Northumbria University NEWCASTLE

A NEW MODEL OF LONG WAVE-SHORT WAVE INTERACTION GENERALISING THE YAJIMA-OIKAWA AND NEWELL SYSTEMS: INTEGRABILITY AND LINEAR STABILITY SPECTRA

MARCOS CASO HUERTA

PhD in Mathematics

2022

A NEW MODEL OF LONG WAVE-SHORT WAVE INTERACTION GENERALISING THE YAJIMA-OIKAWA AND NEWELL SYSTEMS: INTEGRABILITY AND LINEAR STABILITY SPECTRA

MARCOS CASO HUERTA

A thesis submitted in partial fulfilment of the requirements of the University of Northumbria at Newcastle for the degree of Doctor of Philosophy

Faculty of Engineering and Environment

October 2022

Abstract

In this thesis we consider a recent model for resonant interaction between long and short waves that we proposed unifying and generalising those first proposed by Yajima and Oikawa and by Newell, which we call Yajima-Oikawa-Newell (YON) system, which has the remarkable property of remaining integrable for any choice of the two arbitrary, non-rescalable parameters it features. Long wave-short wave systems, which model the propagation of short waves that generate waves of much longer wavelength, appear in many physical settings, especially in fluid dynamics and plasma physics. Throughout the thesis, we introduce this new system through mathematical means and by recalling the physical origin of the Yajima-Oikawa system, and employ various techniques, both general and pertaining to the theory of integrable systems, to study it. In particular, we obtain several types of solutions, including bright and dark solitons, periodic solutions, breathers and rational solutions, by means of a general Ansatz approach and by using Hirota bilinearisation techniques (namely, the theory of τ-functions, which allows us to relate the system with the Kadomtsev-Petviashvili equation), with which we were able to derive the general N-soliton solution both on a zero and non-zero background, and the phase shift corresponding to the collision of two solitons (which, remarkably, only depends on the wave numbers of the two solitons).

Even though a physical derivation of the whole YON system is not available at this point of the research and remains an open problem, the fact that the Yajima-Oikawa system, in itself a subcase of the YON system, can be physically derived encourages us to try to obtain the whole system as a reduction of a physical model of resonantly interacting long and short waves. In case it can be derived in a physical context, the fact that the system features free parameters might be useful to better model and assist experiments.

Furthermore, we introduce a recent technique for the study of stability of solutions of integrable systems, proposed by Degasperis, Lombardo & Sommacal (2018), which makes use of the Lax pair associated to the system to perform a linear stability analysis of the solution by introducing a new object that we refer to as the stability spectrum, defined as an algebraic/topological structure in the complex plane. The geometric properties of this spectrum are linked to the stability or instability of the given solution, which allows us to provide a full classification of the stability behaviours in the parameter space. In the thesis we employ this technique to study the stability of the plane waves of the YON system. We also provide a few conjectures relating the topology

of the stability spectrum and the existence of special kinds of functions, namely dark solitons and rational solutions. These predictions are indeed true for the YON system, as checked with the solutions derived via Hirota bilinearisation.

Contents

List of Figures

List of Tables

Acknowledgements

First and foremost I would like to express my deepest gratitude to my supervisor Dr Matteo Sommacal for his extraordinary mentoring during my doctoral studies. He really went out of his way to help me progress in my career and grow as a researcher, and on the personal side I could not have asked for more. I truly could not have been luckier with my supervisor. I want to extend my gratitude to his family for their great hospitality, and for their big part in making me feel at home at Newcastle.

In that regard, I also have to thank the whole Maths group and my fellow PhD students (former and current) for their very warm welcome to university and their much appreciated friendliness. Of course, my gratitude to Northumbria University for their financial support during my doctoral studies.

I want to thank all the collaborators I have worked with in the last three years for their invaluable help in my research and for sharing with me their deep expertise: Prof. Sara Lombardo, Prof. Antonio Degasperis, Dr Priscila Leal da Silva, Prof. Baofeng Feng, Prof. Kenichi Maruno, and Prof. Cornelis van der Mee.

I cannot but highlight my deep gratitude to the Fundación María Cristina Masaveu Peterson for their critical influence in my career. Without their support to improve my education and better myself during my undergraduate studies I would not be where I am right now.

I would like to thank the Isaac Newton Insitute for Mathematical Sciences for their support and hospitality during the programme on Dispersive Hydrodyamics when part of the work for this thesis was undertaken (supported by the EPSRC Grant Number EP/R014604/1), as well as Loughborough University for their hospitality during the same programme.

Last but not least, I would like to thank my parents, Carmen and César, without whom I would not be who I am today.

Declaration

I declare that the work contained in this thesis has not been submitted for any other award and that it is all my own work. I also confirm that this work fully acknowledges opinions, ideas and contributions from the work of others.

Any ethical clearance for the research presented in this thesis has been approved. Approval has been sought and granted by the Ethics Committee on 1 December 2019.

I declare that the Word Count of this thesis is 45180 words.

Name: Marcos Caso Huerta

Date: 28 October 2022

Chapter 1

Introduction

In this work, we will introduce a new integrable model, the Yajima-Oikawa-Newell system, to describe the interaction between long and short waves, unifying and generalising two very well-known integrable systems (namely, the Yajima-Oikawa and Newell systems), and we will take advantage of its integrability to apply different machinery to derive some of its properties and solutions. With this in mind, in this introductory chapter we will review some of the basic tools and concepts that we will later use throughout the thesis.

In Section 1.1, we will introduce the multiscale method, which enables to reduce a given system into different ones whose behaviour is easier to study, while still keeping some properties and information of the original system.

In Section 1.2, we will give a flavour of the concept of integrability by introducing some of the most popular definitions of integrable system and, in particular, we will introduce the concept of Lax integrability, along with some basic properties of Lax pairs.

In Section 1.3, we will provide a brief overview of the inverse scattering method, which permits to derive solutions of a Cauchy problem for an integrable system by making use of its Lax pair formulation. We will also underline one of the problems one can encounter when applying the theory (which, in fact, appears when dealing with the Yajima-Oikawa-Newell system).

In Section 1.4, we will review some of the main techniques to study the stability of solutions of nonlinear systems, which will be later complemented in Chapter 5, by the introduction of a recently developed technique for the special case of solutions of integrable systems.

