Flexural-torsional behavior of thin-walled closed-section composite box beams

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This paper presents a flexural-torsional analysis of composite box beams. A general analytical model applicable to thin-walled box section composite beams subjected to vertical and torsional load is developed. This model is based on the classical lamination theory, and accounts for the coupling of flexural and torsional responses for arbitrary laminate stacking sequence configuration, i.e. unsymmetric as well as symmetric. Governing equations are derived from the principle of the stationary value of total potential energy. Numerical results are obtained for thin-walled composites beams under vertical and torsional loading, addressing the effects of fiber angle and laminate stacking sequence.

Keywords: thin-walled composites, classical lamination theory, flexural-torsional response, finite element method

I. INTRODUCTION

Fiber-reinforced composite materials have been used over the past few decades in a variety of structures. Composites have many desirable characteristics, such as high ratio of stiffness and strength to weight, corrosion resistance and magnetic transparency. Thin-walled structural shapes made up of composite materials, which are usually produced by pultrusion, are being increasingly used in many engineering fields. In particular, the use of pultruded composites in civil engineering structures await increased attention.

Thin-walled composite structures are often very thin and have complicated material anisotropy. Accordingly, warping and other secondary coupling effects should be considered in the analysis of thin-walled composite structures. The theory of thin-walled closed section members made of isotropic materials was first developed by Vlasov [1] and Gjelsvik [2]. For fiber-reinforced composites, some analyses have been formulated to analyze composite box beams with varying levels of assumptions. Chandra et al. [3] discussed the structural couplings effects for symmetric and anti-symmetric box beams under flexural, torsional, and extensional loads. Song and Librescu [4] focused on the formulation of the dynamic problem of laminated composite thick- and thin-walled, single-cell beams of arbitrary cross-section and on the investigation of their associated free vibration behavior. Puspita et al. [5] have proposed a simplified analytical calculation of composite beams with orthotropic phases, as well as a computer-aided design software based on a finite element method to treat composite beams such as helicopter blades. Jeon et al. [6] developed an analysis model of large deflection for the static and dynamic analysis of composite box beams. Kollar and Pluzsik [7] presented a beam theory for thin-walled open and closed section composite beams with arbitrary layups which neglects the effect of restrained warping and transverse shear deformation, and developed expressions for the stiffness matrix. Salim and Davalos [9] presented the linear analysis of open and closed sections made of general laminated composites by extending Gjelsvic's model [2]. This model accounts for all possible elastic couplings in composite sections, such as extension- and bending-torsion. The effect of warping-torsion on the torsional stiffness of the beam is investigated. Recently, Cortinez and Piovan [10] presented the stability analysis of composite thin-walled beams with open or closed cross-sections. This model is based on the use of the Hellinger-Reissner principle, that considers shear flexibility in a full form, general cross-section shapes and symmetric balanced or especially orthotropic laminates.

In this paper, an analytical model for thin-walled open-section composite beams developed by Lee et al. [8] has been extended to the closed-section composite beams. This model is applicable to the flexural, torsional and flexural-torsional behavior of an box section composite beams subjected to vertical and torsional load. It is based on the classical lamination theory, and accounts for the coupling of flexural and torsional responses for arbitrary laminate stacking sequence configuration, i.e. unsymmetric as well as symmetric. Governing equations are derived from the the

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FIG. 1 Definition of coordinates in thin-walled closed sections

principle of the stationary value of total potential energy. Numerical results are obtained for thin-walled composite beams under vertical and torsional loading, addressing the effects of fiber angle and laminate stacking sequence.

II. KINEMATICS

The theoretical developments presented in this paper require two sets of coordinate systems which are mutually interrelated. The first coordinate system is the orthogonal Cartesian coordinate system (x, y, z), for which the x and y axes lie in the plane of the cross section and the z axis parallel to the longitudinal axis of the beam. The second coordinate system is the local plate coordinate (n, s, z) as shown in Fig.1, wherein the n axis is normal to the middle surface of a plate element, the s axis is tangent to the middle surface and is directed along the contour line of the cross section. The (n, s, z) and (x, y, z) coordinate systems are related through an angle of orientation θ as defined in Fig.1. Point P is called the pole axis, through which the axis parallel to the z axis is called the pole axis.

To derive the analytical model for a thin-walled composite beam, the following assumptions are made:

1. The contour of the thin wall does not deform in its own plane.
2. The linear shear strain $\bar{\gamma}_{sz}$ of the middle surface is to have the same distribution in the contour direction as it does in the St. Venant torsion in each element.
3. The Kirchhoff-Love assumption in classical plate theory remains valid for laminated composite thin-walled beams.

According to assumption 1, the midsurface displacement components $\bar{u}, \bar{v}$ at a point A in the contour coordinate system can be expressed in terms of a displacements U, V of the pole P in the x, y directions, respectively, and the rotation angle Φ about the pole axis,

$$\bar{u}(s, z) = U(z) \sin\theta(s) - V(z) \cos\theta(s) - \Phi(z) q(s) \tag{1a}$$
$$\bar{v}(s, z) = U(z) \cos\theta(s) + V(z) \sin\theta(s) + \Phi(z) r(s) \tag{1b}$$

These equations apply to the whole contour. The out-of-plane shell displacement $\bar{w}$ can now be found from the assumption 2. For each element of middle surface, the shear strain become

$$\bar{\gamma}_{sz} = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial s} = \Phi'(z) \frac{F(s)}{t(s)} \tag{2}$$

where $t(s)$ is thickness of contour box section, $F(s)$ is the St. Venant circuit shear flow.

