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On the occurrence of gauge-dependent secularities in nonlinear gravitational waves

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We study the plane (not necessarily monochromatic) gravitational waves at nonlinear quadratic order on a flat background in vacuum. We show that, in the harmonic gauge, the nonlinear waves are unstable. We argue that, at this order, this instability can not be eliminated by means of a multiscale approach, i.e. introducing suitable long variables, as it is often the case when secularities appear in a perturbative scheme. However, this is a non-physical and gauge-dependent effect that disappears in a suitable system of coordinates. In facts, we show that in a specific gauge such instability does not occur, and that it is possible to solve exactly the second order nonlinear equations of gravitational waves. Incidentally, we note that this gauge coincides with the one used by Belinski and Zakharov to find exact solitonic solutions of Einstein's equations, that is to an exactly integrable case, and this fact makes our second order nonlinear solutions less interesting. However, the important warning is that one must be aware of the existence of the instability reported in this paper, when studying nonlinear gravitational waves in the harmonic gauge.

I. INTRODUCTION

The LIGO-VIRGO collaboration has recently detected two events, GW150914 [1] and GW151226 [2], relative to the coalescence and merger of binary systems of black holes, and the formation of a final black hole. This exceptional achievement has finally directly proven the existence of gravitational waves [17], which is one of the main predictions of general relativity, as pointed out by Einstein in 1916 [3]. The modelling of the gravitational signal produced in the merge of binary systems has played a crucial role in the direct detection of gravitational waves, since the comparison between the data and the theoretically predicted signal allows to discriminate between different types of sources, and to estimate the physical parameters involved, e.g. the black holes masses, spins and distances. This has been achieved combining two (quite different) approaches, which are used together to construct a model of the gravitational signal at different space scales (for instance in the "near-zone" and "far-zone" with respect to the source) and for different parts of the waveform (inspiral phase, merger and ringdown). The first approach consists in the numerical solution of the Einstein's equations, see [7] and references therein for a review of numerical relativity results, which gives an useful description of the merger of the black holes after the inspiral phase. The second approach makes use of a perturbative expansion of the Einstein's equations. The two most popular realizations of this approach are the Post-Newtonian (PN) expansion [8], and the Effective One Body (EOB) formalism [9], which are successful to model the inspiral phase, when the two black holes are well separated.

In this paper we are interested in the stability of the perturbative approach. We consider the Einstein's equations in vacuum for small perturbations h of the Minkowski metric at second order h^2 in the perturbative expansion in powers of h , and we show that plane waves are unstable in the harmonic gauge, which is commonly used to study gravitational waves in both the PN and EOB formalisms. We discuss this instability, and show that it can not be eliminated by means of a multi scale approach [18]. In spite of this fact, the nature of such instability is not physical, instead it is a feature of the harmonic gauge, so it is simply a gauge effect. This conclusion is based on the observation that, in an appropriate reference system explicitly defined, the solutions of the nonlinear quadratic perturbative equations are stable. In facts, in this gauge the second order nonlinear equations are easily integrated, and their explicit solution shows that the quadratic nonlinearities do not change in any significant respect the picture of the gravitational waves obtained from the linearized Einstein's equations. However, the relevance of these perturbative

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solutions is diminished by the fact that the reference system in which they have been found, is the same used in the famous result of Zakharov and Belinski on gravitational solitons [10], so that this gauge choice corresponds to a situation in which the full Einstein's equations in vacuum (and in electro-vacuum) are integrable.

The relevant point here is the existence of such gauge-induced instability of plane waves in vacuum in the harmonic reference system. This fact should be a warning for those who want to study nonlinear gravitational waves using the harmonic gauge in various contexts. In facts, in some cases the choice of the harmonic gauge might not be convenient, due to the secular instability of the perturbation h . For instance, in the PN and EOB formalisms which make use of the harmonic gauge on a flat background, this instability might appear if one studies the evolution of gravitational waves far from the source using the plane wave approximation. We mention that an analysis of this instability for more general wavefronts and for different backgrounds is currently under study, and will be reported elsewhere.

A further interesting question is whether this kind of instability is still present in alternative gravitational models. For instance, in [11] it has been reported an instability of gravitational waves in the bigravity model, which might be due essentially to the same mechanism discussed in this paper. Furthermore, a study of this instability in the case of modified gravity [12] and of nonlocal gravitational models [13] would be also interesting, but it goes beyond the purposes of this paper and will be discussed separately.

In what follows, unless explicitly stated, we will use the following notations: Greek indices run from zero to three, i.e. $\alpha, \beta, \gamma \dots = 0, 1, 2, 3$, Latin indices run from 2 to 3 $i, j, k \dots = 2, 3$, capital Latin indices run from 1 to 3 $A, B, C \dots = 1, 2, 3$, and underlined indices run from 0 to 1 $\underline{a}, \underline{b}, \underline{c} \dots = 0, 1$. Moreover we rise and lower indices with the Minkowski metric with signature $1, -1, -1, -1$. The Ricci tensor is defined as in [16] as $R_{ij} = \partial_k \Gamma_{ij}^k + \dots$

II. EINSTEIN'S EQUATIONS

In this section we will write the Einstein's equations at second order of perturbations. It is well known [15, 16] that the components G^0_β of the Einstein's tensor do not contain second time derivatives $\ddot{g}_{\alpha\beta}$ of the metric tensor, but they contain only the first time derivative \dot{g}_{AB} and no derivatives $\dot{g}_{0\alpha}$, so that

$$G^0_\beta = G^0_\beta(g_{\alpha\beta}, \dot{g}_{AB}, \partial_C g_{\alpha\beta}, \partial_C \partial_D g_{\alpha\beta}). \quad (1)$$

Moreover the derivatives $\ddot{g}_{0\alpha}$ do not appear at all and one has

$$G^A_\gamma = G^A_\gamma(g_{\alpha\beta}, \dot{g}_{0\alpha}, \dot{g}_{AB}, \ddot{g}_{AB}, \partial_C g_{\alpha\beta}, \partial_C \partial_D g_{\alpha\beta}). \quad (2)$$

Therefore the equations $G^0_\beta = 0$ are involutive: if the equations $G^A_B = 0$ are verified at any time and if $G^0_\alpha = 0$ at the initial time t_0 , then $G^0_\alpha = 0$ at any time t . Thus, the equations $G^0_\alpha = 0$ can be viewed as a constraint for initial data. As a consequence, instead of 10 evolutionary equations for the 10 independent components of the metric tensor one has 6 evolutionary equations + 4 constraints on the initial data, which leaves 4 of the 10 components of the metric arbitrary: this arbitrariness corresponds to the arbitrariness in the choice of the reference system [19]. These considerations will be helpful in what follows, when we will discuss the perturbative equations.

Since we want to study the propagation of nonlinear gravitational waves in vacuum, we consider small perturbations h of the flat spacetime metric, that is a metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu}$ the Minkowski tensor and $|h_{\mu\nu}| = O(\epsilon)$, with $0 < \epsilon \ll 1$. From this definition it is possible to expand the Ricci tensor and the Ricci scalar as well as the Einstein's tensor in powers of h , see Appendix B.

For the Ricci tensor one has

$$R_{\mu\nu} = R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + O(h^3) \quad (3)$$

where $R_{\mu\nu}^{(1)}$ and $R_{\mu\nu}^{(2)}$ are respectively the linear and quadratic part of the Ricci tensor, and are given by

$$R_{\mu\nu}^{(1)} = \frac{1}{2} [-\square h_{\mu\nu} + \partial_\mu \partial_\alpha \psi_\nu^\alpha + \partial_\nu \partial_\alpha \psi_\mu^\alpha] = \frac{1}{2} [-h_{\mu\nu}{}^{,\alpha}{}_{,\alpha} - h_{\alpha,\mu\nu}^\alpha + h_{\nu,\alpha\mu}^\alpha + h_{\mu,\alpha\nu}^\alpha] \quad (4)$$

and

$$R_{\mu\nu}^{(2)} = \frac{1}{2} \{ -\psi^{\alpha\rho}{}_{,\alpha} (h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho}) + h^{\alpha\rho} (h_{\mu\nu,\alpha\rho} + h_{\alpha\rho,\mu\nu} - h_{\nu\rho,\alpha\mu} - h_{\mu\rho,\alpha\nu}) + \frac{1}{2} [h^{\alpha\rho}{}_{,\mu} h_{\alpha\rho,\nu} + (h_{\mu\alpha,\rho} - h_{\mu\rho,\alpha}) (h_\nu{}^{\alpha,\rho} - h_\nu{}^{\rho,\alpha})] \}, \quad (5)$$

where we have defined

$$\psi^\alpha_\beta \equiv h^\alpha_\beta - \frac{1}{2} h^\rho_\rho \delta^\alpha_\beta \quad (6)$$

see for instance [15]. The Ricci scalar is given by

$$R = \eta^{\alpha\beta} R_{\alpha\beta}^{(1)} + \eta^{\alpha\beta} R_{\alpha\beta}^{(2)} - h^{\alpha\beta} R_{\alpha\beta}^{(1)} + O(h^3) \quad (7)$$

and the Einstein's tensor up to second order in h is given by

$$G^\alpha_\beta \equiv R^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta R = G^{(1)\alpha}_\beta + G^{(2)\alpha}_\beta + O(h^3) \quad (8)$$

where $G^{(1)\alpha}_\beta$ and $G^{(2)\alpha}_\beta$ are respectively the linear and quadratic parts of the Einstein's tensor, and are given by

$$G^{(1)\alpha}_\beta \equiv R^{(1)\alpha}_\beta - \frac{1}{2} \delta^\alpha_\beta R^{(1)\gamma}_\gamma \quad (9)$$

and

$$G^{(2)\alpha}_\beta \equiv R^{(2)\alpha}_\beta - h^{\alpha\sigma} R^{(1)}_{\sigma\beta} - \frac{1}{2} \delta^\alpha_\beta \left(R^{(2)\gamma}_\gamma - h^{\mu\nu} R^{(1)}_{\mu\nu} \right) \quad (10)$$

In this work we limit our analysis to plane waves and, without loss of generality, we consider waves travelling along the x^1 axis, so that

$$h_{\alpha\beta} = h_{\alpha\beta}(x^0, x^1). \quad (11)$$

Moreover, we choose h as

$$h(x^0, x^1) = \begin{pmatrix} h_{00} & h_{10} & 0 & 0 \\ h_{10} & h_{11} & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & h_{33} \end{pmatrix} \quad (12)$$

We stress that the choice $h_{0i} = 0$ corresponds to a gauge choice, and therefore it does not affect the generality of the solution, while setting $h_{1i} = 0$ is in fact a constraint on the form of the gravitational wave.