Finally, in Section 1.5, we will outline the structure of the thesis, with a brief description of the content of each chapter.

1.1 The multiscale method

The multiscale method is one of the most popular methods to reduce a complicated nonlinear system into multiple simpler models at different scales of resolution, for which one can more readily obtain properties that may be translated into the original system (see [17, 27, 28, 50, 146, 163]). Roughly speaking, this approach is based on the assumption that the behaviour of a nonlinear system can be described separately at different levels of detail, some of them coarser and some finer, in the independent variables on which the unknowns depend.

In our case, the rationale behind this assumption lies in the fact that most of the systems we are considering that describe some physical systems actually model interacting waves. Whenever they represent monochromatic wave packets of a relatively small amplitude, one can consider the characteristic packet size and wavelenght of each of the packets as the different scales of the system. The idea is to focus on one, or at most a few, carrier waves of the linear part of the nonlinear system and study the effect of the nonlinear part on them, which, for the weakly nonlinear regime, will act as an amplitude modulation. Thus, the amplitude of the carrier waves will be studied as a function of coarse-grained space variables and slow time variables.

For the sake of completeness, it is worth noting that the multiscale method is closely related to the averaging method developed by Whitham and others in the 1970s (see [74, 155]). However, the multiscale method is more elementary, and as such it allows one to deepen further not only on the system itself but also on some of its properties, which will be conserved or even reinforced through the method.

In particular, the multiscale method conserves, or “increases”, the integrability of the system (more on integrability in Section 1.2), so the integrability of the derivative system will actually become a necessary condition for the integrability of the original one (see [25]). An additional application of this fact is that, by applying the multiscale method on systems that are known integrable, one can obtain new systems that will also be integrable and have a physical meaning provided the original system also has it. However, the number of integrable systems coming from the application of the method is rather small, and for that reason we will call them universal systems, in the sense that they cover the multiscale behaviour of many different systems [25].

Let us consider a dispersive, nonlinear partial differential equation of the form

$$Du = \frac{\partial^h}{\partial x^h} F(u, u_x, u_{xx}, \ldots; u^*, u_x^*, u_{xx}^*, \ldots) \,, \tag{1.1}$$

where $u = u(x,t)$ is the dependent variable, which will in general be complex, x and t are the original space and time variables, the subindex x denotes partial derivation with respect to x, and the asterisk * denotes complex conjugation. D denotes the linear dispersive differential operator

$$D = \frac{\partial}{\partial t} + i\omega\left(-i\frac{\partial}{\partial x}\right) \,, \tag{1.2}$$

with

$$\omega(k) = \sum_{m=0}^{M} a_m k^m \,, \qquad a_M \neq 0 \,. \tag{1.3}$$

The method can still be applied if higher order derivatives appear in the expression of D or if the dependent variable u is a vector or a matrix rather than a scalar, though we will provide an introduction restricted to this case for simplicity. The adjective "dispersive" applied to D means that we will impose all the constants a_m to be real, and that $M > 1$. This is imposed so that the linear PDE

$$Du = 0 \tag{1.4}$$

admits a solution of the form

$$u(x,t) = e^{i[kx-\omega(k)t]} \tag{1.5}$$

with a real ω for any given real k, thus ensuring that $|u(x,t)|$ is constant for that solution (and in fact equal to 1).

The group velocity $v(k)$ is defined in terms of the frequency $\omega(k)$ as

$$v(k) = \frac{d\omega(k)}{dk} = \sum_{m=1}^{M} m\, a_m k^{m-1} \,. \tag{1.6}$$

The operator $\frac{\partial^h}{\partial x^h}$ in (1.1), where h is a non-negative integer, is introduced for convenience and could actually be absorbed by F or by u, however it is usually more convenient to keep it to simplify the computation. The term F in the right-hand side represents the nonlinear part of the evolution PDE. We will assume that it satisfies the expression

$$F(\varepsilon u, \epsilon u_x, \ldots; \varepsilon u^*, \varepsilon u_x^*, \ldots) = \sum_{m=2}^{\infty} \varepsilon^m F^{(m)}(u, u_x, \ldots; u^*, u_x^*, \ldots) \,, \tag{1.7}$$

where ε denotes a small parameter and $F^{(m)}$ is a homogeneous polynomial of degree m.

We will consider as our Ansatz a superposition of plane waves of the form

$$u(x,t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} e^{inz} \psi_n(\xi, \tau) \,, \tag{1.8}$$

where z is coming from the solution of the linear problem,

$$z = kx - \omega(k)t \,, \tag{1.9}$$

and the amplitude modulation ψ_n depends on the slow variables ξ and τ,

$$\xi = \varepsilon^p (x - Vt) \,, \tag{1.10a}$$

$$\tau = \varepsilon^q t \,, \tag{1.10b}$$

where V denotes the group velocity, $V = v(k)$.

The sum in the right-hand side of (1.8) has an asymptotic character, meaning that we do not need it to converge and we will only consider a few terms (e. g. $|n| \leqslant 2$) for our computations. The constant k, which ultimately plays the role of a wave number can be chosen arbitrarily as it is convenient. One of the important points of the computation are the values of the exponents γ_n, p and q, since they have to be chosen in such a way that we obtain finite, non-trivial solutions in the asymptotic $\varepsilon \to 0$ limit for the evolution of the amplitude modulations $\psi_n(\xi, \tau)$ in the slow variables ξ and τ. The exponents p and q are required to be positive so that the nonlinear effects in the limit of weak nonlinearity $\varepsilon \to 0$ are finite.

Let us remark that when performing the computations for a particular system, further dependences of ε may be required, either in the choice of k (e. g. choosing it as $k = k_0 + \varepsilon^{\lambda} k_1$) or in the form of $\psi_n(\xi, \tau)$ (e. g. via relations like $\psi_n(\xi, \tau) = \psi_n^{(0)}(\xi, \tau) + \varepsilon^{\mu_n} \psi_n^{(1)}(\xi, \tau)$). These further introduction of ε may be necessary for cases where extra cancellations occur in the computations, keeping us from obtaining non-trivial results at the leading order.

The idea is now to introduce the Ansatz (1.8) into (1.1) and compensate terms in the left and right-hand side to present the same exponentials, to then take the limit $\varepsilon \to 0$ to obtain equations for the $\psi_n(\xi, \tau)$, possibly after introducing some rescalings to the dependent and independent variables to reabsorb constants and show the equation in a clearer form.

