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Article

# On the Second-Largest Reciprocal Distance Signless Laplacian Eigenvalue

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**Abstract:** The signless Laplacian reciprocal distance matrix for a simple connected graph  $G$  is defined as  $RQ(G) = \text{diag}(RH(G)) + RD(G)$ . Here,  $RD(G)$  is the Harary matrix (also called reciprocal distance matrix) while  $\text{diag}(RH(G))$  represents the diagonal matrix of the total reciprocal distance vertices. In the present work, some upper and lower bounds for the second-largest eigenvalue of the signless Laplacian reciprocal distance matrix of graphs in terms of various graph parameters are investigated. Besides, all graphs attaining these new bounds are characterized. Additionally, it is inferred that among all connected graphs with  $n$  vertices, the complete graph  $K_n$  and the graph  $K_n - e$  obtained from  $K_n$  by deleting an edge  $e$  have the maximum second-largest signless Laplacian reciprocal distance eigenvalue.

**Keywords:** signless Laplacian reciprocal distance matrix (spectrum); spectral radius; total reciprocal distance vertex; Harary matrix

**MSC:** 05C50; 05C12; 15A18



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## 1. Introduction

Let  $V(G) = \{v_1, v_2, \dots, v_n\}$  be the set of vertices in a connected graph  $G$ . Let  $N(v_i)$  represent the vertices adjacent to  $v_i \in V(G)$ . The degree of  $v_i$ , denoted by  $d(v_i)$  or  $d_i$ , is the number of vertices in  $N(v_i)$ . For the adjacency matrix  $A$  of  $G$ , the signless Laplacian matrix is  $Q(G) = \text{Deg}(G) + A(G)$ , where  $\text{Deg}(G) = \text{diag}(d_1, d_2, \dots, d_n)$  is the diagonal matrix of vertex degrees. This matrix is a real symmetric matrix, so we can arrange its eigenvalues as  $q_1 \geq q_2 \geq \dots \geq q_n$ . The distance between a pair of vertices  $v_i$  and  $v_j$  is signified by  $d_G(v_i, v_j)$  or  $d_{ij}$  and the diameter by  $\text{diam}(G)$ . The distance matrix is defined as  $\tilde{D}(G) = [d_G(v_i, v_j)]_{v_i, v_j \in V(G)}$ .

The total distance and the total reciprocal distance of the vertex  $v$  of graph  $G$  are respectively defined as  $\text{Tr}_G(v) = \sum_{u \in V(G)} d_G(u, v)$  and

$$RH_G(v) = \sum_{u \in V(G)} \frac{1}{d_G(u, v)}, \quad u \neq v.$$

We define the distance of the signless Laplacian matrix of  $G$  as the sum of  $\text{Diag}(\text{Tr}(G))$  and  $\tilde{D}(G)$ , namely,  $D^Q(G) = \text{Diag}(\text{Tr}(G)) + \tilde{D}(G)$ , where  $\text{Tr}(G) = (\text{Tr}_1, \dots, \text{Tr}_n)$  and  $\text{Tr}_i = \sum_{v_j \in V(G)} d_G(v_j, v_i)$ , see [1–6]. Additionally, the Harary matrix  $RD(G)$  or the reciprocal



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distance matrix of  $G$  is a square matrix  $(RD_{i,j})$ , where the  $(i, j)$ -entry is  $\frac{1}{d_{i,j}}$  if  $i \neq j$  and 0 otherwise, see [7,8]. Finally, the Harary index of  $G$  is [9,10]

$$H(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n RD_{i,j} = \sum_{i < j} \frac{1}{d_{i,j}}.$$

Clearly, we have  $H(G) = \frac{1}{2} \sum_{v \in V(G)} RH_G(v)$ . A graph  $G$  is total reciprocal distance regular (trdr) if, and only if for any two vertices  $u$  and  $v$ , it holds true that  $RH_G(v) = RH_G(u)$ . For two adjacent vertices  $u, v \in V(G)$ , suppose  $H_{uv} = \{x \in V(G) | d(u, x) < d(v, x)\}$ . A graph  $G$  is reciprocal distance-balanced, if  $H_{uv} = H_{vu}$  holds for any edge  $uv$  of  $G$ . In [11], Balakrishnan et al. showed that the transmission regular graphs and distance-balanced graphs are the same. Similarly, we can show that, for a connected graph  $G$ , concepts that are total reciprocal distance-regular and reciprocal distance-balanced are the same. A graph  $G$  is called  $r$ -reciprocal distance-balanced if  $RH_G(v) = r$ , for all vertices.

In what follows, suppose  $RH(G)$  is a diagonal matrix, where  $(RH)_{ii} = RH(v_i)$ . The Laplacian reciprocal distance and the signless Laplacian reciprocal distance matrices [12,13] have been defined as  $RL(G) = RH(G) - RD(G)$  and  $RQ(G) = RH(G) + RD(G)$ , respectively.

The signless Laplacian reciprocal distance spectrum of  $G$  is a multiset consisting of the eigenvalues of  $RQ(G)$ . In addition, if  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  are all eigenvalues of  $RQ(G)$ , then  $\lambda_1$  is called the slrd-spectral radius of  $G$ .

Like the spectral radius with respect to different matrices associated with the graph  $G$ , the second-largest eigenvalue is also of much interest. This fact is clear from the works that can be found in the literature regarding the second-largest eigenvalue of the graph with respect to different graph matrices. For some recent works on the second-largest adjacency eigenvalue, we refer to [14]; for the second-largest Laplacian eigenvalue, we refer to [15]; for the second-largest signless Laplacian eigenvalue, we refer to [16]; for the second-largest distance eigenvalue, we refer to [17]; for the second-largest generalized distance eigenvalue, we refer to [18], and so forth. Motivated by these works, we, in this paper, study the second-largest signless Laplacian reciprocal distance eigenvalue of a connected graph.

