problem for anyone trying to use the linearised analysis to map that system's phase portrait. This is because the linear method cannot identify the presence of limit cycles. For any analysis of a planar system to be called complete one must clearly augment the linearised analysis with an hunt for limit cycles.

Since limit cycles represent closed orbital paths for trajectories starting on them one can look for limit cycles by looking for closed orbits. Bendixson's criterion provides a sufficient condition for the non-existence of periodic solutions of a planar system and is as follows [104]:

Theorem B1.2 *Consider the planar system (B.1). If on a simply connected region D in $\mathbb{R}^2$ (that is, D is a closed loop that encloses an area containing no holes and does not intersect itself anywhere) the expression*

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

is not identically zero and does not change sign, then (B.1) has no closed orbits lying entirely in D.

Proof. Suppose that the domain D does in fact contain a closed trajectory of (B.1), then call that trajectory C. Recall that the velocity vector field $\mathbf{f}(\mathbf{x}) = (f(x_1, x_2), g(x_1, x_2))$ is always tangential to the curve C. Let us denote the vector field that describes the normal to the curve C as $\mathbf{n}(\mathbf{x})$ thus we can deduce that $\mathbf{f}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0$, $\forall \mathbf{x} \in C$. Hence the line integral round the curve C

$$\int_C \mathbf{f}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dl = 0$$

and the divergence theorem in the plane tells us that

$$\begin{aligned}\int\int_S \nabla \cdot \mathbf{f}(\mathbf{x})\, dx_1 dx_2 &= \int_C \mathbf{f}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dl \\ &= 0\end{aligned}$$

where S is the interior bounded by the closed loop C. If this integral is equal to zero $\nabla \cdot \mathbf{f}(\mathbf{x})$ must be either identically zero or change sign over C. If it does not then C cannot exist and there are no closed trajectories of (B.1). ■

The presence of limit cycles may also be exposed by the successful application of the Poincaré-Bendixson Theorem which we give below [10].

Theorem B1.3 *Consider the system (B.1) and a closed bounded region of the plane M. If M is an invariant set of the system (that is, all trajectories starting in M remain in M for all time) then M must contain one of the following:*

1. *An equilibrium point.*
2. *A limit cycle.*
3. *A cycle graph.*

N.B. Since ruling in/out the possible existence of a cycle graph in M is simple for most cases we shall look at how one may use the above theorem to identify the presence of limit cycles.

It should be emphasised first that if M is to contain a limit cycle and no equilibria of the system, then M must be punctured since every limit cycle encloses a fixed point.

To make use of the Poincaré-Bendixson Theorem in demonstrating the existence of a limit cycle, one could pick a punctured region M that is known to enclose no equilibria of (B.1) and show that it is an invariant set. This last part can be proven by demonstrating that the vector field $\mathbf{f}(\mathbf{x})$ points inwards to the interior of M at all points on the boundary of M.

The most difficult part of the above procedure is the selection of a suitable region M. One possible approach might be to let M be an annulus that encircles, but does not enclose, a fixed point. Allowing the annular radius of M to be variable one might arrive at conditions on this radius for M to be an invariant set of the system. Succeeding in this would allow one to apply the Poincaré-Bendixson Theorem to conclude that M contains a limit cycle.

What we have discussed here are the most common approaches that are taken in making a first analysis of an autonomous dynamical system of order 2. Obviously there exists a veritable plethora of methods that have been developed over the years for analysing 2nd order autonomous systems with varying degrees of generality and utility, the mention of which we have omitted. We shall, however, encounter some of these alternative approaches later as they appertain directly to results arrived at in this piece of research work.

B.2 3rd Order Systems

It has been stated that the focus of the research presented in this thesis is levelled at equation (1.1). We have already shown that the order of the dynamical system that this equation describes depends on whether or not the function f shows explicit time dependence on t. Assuming this is the case, we are interested in the analysis of dynamical systems of order 3. Namely, the class of systems defined by the matrix DE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{B.8}$$

where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

We have already encountered (and mathematically justified) the relationship between a system's order and its associated trajectories' complexity and variety. It should therefore be obvious that all trajectories that may be exhibited as solutions of 2nd order systems may also be exhibited by those of 3rd order. Bearing this in mind we shall not concern ourselves with reformulating the above 2nd order analyses for the 3rd order case but concentrate instead on the identification and analysis of those new trajectories that appear as a result of the 2nd-3rd order transition. In particular, we shall proceed to look at one of the more conceptually simple methods of analysing and identifying chaotic behaviour in 3rd order systems.