The waveform (12) is compatible with the Einstein's equations, up to the order h^2 . In fact, the equations $R_{i\bar{a}} = R^{(1)}_{i\bar{a}} + R^{(2)}_{i\bar{a}} = 0$ are identically satisfied for $h_{i\bar{a}} = 0$, see (A5) and (A8). Therefore one has that $G^{\bar{a}}_i = g^{\alpha\bar{a}} G_{\alpha i} = (\eta^{\bar{a}b} - h^{\bar{a}b}) (R_{bi} - g_{bi} R/2) = \delta^{\bar{a}}_i R/2 = 0$, where we have used $h^{\bar{a}i} = 0$ and $(\eta^{\bar{a}b} - h^{\bar{a}b}) g_{bi} = \delta^{\bar{a}}_i$. Thus, the equations $G^{\bar{a}}_i = 0$ are identically satisfied for $h_{i\bar{a}} = 0$ and one is left with the four dynamical equations $G^i_j = 0$ and $G^1_1 = 0$ plus the two constraints $G^0_0 = 0$ and $G^0_1 = 0$ for the six nonzero variables h_{ij} and $h_{\bar{a}b}$. Thus, the solution (12) still contains two arbitrary functions, corresponding to the residual gauge freedom, that can be used to fix the gauge. Henceforth we will always set to zero the components $h_{i\bar{a}}$ of the metric.

A. Einstein's equations at first order

We start with the Einstein's equations in vacuum, at first linear perturbative order, and we refer the reader to the Appendix B for explicit calculations. Let us define the following variables which will be useful in the following discussions

$$B \equiv h_{22} + h_{33}; \quad \psi \equiv h_{22} - h_{23}; \quad A \equiv h_{00,11} + h_{11,00} - 2h_{01,01} \quad (13)$$

Using these definitions, the linearized Einstein's equations for the metric (12) read

$$\begin{aligned} G^{(1)0}_0 &= \frac{1}{2}B_{,11} = 0; & G^{(1)0}_1 &= \frac{1}{2}B_{,01} = 0; & G^{(1)1}_1 &= -\frac{1}{2}B_{,00} = 0; & G^{(1)2}_3 &= \frac{1}{2}\square h_{23} = 0 \\ G^{(1)2}_2 &= \frac{1}{2}(\square h_{22} - A - \square B) = 0; & G^{(1)3}_3 &= \frac{1}{2}(\square h_{33} - A - \square B) = 0. \end{aligned} \quad (14)$$

It is convenient to write the Einstein's equations by means of linear combinations of (14) as follows

$$\begin{aligned} G^{(1)0}_0 + G^{(1)1}_1 &= -\frac{1}{2}\square B = 0, & G^{(1)2}_2 + G^{(1)3}_3 - (G^{(1)0}_0 + G^{(1)1}_1) &= -A = 0, \\ G^{(1)2}_3 &= \frac{1}{2}\square h_{23} = 0, & G^{(1)2}_2 - G^{(1)3}_3 &= \frac{1}{2}\square\psi = 0, \end{aligned} \quad (15)$$

together with the two constrains

$$G^{(1)0}_0 = \frac{1}{2}B_{,11} = 0, \quad G^{(1)0}_1 = \frac{1}{2}B_{,01} = 0 \quad (16)$$

From equations (15-16) one has $B_{,00} = B_{,01} = B_{,11} = 0$, so that $B = c_0x^0 + c_1x^1 + c$, where c_0, c_1, c are constants. Since we limit our attention to bounded nonconstant solutions, we set $c_0 = c_1 = c = 0$, thus we have $B = h_{22} + h_{33} = 0$. Moreover, since h_{23} and ψ satisfy the wave equation, the components h_{ij} are a linear wave, that is $h_{ij}(x^0 \pm x^1)$. Furthermore, one can always perform an infinitesimal gauge transformation

$$x'^0 = x^0 + \xi^0(x^0 \pm x^1), \quad x'^1 = x^1 + \xi^1(x^0 \pm x^1), \quad (17)$$

under which the components h_{ab} transform as

$$h'_{00} = h_{00} - 2\xi_{0,0} + O(\epsilon^2); \quad h'_{11} = h_{11} - 2\xi_{1,1} + O(\epsilon^2); \quad h'_{01} = h_{01} - \xi_{0,1} - \xi_{1,0} + O(\epsilon^2), \quad (18)$$

and choose ξ^0 and ξ^1 in such a way that in the new coordinates one has $h'_{ab} \sim \epsilon^2$, so that the gravitational wave reduces to the two well know polarizations

$$h'_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h'_+ & 0 \\ 0 & 0 & 0 & -h'_+ \end{pmatrix}, \quad h'_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h'_\times \\ 0 & 0 & h'_\times & 0 \end{pmatrix} \quad (19)$$

up to quadratic $\sim \epsilon^2$ corrections, where h'_+ and h'_\times are solutions of the wave equation (see [15, 16] for review).

B. Einstein's equations at second order

The Einstein's equations in vacuum at second perturbative order are

$$G^\alpha_\beta = G^{(1)\alpha}_\beta + G^{(2)\alpha}_\beta + O(h^3) = 0. \quad (20)$$

Using equations (7-10) and rearranging the Einstein's equations in the same form as in (15-16), one obtains (see Appendix B) the four dynamical equations for the metric perturbation (12)

$$\begin{aligned} G^0_0 + G^1_1 &= \frac{1}{2}\{-\square B + B_{,00}h_{00} + B_{,11}h_{11} - 2B_{,01}h_{01} + \frac{1}{2}B_{,0}(h_{00,0} + h_{11,0} - 2h_{01,1}) + \\ &+ \frac{1}{2}B_{,1}(h_{00,1} + h_{11,1} - 2h_{01,0}) - h_{22}\square h_{22} - h_{33}\square h_{33} - 2h_{23}\square h_{23} \\ &+ \frac{1}{2}[(\psi_{,1})^2 - (\psi_{,0})^2] + 2[(h_{23,1})^2 - (h_{23,0})^2]\} = 0, \end{aligned} \quad (21a)$$

$$\begin{aligned}
G^2_2 + G^3_3 - (G^0_0 + G^1_1) &= \\
&= \frac{1}{2}\{-2A + (h_{00,1})^2 - (h_{11,0})^2 + (h_{23,0})^2 - (h_{23,1})^2 + \frac{1}{4}[(B_{,1})^2 - (B_{,0})^2 + (\psi_{,0})^2 - (\psi_{,1})^2]\} + \\
&+ 2(h_{00} - h_{11})A + h_{00,0}(h_{11,0} - 2h_{01,1}) - h_{11,1}(h_{00,1} - 2h_{01,0})\} = 0,
\end{aligned} \tag{21b}$$

$$\begin{aligned}
G^2_3 &= \frac{1}{2}\{\square h_{23} - h_{00}h_{23,00} - h_{11}h_{23,11} + 2h_{01}h_{23,01} + h_{22}(h_{23,00} - h_{23,11}) + h_{23}(h_{33,00} - h_{33,11}) + \\
&+ \frac{1}{2}\left[-h_{23,0}\left[(h_{00} + h_{11} - B)_{,0} - 2h_{01,1}\right] - h_{23,1}\left[(h_{00} + h_{11} + B)_{,1} - 2h_{01,0}\right]\right]\} = 0,
\end{aligned} \tag{21c}$$

$$\begin{aligned}
G^2_2 - G^3_3 &= \frac{1}{2}\{\square(h_{22} - h_{33}) + h_{00}(h_{33} - h_{22})_{,00} + 2h_{01}(h_{22} - h_{33})_{,01} + h_{11}(h_{33} - h_{22})_{,11} + \\
&- \frac{1}{2}(h_{22} - h_{33})_{,0}(h_{00,0} + h_{11,0} - B_{,0} - 2h_{01,1}) - \frac{1}{2}(h_{22} - h_{33})_{,1}(h_{00,1} + h_{11,1} + B_{,1} - 2h_{01,0}) \\
&+ h_{22}\square h_{22} - h_{33}\square h_{33}\} = 0,
\end{aligned} \tag{21d}$$

and the two constrains on initial data

$$\begin{aligned}
G^0_0 &= \frac{1}{2}\{B_{,11} + h_{11}B_{,11} - h_{01}B_{,01} + \frac{1}{2}B_{,0}(h_{11,0} - 2h_{01,1}) + \frac{1}{2}B_{,1}h_{11,1} + \\
&+ \frac{1}{2}[(h_{22,1})^2 - (h_{22,0})^2 + 3(h_{23,1})^2 - (h_{23,0})^2 + h_{22,0}B_{,0} + h_{33,1}(h_{33,1} - h_{22,1})] + \\
&+ h_{22}h_{22,11} + h_{33}h_{33,11} + 2h_{23}h_{23,11}\} = 0,
\end{aligned} \tag{21e}$$

$$\begin{aligned}
G^0_1 &= \frac{1}{2}\{B_{,01} + h_{01}B_{,11} - h_{00}B_{,01} + \frac{1}{2}h_{11,0}B_{,1} - \frac{1}{2}h_{00,1}B_{,0} + \\
&+ \frac{1}{2}[h_{22,0}h_{22,1} + h_{33,0}h_{33,1} + 2h_{23,0}h_{23,1}] + h_{22}h_{22,01} + h_{33}h_{33,01} + 2h_{23}h_{23,01}\} = 0.
\end{aligned} \tag{21f}$$