In the remainder of the work, we present some preliminary results in Section 2, which serves as a useful tool box for the rest of the paper. In Section 3, we obtain some upper and lower bounds for  $\lambda_2(RQ(G))$  by employing useful graph structural parameters, and we also characterize some extremal graphs attaining these bounds. Amongst all connected graphs of order  $n$ , it is uncovered that the complete graph  $K_n$ , together with the graph  $K_n - e$  obtained by deleting an edge  $e$  from  $K_n$ , possess the maximum second-largest signless Laplacian reciprocal distance eigenvalue. We explore the effect of some graph operations on  $\lambda_2(RQ(G))$  in the last section.

## 2. Preliminary Results

Some known results in the matrix theory are conveniently collected in this section. The relation between the eigenvalues of a symmetric matrix and its principal submatrix is summarized as below [19]. Some recent applications can be found, for example, in [20,21].

**Lemma 1.** (Interlacing theorem) [19] Assume that  $A$  is a real  $n \times n$  symmetric matrix and  $B$  is a  $s \times s$  principal submatrix of  $A$  with  $s \leq n$ . We have the following interlacing for their eigenvalues:

$$\lambda_{i+n-s}(A) \leq \lambda_i(B) \leq \lambda_i(A), \quad 1 \leq i \leq s.$$

Since the matrix  $RQ(G)$  is a symmetric matrix, the following corollary directly follows from Lemma 1.

**Corollary 1.** Assume that  $G$  is a connected graph with order  $n \geq 3$ . Suppose that  $M$  is the  $(n - 1) \times (n - 1)$  principal submatrix of  $RQ(G)$  with order. We have

$$\lambda_1(RQ(G)) \geq \lambda_1(M) \geq \lambda_2(RQ(G)) \geq \dots \geq \lambda_{n-1}(M) \geq \lambda_n(RQ(G)).$$

The signless Laplacian reciprocal distance eigenvalues of a connected graph  $G$  are linked to its connected spanning subgraph in the following lemma [22].

**Lemma 2.** [22] Suppose that  $G$  is a connected graph with  $n$  vertices and  $m$  edges. Assume that  $m \geq n$ . Let  $G'$  be the connected graph obtained from  $G$  by removing an edge. We have  $\lambda_i(RQ(G)) \geq \lambda_i(RQ(G'))$  ( $i = 1, 2, \dots, n$ ).

The next result was studied in [23] with useful applications found in, for example, [20,24].

**Lemma 3.** Let  $X$  and  $Y$  be two  $n \times n$  Hermitian matrices. Suppose that  $Z = X + Y$ , and we arrange the eigenvalues of a matrix by  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Then, the following inequalities hold true:

$$\begin{aligned} \lambda_k(Z) &\leq \lambda_j(X) + \lambda_{k-j+1}(Y), \quad n \geq k \geq j \geq 1, \\ \lambda_k(Z) &\geq \lambda_j(X) + \lambda_{k-j+n}(Y), \quad n \geq j \geq k \geq 1. \end{aligned}$$

Here,  $\lambda_i$  is the  $i$ -th largest eigenvalue of a given matrix. In any of these inequalities above, equality is attained if, and only if there exists a unit eigenvector associated with each of the three eigenvalues involved.

### 3. Bounds for $\lambda_2(RQ(G))$

In this section, we discuss the relationship between the second-largest signless Laplacian reciprocal distance eigenvalues and the other graph parameters. We show that the complete graph  $K_n$  and the graph  $K_n - e$  obtained from  $K_n$  by deleting an edge  $e$  have the maximum second-largest signless Laplacian reciprocal distance eigenvalue among all connected graphs of order  $n$ .

The following result gives bounds for the second-largest signless Laplacian reciprocal distance eigenvalue  $\lambda_2(RQ(G))$ , in terms of the maximum total reciprocal distance vertex  $RH_{\max}$ , the minimum total reciprocal distance vertex  $RH_{\min}$ , and the second-largest reciprocal distance eigenvalue  $\lambda_2(RD(G))$ .

**Theorem 1.** Suppose that  $G$  is a connected graph of order  $n \geq 3$  having total reciprocal distance vertices  $RH_{\max} = RH_1 \geq RH_2 \geq \dots \geq RH_n = RH_{\min}$ . Then,

$$RH_{\min} + \lambda_2(RD(G)) \leq \lambda_2(RQ(G)) \leq RH_{\max} + \lambda_2(RD(G)). \tag{1}$$

If both the inequalities occur as equalities, then  $G$  is a reciprocal distance-balanced graph.