After substituting for $\bar{v}$ from Eq.(1) and considering the following geometric relations,

$$dx = ds \cos\theta \tag{3a}$$
$$dy = ds \sin\theta \tag{3b}$$

Eq.(2) can be integrated with respect to s from the origin to an arbitrary point on the contour,

$$\bar{w}(s, z) = W(z) - U'(z) x(s) - V'(z) y(s) - \Phi'(z) \omega(s) \tag{4}$$

where differentiation with respect to the axial coordinate z is denoted by primes ($'$); W represents the average axial displacement of the beam in the z direction; x and y are the coordinates of the contour in the (x, y, z) coordinate system; and ω is the so-called sectorial coordinate or warping function given by

$$\omega(s) = \int_{s_\circ}^{s} \Big[r(s) - \frac{F(s)}{t(s)} \Big] ds \tag{5a}$$
$$\oint_i \frac{F(s)}{t(s)} ds = 2A_i \quad i = 1, ..., n \tag{5b}$$

where $r(s)$ is height of a triangle with the base ds; A_i is the area circumscribed by the contour of the i circuit. The explicit forms of $\omega(s)$ and $F(s)$ for box section are given in the Appendix.

The displacement components u, v, w representing the deformation of any generic point on the profile section are given with respect to the midsurface displacements $\bar{u}, \bar{v}, \bar{w}$ by the assumption 3.

$$u(s,z,n) = \bar{u}(s,z) \tag{6a}$$
$$v(s,z,n) = \bar{v}(s,z) - n\frac{\partial \bar{u}(s,z)}{\partial s} \tag{6b}$$
$$w(s,z,n) = \bar{w}(s,z) - n\frac{\partial \bar{u}(s,z)}{\partial z} \tag{6c}$$

The strains associated with the small-displacement theory of elasticity are given by

$$\epsilon_s = \bar{\epsilon}_s + n\bar{\kappa}_s \tag{7a}$$
$$\epsilon_z = \bar{\epsilon}_z + n\bar{\kappa}_z \tag{7b}$$
$$\gamma_{sz} = \bar{\gamma}_{sz} + n\bar{\kappa}_{sz} \tag{7c}$$

where

$$\bar{\epsilon}_s = \frac{\partial \bar{v}}{\partial s}; \quad \bar{\epsilon}_z = \frac{\partial \bar{w}}{\partial z} \tag{8a}$$
$$\bar{\kappa}_s = -\frac{\partial^2 \bar{u}}{\partial z^2}; \quad \bar{\kappa}_z = -\frac{\partial^2 \bar{u}}{\partial z^2}; \quad \bar{\kappa}_{sz} = -2\frac{\partial^2 \bar{u}}{\partial s \partial z} \tag{8b}$$

All the other strains are identically zero. In Eq.(8), $\bar{\epsilon}_s$ and $\bar{\kappa}_s$ are assumed to be zero. $\bar{\epsilon}_z$, $\bar{\kappa}_z$ and $\bar{\kappa}_{sz}$ are midsurface axial strain and biaxial curvature of the shell, respectively. The above shell strains can be converted to beam strain components by substituting Eqs.(1), (4) and (6) into Eq.(8) as

$$\bar{\epsilon}_z = \epsilon_z^{\circ} + x\kappa_y + y\kappa_x + \omega\kappa_\omega \tag{9a}$$
$$\bar{\kappa}_z = \kappa_y \sin\theta - \kappa_x \cos\theta - \kappa_\omega q \tag{9b}$$
$$\bar{\kappa}_{sz} = 2\bar{\chi}_{sz} = \kappa_{sz} \tag{9c}$$

where $\epsilon_z^{\circ}, \kappa_x, \kappa_y, \kappa_\omega$ and κ_{sz} are axial strain, biaxial curvatures in the x and y direction, warping curvature with respect to the shear center, and twisting curvature in the beam, respectively defined as

$$\epsilon_z^{\circ} = W' \tag{10a}$$
$$\kappa_x = -V'' \tag{10b}$$
$$\kappa_y = -U'' \tag{10c}$$
$$\kappa_\omega = -\Phi'' \tag{10d}$$
$$\kappa_{sz} = 2\Phi' \tag{10e}$$

The resulting strains can be obtained from Eqs.(7) and (9) as

$$\epsilon_z = \epsilon_z^{\circ} + (x + n\sin\theta)\kappa_y + (y - n\cos\theta)\kappa_x + (\omega - nq)\kappa_\omega \tag{11a}$$
$$\gamma_{sz} = (n + \frac{F}{2t})\kappa_{sz} \tag{11b}$$

III. VARIATIONAL FORMULATION

Total potential energy of the system is calculated by sum of strain energy and potential energy,

$$\Pi = \mathcal{U} + \mathcal{V} \tag{12}$$

where $\mathcal{U}$ is the strain energy

$$\mathcal{U} = \frac{1}{2}\int_v (\sigma_z \epsilon_z + \sigma_{zs}\gamma_{sz}) dv \tag{13}$$

The strain energy is calculated by substituting Eq.(11) into Eq.(13)

$$\mathcal{U} = \frac{1}{2}\int_v \left\{ \sigma_z \left[\epsilon_z^{\circ} + (x + n\sin\theta)\kappa_y + (y - n\cos\theta)\kappa_x + (\omega - nq)\kappa_\omega \right] + \sigma_{zs}(n + \frac{F}{2t})\kappa_{sz} \right\} dv \tag{14}$$

The variation of strain energy can be stated as

$$\delta\mathcal{U} = \int_0^l (N_z \delta\epsilon_z + M_y \delta\kappa_y + M_x \delta\kappa_x + M_\omega \delta\kappa_\omega + M_t \delta\kappa_{sz}) ds \tag{15}$$

where $N_z, M_x, M_y, M_\omega, M_t$ are axial force, bending moments in the x and y directions, warping moment (bimoment), and tortional moment with respect to the centroid, respectively, defined by integrating over the cross-sectional area A as

$$N_z = \int_A \sigma_z ds dn \tag{16a}$$

$$M_y = \int_A \sigma_z (x + n \sin\theta) ds dn \tag{16b}$$

$$M_x = \int_A \sigma_z (y - n \cos\theta) ds dn \tag{16c}$$

$$M_\omega = \int_A \sigma_z (\omega - nq) ds dn \tag{16d}$$

$$M_t = \int_A \sigma_{zs} (n + \frac{F}{2t}) ds dn \tag{16e}$$

The variation of the work done by external force can be stated as

$$\delta\mathcal{V} = \int_0^l (q\delta V + t\delta\Phi) dz \tag{17}$$

where q is transverse load and t is applied torque. Using the principle that the variation of the total potential energy is zero, the following weak statement is obtained

$$0 = \int_0^l (N_z \delta W' - M_y \delta U'' - M_x \delta V'' - M_\omega \delta\Phi'' + 2M_t \delta\Phi' + q\delta V + t\delta\Phi) ds \tag{18}$$