Let us discuss the properties of the solutions of the system (21). First of all note that (21a, 21e, 21f) imply that $B_{,00} \sim B_{,01} \sim B_{,11} \sim h^2$, so that $B = c + c_0x^0 + c_1x^1 + O(h^2)$. Again, since we are interested in bounded nonconstant waves, we choose $c = c_1 = c_2 = 0$, and therefore $B \sim h^2$.

Also note that $\square h_{23} \sim \square \psi \sim A \sim h^2$. That implies that, since we are doing an analysis at second quadratic order h^2 , we must neglect all the terms $Bh \sim h_{23} \square h \sim Ah \dots \sim h^3$ in the system (21), so that the dynamical equations reduce to the simplified form

$$\begin{aligned}
\square h_{23} &= h_{00}h_{23,00} + h_{11}h_{23,11} - 2h_{01}h_{23,01} + \frac{1}{2}h_{23,0}(h_{00,0} + h_{11,0} - 2h_{01,1}) + \\
&+ \frac{1}{2}h_{23,1}(h_{00,1} + h_{11,1} - 2h_{01,0}),
\end{aligned} \tag{22a}$$

$$\begin{aligned}
\square \psi &= h_{00}\psi_{,00} + h_{11}\psi_{,11} - 2h_{01}\psi_{,01} + \frac{1}{2}\psi_{,0}(h_{00,0} + h_{11,0} - 2h_{01,1}) + \\
&+ \frac{1}{2}\psi_{,1}(h_{00,1} + h_{11,1} - 2h_{01,0}),
\end{aligned} \tag{22b}$$

$$\square B = \frac{1}{2}\left[(\psi_{,1})^2 - (\psi_{,0})^2\right] + 2\left[(h_{23,1})^2 - (h_{23,0})^2\right], \tag{22c}$$

$$\begin{aligned}
A &= \frac{1}{2}\left[(h_{00,1})^2 - (h_{11,0})^2 + (h_{23,0})^2 - (h_{23,1})^2\right] + \frac{1}{8}\left[(\psi_{,0})^2 - (\psi_{,1})^2\right] + \\
&+ \frac{1}{2}h_{00,0}(h_{11,0} - 2h_{01,1}) - \frac{1}{2}h_{11,1}(h_{00,1} - 2h_{01,0}).
\end{aligned} \tag{22d}$$

In the same way, the two constraints on the initial data reduce to

$$B_{,11} + \frac{1}{2} [(h_{22,1})^2 - (h_{22,0})^2 + 3(h_{23,1})^2 - (h_{23,0})^2 + h_{33,1}(h_{33,1} - h_{22,1})] + h_{22}h_{22,11} + h_{33}h_{33,11} + 2h_{23}h_{23,11} = 0, \quad (22e)$$

$$B_{,01} + \frac{1}{2} [h_{22,0}h_{22,1} + h_{33,0}h_{33,1} + 2h_{23,0}h_{23,1}] + h_{22}h_{22,01} + h_{33}h_{33,01} + 2h_{23}h_{23,01} = 0. \quad (22f)$$

Note that the constraints (22e,22f) depend only on the components h_{ij} , which are the physical degrees of freedom of the gravitational wave.

At this point we must fix the gauge, in order to discuss the properties of the solutions of the second order Einstein's equations in the corresponding reference system. In facts, as discussed in section II, due to the residual gauge invariance, the waveform (12) still contains two arbitrary functions, that can be used to impose two constraints on the components of the perturbations $h_{\alpha\beta}$.

III. OCCURRENCE OF THE SECULARITY IN THE HARMONIC GAUGE

In this section we show that, in the harmonic gauge, the plane waves of the form (19) are unstable. The harmonic gauge is defined by the four conditions $\Gamma^\alpha_{\beta\gamma} g^{\beta\gamma} \equiv \Gamma^\alpha = 0$, and it is commonly used to study the generation and propagation of gravitational waves, e.g. in the PN and EOB formalisms [8, 9]. Expanding the equations $\Gamma^\lambda = 0$ at second order in h one has

$$\Gamma^\lambda = \psi^{\lambda\alpha}_{,\alpha} - h^{\lambda\sigma} \psi^\alpha_{\sigma,\alpha} - h^{\alpha\beta} \left(h^\lambda_{\alpha,\beta} - \frac{1}{2} h_{\alpha\beta}{}^{,\lambda} \right) + O(h^3) = 0, \quad (23)$$

which gives

$$\psi^{\lambda\alpha}_{,\alpha} = h^{\lambda\sigma} \psi^\alpha_{\sigma,\alpha} + h^{\alpha\beta} \left(h^\lambda_{\alpha,\beta} - \frac{1}{2} h_{\alpha\beta}{}^{,\lambda} \right) \sim h^2. \quad (24)$$

Since the perturbation $h_{\alpha\beta}$ is in the form (12), equation (24) gives

$$\partial_\alpha \psi^{\alpha 0} = \partial_\alpha \psi^\alpha_0 = \left(\frac{h_{00} + h_{11} + h_{22} + h_{33}}{2} \right)_{,0} - h_{10,1}, \quad (25a)$$

$$\partial_\alpha \psi^{\alpha 1} = -\partial_\alpha \psi^\alpha_1 = \left(\frac{h_{00} + h_{11} - h_{22} - h_{33}}{2} \right)_{,1} - h_{10,0}, \quad (25b)$$

$$\partial_\alpha \psi^\alpha_i = h^a_{i,\underline{a}} = 0; \quad i = 2, 3; \quad \underline{a} = 0, 1. \quad (25c)$$

Note that equations (25c), corresponding to the conditions $\Gamma^i = 0$, are identically satisfied since $h_{\underline{a}i} = 0$ for (12). Moreover, equations (25a-25b), which correspond to the conditions $\Gamma^a = 0$, take the form

$$h_{01,1} - \left(\frac{h_{00} + h_{11}}{2} \right)_{,0} = \frac{1}{2} B_{,0} - h^{\underline{c}\underline{d}} h_{0\underline{c},\underline{d}} + \frac{1}{2} h^{\underline{c}\underline{d}} h_{\underline{c},\underline{d},0} + \frac{1}{2} h^{ij} h_{ij,0}, \quad (26)$$

and

$$h_{01,0} - \left(\frac{h_{00} + h_{11}}{2} \right)_{,1} = -\frac{1}{2} B_{,1} + h^{\underline{c}\underline{d}} h_{1\underline{c},\underline{d}} - \frac{1}{2} h^{\underline{c}\underline{d}} h_{\underline{c},\underline{d},1} - \frac{1}{2} h^{ij} h_{ij,1}, \quad (27)$$

Note that the right hand sides of (26-27) are of order h^2 since, as discussed in the previous section, we have $B \sim h^2$.

For our purposes, it is more convenient to write the Einstein's equations in terms of the Ricci tensor in the form $R_{\mu\nu} = 0$, that, through equations (3-5), gives at the second perturbative order in h

$$\square h_{\mu\nu} = \partial_\mu \partial_\alpha \psi_\nu^\alpha + \partial_\nu \partial_\alpha \psi_\mu^\alpha + 2R_{\mu\nu}^{(2)}. \quad (28)$$

Assuming again that $h_{i\bar{a}} = 0$ and using equations (A11) and (A13), equation (28) gives up to quadratic terms

$$\square h_{ij} = h^{\underline{a}\underline{b}} h_{ij,\underline{a}\underline{b}} \quad (29a)$$

$$\square h_{\underline{a}\underline{b}} = -\frac{1}{2} h^{ij,\underline{a}} h_{ij,\underline{b}} + M(h)_{\underline{a}\underline{b}} \quad (29b)$$

where

$$\begin{aligned} M(h)_{\underline{a}\underline{b}} \equiv & \partial_{\underline{a}} \left[h^{\underline{c}\underline{d}} \left(h_{\underline{b}\underline{c},\underline{d}} - \frac{1}{2} h_{\underline{c}\underline{d},\underline{b}} \right) \right] + \partial_{\underline{b}} \left[h^{\underline{c}\underline{d}} \left(h_{\underline{a}\underline{c},\underline{d}} - \frac{1}{2} h_{\underline{c}\underline{d},\underline{a}} \right) \right] + \\ & + h^{\underline{c}\underline{d}} \left(h_{\underline{a}\underline{b},\underline{c}\underline{d}} + h_{\underline{c}\underline{d},\underline{a}\underline{b}} - h_{\underline{a}\underline{c},\underline{b}\underline{d}} - h_{\underline{b}\underline{c},\underline{a}\underline{d}} \right) + \\ & + \frac{1}{2} \left[h^{\underline{c}\underline{d},\underline{a}} h_{\underline{c}\underline{d},\underline{b}} + \left(h_{\underline{a}\underline{c},\underline{d}} - h_{\underline{a}\underline{d},\underline{c}} \right) \left(h_{\underline{b}}^{\underline{c},\underline{d}} - h_{\underline{b}}^{\underline{d},\underline{c}} \right) \right] \end{aligned} \quad (30)$$