**Proof.** We have  $RQ(G) = RH(G) + RD(G)$ . Taking  $Z = RQ(G), Y = RH(G), X = RD(G)$  and  $k = j = 2$  in Lemma 3, we get

$$\lambda_n(RH(G)) + \lambda_2(RD(G)) \leq \lambda_2(RQ(G)) \leq \lambda_1(RH(G)) + \lambda_2(RD(G)).$$

As  $\lambda_n(RH(G)) = RH_{\min}$  and  $\lambda_1(RH(G)) = RH_{\max}$ , hence, (1) follows. Suppose that both the left and right inequalities in (1) occur as equalities, then by Lemma 3, there exists a common unit vector  $X$  which is an eigenvector to each of the four eigenvalues  $\lambda_n(RH(G)), \lambda_1(RH(G)), \lambda_2(RD(G))$ , and  $\lambda_2(RQ(G))$ . Now, since  $X$  is an eigenvector of the matrix  $RH(G)$  corresponding to the eigenvalues  $\lambda_n(RH(G)) = RH_{\min}$  and  $\lambda_1(RH(G)) = RH_{\max}$  gives that  $RH_{\min}X = RH_{\max}X$ , this, in turn, gives that  $RH_{\min} = RH_{\max}$ . This shows that if

both the left- and right-hand inequalities in (1) occur as equalities, then  $G$  is a reciprocal distance-balanced graph. This completes the proof.  $\square$

From Theorem 1, we know that any lower or upper bound on the second-largest eigenvalue of the reciprocal distance matrix gives a corresponding lower or upper bound for the second-largest signless Laplacian reciprocal distance eigenvalue  $\lambda_2(RQ(G))$ .

An upper bound for  $\lambda_2(RQ(G))$  is presented in the next theorem, where  $n$  is the number of vertices in  $G$ .

**Theorem 2.** Assume that  $G$  is a connected graph of order  $n \geq 4$ . Then,  $\lambda_2(RQ(G)) \leq n - 2$  with an equality if, and only if  $G \cong K_n$  or  $G \cong K_n - e$ .

**Proof.** Let  $G$  be a connected graph of order  $n \geq 4$ . If  $G \cong K_n$ , then using the fact signless Laplacian reciprocal distance spectrum of  $K_n$  is  $\{2n - 2, n - 2^{[n-1]}\}$ , it follows that equality occurs in this case. If  $G \cong K_n - e = K_{n-2} \vee \bar{K}_2$ , where  $e$  is an edge, then it can be seen that (see discussion after Corollary 2.6 in [25]) the signless Laplacian reciprocal distance spectrum of  $K_n - e$  is  $\{n - 2^{[n-3]}, 2x_1, 2x_2, 2x_3\}$ , where  $x_1, x_2$  and  $x_3$  are the zeros of the polynomial

$$f(x) = x^3 - \left(2n - \frac{7}{2}\right)x^2 - \left(\frac{5n^2 - 19n + 3}{4}\right)x - \left(\frac{n^3}{4} - \frac{3n^2}{2} + 3n - 2\right).$$

Since by Lemma 2, we have  $n - 2 = \lambda_2(RQ(K_n)) \geq \lambda_2(RQ(K_n - e))$ , it follows that  $\lambda_2(RQ(K_n - e)) = n - 2$ . This shows that equality occurs in this case also. Assume that  $G$  is a connected graph of order  $n$ , which is neither  $K_n$  nor  $K_n - e$ . Clearly,  $G$  constitutes a spanning subgraph of  $H$  of order  $n$ .  $H$  can be obtained from  $K_n$  by scrapping a couple of edges, say  $e_1$  and  $e_2$ . Using Lemma 2, we have  $\lambda_2(RQ(H)) \geq \lambda_2(RQ(G))$ . From this, it is clear that it remains to show  $\lambda_2(RQ(H)) < n - 2$ . The following two situations are in order:

(i). If the two edges  $e_1$  and  $e_2$  share a vertex, denote by  $v_0$  the common vertex of  $e_1$  and  $e_2$  and denote by  $v_1$  and  $v_2$  the other two end nodes. Suppose that  $M$  forms the matrix indexed by the vertices  $v_0, v_1$ , and  $v_2$ , then the matrix  $RQ$  can be written as

$$RQ(H) = RQ(K_n) + \begin{pmatrix} M & O_{3 \times (n-3)} \\ O_{(n-3) \times 2} & O_{(n-3) \times (n-3)} \end{pmatrix},$$

where  $M = \begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$ . Using Lemma 3, it follows that  $\lambda_2(RQ(H)) \leq \lambda_2(RQ(K_n)) +$

$\lambda_1(M)$ . By direct calculation, it can be seen that the largest eigenvalue of  $M$  is  $\lambda_1(M) = 0$ , giving that

$$\lambda_2(RQ(H)) \leq \lambda_2(RQ(K_n)) = n - 2. \tag{2}$$

We claim that  $\lambda_2(RQ(H)) < n - 2$ . For if  $\lambda_2(RQ(H)) = n - 2$ , then equality occurs in (2), which is so if equality occurs in Lemma 3. Since equality occurs in Lemma 3 if, and only if there is a common unit eigenvector  $X$  for the eigenvalues  $\lambda_2(RQ(H)), n - 2$  and 0 of the matrices  $RQ(H), RQ(K_n)$ , and  $\begin{pmatrix} M & O_{3 \times n-3} \\ O_{n-3 \times 2} & O_{n-3 \times n-3} \end{pmatrix}$ . It is clear that the vector  $X = \frac{1}{3}(1, 1, 1, 0, \dots, 0)^T$  is a unit eigenvector for the eigenvalue 0. Therefore, if  $\lambda_2(RQ(H)) = n - 2$ , then  $X$  must be an eigenvector for the eigenvalue  $n - 2$  of the matrix  $RQ(K_n)$ , which is not so. This proves our claim and the result in this case.