Let us illustrate the method by applying it to an example (the derivation was performed for this work as an exercise; we do not know of a reference including this computation, though it is likely there is some). We will consider the cubic nonlinear Klein-Gordon equation, which appears in several fields of physics, especially in quantum mechanics (see [20]),

$$u_{tt} - u_{xx} + u(1 - uu^*) = 0\,. \tag{1.11}$$

To apply the multiscale method, we will define a slow spatial variable $X = \varepsilon x$ and slow time variables $T = \varepsilon t$ and $\tau = \varepsilon^2 t$, with $\varepsilon \ll 1$ a small positive parameter.

We will look for a solution in the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots\,, \tag{1.12}$$

where the components are of the form $u_i = u_i(x, X, t, T, \tau)$. Then we can transform (1.11) into our new variables by using that

$$u_{xx} = \varepsilon(u_1)_{xx} + \varepsilon^2\Big[(u_2)_{xx} + 2(u_1)_{xX}\Big] + \varepsilon^3\Big[(u_3)_{xx} + 2(u_2)_{xX} + (u_1)_{XX}\Big] + \mathcal{O}(\varepsilon^4)\,, \quad (1.13)$$

and

$$u_{tt} = \varepsilon(u_1)_{tt} + \varepsilon^2\Big[(u_2)_{tt} + 2(u_1)_{tT}\Big] + \varepsilon^3\Big[(u_3)_{tt} + 2(u_2)_{tT} + 2(u_1)_{t\tau} + (u_1)_{TT}\Big] + \mathcal{O}(\varepsilon^4)\,. \quad (1.14)$$

We can define a differential operator $\mathcal{L}$ as

$$\mathcal{L}(\phi) = \phi_{tt} - \phi_{xx} + \phi\,, \quad (1.15)$$

and the nonlinearity as

$$\mathcal{N}(\phi) = \phi^2\phi^*\,. \quad (1.16)$$

We can introduce the expression for u (1.12) and its derivatives (1.13) and (1.14) to rewrite (1.11) as

$$\begin{aligned} 0 = \varepsilon\mathcal{L}(u_1) &+ \varepsilon^2\Big[\mathcal{L}(u_2) + 2(u_1)_{tT} - 2(u_1)_{xX}\Big] \\ &+ \varepsilon^3\Big[\mathcal{L}(u_3) + 2(u_2)_{tT} - 2(u_2)_{xX} + 2(u_1)_{t\tau} + (u_1)_{TT} - (u_1)_{XX} - u_1^2 u_1^*\Big] + \mathcal{O}(\varepsilon^4)\,. \end{aligned} \quad (1.17)$$

At $\mathcal{O}(\varepsilon)$, we have the equation

$$\mathcal{L}(u_1) = 0\,, \quad (1.18)$$

that is,

$$(u_1)_{tt} - (u_1)_{xx} + u_1 = 0\,, \quad (1.19)$$

which, thanks to its linearity, is solved by a function of the form

$$u_1(x, X, t, T, \tau) = A(X, T, \tau)e^{i(kx-\omega t)} + \text{c.c.}\,, \quad (1.20)$$

where c.c. denotes the complex conjugate, and ω has to satisfy the dispersion relation

$$\omega(k) = \sqrt{k^2 + 1}\,. \tag{1.21}$$

Inserting this solution into the equation at $\mathcal{O}(\varepsilon^2)$, we have that

$$\mathcal{L}(u_2) = 2i(\omega A_T + kA_X)e^{i(kx-\omega t)} + \text{c.c.} \tag{1.22}$$

The left hand side has the same structure as the homogeneous problem and hence represents a secularity. Because of that, we shall equal the right-hand side to zero, which gives the relation

$$\omega A_T = -kA_X\,, \tag{1.23}$$

which means that the wave is propagating with group velocity

$$V = \frac{k}{\omega} = \omega'(k) \tag{1.24}$$

in the slow variables. We can then introduce a new variable $\xi = X - VT$ so that $A(X, T, \tau) = A(\xi, \tau)$ and $\mathcal{L}(u_2) = 0$ in the equation above. That means we have

$$u_2 = B(X, T, \tau)e^{i(kx-\omega t)} + \text{c.c.} \tag{1.25}$$

We can now introduce our formulae for u_1 and u_2 into the equation at $\mathcal{O}(\varepsilon^3)$ to obtain

$$\mathcal{L}(u_3) = \Big[2i\omega A_\tau + (1-c^2)A_{\xi\xi} + |A|^2A + 2i(\omega B_T + kB_X)\Big]e^{i(kx-\omega t)} + |A|^2Ae^{3i(kx-\omega t)} + \text{c.c.} \tag{1.26}$$

Again, we have a secular term if the term in front of $e^{i(kx-\omega t)}$ is non-zero. Imposing it to be zero, we get the condition

$$\omega B_T = -kB_X\,, \tag{1.27}$$

so again we can set $B(X, T, \tau) = B(\xi, \tau)$ and the term vanishes. Moreover, from the vanishing of the term above we get the additional condition

$$2i\omega A_\tau + \omega\omega'' A_{\xi\xi} + |A|^2 A = 0\,, \tag{1.28}$$

which happens to be the nonlinear Schrödinger (NLS) equation. Assuming $\omega \neq 0$ (which is true for every real k by means of the dispersion relation (1.21)), we can divide the equation by 2ω and use that $\omega'' = \omega^{-3}$ to get

$$iA_T + \frac{1}{2\omega^3} A_{\xi\xi} + \frac{1}{2\omega}|A|^2 A = 0\,. \tag{1.29}$$

We do not need to worry about the remaining term in (1.26), $|A|^2 A e^{3i(kx-\omega t)}$ since in the general case $\omega(3k) \neq 3\omega(k)$, so the term is not in resonance with the homogeneous solution.

The NLS equation is indeed one of the most ubiquitous integrable equations that can result from a multiscale analysis. A full classification of all the obtainable integrable systems in the simplest case is available in [27]. However, additional equations may appear by allowing for more general operators in the original equation, and more complicated expressions for the elements in the method in terms of ε. This is in fact an active field of current research (e. g. [103]).