Equations (29) imply that, at the linear leading order $\sim \epsilon$, all the components $h_{\alpha\beta}$ are solutions of the d'Alembert equation. Therefore can write a plane nonlinear wave travelling along the x^1 axis in the form

$$h_{\alpha\beta} = \epsilon h_{\alpha\beta}^{(1)}(x^0 - x^1) + \epsilon^2 h_{\alpha\beta}^{(2)}(x^0, x^1) + O(\epsilon^3), \quad (31)$$

where for simplicity we choose waves travelling only in one direction, and where $h_{\alpha\beta}^{(1)}, h_{\alpha\beta}^{(2)} \sim 1$. By means of (31), the system (29) is identically satisfied at linear ϵ order, while at quadratic order one has

$$\square h_{ij}^{(2)} = h^{(1)\underline{a}\underline{b}} h_{ij,\underline{a}\underline{b}}^{(1)} = \left(h_{00}^{(1)} + h_{11}^{(1)} + 2h_{01}^{(1)} \right) h_{ij,00}^{(1)}, \quad (32a)$$

$$\square h_{\underline{a}\underline{b}}^{(2)} = -\frac{1}{2} h^{(1)ij} h_{ij,\underline{a}\underline{b}}^{(1)} + M(h^{(1)})_{\underline{a}\underline{b}} = (-1)^{\underline{a}+\underline{b}+1} \kappa + M(h^{(1)})_{\underline{a}\underline{b}}, \quad (32b)$$

where we have defined

$$\kappa \equiv \frac{1}{2} \left[\left(h_{22,0}^{(1)} \right)^2 + \left(h_{33,0}^{(1)} \right)^2 + 2 \left(h_{23,0}^{(1)} \right)^2 \right]. \quad (33)$$

Note that, equations (26-27,31) imply that $h_{00}^{(1)} + h_{11}^{(1)} + 2h_{01}^{(1)} = c$ with c constant, and since we are not interested in constant solutions, we set $c = 0$. Therefore, having $h_{00}^{(1)} + h_{11}^{(1)} + 2h_{01}^{(1)} = 0$, equation (32a) reads $\square h_{ij}^{(2)} = 0$, so that also the second order components $h_{ij}^{(2)}(x^0 - x^1)$ are plane waves, and therefore they are stable.

In order to study the behavior of the second order components $h_{\underline{a}\underline{b}}^{(2)}$, and without loss of generality, in what follows we consider nonlinear gravitational waves of the form

$$h(x^0, x^1) = \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22}^{(1)} & h_{23}^{(1)} \\ 0 & 0 & h_{23}^{(1)} & -h_{22}^{(1)} \end{pmatrix} + \epsilon^2 h^{(2)}, \quad (34)$$

In facts, it is always possible to use the coordinate transformation (17) to cancel the contribution of the functions $h_{\underline{a}\underline{b}}^{(1)}$ in (31), so that $h_{\underline{a}\underline{b}} \sim \epsilon^2$ as long as $h_{\underline{a}\underline{b}}^{(2)} \sim 1$ [20]. Therefore, assuming that the gravitational wave is in the form (34), which implies that $M(h^{(1)}) = 0$, the equation (32b) for the components $h_{\underline{a}\underline{b}}^{(2)}$ becomes

$$\square h_{\underline{a}\underline{b}}^{(2)} = (-1)^{a+b+1} \kappa. \quad (35)$$

Note that, equation(33) implies that $\kappa > 0$ in presence of the gravitational wave (34). Moreover, from (35) one immediately recognizes that κ is a resonant forcing for the components $h_{\underline{a}\underline{b}}^{(2)}$, since from equation (33) follows that κ is a function of $(x^0 - x^1)$, and therefore a solution of the equation $\square \kappa = 0$. In conclusion, equation (35) contains the resonant forcing κ and it is secular, so that the components $h_{\underline{a}\underline{b}}^{(2)}$ grow with time, thus the perturbative expansion in h loses its asymptotic character in the long time regime. That means that, in the harmonic gauge, the linearized gravitational waves of the form (19) are unstable, when second order nonlinearities are considered.

Incidentally we also note that equation (35) implies that

$$\square \left(h_{00}^{(2)} + h_{11}^{(2)} + 2h_{01}^{(2)} \right) = 0, \quad (36)$$

thus the quantity $h_{00} + h_{11} + 2h_{01}$ is a solution of the d'Alembert equation and it remains always finite.

Sometimes in perturbation theory, the presence of secularities is due to a wrong perturbative approach, in problems in which the solutions depend simultaneously on widely different scales. In such cases, the secularities can be eliminated introducing suitable long variables; i.e., dealing with multiscale expansions. However, our example is not curable in this way, and this seems to be related to the fact that the instability found here is a feature of the harmonic gauge, and the right way to treat it, is to change the reference system.

Indeed, one can use a multiscale expansion

$$h_{ij} = \epsilon h_{ij}^{(1)} + \epsilon^2 h_{ij}^{(2)} + O(\epsilon^r), \quad h_{\underline{a}\underline{b}} = \epsilon h_{\underline{a}\underline{b}}^{(1)} + \epsilon^2 h_{\underline{a}\underline{b}}^{(2)} + O(\epsilon^r), \quad (37)$$

with $r > 2$, introducing the slow variable τ as follows

$$\begin{aligned} h_{ij}^{(1)} &= h_{ij}^{(1)}(\xi, \tau), & h_{\underline{a}\underline{b}}^{(1)} &= h_{\underline{a}\underline{b}}^{(1)}(\xi, \tau), \\ \xi &= x^0 - x^1, & \tau &= \epsilon^N x^0. \end{aligned} \quad (38)$$

We do so, since we are interested in studying the solutions in the longtime regime $\xi = O(1)$ and $\tau = O(1)$. With this assumptions, equation (29a) reads

$$\epsilon \square h_{ij}^{(1)} + \epsilon^2 \square h_{ij}^{(2)} = \epsilon^2 \left(h_{00}^{(1)} + h_{11}^{(1)} + 2h_{01}^{(1)} \right) \partial_\xi^2 h_{ij}^{(1)} + O(\epsilon^3) + 2\epsilon^{2+N} \left(h_{00}^{(1)} + h_{01}^{(1)} \right) \partial_\xi \partial_\tau h_{ij}^{(1)}. \quad (39)$$

We have previously shown that the gauge conditions (26-27,31) imply $h_{00}^{(1)} + h_{11}^{(1)} + 2h_{01}^{(1)} = O(\epsilon)$, thus the first term in (39) is $O(\epsilon^3)$. The second term in (39) is resonant, but we can avoid it imposing $N > 0$, so that (39) reduces to (32a). Let us stress that, doing so, the τ dependence of the components $h_{ij}^{(1)}(\xi, \tau)$, and consequently the τ dependence of $\kappa(\xi, \tau)$, is not determined at order ϵ^2 . We also emphasize that (32a) does not need a multiscale expansion, indeed the components h_{ij} are not secular, even in the standard perturbative expansion. Instead, the appearance of the second resonant term in (39) is related to the introduction of the slow variable τ , so that it can be avoided if τ is not introduced at all. However, let us continue to use our multiscale approach, to show that it is not useful to cure the secularities in the components $h_{\underline{a}\underline{b}}$.

Let us consider the equation (29b), which at $O(\epsilon^2)$ reads

$$2\epsilon^{N+1} \partial_\xi \partial_\tau h_{\underline{a}\underline{b}}^{(1)}(\xi, \tau) + \epsilon^2 \square h_{\underline{a}\underline{b}}^{(2)} = \epsilon^2 \left[(-1)^{a+b+1} \kappa(\xi, \tau) + M_{\underline{a}\underline{b}} \left(h^{(1)}(\xi, \tau) \right) \right]. \quad (40)$$

For the principle of maximum balance one has $N = 1$, and using the fact that $M_{\underline{a}\underline{b}} \left(h^{(1)}(\xi, \tau) \right) = 0$ for the waveform (34), one has

$$\square h_{\underline{a}\underline{b}}^{(2)} = -2\partial_\xi \partial_\tau h_{\underline{a}\underline{b}}^{(1)}(\xi, \tau) + (-1)^{a+b+1} \kappa(\xi, \tau). \quad (41)$$

Since the r.h.s. of this equation is a secular forcing for $h_{ab}^{(2)}$, we should set it to 0 in order to avoid the secularity, also defining in that way the τ dependence of $h_{ab}^{(1)}$ through the equation

$$2\partial_\xi\partial_\tau h_{ab}^{(1)}(\xi, \tau) = (-1)^{a+b+1}\kappa(\xi, \tau), \quad (42)$$

valid in the longtime regime $\xi = O(1)$ and $\tau = \epsilon x^0 = O(1)$. However, since the τ dependence of $\kappa(\xi, \tau)$ is not defined by (39), it is not possible to obtain the τ dependence of $h_{ab}^{(1)}(\xi, \tau)$ through the equation (42). Therefore, the multiscale method does not fix the dependence of $h_{\mu\nu}$ on the slow variable τ at order ϵ^2 , and we conclude that, at this order, the secularity in the components h_{ab} can not be eliminated by means of a multiscale perturbative expansion.