(ii). Suppose that  $e_1$  and  $e_2$  have no common vertex. Let the end vertices of  $e_1$  and  $e_2$  be  $v_0, v_1$  and  $v_2, v_3$ , respectively. Let  $M$  be the matrix indexed by the vertices  $v_0, v_1, v_2$  and  $v_3$ ; then, the matrix  $RQ(H)$  can be written as

$$RQ(H) = RQ(K_n) + \begin{pmatrix} M & O_{4 \times (n-4)} \\ O_{(n-4) \times 4} & O_{(n-4) \times (n-4)} \end{pmatrix},$$

where  $M = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ . Using Lemma 3, it follows that  $\lambda_2(RQ(H)) \leq$

$\lambda_2(RQ(K_n)) + \lambda_1(M)$ . By direct calculation, it can be seen that the largest eigenvalue of  $M$  is  $\lambda_1(M) = 0$ , giving that  $\lambda_2(RQ(H)) \leq \lambda_2(RQ(K_n)) = n - 2$ . Now, proceeding similarly as in the above case, it can be shown that  $\lambda_2(RQ(H)) < n - 2$ . This completes the proof.  $\square$

The above theorem gives that among all the connected graphs of order  $n \geq 4$ , the complete graph  $K_n$  and the graph  $K_n - e$  attains the maximum value for second-largest signless Laplacian reciprocal distance eigenvalues.

In the following result, we will establish a relationship between the second-largest signless Laplacian reciprocal distance eigenvalue of the graph  $G$  of diameter 2 with the smallest, and the second-smallest signless Laplacian eigenvalues of the complement  $\bar{G}$  of the graph  $G$ . Further, for graphs of diameter greater than or equal to 3, we give a relationship between second-largest signless Laplacian reciprocal distance eigenvalues with the second-largest signless Laplacian eigenvalue.

**Theorem 3.** Suppose that  $G$  is a connected graph over  $n \geq 4$  vertices with the diameter being  $d$ . Denote by  $\bar{G}$  the complement of  $G$ , and arrange the signless Laplacian eigenvalues of  $\bar{G}$  as  $\bar{q}_1 \geq \bar{q}_2 \geq \dots \geq \bar{q}_n$ .

(i) If  $d = 2$ , then

$$n - 2 - \frac{1}{2}\bar{q}_{n-1} \leq \lambda_2(RQ(G)) \leq n - 2 - \frac{1}{2}\bar{q}_n. \tag{3}$$

(ii) If  $d \geq 3$ , then

$$\lambda_2(RQ(G)) \geq \frac{1}{2}(n + q_2 - 2).$$

**Proof.** Suppose we have a connected graph  $G$  over more than or equal to 4 nodes, and have diameter  $d$ .  $Deg(\bar{G}) = diag(n - 1 - d_1, n - 1 - d_2, \dots, n - 1 - d_n) = (n - 1)I - Deg(G)$  is the diagonal matrix of vertex degrees of  $\bar{G}$ . If  $G$  has diameter two, the total reciprocal distance vertex  $RH_i$  of each vertex  $v_i$  is given by  $RH_i = \frac{1}{2}(n + d_i - 1)$ . Since the diameter of  $G$  is two, and any two vertices are either adjacent in  $G$  or in  $\bar{G}$ , we see that  $RD(G) = A + \frac{1}{2}\bar{A}$  becomes the reciprocal distance matrix of  $G$ . Here,  $A$  is the adjacency matrix of  $G$ , and  $\bar{A}$  is the counterpart for  $\bar{G}$ . We have

$$\begin{aligned} RQ(G) &= RH(G) + RD(G) = \frac{1}{2}(n - 1)I + \frac{1}{2}Deg(G) + A + \frac{1}{2}\bar{A} \\ &= \frac{1}{2}(n - 1)I + \frac{1}{2}Deg(G) + \bar{A} + A + \frac{1}{2}Deg(\bar{G}) - \frac{1}{2}(\bar{A} + Deg(\bar{G})) \\ &= \frac{1}{2}(n - 1)I + \frac{1}{2}(Deg(G) + Deg(\bar{G})) + A + J - I - A - \frac{1}{2}Q(\bar{G}) \\ &= (n - 2)I + J - \frac{1}{2}Q(\bar{G}), \end{aligned}$$

where  $I$  is the  $n \times n$  identity matrix and  $J$  is the  $n \times n$  all one matrix. Taking  $Z = RQ(G)$ ,  $Y = (n - 2)I - \frac{1}{2}Q(\bar{G})$ ,  $X = J, k = 2, j = 2$  in the first inequality and  $k = 2, j = n$  in the second inequality of Lemma 3, and recalling that  $J$  has a single eigenvalue  $n$  together with  $0$  with multiplicity  $n - 1$ , we obtain

$$n - 2 - \frac{1}{2}\bar{q}_{n-1} \leq \lambda_2(RQ(G)) \leq n - 2 - \frac{1}{2}\bar{q}_n.$$

This proves the first part of the theorem.

If  $d \geq 3$ , we define the matrix  $M = (m_{ij})$  of order  $n$ , where  $m_{ij} = \min\{0, \frac{1}{2} - \frac{1}{d_{ij}}\}$ ,  $d_{ij}$  is the distance between the vertices  $v_i$  and  $v_j$ . The total reciprocal distance of a vertex  $v_i$  can be written as  $RH_i = d_i + \frac{1}{2}\bar{d}_i + RTr'_i$ , where  $RH'_i = \sum_{d_{ij} \geq 3} (\frac{1}{2} - \frac{1}{d_{ij}})$ , is the contribution from the vertices which are at a distance of more than two from  $v_i$ . For  $RH'(G) = \text{diag}(RH'_1, RH'_2, \dots, RH'_n)$ , we have

$$\begin{aligned} RQ(G) &= RH(G) + RD(G) = \text{Deg}(G) + \frac{1}{2}\text{Deg}(\bar{G}) + RH'(G) + A + \frac{1}{2}\bar{A} + M \\ &= \frac{1}{2}(\text{Deg}(G) + \text{Deg}(\bar{G}) + A + \bar{A}) + \frac{1}{2}(A + \text{Deg}(G)) + RH'(G) + M \\ &= \frac{1}{2}Q(K_n) + \frac{1}{2}Q(G) + M', \end{aligned}$$

where  $M' = RH'(G) + M$ . Taking  $X = \frac{1}{2}Q(K_n), Y = \frac{1}{2}Q(G) + M', k = 2$  and  $j = n$  in the second inequality of Lemma 3, we obtain

$$\lambda_2(RQ(G)) \geq \frac{1}{2}(n - 2) + \lambda_2(\frac{1}{2}Q(G) + M'). \tag{4}$$