So far we have offered a definition of chaotic behaviour described in terms of the properties of those trajectories that are classified as chaotic. In order to identify a trajectory as chaotic we must carry out some analysis. Proceeding as in [93] we recall that one of the properties of chaotic systems is the exponentially rapid divergence of initially nearby trajectories and let $\mathbf{y}(t) = (y_1, y_2, y_3)^{\mathrm{T}}$ and $\mathbf{z}(t) = (z_1, z_2, z_3)^{\mathrm{T}}$ be two distinct trajectories of the 3rd order system (B.8). Denoting the separation between these solutions by $\delta\mathbf{x} \in \mathbb{R}^3$ we write

$$\delta x_i = y_i - z_i$$

where $i = 1, 2, 3$. Differentiating this expression with respect to the time gives

$$\frac{d}{dt}\delta x_i = f_i(z_1 + \delta x_1, z_2 + \delta x_2, z_3 + \delta x_3) - f_i(z_1, z_2, z_3)$$

Given that we are interested in the rate of separation of initially nearby trajectories we may assume that $\delta\mathbf{x}$ is initially small. We may therefore make the approximated Taylor expansion

$$f_i(z_1 + \delta x_1, z_2 + \delta x_2, z_3 + \delta x_3) \approx f_i(z_1, z_2, z_3) + \sum_{j=1}^{3} \delta x_j \left.\frac{\partial}{\partial x_j} f_i(\mathbf{x})\right|_{\mathbf{x}=\mathbf{z}(t)}$$

hence

$$\frac{d}{dt}\delta x_i \approx \sum_{j=1}^{3} \delta x_j \left.\frac{\partial}{\partial x_j} f_i(\mathbf{x})\right|_{\mathbf{x}=\mathbf{z}(t)}$$

Expressing this result in matrix form we have

$$\frac{d}{dt}\delta\mathbf{x} \approx \mathbf{M}\delta\mathbf{x} \tag{B.9}$$

where the matrix

$$\mathbf{M} = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}\right|_{\mathbf{x}=\mathbf{z}(t)}$$

Since it is our aim to test for the exponential divergence of $\mathbf{y}(t)$ and $\mathbf{z}(t)$ we impose the relation

$$\|\delta\mathbf{x}(t)\| = \|\delta\mathbf{x}_0\| \, e^{\lambda(t-t_0)} \tag{B.10}$$

where $\delta \mathrm{x}_0$ is the initial separation and λ is the *Lyapunov characteristic exponent* (LCE). Clearly, the motion of two distinct but initially close trajectories on a strange attractor (i.e. chaotic motion) will be characterised by a positive LCE. Rearranging (B.10) we have the relation

$$\lambda = \frac{1}{t - t_0} \ln \frac{\|\delta \mathrm{x}(t)\|}{\|\delta \mathrm{x}_0\|} \tag{B.11}$$

which along with the criterion $\lambda > 0$ constitutes a test for chaos. It is worth noting here that this is a simplified analysis since there are in fact n LCEs for a system of order n. What equation (B.11) gives is the largest LCE of the system. However, for a 3rd order system to be identified as chaotic it is required that just one of the LCEs be positive, hence the test defined by (B.11) is quite sufficient for our ends.

It should also be noted that, defining λ by equation (B.11) causes problems at large values of $t - t_0$. This is because, as mentioned earlier, chaotic trajectories do not adhere to the divergence law (B.10) indefinitely. We must therefore limit the duration of our exponential divergence test to suitably sized time intervals. This however presents its own problems since we want to test the system for continued adherence to (B.10) in order to obtain trustworthy results (recall that saddle nodes generate exponentially divergent trajectories in the short term and as such could be mistaken for chaotic orbits if one does not test properly). The solution to this conundrum is to make numerous short interval tests in sequence and thereby obtain several values for λ that one can average. Defining a constant time step $\tau = t_{k+1} - t_k$ where $k = 1, 2, 3, \ldots$ we would write the largest LCE for test number k as

$$\lambda_k = \frac{1}{\tau} \ln \frac{\|\delta \mathrm{x}(t_{k+1})\|}{\|\delta \mathrm{x}_k\|} \tag{B.12}$$

where $\|\delta \mathrm{x}_k\|$ is the IC of the trajectory that passes through the point $\delta \mathrm{x}(t_{k+1})$ ($\|\delta \mathrm{x}_k\|$ is always taken to be small). The average of N such values is clearly

$$\bar{\lambda} = \frac{1}{N\tau} \sum_{k=1}^{N} \ln \frac{\|\delta \mathrm{x}(t_{k+1})\|}{\|\delta \mathrm{x}_k\|} \tag{B.13}$$