IV. STABILITY OF THE PERTURBATIVE SCHEME IN A DIFFERENT GAUGE

In order to prove that the instability discussed in the previous section is gauge dependent, in what follows we show that, in an opportune gauge, the evolution of gravitational waves at second perturbative order is stable. Therefore, we come back to the waveform (12), which contains two arbitrary functions, and we fix such functions imposing different gauge conditions from (23), which characterize the harmonic gauge. Since we want to exploit the properties of the d'Alembert equation, we impose the following gauge conditions

$$h_{00} = -h_{11}; \quad h_{01} = 0 \quad (43)$$

which imply that

$$A = h_{00,11} + h_{11,00} - 2h_{01,01} = \square h_{11} \quad (44)$$

Using (43) and neglecting the terms $O(h^3)$, the equations (22a) and (22b) reduce to the homogeneous wave equations

$$\square h_{23} = 0, \quad \square \psi = 0. \quad (45)$$

Since h_{23} and ψ are plane waves solutions of the d'Alembert equation, it is immediate to recognize that the right hand side of equation (22c) is null, so that also B is solution of the homogeneous wave equation

$$\square B = 0. \quad (46)$$

Therefore the functions h_{23} , B and ψ are plane waves travelling along the x^1 axis

$$\begin{aligned} h_{23} &= h_{23}^+(x^0 - x^1) + h_{23}^-(x^0 + x^1) \sim h \\ \psi &= \psi^+(x^0 - x^1) + \psi^-(x^0 + x^1) \sim h \\ B &= B^+(x^0 - x^1) + B^-(x^0 + x^1) \sim h^2, \end{aligned} \quad (47)$$

and are such that they satisfy the constraints (22e,22f) on their initial values. Moreover, evaluating equation(22d) over the solutions (47) in the gauge (43) one has

$$\square h_{11} = (h_{11,1})^2 - (h_{11,0})^2 \quad (48)$$

whose general solution is given by

$$h_{11} = \ln(1 + \alpha(x^0 - x^1) + \beta(x^0 + x^1)) \quad (49)$$

where $\alpha(x^0 - x^1) \sim \epsilon$ and $\beta(x^0 + x^1) \sim \epsilon$ so that $h_{11} \sim \epsilon$ (see Appendix C for the solution of the equation (48)). We conclude that equations (47,49) gives the general solution of the Einstein's equations in vacuum, at second order of perturbations h^2 , in the gauge (43).

At this point it is worth to notice that our gauge choice (43) coincides with the one introduced by Zakharov and Belinski in their studies on gravitational solitons [10], in which the metric tensor is in the form

$$ds^2 = f(x^0, x^1) ((dx^0)^2 - (dx^1)^2) + g_{ij} dx^i dx^j. \quad (50)$$

Therefore, in the gauge (43), the Einstein's equations can be resolved exactly with the inverse scattering technique, so that our solution is just a perturbative result in an exactly integrable context. However, here we were interested in showing that the secularity discussed in section III is gauge-dependent, and it is a feature of the harmonic reference system.

To conclude this section, let us show that the solution (47,49) can be recast to coincide with (19) at linear order $\sim \epsilon$. Since $\alpha \sim \beta \sim \epsilon$, from (49) one has

$$h_{00} = \alpha(x^0 - x^1) + \beta(x^0 + x^1) + O(\epsilon^2) \quad (51)$$

Under an infinitesimal change of coordinates of the type $x'^0 = x^0 + \xi^0(x^0, x^1)$, $x'^1 = x^1 + \xi^1(x^0, x^1)$, the components h_{ab} transform as

$$h'_{00} = h_{00} - 2\xi_{0,0} + O(\epsilon^2); \quad h'_{11} = h_{11} - 2\xi_{1,1} + O(\epsilon^2); \quad h'_{01} = h_{01} - \xi_{0,1} - \xi_{1,0} + O(\epsilon^2). \quad (52)$$

Choosing ξ_0 and ξ_1 as

$$\begin{aligned} \xi_0 &= \xi^+(x^0 - x^1) + \xi^-(x^0 + x^1), & \xi_1 &= \xi^+(x^0 - x^1) - \xi^-(x^0 + x^1) \\ \xi^+(r) &\equiv \frac{1}{2} \int^r \alpha(r') dr', & \xi^-(s) &\equiv \frac{1}{2} \int^s \beta(s') ds', \end{aligned} \quad (53)$$

from (52) one has $h'_{00} = O(\epsilon^2)$, and $h'_{11} = O(\epsilon^2)$. Moreover, using (53) and the fact that $h_{01} = 0$, one has $h'_{01} = -\xi_{0,1} - \xi_{1,0} + O(\epsilon^2) = O(\epsilon^2)$, therefore all the components h'_{ab} are of order $\sim \epsilon^2$, and they can be always neglected at the dominant order ϵ . For the components h_{ij} , using (47) one has

$$\begin{aligned} h'_{ij} &= h_{ij}^+ (x^0 - x^1 + \xi^0 - \xi^1) + h_{ij}^- (x^0 + x^1 + \xi^0 + \xi^1) = \\ &= h_{ij}^+ (x^0 - x^1 + 2\xi^+(x^0 - x^1)) + h_{ij}^- (x^0 + x^1 + 2\xi^-(x^0 + x^1)) \sim \epsilon, \end{aligned} \quad (54)$$

which shows how the nonlinearity mildly changes the waveform of the plane wave. In conclusion, in the appropriate gauge (43), the nonlinear gravitational waves will be of the usual form

$$h(x^0, x^1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & -h_{22} \end{pmatrix} + O(\epsilon^2) \quad (55)$$

with h_{22} and h_{23} given by (54).

V. CONCLUSIONS

In this paper we have studied the properties of nonlinear plane (but non necessarily monochromatic) gravitational waves. More specifically, we have analyzed the evolution of small perturbations h of the Minkowski metric of the form (11-12), at quadratic perturbative level, in vacuum. We have shown that, in the harmonic gauge, which is usually used to study gravitational waves, the components (for a plane wave moving in the direction x^1) h_{00} , h_{11} and h_{11} grow with time, thus gravitational waves of the form (19) are unstable. Therefore, in some cases, the harmonic gauge might not be the best choice to study the evolution of gravitational waves perturbatively. We mention that the instability reported in this paper resembles that appearing in [14], where the authors have studied the effect of quadratic nonlinearities on monochromatic gravitational waves. However, in [14] the authors do not infer that the

growth of the components h_{ab} represents a failure of the perturbative analysis, and they do not provide a solution or interpretation of this fact.

Finally, we have argued that the instability found here, is characteristic of the harmonic gauge, thus it is a gauge-dependent feature. In fact, we have shown that, in the gauge (43), the waveform (11-12) is stable, and the dynamic and the properties of nonlinear gravitational waves are essentially the same of linear gravitational waves, given by the two polarizations in (19).

APPENDIX A: RICCI TENSOR

In this appendix we calculate the Ricci tensor and the Ricci scalar at linear and quadratic orders as defined in equations (3-7), assuming that $h_{\mu\nu}$ is a plane wave travelling along the x^1 axis, of the form (11).

1. First order contribution to the Ricci tensor

For the linear contribution to the Ricci tensor (4) one has

$$R_{00}^{(1)} = \frac{1}{2} \{ h_{00,11} + h_{11,00} + (h_{22} + h_{33})_{,00} - 2h_{10,10} \} \quad (\text{A1})$$

$$R_{01}^{(1)} = \frac{1}{2} (h_{22} + h_{33})_{,01} \quad (\text{A2})$$

$$R_{11}^{(1)} = \frac{1}{2} \{ -(h_{00,11} + h_{11,00}) + (h_{22} + h_{33})_{,11} + 2h_{10,10} \} \quad (\text{A3})$$

Together with

$$R_{ij}^{(1)} = -\frac{1}{2} \square h_{ij} \quad (\text{A4})$$

and

$$R_{\underline{a}i}^{(1)} = \frac{1}{2} \{ -\square h_{\underline{a}i} - h^b_{i,\underline{b}\underline{a}} \} \quad (\text{A5})$$

Note that, if $h_{\underline{a}i}$ vanishes, the components $R_{\underline{a}i}^{(1)}$ are identically zero. Therefore, setting $h_{\underline{a}i} = 0$ is allowed by the linearized Einstein's equations.