Again taking  $X = \frac{1}{2}Q(G), Y = M', k = j = 2$  in the second inequality of Lemma 3, it follows from inequality (4) that

$$\lambda_2(RQ(G)) \geq \frac{1}{2}(n - 2) + \frac{1}{2}q_2 + \lambda_n(M').$$

It is easy to see that the matrix  $M'$  is a positive semi-definite. Therefore, we obtain

$$\lambda_2(RQ(G)) \geq \frac{1}{2}(n + q_2 - 2).$$

This completes the proof.  $\square$

A lower bound for  $\lambda_2(RQ(G))$  is presented below by employing the maximum total reciprocal distance vertex  $RH_{\max}$ , and the second maximum total reciprocal distance vertex  $RH'_{\max}$ , of the connected graph  $G$ .

**Theorem 4.** Suppose that  $G$  is a connected graph over  $n \geq 3$  vertices. Let  $v_1$  and  $v_2$  be the vertices with the maximum total reciprocal distance vertex  $RH_{\max}$  and the second maximum total reciprocal distance vertex  $RH'_{\max}$ , respectively. If  $d(v_1, v_2) = t$ , then

$$\lambda_2(RQ(G)) \geq \frac{RH_{\max} + RH'_{\max} - \sqrt{(RH_{\max} - RH'_{\max})^2 + (\frac{2}{t})^2}}{2}.$$

**Proof.** Let  $G$  be such a graph. Suppose that  $v_1$  and  $v_2$  are the vertices with maximum total reciprocal distance vertex  $RH_{\max}$  and the second maximum total reciprocal distance vertex  $RH'_{\max}$ , respectively. The following scenarios are in order:



(i). Suppose that  $v_1$  and  $v_2$  are adjacent. Clearly,  $d(v_1, v_2) = 1$ .  $M = \begin{pmatrix} RH_{\max} & 1 \\ 1 & RH'_{\max} \end{pmatrix}$  is the submatrix of  $RQ(G)$  indexed by  $v_1$  and  $v_2$ . Employing Lemma 1, we see  $\lambda_1(RQ(G)) \geq x_1$  and  $\lambda_2(RQ(G)) \geq x_2$ , where  $x_1 \geq x_2$  are given by

$$|M - xI_2| = \begin{vmatrix} RH_{\max} - x & 1 \\ 1 & RH'_{\max} - x \end{vmatrix} = 0.$$

Then,

$$\lambda_1(RQ(G)) \geq \frac{RH_{\max} + RH'_{\max} + \sqrt{(RH_{\max} - RH'_{\max})^2 + 4}}{2},$$

and

$$\lambda_2(RQ(G)) \geq \frac{RH_{\max} + RH'_{\max} - \sqrt{(RH_{\max} - RH'_{\max})^2 + 4}}{2}.$$

(ii). If  $v_1$  and  $v_2$  are not adjacent, then  $d(v_1, v_2) = t > 1$ . Again, consider the submatrix  $M = \begin{pmatrix} RH_{\max} & \frac{1}{t} \\ \frac{1}{t} & RH'_{\max} \end{pmatrix}$  of  $RQ(G)$  indexed by the vertices  $v_1$  and  $v_2$ . By Lemma 1, we have  $\lambda_1(RQ(G)) \geq x_1$  and  $\lambda_2(RQ(G)) \geq x_2$ , where  $x_1 \geq x_2$  are given by

$$|M - xI_2| = \begin{vmatrix} RH_{\max} - x & \frac{1}{t} \\ \frac{1}{t} & RH'_{\max} - x \end{vmatrix} = 0.$$

Then,

$$\lambda_1(RQ(G)) \geq \frac{RH_{\max} + RH'_{\max} + \sqrt{(RH_{\max} - RH'_{\max})^2 + \left(\frac{2}{t}\right)^2}}{2}$$

and

$$\lambda_2(RQ(G)) \geq \frac{RH_{\max} + RH'_{\max} - \sqrt{(RH_{\max} - RH'_{\max})^2 + \left(\frac{2}{t}\right)^2}}{2}.$$

The desired result now follows.  $\square$

Let  $v_i$  and  $v_j$  be the vertices in  $G$  with total reciprocal distance vertices  $RH_i$  and  $RH_j$ . Let  $M(i, j)$  be the principal submatrix of  $RQ(G)$  indexed by the vertices  $v_i$  and  $v_j$ . Since  $1 \leq i \neq j \leq n$ , it follows that there are  $\binom{n}{2}$  such submatrices in  $RQ(G)$ . Therefore, by Lemma 1, we have  $\lambda_2(RQ(G)) \geq \lambda_2(M(i, j))$ , for all  $i \neq j$ .