We may therefore define the largest LCE of a system to be the limit

$$\lambda = \lim_{N \to +\infty} \frac{1}{N\tau} \sum_{k=1}^{N} \ln \frac{\|\delta \mathrm{x}(t_{k+1})\|}{\|\delta \mathrm{x}_k\|} \tag{B.14}$$

In general it is not possible to solve (B.9) for $\delta \mathrm{x}(t)$ or to find the function $\mathrm{z}(t)$, the trajectory along which M must be evaluated. Consequently, the test defined by relation (B.14) may only be made by resorting to an examination of data obtained either by experiment or simulation. One procedure (similar to that given in [62]) for making use of a numerical solver to evaluate the largest LCE is as follows:

1. Solve (B.8) with the IC $\mathrm{x}(t_0) = \mathrm{z}(t_0)$ to find $\mathrm{z}(t)\ \forall t \in [t_0, T]$ where $T - t_0$ is the prescribed duration of the simulation.

2. Allowing time for possible transients in $\mathrm{z}(t)$ to die away, run a solution of (B.9) using the IC $\delta \mathrm{x}_1$ for the interval $t \in [t_1, t_2]$ where $\tau = t_2 - t_1$ is small enough for (B.10) to be obeyed by possible chaotic trajectories throughout the interval. The direction of the IC $\delta \mathrm{x}_1$ may be arbitrarily chosen however we do require that $\|\delta \mathrm{x}_1\| = \varepsilon$ is small.

3. Evaluate a largest LCE using the formula (B.12) with $k = 1$.

4. Start a second solution of (B.9) using the IC $\delta \mathrm{x}_2 = \dfrac{\varepsilon \delta \mathrm{x}(t_2)}{\|\delta \mathrm{x}(t_2)\|}$ (i.e. same size as $\delta \mathrm{x}_1$ and same direction as $\delta \mathrm{x}(t_2)$) for the interval $t \in [t_2, t_3]$. The new largest LCE is given by (B.12) with $k = 2$.

5. Iterating this procedure one can compute the average LCE using (B.13) with $N = \frac{T}{\tau}$ and where

$$\delta \mathrm{x}_j = \frac{\varepsilon \delta \mathrm{x}\left(t_j\right)}{\left\|\delta \mathrm{x}\left(t_j\right)\right\|}$$

$\forall j \in \left[2, \frac{T}{\tau}\right]$. Clearly, this computed LCE will approach that which is defined in equation (B.14) for large values of T.

The method just outlined is not the only means of diagnosing chaotic orbits, nor is this diagnosis the only way of identifying chaotic systems. Among the other diagnostic techniques such as *fractal dimension* of the Poincaré map and computation of the *Kolmogorov entropy* [78] there are predictive techniques that use commonly observed dynamic behaviour to foretell the onset of chaos. A few examples of these transitional behaviours are period doubling, intermittent chaos and transient chaos (see [62] for details).

Appendix C
Derivation of $f\left(t\right)$

In the proof of Theorem 2.4 it was considered judicious to omit, for the sake of brevity, how we get from the ODE (2.47) to the functions given in (2.40). However, since the steps involved lead us to define the function v i.e. (2.41), which plays an essential role in the solution structure, we shall submit the details here.

Recalling the ODE (2.47) we are required to solve the separable equation

$$\dot{f}\left(t\right) = \left(f^2\left(t\right) + Kf\left(t\right) + \sigma\right)\sqrt{q\left(t\right)}$$

Separating variables and integrating both sides between the limits t and t_0 we have

$$\int_{f(t_0)}^{f(t)} \frac{df}{f^2 + Kf + \sigma} = \int_{t_0}^{t} \sqrt{q\left(\tau\right)}d\tau \tag{C.1}$$