The Ricci scalar at first order is

$$R^{(1)\alpha}_{\alpha} = h_{11,00} + h_{00,11} - 2h_{01,01} + (h_{22} + h_{33})_{,00} - (h_{22} + h_{33})_{,11} \quad (\text{A6})$$

2. Second order contribution to the Ricci tensor

Let us calculate the second order contribution to the Ricci tensor as given in (5)

a. Components $i\underline{a}$

Let us calculate the components $R^{(2)}_{i\underline{a}}$ with $i = 2, 3$ and $\underline{a} = 0, 1$. One has

$$\begin{aligned}
-\psi^{\alpha\rho}{}_{,\alpha} (h_{i\rho,\underline{a}} + h_{\underline{a}\rho,i} - h_{i\underline{a},\rho}) &= -\psi^{\underline{c}\underline{b}}{}_{,\underline{c}} (h_{i\underline{b},\underline{a}} - h_{i\underline{a},\underline{b}}) - h^{\underline{b}j}{}_{,\underline{b}} h_{ij,\underline{a}} \\
h^{\alpha\rho} (h_{i\underline{a},\alpha\rho} + h_{\alpha\rho,i\underline{a}} - h_{\underline{a}\rho,\alpha i} - h_{i\rho,\alpha\underline{a}}) &= h^{\underline{b}\underline{c}} (h_{i\underline{a},\underline{b}\underline{c}} - h_{i\underline{c},\underline{b}\underline{a}}) - h^{\underline{b}h} h_{ih,\underline{b}\underline{a}} \\
h^{\alpha\rho}{}_{,i} h_{\alpha\rho,\underline{a}} &= 0 \\
\frac{1}{2} (h_{i\underline{a},\rho} - h_{i\rho,\underline{a}}) (h_{\underline{a}}^{\alpha,\rho} - h_{\underline{a}}^{\rho,\alpha}) &= \frac{1}{2} (h_{i\underline{c},\underline{b}} - h_{i\underline{b},\underline{c}}) (h_{\underline{a}}^{\underline{c},\underline{b}} - h_{\underline{a}}^{\underline{b},\underline{c}}) + 2 h_{ih,\underline{b}} h_{\underline{a}}^{\underline{h},\underline{b}}
\end{aligned} \tag{A7}$$

which gives

$$\begin{aligned}
R_{\underline{a}}^{(2)} &= \frac{1}{2} \{ -\psi^{\underline{c}\underline{b}}{}_{,\underline{c}} (h_{i\underline{b},\underline{a}} - h_{i\underline{a},\underline{b}}) + h^{\underline{b}\underline{c}} (h_{i\underline{a},\underline{b}\underline{c}} - h_{i\underline{c},\underline{b}\underline{a}}) + \\
&+ \frac{1}{2} (h_{i\underline{c},\underline{b}} - h_{i\underline{b},\underline{c}}) (h_{\underline{a}}^{\underline{c},\underline{b}} - h_{\underline{a}}^{\underline{b},\underline{c}}) + 2 h_{ih,\underline{b}} h_{\underline{a}}^{\underline{h},\underline{b}} - h^{\underline{b}j}{}_{,\underline{b}} h_{ij,\underline{a}} - h^{\underline{b}h} h_{ih,\underline{b}\underline{a}} \}
\end{aligned} \tag{A8}$$

Note that if the components $h_{i\underline{a}}$ are null, the Einstein's equations for the components $i\underline{a}$, namely $R^{(1)}{}_{i\underline{a}} + R^{(2)}{}_{i\underline{a}} = 0$ are identically satisfied (in facts $R^{(1)}{}_{i\underline{a}} = 0$ and $R^{(2)}{}_{i\underline{a}} = 0$ separately), so that the choice $h_{i\underline{a}} = 0$ is allowed by the Einstein's equations. Therefore, henceforth we will keep this assumption in our equations and we will always neglect the components $h_{i\underline{a}}$ of the metric.

b. Components i, j

Let us calculate the components $R_{i,j}^{(2)}$ with $i, j = 2, 3$. All the derivatives with respect to x^i and x^j will be zero, so

$$\begin{aligned}
-\psi^{\alpha\rho}{}_{,\alpha} (h_{i\rho,j} + h_{j\rho,i} - h_{ij,\rho}) &= \psi^{\underline{a}\underline{b}}{}_{,\underline{a}} h_{ij,\underline{b}} \\
h^{\alpha\rho} (h_{ij,\alpha\rho} + h_{\alpha\rho,ij} - h_{j\rho,\alpha i} - h_{i\rho,\alpha j}) &= h^{\underline{a}\underline{b}} h_{ij,\underline{a}\underline{b}} \\
h^{\alpha\rho}{}_{,i} h_{\alpha\rho,j} &= 0 \\
\frac{1}{2} (h_{i\underline{a},\rho} - h_{i\rho,\underline{a}}) (h_j^{\alpha,\rho} - h_j^{\rho,\alpha}) &= \frac{1}{2} (h_{i\underline{a},\underline{b}} - h_{i\underline{b},\underline{a}}) (h_j^{\underline{a},\underline{b}} - h_j^{\underline{b},\underline{a}}) + h_{ih,\underline{a}} h_j^{\underline{h},\underline{a}}
\end{aligned} \tag{A9}$$

Therefore

$$R_{ij}^{(2)} = \frac{1}{2} \{ \psi^{\underline{a}\underline{b}}{}_{,\underline{a}} h_{ij,\underline{b}} + h^{\underline{a}\underline{b}} h_{ij,\underline{a}\underline{b}} + \frac{1}{2} (h_{i\underline{a},\underline{b}} - h_{i\underline{b},\underline{a}}) (h_j^{\underline{a},\underline{b}} - h_j^{\underline{b},\underline{a}}) + h_{ih,\underline{a}} h_j^{\underline{h},\underline{a}} \} \tag{A10}$$

Under the condition $h_{i\underline{a}} = 0$ one has

$$R_{ij}^{(2)} = \frac{1}{2} \{ \psi^{\underline{a}\underline{b}}{}_{,\underline{a}} h_{ij,\underline{b}} + h^{\underline{a}\underline{b}} h_{ij,\underline{a}\underline{b}} + h_{ih,\underline{a}} h_j^{\underline{h},\underline{a}} \} \tag{A11}$$

c. Components $\underline{a}\underline{b}$

Let us calculate the components $R^{\underline{a}\underline{b}}{}_{(2)}$. One has

$$\begin{aligned}
-\psi^{\alpha\rho}{}_{,\alpha} (h_{\underline{b}\rho,\underline{a}} + h_{\underline{a}\rho,\underline{b}} - h_{\underline{b}\underline{a},\rho}) &= -\psi^{\underline{d}\underline{c}}{}_{,\underline{d}} (h_{\underline{b}\underline{c},\underline{a}} + h_{\underline{a}\underline{c},\underline{b}} - h_{\underline{b}\underline{a},\underline{c}}) \\
h^{\alpha\rho} (h_{\underline{b}\underline{a},\alpha\rho} + h_{\alpha\rho,\underline{b}\underline{a}} - h_{\underline{a}\rho,\alpha\underline{b}} - h_{\underline{b}\rho,\alpha\underline{a}}) &= h^{ij} h_{ij,\underline{a}\underline{b}} + h^{\underline{c}\underline{d}} (h_{\underline{b}\underline{a},\underline{c}\underline{d}} + h_{\underline{c}\underline{d},\underline{b}\underline{a}} - h_{\underline{a}\underline{c},\underline{d}\underline{b}} - h_{\underline{b}\underline{c},\underline{d}\underline{a}}) \\
\frac{1}{2} h^{\alpha\rho}{}_{,\underline{b}} h_{\alpha\rho,\underline{a}} &= \frac{1}{2} h^{\underline{c}\underline{d}}{}_{,\underline{b}} h_{\underline{c}\underline{d},\underline{a}} + \frac{1}{2} h^{ij}{}_{,\underline{b}} h_{ij,\underline{a}} \\
\frac{1}{2} (h_{\underline{b}\alpha,\rho} - h_{\underline{b}\rho,\alpha}) (h_{\underline{a}}^{\alpha,\rho} - h_{\underline{a}}^{\rho,\alpha}) &= \frac{1}{2} (h_{\underline{b}\underline{c},\underline{d}} - h_{\underline{b}\underline{d},\underline{c}}) (h_{\underline{a}}^{\underline{c},\underline{d}} - h_{\underline{a}}^{\underline{d},\underline{c}})
\end{aligned} \tag{A12}$$

so that

$$R_{\underline{b}\underline{a}}^{(2)} = \frac{1}{2} \{ -\psi_{,\underline{d}}^{\underline{d}\underline{c}} (h_{\underline{b}\underline{c},\underline{a}} + h_{\underline{a}\underline{c},\underline{b}} - h_{\underline{b}\underline{a},\underline{c}}) + h^{\underline{c}\underline{d}} (h_{\underline{b}\underline{a},\underline{c}\underline{d}} + h_{\underline{c}\underline{d},\underline{b}\underline{a}} - h_{\underline{a}\underline{c},\underline{d}\underline{b}} - h_{\underline{b}\underline{c},\underline{d}\underline{a}}) + \frac{1}{2} (h_{\underline{b}\underline{c},\underline{d}} - h_{\underline{b}\underline{d},\underline{c}}) (h_{\underline{a}}^{\underline{c},\underline{d}} - h_{\underline{a}}^{\underline{d},\underline{c}}) + \frac{1}{2} h^{\underline{c}\underline{d}}_{,\underline{b}} h_{\underline{c}\underline{d},\underline{a}} + \frac{1}{2} h^{ij}_{,\underline{b}} h_{ij,\underline{a}} + h^{ij} h_{ij,\underline{a}\underline{b}} \} \quad (\text{A13})$$