1.2 Fundamentals of integrability and the Lax pair

The study of integrable systems has its roots at the very beginning of classical mechanics. Since Newton formulated his well-known laws, both physicists and mathematicians have tried to find exact solutions to the problems they model.

Newton himself managed to solve Kepler's problem, but until several centuries later, only a handful of simple problems were solved. It was not until the 19th century that Liouville made a qualitative leap in the study of integrable Hamiltonian systems, giving a general framework in what is called today the Liouville-Arnold theorem, allowing to find a primitive that defines the dynamics of the system (see [10, 46, 118]). However, it took another century to develop quite systematic methods to carry out this task.

One of the earliest and most popular methods for that is the so-called classical inverse scattering method, developed by Gardner, Greene, Kruskal and Miura in 1967, and first applied to solve the Korteweg-De Vries (or KdV) equation (see [78, 126]). It can be seen as a nonlinear analogue and in some sense a generalisation of Fourier analysis, and it is applicable to many completely integrable infinite-dimensional systems. Methods for investigating integrability in a quantum context were developed during the following decade by the Leningrad-St. Petersburg school, headed by L. Faddeev along with several of his students, notably Korepin, Kulish, Reshetikhin, Sklyanin or Semenov-Tian-Shansky (see [144]). Their work, connected with the theory of quantum groups by Drinfeld and Jimbo, paved the way for the algebraic formulation of the problem (in fact, solving the equations of motion became equivalent to solving the factorisation problem in the corresponding group, see [11]). This new formulation made it possible to unify in a single mathematical framework the study of integrable quantum field theories and spin-lattice systems.

The definition of what makes a system integrable is however not unique, and different authors and trends within the field use different definitions, which depending on the setting may or may not be equivalent (see e. g. [94]).

The Liouville-Arnold theorem deals with integrability in the sense of dynamical systems, meaning that there exist invariant, regular foliations, i. e., foliations whose leaves are embedded submanifolds of the smallest possible dimension, that are invariant under the flow. For Hamiltonian systems this condition is equivalent to what is know as complete integrability, or integrability in the sense of Liouville, which consists of having a maximal set of conserved quantities in involution, that is, such that the Poisson brackets between the different conserved quantities vanish. For systems where the phase space is finite-dimensional and symplectic (which is equivalent to the Poisson bracket being non-degenerate), the dimension is always even, say, $2n$ (where, conceptually each position coordinate is associated to a momentum coordinate). In that case, the maximum number of conserved quantities in involution, including the Hamiltonian itself if the system is autonomous, is n, so that if that amount is achieved then the system is completely integrable. The leaves of the maximal foliation associated to the integrable system are totally isotropic, and the foliation becomes Lagrangian (which in certain fields of physics is also known as a foliation of branes). In the infinite-dimensional case, the system is required to have an infinite number of conserved quantities in involution to be considered completely integrable.

In the case of autonomous Hamiltonian systems, the leaves of the foliation happen to be tori, allowing us to use the natural coordinates of the torus to define canonical coordinates known as action-angle coordinates that decouple the dynamical system, thus allowing to solve the equations of movement by quadratures (see e. g. [11, 46, 106]). For other cases of complete integrability these coordinates can also be achieved, with the flow parameters acting as coordinates of the phase space, and the system is still solvable by quadratures. This leads us to an alternate definition of integrability, often called solvability, consisting on the ability to find exact solutions in a closed form. Even though the solutions of completely integrable systems can always be obtained by quadratures, said solutions need not be in closed form as generally understood, and thus the concepts of completely integrable system and solvable system are not equivalent. Note here that some authors employ the term “exactly solvable” to refer to completely integrable systems, especially in the infinite-dimensional case, while others employ it with the vague definition that the solutions can be expressed in terms of “known” functions. Due to its ambiguity we will avoid using this terminology and keep the distinction between completely integrable and solvable systems.

The geometric conditions imposed for integrability in the sense of dynamical systems can be relaxed by allowing a definition of integrability in terms of local properties, instead of global ones. Thus, a system is called Frobenius integrable if it features locally a foliation by maximal integral manifolds. Through the Frobenius theorem, the condition for Frobenius integrability is a necessarily and sufficient condition for the existence of compatible coordinate grids coming from the different local manifolds (see [107]).

Another definition of integrability is the concept of Painlevé integrability (see [1, 5, 95]). A system is said to be Painlevé integrable if it exhibits the Painlevé property, which means that all of its movable singularities, that is, singularities whose location depend on the initial condition, are poles of the solution. All other critical points, including logarithmic branch points and essential singularities, must be fixed no matter the initial condition. This can be checked by means of the Painlevé test (see [153]). Painlevé integrability is a necessary condition for complete integrability and for Lax integrability (explained in the next paragraph), although it is not a sufficient condition. The Laurent series employed in the Painlevé test can be used to construct auto-Bäcklund and Darboux transformations, and is useful to identify the symmetry algebras of the PDEs, which are in turn subalgebras of the Kac-Moody algebra and the Virasoro algebra [12].

A further definition of integrability, which will be the one we refer to throughout the thesis simply as “integrability”, is the definition of Lax integrability, which consists on the existence of a Lax pair (see [26, 63, 108, 109]). The classical definition of a Lax pair, which we will refer to as a Lax pair in operator form, is a pair of matrices $L(t)$, $P(t)$ whose entries are allowed to further depend on the variables of the phase space and usually on an additional arbitrary parameter known as the spectral parameter and usually denoted as λ, and such that the equations of motion of the system can be written as

$$\frac{dL}{dt} = [P, L]\,, \tag{1.30}$$

where $[P, L] = PL - LP$ denotes the commutator and P is an anti-Hermitian matrix. A remarkable property is that the matrix L generates the set of conserved quantities

$$C_k = \operatorname{tr} L^k\,. \tag{1.31}$$

This follows from the fact that

$$\frac{dC_k}{dt} = k \operatorname{tr}\left(L^{k-1}\frac{dL}{dt}\right) = k \operatorname{tr}\left(L^{k-1}[P, L]\right) = 0\,, \tag{1.32}$$

due to the cyclicity of the trace. This also implies that the eigenvalues of $L(t)$ are isospectral as t varies. Because of that, all $L(t)$ are similar for every t, so that

$$L(t) = U(t, s)L(s)U(t, s)^{-1}\,, \tag{1.33}$$

for any t and s, where the unitary matrix $U(t,s)$ is the solution of the Cauchy problem

$$\frac{d}{dt}U(t,s) = P(t)U(t,s)\,, \qquad U(s,s) = \mathbb{1}\,, \tag{1.34}$$

where $\mathbb{1}$ denotes the identity matrix. This fact is the basis of the inverse scattering method (further developed in Section 1.3), which roughly consists on solving the eigenvalue problem for L when $t = 0$ (which will be associated to an initial condition for the system) to then let the eigenvalues and eigenstate evolve using (1.33) up to an arbitrary t and use the solution to generate the corresponding solution for the system at time t.