IV. CONSTITUTIVE EQUATIONS

The constitutive equations of a k^{th} orthotropic lamina in the laminate co-ordinate system of box section are given by

$$\left\{ \begin{array}{c} \sigma_z \\ \sigma_{sz} \end{array} \right\}^k = \left[\begin{array}{cc} \bar{Q}^*_{11} & \bar{Q}^*_{16} \\ \bar{Q}^*_{16} & \bar{Q}^*_{66} \end{array} \right]^k \left\{ \begin{array}{c} \epsilon_z \\ \gamma_{sz} \end{array} \right\} \tag{19}$$

where $\bar{Q}^*_{ij}$ are transformed reduced stiffnesses. The transformed reduced stiffnesses can be calculated from the transformed stiffnesses based on the plane stress assumption and plane strain assumption. More detailed explanation can be found in Ref.[12]

The constitutive equations for bar forces and bar strains are obtained by using Eqs.(11), (16) and (19)

$$\left\{ \begin{array}{c} N_z \\ M_y \\ M_x \\ M_\omega \\ M_t \end{array} \right\} = \left[\begin{array}{ccccc} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} \\ & E_{22} & E_{23} & E_{24} & E_{25} \\ & & E_{33} & E_{34} & E_{35} \\ & & & E_{44} & E_{45} \\ \text{sym.} & & & & E_{55} \end{array} \right] \left\{ \begin{array}{c} \epsilon^\circ_z \\ \kappa_y \\ \kappa_x \\ \kappa_\omega \\ \kappa_{sz} \end{array} \right\} \tag{20}$$

where E_{ij} are stiffnesses of the thin-walled composite, and can be defined by

$$E_{11} = \int_s A_{11} ds \tag{21a}$$

$$E_{12} = \int_s (A_{11}x + B_{11}\sin\theta) ds \tag{21b}$$

$$E_{13} = \int_s (A_{11}y - B_{11}\cos\theta) ds \tag{21c}$$

$$E_{14} = \int_s (A_{11}\omega - B_{11}q) ds \tag{21d}$$

$$E_{15} = \int_s (A_{16}\frac{F}{2t} + B_{16}) ds \tag{21e}$$

$$E_{22} = \int_s (A_{11}x^2 + 2B_{11}x\sin\theta + D_{11}\sin^2\theta) ds \tag{21f}$$

$$E_{23} = \int_s \Big[A_{11}xy + B_{11}(y\sin\theta - x\cos\theta) - D_{11}\sin\theta\cos\theta\Big] ds \tag{21g}$$

$$E_{24} = \int_s \Big[A_{11}x\omega + B_{11}(\omega\sin\theta - qx) - D_{11}q\sin\theta\Big] ds \tag{21h}$$

$$E_{25} = \int_s \Big[A_{16}\frac{F}{2t}x + B_{16}(x + \frac{F\sin\theta}{2t}) + D_{16}\sin\theta\Big] ds \tag{21i}$$

$$E_{33} = \int_s (A_{11}y^2 - 2B_{11}y\cos\theta + D_{11}\cos^2\theta) ds \tag{21j}$$

$$E_{34} = \int_s \Big[A_{11}y\omega - B_{11}(\omega\cos\theta + qy) + D_{11}q\cos\theta\Big] ds \tag{21k}$$

$$E_{35} = \int_s \Big[A_{16}\frac{F}{2t}y + B_{16}(y - \frac{F\cos\theta}{2t}) - D_{16}\cos\theta\Big] ds \tag{21l}$$

$$E_{44} = \int_s (A_{11}\omega^2 - 2B_{11}\omega q + D_{11}q^2) ds \tag{21m}$$

$$E_{45} = \int_s \Big[A_{16}\frac{F}{2t}\omega + B_{16}(\omega - \frac{Fq}{2t}) - D_{16}q\Big] ds \tag{21n}$$

$$E_{55} = \int_s (A_{66}\frac{F^2}{4t^2} + B_{66}\frac{F}{t} + D_{66}) ds \tag{21o}$$

where A_{ij}, B_{ij} and D_{ij} matrices are extensional, coupling and bending stiffness, respectively, defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \int \bar{Q}_{ij}(1, n, n^2) dn \tag{22}$$

It appears that the laminate stiffnesses E_{ij} depend on the cross section of the composites. The explicit forms of the laminate stiffnesses E_{ij} can be calculated for composite box section and given in the Appendix.

V. GOVERNING EQUATIONS

The equilibrium equations of the present study can be obtained by integrating the derivatives of the varied quantities by parts and collecting the coefficients of $\delta U, \delta V, \delta W$ and $\delta\Phi$

$$N_z' = 0 \tag{23a}$$

$$M_y'' = 0 \tag{23b}$$

$$M_x'' + q = 0 \tag{23c}$$

$$M_\omega'' + 2M_t' + t = 0 \tag{23d}$$

By substituting Eq.(10) and (20) into Eq.(23)the explicit form of the governing equations can be expressed with respect to the laminate stiffnesses E_{ij} as

$$E_{11}W'' - E_{12}U''' - E_{13}V''' - E_{14}\Phi''' + 2E_{15}\Phi'' = 0 \tag{24a}$$
$$E_{12}W''' - E_{22}U^{iv} - E_{23}V^{iv} - E_{24}\Phi^{iv} + 2E_{25}\Phi''' = 0 \tag{24b}$$
$$E_{13}W''' - E_{23}U^{iv} - E_{33}V^{iv} - E_{34}\Phi^{iv} + 2E_{35}\Phi''' + q = 0 \tag{24c}$$
$$E_{14}W''' + 2E_{15}W'' - E_{24}U^{iv} - 2E_{25}U''' - E_{34}V^{iv} - 2E_{35}V''' - E_{44}\Phi^{iv} + 4E_{55}\Phi'' + t = 0 \tag{24d}$$

Eq.(24) is most general form for flexural, torsional behavior of a thin-walled laminated composite with a box section, and the dependent variables, U, V, W and Φ are fully coupled.