3. Explicit expressions

$$R^{(2)}_{00} = \frac{1}{2} \{ [h_{01,1} - (\frac{h_{00}+h_{11}+h_{22}+h_{33}}{2})_{,0}] h_{00,0} + [h_{10,0} - (\frac{h_{00}+h_{11}-h_{22}-h_{33}}{2})_{,1}] (2h_{01,0} - h_{00,1}) + h_{11} (h_{00,11} + h_{11,00} - 2h_{01,01}) - (h_{01,0} - h_{00,1})^2 + \frac{1}{2} [(h_{00,0})^2 + (h_{11,0})^2 - 2(h_{01,0})^2] + \frac{1}{2} [(h_{22,0})^2 + (h_{33,0})^2 + 2(h_{23,0})^2] + [h_{22}h_{22,00} + h_{33}h_{33,00} + 2h_{23}h_{23,00}] \} \quad (\text{A14})$$

$$R^{(2)}_{11} = \frac{1}{2} \{ [h_{01,1} - (\frac{h_{00}+h_{11}+h_{22}+h_{33}}{2})_{,0}] (2h_{01,1} - h_{11,0}) + [h_{10,0} - (\frac{h_{00}+h_{11}-h_{22}-h_{33}}{2})_{,1}] h_{11,1} + h_{00} (h_{00,11} + h_{11,00} - 2h_{01,01}) - (h_{01,1} - h_{11,0})^2 + \frac{1}{2} [(h_{00,1})^2 + (h_{11,1})^2 - 2(h_{01,1})^2] + \frac{1}{2} [(h_{22,1})^2 + (h_{33,1})^2 + 2(h_{23,1})^2] + [h_{22}h_{22,11} + h_{33}h_{33,11} + 2h_{23}h_{23,11}] \} \quad (\text{A15})$$

$$R^{(2)}_{01} = \frac{1}{2} \{ h_{01} (h_{00,11} + h_{11,00} - 2h_{01,01}) - (\frac{h_{22}+h_{33}}{2})_{,0} h_{001} + (\frac{h_{22}+h_{33}}{2})_{,1} h_{110} + \frac{1}{2} [h_{22,0}h_{22,1} + h_{33,0}h_{33,1} + 2h_{23,0}h_{23,1}] + [h_{22}h_{22,01} + h_{33}h_{33,01} + 2h_{23}h_{23,01}] \} \quad (\text{A16})$$

$$R^{(2)}_{ij} = \frac{1}{2} \{ [(\frac{h_{00}+h_{11}+h_{22}+h_{33}}{2})_{,0} - h_{01,1}] h_{ij,0} + [(\frac{h_{00}+h_{11}-h_{22}-h_{33}}{2})_{,1} - h_{10,0}] h_{ij,1} + -h_{i2,0}h_{j2,0} - h_{i3,0}h_{j3,0} + h_{i2,1}h_{j2,1} + h_{i3,1}h_{j3,1} + h_{00}h_{ij,00} + h_{11}h_{ij,11} - 2h_{01}h_{ij,01} \} \quad (\text{A17})$$

APPENDIX B: EINSTEIN'S TENSOR

In this appendix we calculate the Einstein's tensor at linear and second order of perturbations. Using (A1-A5) and the definition (9), one easily obtains the formulas (14) for the Einstein's tensor at first order. In the same way one obtains the components of the Einstein's tensor at second order, defined in (10) as

$$G^{(2)0}_0 = \frac{1}{2} \{ h_{11}B_{,11} - h_{01}B_{,01} + \frac{1}{2}B_{,0} (h_{11,0} - 2h_{01,1}) + \frac{1}{2}B_{,1}h_{11,1} + \frac{1}{2} [(h_{22,1})^2 - (h_{22,0})^2 + 3(h_{23,1})^2 - (h_{23,0})^2 + h_{22,0}B_{,0} + h_{33,1}(h_{33,1} - h_{22,1})] + h_{22}h_{22,11} + h_{33}h_{33,11} + 2h_{23}h_{23,11} \} \quad (\text{B1a})$$

$$G^{(2)0}_1 = \frac{1}{2} \{ h_{01}B_{,11} - h_{00}B_{,01} + \frac{1}{2}h_{11,0}B_{,1} - \frac{1}{2}h_{00,1}B_{,0} + \frac{1}{2} [h_{22,0}h_{22,1} + h_{33,0}h_{33,1} + 2h_{23,0}h_{23,1}] + h_{22}h_{22,01} + h_{33}h_{33,01} + 2h_{23}h_{23,01} \} \quad (\text{B1b})$$

$$G^{(2)1}_1 = \frac{1}{2} \{ h_{00}B_{,00} - h_{01}B_{,01} + \frac{1}{2}B_{,0}h_{00,0} + \frac{1}{2}B_{,1} (h_{00,1} - 2h_{01,0}) + \frac{1}{2} [(h_{33,1})^2 - (h_{33,0})^2 + (h_{23,1})^2 - 3(h_{23,0})^2 + h_{22,0}(h_{33} - h_{22})_{,0} - h_{33,1}B_{,1}] + -h_{22}h_{22,00} - h_{33}h_{33,00} - 2h_{23}h_{23,00} \} \quad (\text{B1c})$$

$$\begin{aligned}
G^{(2)2}_2 &= \frac{1}{2}\{h_{00}(A + h_{33,0}) - h_{11}(A - h_{33,11}) + h_{01}(h_{23,11} - h_{23,00} - 2h_{33,01}) + \\
&+ \frac{1}{2}h_{00,0}(h_{11,0} + h_{33,0} - 2h_{01,1}) + \frac{1}{2}h_{00,1}(h_{00,1} + h_{33,1} - h_{11,1}) + h_{01,0}(h_{11,1} - h_{33,1}) - h_{01,1}h_{33,0} + \\
&+ 2h_{23}(h_{23,11} - h_{23,00}) + h_{33}(h_{33,11} - h_{33,00}) + \\
&+ \frac{1}{2}[(h_{23,1})^2 - (h_{23,0})^2 + (h_{33,1})^2 - (h_{33,0})^2 + (h_{11,1})^2 - (h_{11,0})^2 + h_{11,0}h_{33,0} + h_{11,1}(h_{33,1} - h_{11,1})]\}
\end{aligned} \tag{B1d}$$

$$\begin{aligned}
G^{(2)2}_3 &= \frac{1}{2}\{-h_{00}h_{23,00} - h_{11}h_{23,11} + 2h_{01}h_{23,01} + h_{22}(h_{23,00} - h_{23,11}) + h_{23}(h_{33,00} - h_{33,11}) + \\
&+ \frac{1}{2}\left[-h_{23,0}\left[(h_{00} + h_{11} - B)_{,0} - 2h_{01,1}\right] - h_{23,1}\left[(h_{00} + h_{11} + B)_{,1} - 2h_{01,0}\right]\right\}
\end{aligned} \tag{B1e}$$

$$\begin{aligned}
G^{(2)3}_3 &= \frac{1}{2}\{h_{00}(A + h_{22,00}) - 2h_{01}h_{22,01} + h_{11}(h_{22,11} - A) + \frac{1}{2}h_{00,1}(h_{00,1} + h_{22,1} - h_{11,1}) + \\
&+ \frac{1}{2}h_{00,0}(h_{11,0} + h_{22,0} - 2h_{01,1}) + \frac{1}{2}h_{11,0}(h_{22,0} - h_{11,0}) + \frac{1}{2}h_{11,1}(2h_{01,0} + h_{22,1}) + \\
&+ h_{22}(h_{22,11} - h_{22,00}) + h_{23}(h_{23,11} - h_{23,00}) + \\
&+ \frac{1}{2}\left[-(h_{23,0})^2 + (h_{23,1})^2 + (h_{22,1})^2 - (h_{22,0})^2 - 2h_{22,0}h_{01,1} - 2h_{01,0}h_{22,1}\right\}
\end{aligned} \tag{B1f}$$

where we have used the definitions (13).

1. Einstein's equations at second order

The Einstein's equations at second order are given by

$$G^{(1)\alpha}_\beta + G^{(2)\alpha}_\beta = 0 \tag{B2}$$

One has

$$\begin{aligned}
G^0_0 &= \frac{1}{2}\{B_{,11} + h_{11}B_{,11} - h_{01}B_{,01} + \frac{1}{2}B_{,0}(h_{11,0} - 2h_{01,1}) + \frac{1}{2}B_{,1}h_{11,1} + \\
&+ \frac{1}{2}[(h_{22,1})^2 - (h_{22,0})^2 + 3(h_{23,1})^2 - (h_{23,0})^2 + h_{22,0}B_{,0} + h_{33,1}(h_{33,1} - h_{22,1})] + \\
&+ h_{22}h_{22,11} + h_{33}h_{33,11} + 2h_{23}h_{23,11}\} = 0
\end{aligned} \tag{B3a}$$

$$\begin{aligned}
G^0_1 &= \frac{1}{2}\{B_{,01} + h_{01}B_{,11} - h_{00}B_{,01} + \frac{1}{2}h_{11,0}B_{,1} - \frac{1}{2}h_{00,1}B_{,0} + \\
&+ \frac{1}{2}[h_{22,0}h_{22,1} + h_{33,0}h_{33,1} + 2h_{23,0}h_{23,1}] + h_{22}h_{22,01} + h_{33}h_{33,01} + 2h_{23}h_{23,01}\} = 0
\end{aligned} \tag{B3b}$$

$$\begin{aligned}
G^1_1 &= \frac{1}{2}\{-B_{,00} + h_{00}B_{,00} - h_{01}B_{,01} + \frac{1}{2}B_{,0}h_{00,0} + \frac{1}{2}B_{,1}(h_{00,1} - h_{01,0}) + \\
&+ \frac{1}{2}[(h_{33,1})^2 - (h_{33,0})^2 + (h_{23,1})^2 - 3(h_{23,0})^2 + h_{22,0}(h_{33} - h_{22})_{,0} - h_{33,1}B_{,1}] + \\
&- h_{22}h_{22,00} - h_{33}h_{33,00} - 2h_{23}h_{23,00}\} = 0
\end{aligned} \tag{B3c}$$