It is important to note that Lax pairs for an integrable system are not unique, and a system can even be described by Lax pairs of different dimensions.

We will however employ a different, more modern approach to Lax pairs, usually referred to as AKNS form (due to the renown paper [2]) and which we will simply call Lax pair in matrix form. Let us consider a 1+1 dimensional PDE, that is, a PDE where the dependent variables depend on two independent variables x and t (extensions of this machinery for systems in higher dimensions exist, but we will ignore them for the sake of simplicity). Let us also consider a pair of matrices X and T whose entries are expressed in terms of the dependent and independent variables and possibly of an arbitrary parameter λ, which again we will call the spectral parameter. We define the Lax equations associated to the pair X, T as the pair of ODEs

$$\Psi_x = X\Psi\,, \tag{1.35a}$$

$$\Psi_t = T\Psi\,, \tag{1.35b}$$

where we have introduced a new dependent, matrix-valued variable $\Psi(x,t,\lambda)$. We will say that X, T is a Lax pair for our PDE if it emerges as the compatibility condition of the Lax equations,

$$\Psi_{xt} = \Psi_{tx}\,, \tag{1.36}$$

that is, our PDE is equivalent to the equality

$$X_t - T_x + [X,T] = 0\,, \tag{1.37}$$

which is also referred to as the zero curvature condition. As it happened with the Lax pairs in operator form, Lax pairs in matrix form are not unique. In fact, there is a direct correspondence between both formalisms, and one can be readily transformed into the other (see [125]).

The Lax pairs in matrix form are also employed in inverse scattering techniques. A brief introduction to those techniques will be provided in Section 1.3.

1.3 The inverse scattering method

As explained in the section before, the inverse scattering method is one of the most popular methods to obtain solutions for integrable systems. It makes use of the Lax pair formulation of the system (either in operator or matrix form) to evolve in time some parameters of the system (commonly called the scattering parameters) that then allow us to reconstruct the solution of the system (see e. g. [1, 2, 3, 6, 26, 57, 131, 139]). Let us note here that we will not make use of the inverse scattering machinery throughout the thesis, however it is presented here for the sake of completeness and to illustrate some issues that do in fact occur when treating the Yajima-Oikawa-Newell system. A basic scheme of the method would be

$$\begin{tikzcd}
\text{Initial condition at } t=0 \arrow[r, "\text{Direct scattering}"] \arrow[d, "\text{Solution to the IVP}"] & \text{Scattering data at } t=0 \arrow[d, "\text{Time evolution}"] \\
\text{Solution at } t=t_1 & \text{Scattering data at } t=t_1 \arrow[l, "\text{Inverse scattering}"']
\end{tikzcd}$$

Let us consider a Lax pair X, T defining the Lax equations (1.35). For simplicity, we will assume that X and T have the form

$$X(\lambda) = i\lambda\Sigma + Q\,, \qquad T(\lambda) = (i\lambda)^2 T_2 + i\lambda T_1 + T_0\,, \tag{1.38}$$

where λ is the spectral parameter, Σ, Q, T_2, T_1 and T_0 are λ-independent matrices, Σ is a constant, diagonal matrix, and $Q(x,t)$ is an off-diagonal matrix. The compatibility condition of the Lax pair (1.37) entails the PDE or system of PDEs under scrutiny.

We can start the method by looking at the solution of the first Lax equation, using the matrix X. With our choice of matrices it reads

$$\Psi_x = (i\lambda\Sigma + Q)\Psi\,, \tag{1.39}$$

which, provided the matrix Σ is invertible, can be rewritten as an eigenvalue problem,

$$\left(i\Sigma^{-1}Q - i\Sigma^{-1}\partial_x\right)\Psi = \lambda\Psi\,. \tag{1.40}$$

We will then introduce the so-called Jost solutions $\psi(x,\lambda)$ and $\phi(x,\lambda)$ of this ODE, which are uniquely determined by the asymptotic conditions

$$\psi(x,\lambda) = e^{i\lambda x}\mathbb{1} + \mathcal{O}(1)\,, \qquad x \to +\infty\,, \tag{1.41a}$$

$$\phi(x,\lambda) = e^{-i\lambda x}\mathbb{1} + \mathcal{O}(1)\,, \qquad x \to -\infty\,, \tag{1.41b}$$

where $\mathbb{1}$ denotes the identity matrix, via Volterra integral equations (see [3, 63, 150]). From the Jost functions one can also define the Faddeev functions $M(x,\lambda)$ and $N(x,\lambda)$ as

$$M(x,\lambda) = \psi(x,\lambda)e^{-i\lambda x}\,, \tag{1.42a}$$

$$N(x,\lambda) = \phi(x,\lambda)e^{i\lambda x}\,. \tag{1.42b}$$

The individual columns of $M(x,\lambda)$ and $N(x,\lambda)$ provide us with a basis to construct the so-called scattering matrix, which in turn we will be able to evolve in time using relatively simple rules.

One of the delicate points, however, is the fact that the columns of $M(x,\lambda)$ behave well for $x \in \mathbb{R}^+$, while the columns of $N(x,\lambda)$ behave well for $x \in \mathbb{R}^-$. We have then the task of "gluing" both solutions together to get a solution that behaves well everywhere. This process is usually referred to as the Riemann-Hilbert problem for the Jost functions.

Once the scattering matrix for an arbitrary time t is obtained, then the corresponding solution at time t is usually obtained through some variation of a Marchenko integral equation, also known as Gelfand-Levitan equation (see [123]).

Since we will not employ inverse scattering analysis throughout the thesis, we will not delve deeper into the theory and will refer to the references provided. However, let us note that all the theory presented in this section works only when the matrix Σ is invertible.