VI. FINITE ELEMENT FORMULATION

The present theory for thin-walled composite beams described in the previous section was implemented via a displacement based finite element method. The generalized displacements are expressed over each element as a linear combination of the one-dimensional Lagrange interpolation function Ψ_j and Hermite-cubic interpolation function ψ_j associated with node j and the nodal values

$$W = \sum_{j=1}^{n} w_j \Psi_j \tag{25a}$$
$$U = \sum_{j=1}^{n} u_j \psi_j \tag{25b}$$
$$V = \sum_{j=1}^{n} v_j \psi_j \tag{25c}$$
$$\Phi = \sum_{j=1}^{n} \phi_j \psi_j \tag{25d}$$

Substituting these expressions into the weak statement in Eq.(18), the finite element model of a typical element can be expressed as

$$[K]\{\Delta\} = \{f\} \tag{26}$$

where $[K]$ is the element stiffness matrix and $[f]$ is the element force vector

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ & K_{22} & K_{23} & K_{24} \\ & & K_{33} & K_{34} \\ \text{sym.} & & & K_{44} \end{bmatrix} \tag{27}$$

$$\{f\} = \{0\;0\;f_3\;f_4\}^{\mathrm{T}} \tag{28}$$

FIG. 2 A cantilever composite box beam under axial load

TABLE I The axial displacement and the angle of twist of a cantilever beam at the free end under axial load

	Ref.[7]	Present	
		Plane strain	Plane stress
Axial displacement	0.616 mm	0.603 mm	0.620 mm
Angle of twist	9.948×10^{-3} rad	16.400×10^{-3} rad	9.113×10^{-3} rad

The explicit forms of $[K]$ and $[f]$ are given by

$$K_{ij}^{11} = \int_0^l E_{11}\Psi_i'\Psi_j' dz \tag{29a}$$

$$K_{ij}^{12} = -\int_0^l E_{12}\Psi_i'\psi_j'' dz \tag{29b}$$

$$K_{ij}^{13} = -\int_0^l E_{13}\Psi_i'\psi_j'' dz \tag{29c}$$

$$K_{ij}^{14} = \int_0^l (2E_{15}\Psi_i'\psi_j' - E_{14}\Psi_i'\psi_j'') dz \tag{29d}$$

$$K_{ij}^{22} = \int_0^l E_{22}\psi_i''\psi_j'' dz \tag{29e}$$

$$K_{ij}^{23} = \int_0^l E_{23}\psi_i''\psi_j'' dz \tag{29f}$$

$$K_{ij}^{24} = \int_0^l (E_{24}\psi_i''\psi_j'' - 2E_{25}\psi_i''\psi_j') dz \tag{29g}$$

$$K_{ij}^{33} = \int_0^l E_{33}\psi_i''\psi_j'' dz \tag{29h}$$

$$K_{ij}^{34} = \int_0^l (E_{34}\psi_i''\psi_j'' - 2E_{35}\psi_i''\psi_j') dz \tag{29i}$$

$$K_{ij}^{44} = \int_0^l (E_{44}\psi_i''\psi_j'' - 2E_{45}(\psi_i'\psi_j'' + \psi_i''\psi_j') + 4E_{55}\psi_i'\psi_j') dz \tag{29j}$$

$$f_i^3 = \int_0^l q\psi_i dz \tag{29k}$$

$$f_i^4 = \int_0^l t\psi_i dz \tag{29l}$$

In Eq.(26), $\{\Delta\}$ is the unknown nodal displacements

$$\{\Delta\} = \{W\ U\ V\ \Phi\}^{\mathrm{T}} \tag{30}$$

VII. NUMERICAL EXAMPLES

For verification purpose, a cantilever composite box beam with length $L = 1m$, and the cross section shown in Fig.2 is subjected to an axial load of 24 KN with stacking sequence $[0_{10}/45_{10}]$. Plane stress ($\sigma_s = 0$) and plane strain ($\epsilon_s = 0$) assumptions are made in the analysis. The following material properties are used

$$E_1 = 148\text{GPa}\ , E_2 = 9.65\text{GPa}\ , G_{12} = 4.55\text{GPa}\ , \nu_{12} = 0.34 \tag{31}$$

TABLE II The maximum deflection and the angle of twist at the mid-span of a clamped beam under eccentric uniform load

	Ref.[7]	Present	
		Plane strain	Plane stress
Maximum deflection	0.488 mm	0.438 mm	0.494 mm
Maximum angle of twist	2.760×10^{-3} rad	2.678×10^{-3} rad	6.427×10^{-3} rad

FIG. 3 A clamped composite box beam under an eccentric uniform load

The results using the present analysis are compared with previously available results Ref.[7] in Table I. It is seen that the results of the present finite element analysis for plane stress are in good agreement with the solution in Ref.[7].

The next example is a clamped composite box beam with the same cross section as previous example except stacking sequence $[\pm 45_5/0_{10}]$, which is subjected to an eccentric uniform load p=-6.5 KN/m acting at the midplane of the left web as shown in Fig.3. The maximum angle of twist and the deflection are given in Table II. It is also shown that the solution based on the plane stress assumption ($\sigma_s = 0$), yields more accurate result. It seems that the angle of twist in Ref.[7] was calculated by using plane strain assumption.