$$\begin{aligned}
G^2_2 &= \frac{1}{2}\{\square h_{22} - A - B_{00} + B_{11} + h_{00}(A + h_{33,0}) - h_{11}(A - h_{33,11}) + h_{01}(h_{23,11} - h_{23,00} - 2h_{33,01}) + \\
&+ \frac{1}{2}h_{00,0}(h_{11,0} + h_{33,0} - 2h_{01,1}) + \frac{1}{2}h_{00,1}(h_{00,1} + h_{33,1} - h_{11,1}) + h_{01,0}(h_{11,1} - h_{33,1}) - h_{01,1}h_{33,0} + \\
&+ 2h_{23}(h_{23,11} - h_{23,00}) + h_{33}(h_{33,11} - h_{33,00}) + \\
&+ \frac{1}{2}[(h_{23,1})^2 - (h_{23,0})^2 + (h_{33,1})^2 - (h_{33,0})^2 + (h_{11,1})^2 - (h_{11,0})^2 + h_{11,0}h_{33,0} + h_{11,1}(h_{33,1} - h_{11,1})]\} = 0
\end{aligned} \tag{B3d}$$

$$G^2_3 = \frac{1}{2}\{\square h_{23} - h_{00}h_{23,00} - h_{11}h_{23,11} + 2h_{01}h_{23,01} + h_{22}(h_{23,00} - h_{23,11}) + h_{23}(h_{33,00} - h_{33,11}) +$$

$$+ \frac{1}{2}\left[-h_{23,0}\left[(h_{00} + h_{11} - B)_{,0} - 2h_{01,1}\right] - h_{23,1}\left[(h_{00} + h_{11} + B)_{,1} - 2h_{01,0}\right]\right\} = 0 \quad (\text{B3e})$$

$$G^3_3 = \frac{1}{2}\{\square h_{33} - A - B_{00} + B_{11} + h_{00}(A + h_{22,00}) - 2h_{01}h_{22,01} + h_{11}(h_{22,11} - A) +$$

$$+ \frac{1}{2}h_{00,1}(h_{00,1} + h_{22,1} - h_{11,1}) + \frac{1}{2}h_{00,0}(h_{11,0} + h_{22,0} - 2h_{01,1}) + \frac{1}{2}h_{11,0}(h_{22,0} - h_{11,0}) +$$

$$+ \frac{1}{2}h_{11,1}(2h_{01,0} + h_{22,1}) + h_{22}(h_{22,11} - h_{22,00}) + h_{23}(h_{23,11} - h_{23,00}) +$$

$$+ \frac{1}{2}\left[-(h_{23,0})^2 + (h_{23,1})^2 + (h_{22,1})^2 - (h_{22,0})^2 - 2h_{220}h_{01,1} - 2h_{01,0}h_{22,1}\right\} = 0 \quad (\text{B3f})$$

and the following relations

$$G^2_2 - G^3_3 = \frac{1}{2}\{\square(h_{22} - h_{33}) + h_{00}(h_{33} - h_{22})_{,00} + 2h_{01}(h_{22} - h_{33})_{,01} + h_{11}(h_{33} - h_{22})_{,11} +$$

$$- \frac{1}{2}(h_{22} - h_{33})_{,0}(h_{00,0} + h_{11,0} - B_{,0} - 2h_{01,1}) - \frac{1}{2}(h_{22} - h_{33})_{,1}(h_{00,1} + h_{11,1} + B_{,1} - 2h_{01,0})$$

$$+ h_{22}\square h_{22} - h_{33}\square h_{33}\} = 0 \quad (\text{B3g})$$

$$G^2_2 + G^3_3 = \frac{1}{2}\{-2A - \square B + h_{00,0}(h_{00,0} - 2h_{01,1}) - h_{11,1}(h_{00,1} - 2h_{01,0}) + h_{11,0}(h_{00,0} - h_{11,0}) +$$

$$+ \frac{1}{2}B_{,0}(h_{00,0} + h_{11,0} - 2h_{01,1}) + \frac{1}{2}B_{,1}(h_{00,1} + h_{11,1} - 2h_{01,0}) + h_{00}(B_{,00} + 2A) + h_{11}(B_{,11} - 2A)$$

$$- 2h_{01}B_{,01} - h_{22}\square h_{22} - h_{33}\square h_{33} - 2h_{23}\square h_{23} + (h_{00,1})^2 - (h_{00,0})^2 + (h_{23,1})^2 - (h_{23,0})^2 +$$

$$+ \frac{1}{2}\left[(h_{22,1})^2 - (h_{22,0})^2 + (h_{33,1})^2 - (h_{33,0})^2\right]\} =$$

$$= \frac{1}{2}\{-2A - \square B + h_{00,0}(h_{00,0} - 2h_{01,1}) - h_{11,1}(h_{00,1} - 2h_{01,0}) + h_{11,0}(h_{00,0} - h_{11,0}) +$$

$$+ \frac{1}{2}B_{,0}(h_{00,0} + h_{11,0} - 2h_{01,1}) + \frac{1}{2}B_{,1}(h_{00,1} + h_{11,1} - 2h_{01,0}) + h_{00}(B_{,00} + 2A) + h_{11}(B_{,11} - 2A)$$

$$- 2h_{01}B_{,01} - h_{22}\square h_{22} - h_{33}\square h_{33} - 2h_{23}\square h_{23} + (h_{00,1})^2 - (h_{00,0})^2 + (h_{23,1})^2 - (h_{23,0})^2 +$$

$$+ \frac{1}{4}\left[(B_{,1})^2 - (B_{,0})^2 + (\psi_{,1})^2 - (\psi_{,0})^2\right]\} = 0 \quad (\text{B3h})$$

$$G^0_0 + G^1_1 = \frac{1}{2}\{-\square B + B_{,00}h_{00} + B_{,11}h_{11} - 2B_{,01}h_{01} + \frac{1}{2}B_{,0}(h_{00,0} + h_{11,0} - 2h_{01,1}) +$$

$$+ \frac{1}{2}B_{,1}(h_{00,1} + h_{11,1} - 2h_{01,0}) - h_{22}\square h_{22} - h_{33}\square h_{33} - 2h_{23}\square h_{23}$$

$$+ \frac{1}{2}\left[(\psi_{,1})^2 - (\psi_{,0})^2\right] + 2\left[(h_{23,1})^2 - (h_{23,0})^2\right]\} = 0 \quad (\text{B3i})$$

$$G^2_2 + G^3_3 - (G^0_0 + G^1_1) =$$

$$= \frac{1}{2}\{-2A + (h_{00,1})^2 - (h_{11,0})^2 + (h_{23,0})^2 - (h_{23,1})^2 + \frac{1}{4}\left[(B_{,1})^2 - (B_{,0})^2 + (\psi_{,0})^2 - (\psi_{,1})^2\right] +$$

$$+ 2(h_{00} - h_{11})A + h_{00,0}(h_{11,0} - 2h_{01,1}) - h_{11,1}(h_{00,1} - 2h_{01,0})\} = 0 \quad (\text{B3j})$$

APPENDIX C: SOLUTION OF THE EQUATION (49)

The equation (49), which governs the evolution of h_{11} , is of the type

$$\square U = U_x^2 - U_t^2 \quad (\text{C1})$$

This equation is obtained by the following lagrangian

$$L = (U_t^2 - U_x^2) \exp(2U) \quad (\text{C2})$$

and the corresponding hamiltonian is

$$H = (U_t^2 + U_x^2) \exp(2U) \quad (\text{C3})$$

Equation (C1) is solved by plane waves of the type $U = U(t - x)$ or $U = U(t + x)$, but a linear combination of these two solutions is not a solution itself.

The general solution of (C1) is obtained by the simple observation that, defining

$$\psi \equiv \exp(U) \quad (\text{C4})$$

the lagrangian (C2) becomes

$$L = (\psi_t)^2 - (\psi_x)^2 \quad (\text{C5})$$

which gives the standard wave equation

$$\square \psi = 0 \quad (\text{C6})$$

whose general solution is

$$\psi(x^0, x^1) = A(x^0 + x^1) + B(x^0 - x^1) \quad (\text{C7})$$

Therefore the solution of (C1) is

$$U(x^0, x^1) = \ln(A(x^0 + x^1) + B(x^0 - x^1)) \quad (\text{C8})$$

Since, in the case of equation (48), one has $h_{11} \sim \epsilon$, one has

$$h_{11}(x^0, x^1) = \ln(1 + \alpha(x^0 + x^1) + \beta(x^0 - x^1)) \quad (\text{C9})$$

with $\alpha(x^0 + x^1) \sim \beta(x^0 - x^1) \sim \epsilon$.

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- [17] It is worth to mention that the existence of gravitational waves has been indirectly proved by the observation of the inspiral motion of binary systems, as the famous Hulse-Taylor Binary Pulsar PSR B1913+16 [4], which are characterized by a loss of kinetic energy due to the emission of gravitational waves (see [5] for a review). We mention that gravitational waves are also important in cosmology. For instance, measurements of the cosmic microwave background (CMB) anisotropy [6] aim to reveal the existence of primordial gravitational waves produced during the big bang.
- [18] The occurrence of secular instabilities in perturbation theory is often due to the use of a bad perturbative scheme, in problems in which the solutions depend simultaneously on widely different scales, so that in such cases the secularities can be avoided by means of a multi-scale perturbative expansion.
- [19] We refer to this arbitrariness as gauge freedom, and to the choice of the reference frame as gauge fixing.
- [20] This is exactly the same as in the case of the linearized Einstein's equations discussed in section II A.