Quite surprisingly, to the best of our knowledge there is no literature attacking the problem of performing inverse scattering when Σ is a singular matrix, so that the problem cannot be transformed into an eigenvalue problem, but a generalised eigenvalue problem. However, that is precisely the case for the main systems we will treat throughout the thesis (the Yajima-Oikawa and Newell systems, and its generalisation in the Yajima-Oikawa-Newell system). In such situation, the Jost matrices can still be uniquely defined with similar asymptotics as solutions of the generalised eigenvalue problem. However, when obtaining the columns of the Faddeev functions to construct the scattering matrix, the columns corresponding to zeros of the matrix Σ vanish too, so one does not get enough vectors to cover for the dimension of the solution space. Newell himself does perform an inverse scattering analysis for his eponymous system in [128], however we were unable to reproduce the results in that work, and thus it remains unclear for us whether he employed a general method that can be applied to general cases.

A way to circumvent that limitation requires heavy mathematical machinery of functional analysis, and is currently work in progress jointly with Cornelis van der Mee [32].

1.4 Introduction to stability

The stability of solutions of PDEs, that is, the property of a given solution to remain close to the initial setting under small perturbations, is a classical problem in analysis and dynamical systems, of great relevance from both the theoretical and applied points of view. For linear systems, there exists a plethora of well-established results and techniques. For example, using Schur analysis, the study of the stability of solutions of a linear system can be translated into studying the eigenvalue problem of a matrix associated to the system (see e. g. [18, 96, 151].

The stability analysis of solutions of nonlinear evolution systems of integrable and non-integrable type has been developed much more recently, and due to its complexity only certain kinds of perturbations of certain kinds of systems have been studied. For integrable nonlinear systems, one of the main techniques makes use of the Floquet-Lyapunov theory to map the problem into a linear one, that can then be more easily solved (e. g. [4, 24, 138, 142]). However, for this analysis to be applied, it is required that both the solutions and the perturbations be periodic.

Another popular line of research focuses on modulational instability, looking at the reinforcement of the perturbations due to the nonlinearity and how this asymptotically affects wave trains. In this context, modulational instability has been identified as a potential mechanism for the onset of rogue waves [13]. However, again only certain kinds of perturbations have been systematically studied: periodic perturbations (e. g. [52, 81, 83, 84]), random perturbations (e. g. [14]) or localised perturbations (e. g. [19]).

Let us illustrate a simple approach of this kind by applying it to a system we will further study in Chapter 2, the Yajima-Oikawa system,

$$iS_t + S_{xx} - LS = 0\,, \tag{1.43a}$$

$$L_t = 2(|S|^2)_x\,, \tag{1.43b}$$

where S is a complex variable and L a real one, and the subindices denote partial differentiation. The Yajima-Oikawa system admits plane wave solutions of the form

$$S(x,t) = ae^{i\theta}\,, \qquad L = b\,, \qquad \theta = qx - \nu t\,, \qquad \nu = q^2 + b\,, \tag{1.44}$$

where a and b are real, constant amplitudes and q is the wave number. We will introduce a perturbation in this solution by setting

$$S(x,t) = ae^{i\theta}(1+b\eta)\,, \qquad L = b(1+\mu)\,, \tag{1.45}$$

where $\eta(x,t)$ and $\mu(x,t)$ satisfy the system of PDEs with constant coefficients

$$i\eta_t + \eta_x x + 2iq\eta_x - \mu = 0\,, \qquad \mu_t - 2a^2(\eta_x + \eta_x^*) = 0\,. \tag{1.46}$$

Through standard Fourier analysis, we can construct solutions of this system of the form

$$\eta(x,t) = A_+ e^{i\phi} + A_-^* e^{-i\phi^*}\,, \qquad \mu(x,t) = Be^{i\phi} + B^* e^{-i\phi^*}\,, \qquad \phi = kx - \omega t\,, \tag{1.47}$$

where A_+, A_- and B are complex amplitudes, and where the dispersion relation $\omega(k)$ is given by the three roots of the polynomial

$$P(\rho) = \rho(\rho-2)^2 - c_1\rho + c_0\,, \qquad \rho = \frac{\omega}{qk}\,, \tag{1.48}$$

with

$$c_1 = \left(\frac{k}{q}\right)^2 > 0\,, \qquad c_0 = 4\frac{a^2}{q^3}\,, \qquad -\infty < c_0 < +\infty\,. \tag{1.49}$$

Since it is a cubic polynomial, $P(\rho)$ always has a real solution, and the other two solutions can be either real or complex conjugate depending on the value of c_1 and c_0 (and thus on the value of a, q and k). Based on our solution (1.47), we require ω to be real for all real values of k (otherwise, it would give rise to a real exponential that would consequently diverge for large t). Hence, we will say that the system is stable for a given value of a and q if all three branches of the dispersion relation $\omega(k)$ are real for every real k. Otherwise, we will define the band of instability as the set of values of k for which $P(\rho)$ has complex conjugate roots. The endpoints of the intervals in the instability band correspond to the zeros of the discriminant of $P(\rho)$ with respect to ρ,

$$D_\rho(c_0,c_1) = 4c_1^3 - 32c_1^2 + (72c_0+64)c_1 - 27c_0^2 - 32c_0\,. \tag{1.50}$$

All of these methods, however, are fit to study scalar models of nonlinear waves, but they do not translate well to the multicomponent case, either because they become hugely convoluted or because their very formulation is only properly defined in the scalar case. Thus, very little is known in general about the stability of solutions of multicomponent systems. Moreover, they do not take advantage of the potential integrability of the system, which could in turn simplify or improve the calculations by using non-general machinery. In Chapter 5, we will introduce an algebraic method which makes use of integrability properties of the system (in particular, of its Lax pair) to investigate the stability of solutions in the linear stage, and which extends naturally to the multicomponent case. A further advantage of this method is that, from the geometric features of the resulting object, one appears to be able to predict the range of parameters for which special kinds of solutions exist (in particular, solitons and rational solutions).

It is worth remarking an also fairly recent method for stability developed by B. Deconinck and collaborators (see [47, 48, 49, 149]), in which they also use the Lax pair formulation to study the stability of integrable systems in the linear stage. However, it is unclear how this method relates to the one introduced in Chapter 5, or whether it can be readily extended to multicomponent systems. This is an open line of research we are currently working on.