For convenience, the following nondimensional values of angle of twist and vertical displacements are used

$$\bar{\phi} = \begin{cases} \dfrac{\phi p L}{G_{12} b_1 t^2} & \text{for uniform load} \\ \\ \dfrac{\phi P}{G_{12} b_1 t^2} \times 10^8 & \text{for concentrate load} \end{cases} \tag{32a}$$

$$\bar{v} = \begin{cases} \dfrac{v p L^3}{E_2 b_1^3 t} & \text{for uniform load} \\ \\ \dfrac{v P L^2}{E_2 b_1^3 t} \times 10^8 & \text{for concentrate load} \end{cases} \tag{32b}$$

In order to investigate the effects of fiber orientation, a clamped composite box beam is subjected to an eccentric uniform load as shown in Fig.3. Two layers with equal thicknesses are considered as anti-symmetric angle-ply laminates $[\theta/-\theta]$ in the flanges and webs. For all the analysis, the assumption $\sigma_s = 0$ is made. The coupling stiffnesses $E_{13}, E_{14}, E_{23}, E_{24}, E_{35}$ are zero, but E_{15} and E_{45} do not vanish due to unsymmetric stacking sequence of the webs and flanges. Variation of the torsional and vertical displacements at mid-span with respect to fiber angle change in the flanges and webs are shown in Figs.4 and 5. The maximum angle of twist occurs near $\theta = 25°$, that is, because the torsional rigidity E_{55} becomes maximum value at $\theta = 25°$.

The next example is a cantilever composite box beam under point load shown in Fig.6. Four layers with equal thickness are considered as an anti-symmetric angle-ply laminate in the flanges and webs. Stacking sequence of top and bottom flanges are $[\pm\theta_2]$ and $[\theta_4]$ respectively, left and right webs are $[\theta_2/-\theta_2]$, and thus, exhibit flexural torsional coupling. The vertical displacements at the free end are shown in Fig.7 with respect to fiber angle variation. It shows that the load eccentricity does not affect the vertical displacements. On the other hand, the maximum torsional displacement shows substantial changes for eccentricity with respect to fiber angle variation as shown in Fig.8. Even for no eccentricity ($e/b = 0$), the torsional displacement becomes nonzero as fiber angle goes off-axis implying that the coupling stiffnesses E_{15} and E_{45} drive flexural-torsional coupling. Vice versa, for ($e/b = 0.25$), the torsional displacement can vanish for specific value of fiber angle (near 3° and 68°) implying that the angle of twist can be suppressed with carefully tailored stacking sequence even for applied torque.

VIII. CONCLUDING REMARKS

An analytical model was developed to study the flexural torsional behavior of a laminated composite beam with box section. The model is capable of predicting accurate deflection as well as angle of twist for various configuration including boundary conditions, laminate stacking sequence and fiber angle. To formulate the problem, a one-dimensional

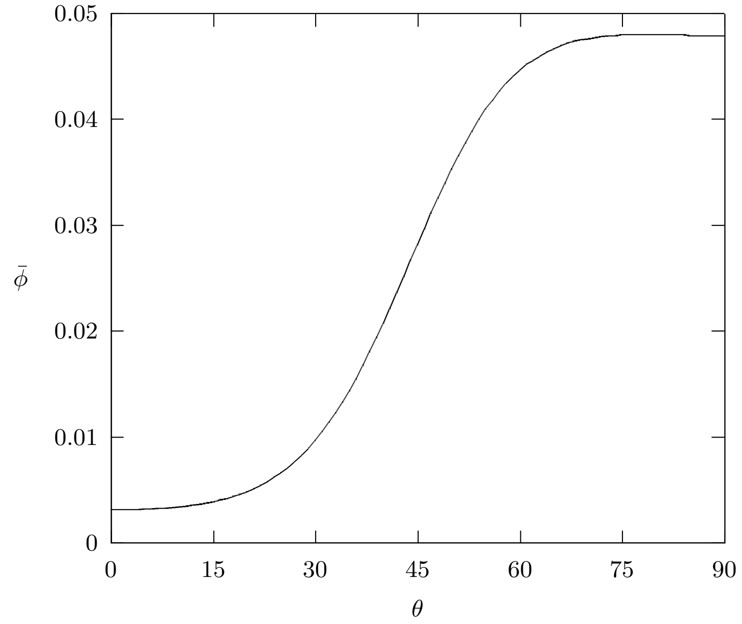

FIG. 4 Variation of angle of twist at mid-span with respect to fiber angle change for a clamped composite box beams under an eccentric uniform load

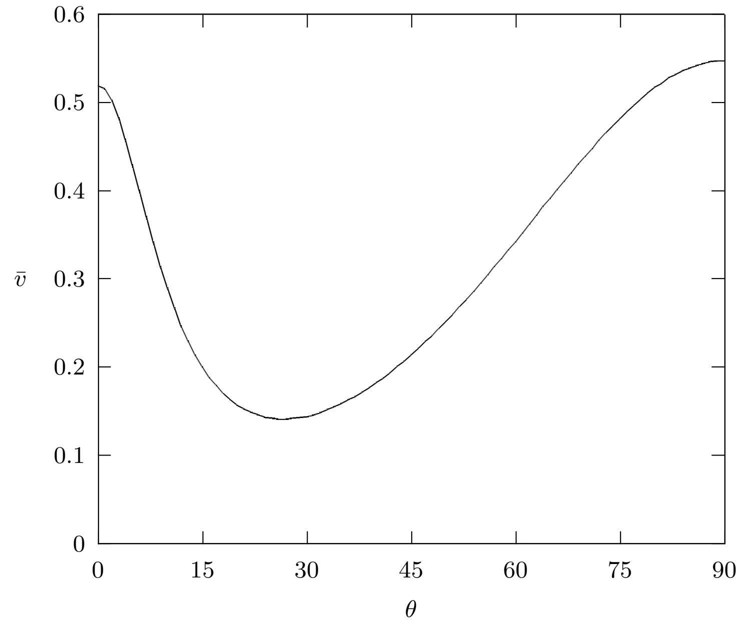

FIG. 5 Variation of the vertical displacements at mid-span with respect to fiber angle change for a clamped composite box beams under an eccentric uniform load

displacement-based finite element method is employed. The assumption that normal stress in contour direction vanishes ($\sigma_s = 0$) seems more appropriate than the free strain assumption in contour direction. The model presented is found to be appropriate and efficient in analyzing flexural torsional problem of a thin-walled box-section laminated composite beam.