**Theorem 5.** Let  $G$  be any connected graph of order  $n \geq 3$ . If the total reciprocal distance vertex sequence of  $G$  is  $\{RH_1, RH_2, \dots, RH_n\}$ , then

$$\lambda_2(RQ(G)) \geq \max_{1 \leq i \neq j \leq n} \left\{ \frac{RH_i + RH_j - \sqrt{(RH_i - RH_j)^2 + \left(\frac{2}{d_{ij}}\right)^2}}{2} \right\},$$

where  $d_{ij}$  is the distance between the vertices  $v_i$  and  $v_j$ .

Next, we present a lower bound for  $\lambda_2(RQ(G))$  by using the second maximum total reciprocal distance vertex  $RH'_{\max}$ .



**Theorem 6.** Assume that  $G$  is a connected graph of order  $n \geq 3$ , and let  $v_1$  and  $v_2$  be the vertices with maximum total reciprocal distance vertex  $RH_{\max}$  and the second maximum total reciprocal distance vertex  $RH'_{\max}$ , respectively. If  $d(v_1, v_2) = t$ , then

$$\lambda_2(RQ(G)) \geq RH'_{\max} - \frac{1}{t}. \tag{5}$$

If  $G \cong K_n$ , then equality always holds in (5). If  $G \not\cong K_n$  and equality holds in (5) then  $RH_{\max} = RH'_{\max}$ .

**Proof.** Let  $v_1$  and  $v_2$  respectively be the vertices with maximum total reciprocal distance vertex  $RH_{\max}$ , and the second maximum total reciprocal distance vertex  $RH'_{\max}$ . In Theorem 4, we have shown that

$$\lambda_2(RQ(G)) \geq \frac{RH_{\max} + RH'_{\max} - \sqrt{(RH_{\max} - RH'_{\max})^2 + (\frac{2}{t})^2}}{2}.$$

To obtain the inequality (5), it suffices to show that

$$\frac{RH_{\max} + RH'_{\max} - \sqrt{(RH_{\max} - RH'_{\max})^2 + (\frac{2}{t})^2}}{2} \geq RH'_{\max} - \frac{1}{t}, \tag{6}$$

that is,

$$\left(RH_{\max} - RH'_{\max} + \frac{2}{t}\right)^2 \geq (RH_{\max} - RH'_{\max})^2 + \left(\frac{2}{t}\right)^2,$$

that is,

$$\frac{4}{t}(RH_{\max} - RH'_{\max}) \geq 0,$$

which is always true. Thus, the inequality (5) follows.

Now, suppose that  $\lambda_2(RQ(G)) = RH'_{\max} - \frac{1}{t}$ . Then, we must have

$$\lambda_2(RQ(G)) = \frac{RH_{\max} + RH'_{\max} - \sqrt{(RH_{\max} - RH'_{\max})^2 + (\frac{2}{t})^2}}{2} = RH'_{\max} - \frac{1}{t}. \tag{7}$$

If  $G \cong K_n$ , then  $\lambda_2(RQ(G)) = n - 2$ ,  $RH_{\max} = RH'_{\min} = n - 1$  and so equality holds in (5). Assume that  $G \not\cong K_n$ . From the inequality (7), it is clear that if equality holds in (5), then we must have  $RH_{\max} = RH'_{\max}$ . This completes the proof.  $\square$

The following result gives a lower bound for  $\lambda_2(RQ(G))$ , in terms of the order  $n$  and the Harary index  $H(G)$  of the graph  $G$ .

**Theorem 7.** Assume that  $G$  is a connected graph of order  $n \geq 3$  having Harary index  $H(G)$ . Let  $v_1$  and  $v_2$  be the vertices with maximum total reciprocal distance vertex  $RH_{\max}$  and the second maximum total reciprocal distance vertex  $RH'_{\max}$ , respectively. If  $d(v_1, v_2) = t$ , then

$$\lambda_2(RQ(G)) \geq \frac{2H(G)}{n} - \frac{1}{t}, \tag{8}$$

provided that  $t \leq \beta$ . If  $G \cong K_n$ , the equality occurs in (8). If  $G \not\cong K_n$  and the equality occurs in (8), then  $t = \beta$  and  $G$  is a graph with maximum total reciprocal distance vertex  $RH_{\max}$  and  $n - 1$  vertices having total reciprocal distance vertex  $RH'_{\max}$ , such that the vertices  $v_1$  and  $v_2$  are adjacent; or  $t \neq \beta$  and  $G$  is a  $RH_{\min}$ -reciprocal distance-balanced graph, where  $\beta = \frac{n(n-2)}{(n-1)(RH_{\max} - RH'_{\max})}$ .

**Proof.** Suppose that  $RH_{\max}$  and  $RH'_{\max}$  are, respectively, the maximum total reciprocal distance vertex and the second maximum total reciprocal distance vertex of the graph  $G$ . We have

$$2H(G) = \sum_{i=1}^n RH_i \leq RH_{\max} + (n - 1)RH'_{\max}. \tag{9}$$

From Theorem 4 and inequality (9), we have

$$\begin{aligned} \lambda_2(RQ(G)) &= \frac{2H(G)}{n} \\ &\geq \frac{RH_{\max} + RH'_{\max} - \sqrt{(RH_{\max} - RH'_{\max})^2 + \left(\frac{2}{t}\right)^2}}{2} - \frac{RH_{\max} + (n - 1)RH'_{\max}}{n} \\ &= \frac{(n - 2)(RH_{\max} - RH'_{\max}) + \frac{2n}{t} - n\sqrt{(RH_{\max} - RH'_{\max})^2 + \left(\frac{2}{t}\right)^2}}{2n} - \frac{1}{t}. \end{aligned}$$