1.5 Aims and scopes

The present thesis covers several mathematical machinery for integrable systems by applying it to a new integrable system we discovered and introduced during the PhD studies, which generalises two very well-known integrable systems describing the interaction between long and short waves: the Yajima-Oikawa system and the Newell system.

Some of the methods employed are newly developed, and at the present time our research on them is still ongoing, trying to understand them in more depth and extend their applicability.

In Chapter 2, we introduce the Yajima-Oikawa system by reproducing its derivation from physical principles, paying particular attention to its derivation from special resonant conditions through the multiscale method, and we provide a systematic review on previously obtained properties and solutions of the system, especially on those related to integrability.

In Chapter 3, we introduce the Newell system and reproduce its original derivation, which, contrary to the Yajima-Oikawa case, does not rely on physical principles but is rather inspired by equations of physical relevance. We then provide a review of results and solutions relating to its integrability.

In Chapter 4, we combine the Yajima-Oikawa system and the Newell system into a more general integrable system, which we call the Yajima-Oikawa-Newell system, by combining their Lax pairs into a single one. Then, we employ an Ansatz approach to derive periodic solutions (expressed as elliptic functions) and traveling waves (both solitary waves and rational solutions) for the system, and derive a few of its symmetries and conservation laws.

In Chapter 5, we present the theory for our algebraic method to study the stability of solutions of integrable systems, and apply it to study the stability of the plane waves of the Yajima-Oikawa-Newell system, which turn out to be unstable for almost every choice of parameters. Some conjectures are also proposed relating the geometric or topological properties of the stability spectra to the existence of various kinds of solutions. Part of the proofs for this chapter are presented separately in the Appendix A due to their length.

In Chapter 6, we give an overview of the theory for the Hirota bilinear method, and in particular we introduce the method of τ-functions, which we then apply to obtain several special solutions for the Yajima-Oikawa-Newell system, including bright and dark solitons, breathers, and rational solutions.

Then in Chapter 7 we provide a review of the different results presented throughout the thesis and how they related with our current lines of research.

Chapter 2

Yajima-Oikawa Long Wave-Short Wave System

The first model of long wave-short wave interaction that we are going to treat is a system proposed by Yajima and Oikawa in 1976 [159] in the context of sonic-Langmuir waves, that is, waves in plasma that interact with acoustic-type waves [99].

We will write the Yajima-Oikawa system as

$$\begin{aligned} iS_t + S_{xx} - LS &= 0\,, \\ L_t &= 2(|S|^2)_x\,, \end{aligned} \tag{2.1}$$

where S is a complex variable representing the short wave, L is real and represents the amplitude of the long wave, and the subindex x (respectively t) denotes partial derivation with respect to the variable x (respectively t).

For the sake of completeness, let us briefly reproduce the original derivation of the system in [159].

2.1 Physical derivation

The initial point of the computation are the equations for Langmuir waves introduced by Zakharov in [162], that is, equations for an ion sound wave under the action of a high-frequency field:

$$\begin{aligned} iE_t + \frac{1}{2}E_{xx} - nE &= 0\,, \\ n_{tt} - n_{xx} - 2(|E|^2)_{xx} &= 0\,, \end{aligned} \tag{2.2}$$

where $E(x,t)$ is the normalised complex amplitude of the electric field of the Langmuir oscillation, $n(x,t)$ is the normalised density perturbation due to the acoustic wave, and the space and time variables x and t are also properly normalised to absorb additional constants in the equation.

The key assumption for deriving the original form of the Yajima-Oikawa system is to consider sound waves propagating in only one direction, e.g. in the positive x-direction, and with constant sound speed, so that we can approximate

$$n_t \cong -n_x\,, \tag{2.3}$$

where the sound speed has been normalised to 1. From the approximation (2.3) we obtain the relation

$$n_{tt} - n_{xx} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)(n_t + n_x) \cong -2\frac{\partial}{\partial x}(n_t + n_x) \tag{2.4}$$

which we can introduce into the second equation in (2.2) to rewrite it as

$$n_t + n_x + (|E|^2)_x = 0\,. \tag{2.5}$$

In later papers, the n_x term in (2.5) was dropped, thus leading to the form of the system we introduced in (2.1).

Just a year after its introduction by Yajima and Oikawa in the context of sonic-Langmuir interaction [159], the same system was introduced again independently in the context of capillary-gravity waves by Djordjevic and Redekopp [58].

In that case, the starting point is Laplace's equation for an irrotational fluid,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \,, \quad -h < z < \zeta \,, \tag{2.6}$$

subject to boundary conditions

$$\begin{aligned} \phi_z = 0 \quad &\text{at} \quad z = -h \,, \\ \phi_z = \zeta_t + \phi_x \zeta_x + \phi_y \zeta_y \quad &\text{at} \quad z = \zeta \,, \end{aligned} \tag{2.7}$$

where $\phi(x, y, z, t)$ denotes the velocity potential, h is the (constant) depth of the liquid, and $\zeta(x, y, t)$ is the position of the free surface of the fluid.

A multiscale approach (as explained in Section 1.1) was applied to this system, by assuming an initial condition for ζ of the form

$$\zeta(x, y, t = 0) = \varepsilon \frac{i\omega}{g + k^2 T} (A e^{ikx} - A^* e^{-ikx}) \,, \tag{2.8}$$

where, physically, g is the gravitational acceleration, T is the ratio between the surface tension coefficient and the fluid density, and $\varepsilon \ll 1$ is a small (assuming weak nonlinearity), non-dimensional parameter measuring the slope of the wave surface (hence giving the scale of the wave envelope). The envelope A depends only on slow variables, and the frequency ω depends on the wave number k through the dispersion relation

$$\omega = \sqrt{gk \left(1 + \frac{k^2 T}{g}\right) \tanh(kh)} \,, \tag{2.9}$$

obtained by substituting the Ansatz (2.8) back into the equation.

Choosing the right scale for the parameters

$$\xi = \varepsilon^{\frac{2}{3}} (x - c_g t) \,, \qquad \tau = \varepsilon^{\frac{4}{3}} t \,,$$

where c_g is the group velocity,

$$c_g = \partial\omega/\partial k\,,$$

and reducing the study to one-dimensional waves allows us to discern for ϕ the behaviour of the free wave at the leading order with a long wave at $O(\varepsilon^{\frac{2}{3}})$, that we denote by $B(\xi,\tau)$, and the short-wave contribution of the first harmonic at $O(\varepsilon)$, which is governed by the envelope $A(\xi,\tau)$.