FIG. 6 A cantilever composite box beam under an eccentric load at free end

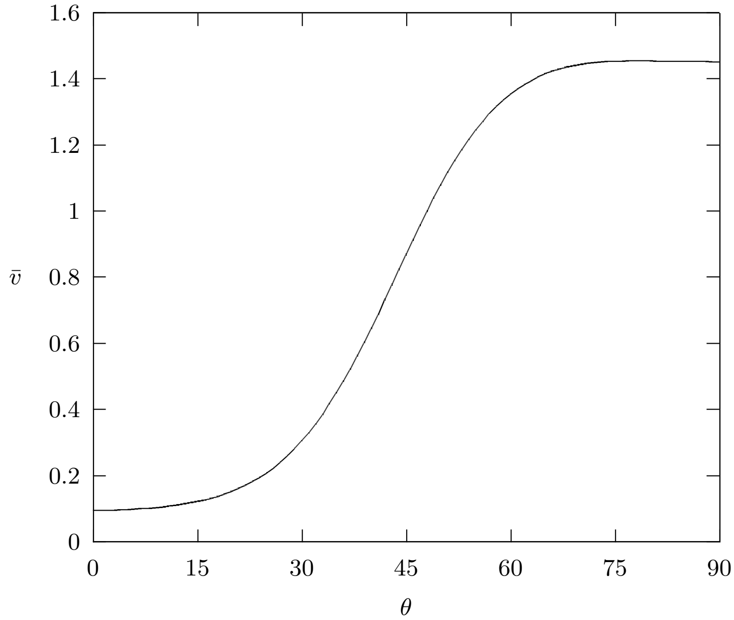

FIG. 7 Variation of the vertical displacements at free end with respect to the fiber angle change of a cantilever composite box beam

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APPENDIX

The St. Venant circuit shear flow of the box section in Fig.9 is given by

$$F = \frac{2b_1b_2}{b_1(\frac{1}{t_1}+\frac{1}{t_3})+b_2(\frac{1}{t_2}+\frac{1}{t_4})} \tag{33}$$

Warping functions with respect to the shear center of side 1, 2, 3, 4 are defined by

$$\omega_1(s_1) = (-x_1+x_p-\frac{F}{t_1})s_1+C = A_1s_1+C \tag{34a}$$

$$\omega_2(s_2) = (-y_2+y_p-\frac{F}{t_2})s_2+(-x_1+x_p-\frac{F}{t_1})b_1+C = A_2s_2+A_1b_1+C \tag{34b}$$

$$\omega_3(s_3) = (x_3-x_p-\frac{F}{t_3})s_3+(-x_1+x_p-\frac{F}{t_1})b_1+(-y_2+y_p-\frac{F}{t_2})b_2+C = A_3s_3+A_1b_1+A_2b_2+C \tag{34c}$$

$$\begin{aligned}\omega_4(s_4) &= (y_4-y_p-\frac{F}{t_4})s_4+(-x_1+x_p-\frac{F}{t_1})b_1+(-y_2+y_p-\frac{F}{t_2})b_2+(x_3-x_p-\frac{F}{t_3})b_1+C \\ &= A_4s_4+A_1b_1+A_2b_2+A_3b_1+C\end{aligned} \tag{34d}$$

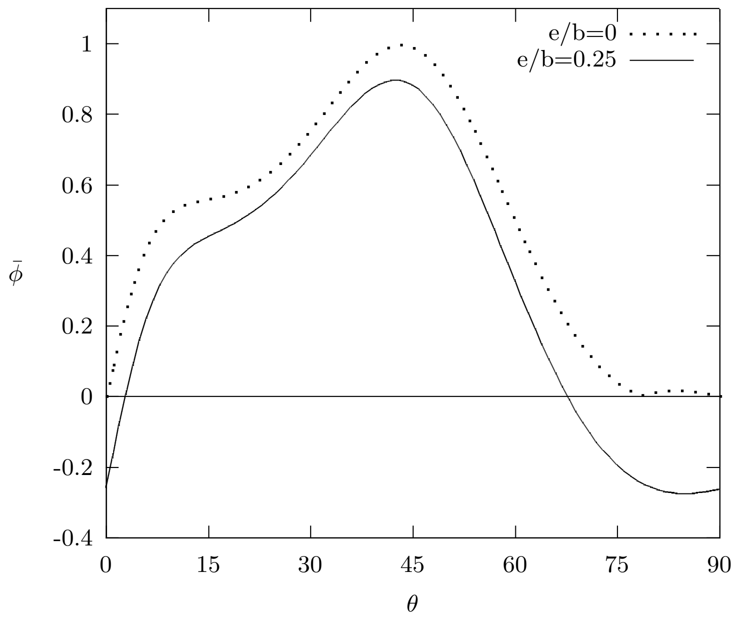

FIG. 8 Variation of the torsional displacement at free end with respect to the fiber angle change of a cantilever composite box beam

FIG. 9 Geometry of thin-walled composite box section

where constant C can be determined by condition

$$0 = \int_A \omega dA \tag{35a}$$

$$0 = \int_0^{b_1} \omega_1(s_1)t_1 ds_1 + \int_0^{b_2} \omega_2(s_2)t_2 ds_2 + \int_0^{b_1} \omega_3(s_3)t_3 ds_3 + \int_0^{b_2} \omega_4(s_4)t_4 ds_4 \tag{35b}$$

$$C = -\frac{b_1^2\{A_1(t_1+2t_3)+A_3t_3\} + b_2^2\{A_2(t_2+2t_4)+A_4t_4\} + 2b_1b_2\{A_1(t_2+t_4)+A_3t_4+A_2t_3\}}{2\{b_1(t_1+t_3)+b_2(t_2+t_4)\}} \tag{35c}$$