To see (8), it remains to show

$$(n - 2)(RH_{\max} - RH'_{\max}) + \frac{2n}{t} - n\sqrt{(RH_{\max} - RH'_{\max})^2 + \left(\frac{2}{t}\right)^2} \geq 0,$$

that is,

$$\begin{aligned} &(n - 2)^2(RH_{\max} - RH'_{\max})^2 + n^2\left(\frac{2}{t}\right)^2 + \frac{4n(n - 2)}{t}(RH_{\max} - RH'_{\max}) \\ &\geq n^2(RH_{\max} - RH'_{\max})^2 + n^2\left(\frac{2}{t}\right)^2, \end{aligned}$$

that is,

$$4\left(RH_{\max} - RH'_{\max}\right)\left(\frac{n(n - 2)}{t} - (n - 1)(RH_{\max} - RH'_{\max})\right) \geq 0, \tag{10}$$

which is always true for  $t \leq \beta$ , where  $\beta = \frac{n(n - 2)}{(n - 1)(RH_{\max} - RH'_{\max})}$ . Thus, the inequality (8) follows.

If the equality in (8) is attained, all the inequalities in the above are forced to be equalities. If  $G \cong K_n$  is true,  $\lambda_2(RQ(G)) = n - 2$ ,  $RH_{\max} = RH'_{\max} = n - 1$ ,  $H(G) = \frac{n(n - 1)}{2}$ , and so, it is easy to see that equality holds in (8). Assume that  $G \not\cong K_n$ . Let  $v_1$  and  $v_2$  be, respectively, the vertices with maximum total reciprocal distance vertex  $RH_{\max}$  and the second maximum total reciprocal distance vertex  $RH'_{\max}$  in  $G$ . It is apparent that equality holds in (10) for  $t = \beta$  or  $t \neq \beta$  and  $RH_{\max} = RH'_{\max}$ . From the equality in (9), we get  $RH'_{\max} = RH_{\min}$ , where  $RH_{\min}$  is the minimum total reciprocal distance vertex in  $G$ . This shows that equality occurs in (8) if  $t = \beta$  and  $G$  is a graph with maximum total reciprocal distance vertex  $RH_{\max}$  and  $n - 1$  vertices having total reciprocal distance vertex  $RH'_{\max}$ ;  $t \neq \beta$  and  $G$  is a  $RH_{\min}$ -reciprocal distance-balanced graph. If the vertices  $v_1$  and  $v_2$  are not adjacent, then there does exist any connected graph  $G$  having the total reciprocal distance vertex sequence  $\{RH_{\max}, RH'_{\max}, \dots, RH'_{\max}\}$ . Therefore, it follows that if  $G \not\cong K_n$  and equality holds in (8), then  $t = \beta$  and  $G$  is a graph with maximum total reciprocal distance vertex  $RH_{\max}$  and  $n - 1$  vertices having total reciprocal distance vertex  $RH'_{\max}$ , such that the vertices  $v_1$  and  $v_2$  are adjacent;  $t \neq \beta$  and  $G$  is a  $RH_{\min}$ -reciprocal distance-balanced graph. This completes the proof.  $\square$

#### 4. Effect of Some Graph Operations on $\lambda_2(RQ(G))$

Consider  $A = [a_{ij}]$  and  $B = [b_{ij}]$ , both of which are  $n \times n$  matrices. Denote by  $A \leq B$  if  $a_{ij} \leq b_{ij}$  for all  $1 \leq i, j \leq n$ . Similarly, we denote by  $A < B$  if  $a_{ij} < b_{ij}$  for all  $1 \leq i, j \leq n$ . Two vertices  $u, v \in V(G)$  are referred to as *multiplicate vertices* if  $N_G(u) = N_G(v)$ . Two adjacent vertices  $u$  and  $v$  are called *quasi-multiplicate vertices* if  $N_{G-v}(u) = N_{G-u}(v)$ . We also say a subset  $S \subset V(G)$  is a *multiplicate vertex set* when  $N_G(u) = N_G(v)$  holds for all  $u, v \in S$ . A subset  $C \subset V(G)$  is *quasi-multiplicate vertex set* when the vertices of  $C$  induce a clique and  $N_G(u) - C = N_G(v) - C$  for all  $u, v \in C$ . It is easy to see that attaching edges to any pair of vertices in a multiplicate vertex set makes it quasi-multiplicate.

**Theorem 8.** *Let  $v$  be a pendent vertex of  $G$ , which has diameter  $d$ . Then,*

$$\lambda_3(RQ(G)) - 1 \leq \lambda_2(RQ(G - v)) \leq \lambda_2(RQ(G)) - \frac{1}{d}.$$

**Proof.** Since  $v$  is a vertex with degree one, we get  $d_{G-v}(x, y) = d_G(x, y)$  for  $x, y \in V(G - v)$ , and  $1 \leq d_G(v, z) \leq d$  for  $z \in V(G - v)$ . Removing the row and column of  $RH_G(z) > RH_{G-v}(z)$  with respect to the vertex  $v$  gives the principal submatrix, say,  $M$ . Clearly,  $M > RQ(G - v)$ . Let  $S = M - RQ(G - v)$ . Therefore,  $S = \text{diag}(a_1, a_2, \dots, a_{n-1})$ , where  $\frac{1}{d} \leq a_i \leq 1$ , for  $i = 1, \dots, n - 1$ , hence  $\frac{1}{d} \leq \lambda_i(S) \leq 1$ . Thus, by Lemma 3, we get

$$\lambda_2(RQ(G - v)) + \frac{1}{d} \leq \lambda_2(M) \leq \lambda_2(RQ(G - v)) + 1. \tag{11}$$