Dealing first with the LHS of (C.1), it can be shown [73] that

$$\int_{f(t_0)}^{f(t)} \frac{df}{f^2 + Kf + \sigma} = \begin{cases} \dfrac{1}{\alpha}\ln\left(\dfrac{2f + K - \alpha}{2f + K + \alpha}\right) + a_1 & \textit{if } K^2 > 4, \sigma = 1 \\ \dfrac{2}{\beta}\tan^{-1}\left(\frac{2f+K}{\beta}\right) + a_2 & \textit{if } K^2 < 4, \sigma = 1 \\ -\dfrac{2}{2f + K} + a_3 & \textit{if } K^2 = 4, \sigma = 1 \\ \dfrac{1}{\gamma}\ln\left(\dfrac{2f + K - \gamma}{2f + K + \gamma}\right) + a_4 & \textit{if } \sigma = -1 \end{cases} \tag{C.2}$$

As for the integral on the RHS of (C.1), we recall the definition (2.38) and write

$$\int_{t_0}^{t} \sqrt{q\left(\tau\right)}d\tau = \int_{t_0}^{t} \frac{\sqrt{q_0}e^{-\int_{t_0}^{\tau} p(\lambda)d\lambda}}{1 + K\sqrt{q_0}\int_{t_0}^{\tau} e^{-\int_{t_0}^{\lambda} p(\mu)d\mu}d\lambda}d\tau$$

Defining v as the denominator of the integrand we notice that

$$\begin{aligned}\int_{t_0}^{t} \frac{\sqrt{q_0}e^{-\int_{t_0}^{\tau} p(\lambda)d\lambda}}{1+K\sqrt{q_0}\int_{t_0}^{\tau} e^{-\int_{t_0}^{\lambda} p(\mu)d\mu}d\lambda}d\tau &= \frac{1}{K}\int_{t_0}^{t}\frac{1}{v}\frac{dv}{d\tau}d\tau \\ &= \frac{1}{K}\ln\left(\frac{v}{v(t_0)}\right)\end{aligned}$$

and since $v(t_0) = 1$ we have

$$\int_{t_0}^{t} \sqrt{q(\tau)}d\tau = \frac{1}{K}\ln v \tag{C.3}$$

which, when solved for v, yields the definition (2.41).

Substituting (C.2) and (C.3) into (C.1) we obtain the relation

$$\left.\begin{array}{ll} \frac{1}{\alpha}\ln\left(\frac{2f+K-\alpha}{2f+K+\alpha}\right) - a_1 & \textit{if } K^2 > 4, \sigma = 1 \\ \frac{2}{\beta}\tan^{-1}\left(\frac{2f+K}{\beta}\right) - a_2 & \textit{if } K^2 < 4, \sigma = 1 \\ -\frac{2}{2f+K} - a_3 & \textit{if } K^2 = 4, \sigma = 1 \\ \frac{1}{\gamma}\ln\left(\frac{2f+K-\gamma}{2f+K+\gamma}\right) - a_4 & \textit{if } \sigma = -1 \end{array}\right\} = \frac{1}{K}\ln v$$

where the a_n for $n = 1, 2, 3, 4$ are constants of integration. Finally, solving the above for f in each of the four cases yields the piecewise function

$$f(t) = \begin{cases} -\frac{K}{2} + \frac{\alpha\left(1+e^{\alpha a_1}v^{\frac{\alpha}{K}}\right)}{2\left(1-e^{\alpha a_1}v^{\frac{\alpha}{K}}\right)} & \textit{if } K^2 > 4, \sigma = 1 \\ -\frac{K}{2} + \frac{\beta}{2}\tan\left(\frac{\beta}{2}\left(a_2 + \frac{1}{K}\ln v\right)\right) & \textit{if } K^2 < 4, \sigma = 1 \\ -\frac{K}{2} - \frac{K}{Ka_3 + \ln v} & \textit{if } K^2 = 4, \sigma = 1 \\ -\frac{K}{2} + \frac{\gamma\left(1+e^{\gamma a_4}v^{\frac{\gamma}{K}}\right)}{2\left(1-e^{\gamma a_4}v^{\frac{\gamma}{K}}\right)} & \textit{if } \sigma = -1 \end{cases}$$

which is precisely what we have given in (2.40) with

$$\begin{aligned} c_1 &= e^{\alpha a_1} \\ c_2 &= \frac{\beta a_2}{2} \\ c_3 &= a_3 \\ c_4 &= e^{\alpha a_4} \end{aligned}$$

and thus the derivation is complete.

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