The explicit forms of the laminate stiffnesses E_{ij} for composite box section can be defined by

$$E_{11} = A^1_{11}b_1 + A^2_{11}b_2 + A^3_{11}b_1 + A^4_{11}b_2 \tag{36a}$$

$$E_{12} = A^1_{11}x_1b_1 - B^1_{11}b_1 + \frac{1}{2}A^2_{11}b_2^2 + A^2_{11}x_1b_2 + A^3_{11}x_3b_1 + B^3_{11}b_1 - \frac{1}{2}A^4_{11}b_2^2 + A^4_{11}x_3b_2 \tag{36b}$$

$$E_{13} = -\frac{1}{2}A^1_{11}b_1^2 + A^1_{11}y_4b_1 + A^2_{11}y_2b_2 - B^2_{11}b_2 + \frac{1}{2}A^3_{11}b_1^2 + A^3_{11}y_2b_1 + A^4_{11}y_4b_2 + B^4_{11}b_2 \tag{36c}$$

$$\begin{aligned} E_{14} &= (A^1_{11}A_1 - B^1_{11})\frac{b_1^2}{2} + A^1_{11}Cb_1 + (A^2_{11}A_2 - B^2_{11})\frac{b_2^2}{2} + A^2_{11}(A_1b_1 + C)b_2 + (A^3_{11}A_3 - B^3_{11})\frac{b_1^2}{2} \\ &+ A^3_{11}(A_1b_1 + A_2b_2 + C)b_1 + (A^4_{11}A_4 - B^4_{11})\frac{b_2^2}{2} + A^4_{11}(A_1b_1 + A_2b_2 + A_3b_1 + C)b_2 \end{aligned} \tag{36d}$$

$$E_{15} = b_1(B^1_{16} + A^1_{16}\frac{F}{2t_1}) + b_2(B^2_{16} + A^2_{16}\frac{F}{2t_2}) + b_1(B^3_{16} + A^3_{16}\frac{F}{2t_3}) + b_2(B^4_{16} + A^4_{16}\frac{F}{2t_4}) \tag{36e}$$

$$\begin{aligned} E_{22} &= A^1_{11}x_1^2b_1 - 2B^1_{11}x_1b_1 + D^1_{11}b_1 + \frac{1}{3}A^2_{11}b_2^3 + A^2_{11}x_1b_2^2 + A^2_{11}x_1^2b_2 \\ &+ A^3_{11}x_3^2b_1 + 2B^3_{11}x_3b_1 + D^3_{11}b_1 + \frac{1}{3}A^4_{11}b_2^3 - A^4_{11}x_3b_2^2 + A^4_{11}x_3^2b_2 \end{aligned} \tag{36f}$$

$$E_{23} = \frac{1}{2}(-A^1_{11}x_1 + B^1_{11})b_1^2 + A^1_{11}x_1y_4b_1 - B^1_{11}y_4b_1 + \frac{1}{2}(A^2_{11}y_2 - B^2_{11})b_2^2 + A^2_{11}x_1y_2b_2 - B^2_{11}x_1b_2$$