Then, by Corollary 1 and the left inequality of (11), we have  $\lambda_2(RQ(G - v)) + \frac{1}{d} \leq \lambda_2(RQ(G))$ . Similarly, by Corollary 1 and the right inequality of (11), we get  $\lambda_3(RQ(G)) \leq \lambda_2(RQ(G - v)) + 1$ . This completes the proof.  $\square$

**Corollary 2.** *Let  $G$  have  $n$  vertices and diameter  $d = 2$ . Assume that  $v \in G$  is adjacent to any other vertex of  $G$ . Moreover,  $G - v$  is connected with  $d(G - v) = d(G)$ . Then, we have*

$$\lambda_3(RQ(G)) - 1 \leq \lambda_2(RQ(G - v)) \leq \lambda_2(RQ(G)) - 1.$$

**Proof.** Using the given assumptions, we obtain  $d_{G-v}(x, y) = d_G(x, y)$  for  $x, y \in V(G - v)$ , hence,  $RH_G(z) = RH_{G-v}(z) + 1$ . Removing the row and column with respect to the vertex  $v$  gives the principal submatrix of  $RQ(G)$ , and we denote it by  $M$ . Similarly, we have  $S = M - RQ(G - v) = I$ . By Lemma 3, we get

$$\lambda_2(RQ(G - v)) + 1 \leq \lambda_2(M) \leq \lambda_2(RQ(G - v)) + 1.$$

Hence, similar to the Theorem 8, we get the desired result.  $\square$

**Corollary 3.** *Suppose that  $G$  is a graph over  $n$  vertices and  $u, v$  are two vertices. If  $u$  and  $v$  are multiplicates (or quasimultiplicates) vertices, then*

$$\lambda_3(RQ(G)) - 1 \leq \lambda_2(RQ(G - v)) \leq \lambda_2(RQ(G)) - \frac{1}{d}.$$

The following lemma characterizes the behaviour of second-largest signless Laplacian reciprocal distance eigenvalues in the case of scrapping the edge connecting two quasi-multiplicate vertices.

**Lemma 4.** *Suppose that  $G$  is a graph of order  $n \geq 3$ . Furthermore, suppose that  $x$  and  $y$  are quasi-multiplicate vertices of  $G$  and  $e = xy$ . We have*

$$\lambda_2(RQ(G - e)) \leq \lambda_2(RQ(G)) \leq \lambda_2(RQ(G - e)) + 2.$$

**Proof.** Suppose that  $x$  and  $y$  are quasi-multiply vertices. Although  $d(x, y) = 1$  is changed to  $d(x, y) = 2$ , the distances of other vertices are fixed. Thus,  $RQ(G - e) < RQ(G)$ . Let  $S = RQ(G) - RQ(G - e)$ . Then,  $S$  can be partitioned into  $S = \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}$ . Hence, the eigenvalues of  $S$  are  $\{2, 0^{[n-1]}\}$ . Thus, the conclusion follows by Lemma 3.  $\square$

The following gives the behaviour of second-largest signless Laplacian reciprocal distance eigenvalues when the edges between the vertices in a quasi-multiply set are deleted.

**Theorem 9.** Let  $U \subset V(G)$  be a quasi-multiply set of  $G$ . Suppose that  $2 \leq z = |U| < n = |V(G)|$ . Let  $G^c$  be the graph obtained by dropping any edge-linking vertices of  $U$ . Then, we have

$$\lambda_2(RQ(G^c)) \leq \lambda_2(RQ(G)) \leq \lambda_2(RQ(G^c)) + z - 1.$$

**Proof.** We know that  $U$  forms a multiply set in  $G^c$ . Similar to Lemma 4, while edges are deleted, only the distances of vertices in  $U$  are boosted from one to two. Denote by  $S = RQ(G) - RQ(G^c)$ . Then,  $S$  can be partitioned into  $S = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$ , where  $R = \frac{1}{2}(J_z + (z - 2)I)$ . Hence, the eigenvalues of  $R$  are  $\{z - 1, \frac{1}{2}(z - 2)^{[z-1]}\}$ . Then, the eigenvalues of  $S$  are  $\{z - 1, \frac{1}{2}(z - 2)^{[z-1]}, 0^{[n-z]}\}$ . Thus, the result follows from Lemma 3.  $\square$

### 5. Conclusions

In this work, we have studied the second-largest signless Laplacian reciprocal distance eigenvalue of a connected graph. The main results of this work lie mostly in Section 3, where we established some upper and lower bounds for  $\lambda_2(RQ(G))$  by employing useful graph structural parameters, and we also characterized some extremal graphs attaining these bounds. Additionally, in the same Section, we have shown that amongst all connected graphs of order  $n$ , the complete graph  $K_n$ , together with the graph  $K_n - e$  obtained by deleting an edge  $e$  from  $K_n$  possess the maximum second-largest signless Laplacian reciprocal distance eigenvalue. Further, in Section 4, we explored the effect of some graph operations on  $\lambda_2(RQ(G))$ . These types of results have been already considered for other graph matrices, like the generalized distance matrix associated with the graph  $G$ . The signless Laplacian reciprocal distance matrix is a different matrix using the structural properties of the graph which are not considered in the other graph matrices. Therefore, it is of interest to explore the spectral properties already done for other graph matrices for this particular matrix of  $G$ , and see how it behaves under those spectral conditions. This is actually the main aim for the spectral study of graphs. Further, from a Matrix theory point of view the spectral study of the signless Laplacian reciprocal distance matrix makes sense.

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