$$
\begin{aligned}
&+ \frac{1}{2}(A_{11}^3 x_3 + B_{11}^3) b_1^2 + A_{11}^3 x_3 y_2 b_1 + B_{11}^3 y_2 b_1 + \frac{1}{2}(-A_{11}^4 y_4 - B_{11}^4) b_2^2 + A_{11}^4 x_3 y_4 b_2 + B_{11}^4 x_3 b_2 &\text{(36g)}\\
E_{24} &= \frac{1}{2}\{A_{11}^1 x_1 A_1 + B_{11}^1(-A_1 - x_1) + D_{11}^1\} b_1^2 + A_{11}^1 x_1 C b_1 - B_{11}^1 C b_1\\
&+ \frac{1}{3}(A_{11}^2 A_2 - B_{11}^2) b_2^3 + \frac{1}{2}\{A_{11}^2 x_1 A_2 + A_{11}^2(A_1 b_1 + C) - B_{11}^2 x_1\} b_2^2 + A_{11}^2 x_1 (A_1 b_1 + C) b_2\\
&+ \frac{1}{2}\{A_{11}^3 x_3 A_3 + B_{11}^3 (A_3 - x_3) - D_{11}^3\} b_1^2 + A_{11}^3 x_3 (A_1 b_1 + A_2 b_2 + C) b_1 + B_{11}^3 (A_1 b_1 + A_2 b_2 + C) b_1\\
&+ \frac{1}{3}(-A_{11}^4 A_4 + B_{11}^4) b_2^3 + \frac{1}{2}\{A_{11}^4 x_3 A_4 - A_{11}^4 (A_1 b_1 + A_2 b_2 + A_3 b_1 + C) - B_{11}^4 x_3\} b_2^2\\
&+ A_{11}^4 x_3 (A_1 b_1 + A_2 b_2 + A_3 b_1 + C) b_2 &\text{(36h)}\\
E_{25} &= B_{16}^1 (x_1 - \frac{F}{2t_1}) b_1 - D_{16}^1 b_1 + A_{16}^1 x_1 \frac{F}{2t_1} b_1 + \frac{1}{2}(B_{16}^2 + A_{16}^2 \frac{F}{2t_2}) b_2^2 + B_{16}^2 x_1 b_2 + A_{16}^2 x_1 \frac{F}{2t_2} b_2\\
&+ B_{16}^3 (x_3 + \frac{F}{2t_3}) b_1 + D_{16}^3 b_1 + A_{16}^3 x_3 \frac{F}{2t_3} b_1 + \frac{1}{2}(-B_{16}^4 - A_{16}^4 \frac{F}{2t_4}) b_2^2 + B_{16}^4 x_3 b_2 + A_{16}^4 x_3 \frac{F}{2t_4} b_2 &\text{(36i)}\\
E_{33} &= \frac{1}{3} A_{11}^1 b_1^3 - A_{11}^1 y_4 b_1^2 + A_{11}^1 y_4^2 b_1 + A_{11}^2 y_2^2 b_2 - 2B_{11}^2 y_2 b_2 + D_{11}^2 b_2\\
&+ \frac{1}{3} A_{11}^3 b_1^3 + A_{11}^3 y_2 b_1^2 + A_{11}^3 y_2^2 b_1 + A_{11}^4 y_4^2 b_2 + 2B_{11}^4 y_4 b_2 + D_{11}^4 b_2 &\text{(36j)}\\
E_{34} &= \frac{1}{3}(-A_{11}^1 A_1 + B_{11}^1) b_1^3 + \frac{1}{2}(A_{11}^1 y_4 A_1 - A_{11}^1 C - B_{11}^1 y_4) b_1^2 + A_{11}^1 y_4 C b_1\\
&+ \frac{1}{2}\{A_{11}^2 y_2 A_2 - B_{11}^2 (A_2 + y_2) + D_{11}^2\} b_2^2 + A_{11}^2 y_2 (A_1 b_1 + C) b_2 - B_{11}^2 (A_1 b_1 + C) b_2\\
&+ \frac{1}{3}(A_{11}^3 A_3 - B_{11}^3) b_1^3 + \frac{1}{2}\{A_{11}^3 y_2 A_3 + A_{11}^3 (A_1 b_1 + A_2 b_2 + C) - B_{11}^3 y_2\} b_1^2 + A_{11}^3 y_2 (A_1 b_1 + A_2 b_2 + C) b_1\\
&+ \frac{1}{2}\{A_{11}^4 y_4 A_4 - B_{11}^4 (-A_4 + y_4) - D_{11}^4\} b_2^2 + A_{11}^4 y_4 (A_1 b_1 + A_2 b_2 + A_3 b_1 + C) b_2\\
&+ B_{11}^4 (A_1 b_1 + A_2 b_2 + A_3 b_1 + C) b_2 &\text{(36k)}\\
E_{35} &= \frac{1}{2}(-B_{16}^1 - A_{16}^1 \frac{F}{2t_1}) b_1^2 + B_{16}^1 y_4 b_1 + A_{16}^1 y_4 \frac{F}{2t_1} b_1 + B_{16}^2 (y_2 - \frac{F}{2t_2}) b_2 - D_{16}^2 b_2 + A_{16}^2 y_2 \frac{F}{2t_2} b_2\\
&+ \frac{1}{2}(B_{16}^3 + A_{16}^3 \frac{F}{2t_3}) b_1^2 + B_{16}^3 y_2 b_1 + A_{16}^3 y_2 \frac{F}{2t_3} b_1 + B_{16}^4 (y_4 + \frac{F}{2t_4}) b_2 + D_{16}^4 b_2 + A_{16}^4 y_4 \frac{F}{2t_4} b_2 &\text{(36l)}\\
E_{44} &= b_1 A_{11}^1 \{b_1 A_1 (C + A_1 \frac{b_1}{3}) + C^2\} - b_1 B_{11}^1 (C + 2A_1 \frac{b_1}{3})\\
&+ b_2 A_{11}^2 \{b_2 A_2 (A_1 b_1 + C + A_2 \frac{b_2}{3}) + (A_1 b_1 + C)^2\} - b_2 B_{11}^2 (A_1 b_1 + C + 2A_2 \frac{b_2}{3})\\
&+ b_1 A_{11}^3 \{b_1 A_3 (A_1 b_1 + A_2 b_2 + C + A_3 \frac{b_1}{3}) + (A_1 b_1 + A_2 b_2 + C)^2\} - b_1 B_{11}^3 (A_1 b_1 + A_2 b_2 + C + 2A_3 \frac{b_1}{3})\\
&+ b_2 A_{11}^4 \{b_2 A_4 (A_1 b_1 + A_2 b_2 + A_3 b_1 + C + A_4 \frac{b_2}{3}) + (A_1 b_1 + A_2 b_2 + A_3 b_1 + C)^2\}\\
&- b_2 B_{11}^4 (A_1 b_1 + A_2 b_2 + A_3 b_1 + C + 2A_4 \frac{b_2}{3}) &\text{(36m)}\\
E_{45} &= B_{66}^1 b_1 \{(A_1 - \frac{F}{2t_1}) \frac{b_1}{2} + C\} + A_{16}^1 b_1 \frac{F}{2t_1} \{A_1 b_1 + C\} - D_{16}^1 \frac{b_1^2}{2}\\
&+ B_{66}^2 b_2 \{(A_2 - \frac{F}{2t_2}) \frac{b_2}{2} + A_1 b_1 + C\} + A_{16}^2 b_2 \frac{F}{2t_2} \{A_2 b_2 + A_1 b_1 + C\} - D_{16}^2 \frac{b_2^2}{2}\\
&+ B_{66}^3 b_1 \{(A_3 - \frac{F}{2t_3}) \frac{b_1}{2} + A_1 b_1 + A_2 b_2 + C\} + A_{16}^3 b_1 \frac{F}{2t_3} \{A_3 b_1 + A_1 b_1 + A_2 b_2 + C\} - D_{16}^3 \frac{b_1^2}{2}\\
&+ B_{66}^4 b_2 \{(A_4 - \frac{F}{2t_4}) \frac{b_2}{2} + A_1 b_1 + A_2 b_2 + A_3 b_1 + C\}\\
&+ A_{16}^4 b_2 \frac{F}{2t_4} \{A_4 b_2 + A_1 b_1 + A_2 b_2 + A_3 b_1 + C\} - D_{16}^4 \frac{b_2^2}{2} &\text{(36n)}
\end{aligned}
$$

$$
\begin{aligned}
E_{55} &= D_{66}^{1} b_1 + A_{66}^{1} b_1 \frac{F^2}{4t_1^2} + 2B_{66}^{1} \frac{F}{2t_1} + D_{66}^{2} b_2 + A_{66}^{2} b_2 \frac{F^2}{4t_2^2} + 2B_{66}^{2} \frac{F}{2t_2} \\
&+ D_{66}^{3} b_1 + A_{66}^{3} b_1 \frac{F^2}{4t_3^2} + 2B_{66}^{3} \frac{F}{2t_3} + D_{66}^{4} b_2 + A_{66}^{4} b_2 \frac{F^2}{4t_4^2} + 2B_{66}^{4} \frac{F}{2t_4} \qquad (36o)
\end{aligned}
$$

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