Further analysis (see [58]) shows that the evolution equations for the system are singular in the resonant case $c_g^2 = gh$, and the consequence is that, in order to avoid secular terms in the multiscale expansion, the long wave evolves forced by the self-interaction of the short wave, while the short wave is modulated by the long one. The equations for this dependence happen to be

$$\begin{aligned} iA_\tau + \rho A_{\xi\xi} &= BA\,,\\ B_\tau &= -\alpha(|A|^2)_\xi\,, \end{aligned} \tag{2.10}$$

where ρ is a dispersion coefficient

$$\rho = \frac{1}{2}\omega''(k)\,, \tag{2.11}$$

and α is defined as

$$\alpha = \frac{1}{2}k^3\Big(1-\tanh^2(kh)\Big)\left[1+\frac{c_g k}{2\omega}\Big(1-\tanh^2(kh)\Big)\left(1+\frac{k^2T}{g}\right)\right]\,. \tag{2.12}$$

System (2.10) coincides with system (2.1) introduced before up to rescaling of the parameters.

This last computation already shows us that Yajima-Oikawa can appear as a result of a multiscale approach of a different system. Indeed, this was not by chance, and Yajima-Oikawa is a system that one can call general in the sense that it is the byproduct of multiscale analysis with a rather general resonance condition, as explained in [59], among others. Let us reproduce their computation.

Let us consider a system with a dispersion relation with two branches. A good example of this would be systems with an upper branch (e. g. of optical nature) and a lower branch (e. g. of acoustic nature), which already reminds us of the capillary-gravity interaction.

We can think of our problem as two waves with very close wave numbers $k + \varepsilon\kappa$ and $k - \varepsilon\kappa$, with $\varepsilon \ll 1$, that play the role of side-bands of the main wave k beating together to form a slow variation. We can write this as a solution

$$\phi = 4a\,\cos(kx - \omega t)\cos[\varepsilon(\kappa x - \Omega t)]\,, \tag{2.13}$$

where ϕ is the variable for the upper branch, k and ω are the wave number and frequency of the fast main wave and κ and Ω are the wave number and frequency of the slow modulation of the side-bands.

The fast component of the upper branch is too fast to affect the lower branch, but the slow component created by the beat of the side-bands can excite the lower branch. This in turn causes the two branches, which would usually be independent, to be actually coupled.

Let us apply multiscale analysis to this system. We can write the equation of motion coming from the upper branch as

$$L^{(u)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) = f(\phi, N)\,, \tag{2.14}$$

where $L^{(u)}$ is a scalar differential operator in $\partial/\partial x$ and $\partial/\partial t$, and N is the variable corresponding to the lower branch. To apply multiscale, we expand ϕ and N as

$$\begin{aligned} \phi &= \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \dots \\ N &= \varepsilon^2 n + \dots \end{aligned} \tag{2.15}$$

and define slow variables $X = \varepsilon x$, $T = \varepsilon t$ and $\tau = \varepsilon^2 t$. Expanding the equations of motion in a Taylor series, we can separate the equations for the different orders in ε. The first equation we get is

$$L^{(u)}\phi^{(1)} = 0\,, \tag{2.16}$$

so we can take $\phi^{(1)}$ as

$$\phi^{(1)} = A(X, T, \tau)e^{i\theta} + \text{c.c.}\,, \tag{2.17}$$

where c.c. denotes the complex conjugate of the part before it. The second equation we get is

$$L^{(u)}\phi^{(2)} = -\left(L_1^{(u)}\frac{\partial}{\partial T} + L_2^{(u)}\frac{\partial}{\partial X}\right)\phi^{(1)}\,, \tag{2.18}$$

where $L_1^{(u)}$ and $L_2^{(u)}$ denote the partial derivatives of $L^{(u)}$ with respect to the first and second positions, respectively. As usual when performing multiscale analysis, in order to remove secular terms, we need to move to the group velocity travelling frame, for which we define a new variable $\xi = \varepsilon(x - c_g t)$. By introducing this new variable into (2.18) (see Section 1.1 for more detail), we can rewrite it as

$$L^{(u)}\phi^{(2)} = -\frac{1}{2}\left(\beta\frac{\partial^2}{\partial\xi^2} + i\frac{\partial}{\partial\tau}\right)\phi^{(1)} + \gamma n\phi^{(1)}\,, \tag{2.19}$$

where β and γ are constants and the last term comes from quadratic terms coupling ϕ and N in (2.16), if they exist. Introducing the form for $\phi^{(1)}$ in (2.17) into (2.19), we get the relation

$$iA_\tau + \tilde{\beta}A_{\xi\xi} = \tilde{\gamma}An\,, \tag{2.20}$$

where $\tilde{\beta}$ and $\tilde{\gamma}$ are also constants.

Treating the equation of motion coming from the lower branch is more complicated. For that reason, we will study as an example a case that appears for many optic-acoustic systems, such as models in plasmas, where it behaves as a wave equation coupled to the upper branch through a nonlinear driving term, so that the corresponding differential operator for the lower branch takes the form

$$L^{(\ell)} = \frac{\partial^2}{\partial t^2} - c_p^2\frac{\partial^2}{\partial x^2}\,, \tag{2.21}$$

where c_p is the phase speed for the waves in the lower branch. Changing variables from x and t

to ξ and τ as introduced above, it becomes

$$L^{(\ell)} = \varepsilon^2 (c_g^2 - c_p^2) \frac{\partial^2}{\partial \xi^2} - 2\varepsilon^3 c_g \frac{\partial^2}{\partial \xi \partial \tau} + \varepsilon^4 \frac{\partial^2}{\partial \tau^2} \,. \tag{2.22}$$

Now, since N is of order ε^2, when we study the system at $O(\varepsilon^4)$, looking at the right-hand side of (2.22) the only terms that match the order are terms featuring $\partial^2 |\phi^{(1)}|^2 / \partial \xi^2$. The consequence of this is that n would be proportional to $|A|^2$ and hence the lower branch would just follow the motion of the upper branch and would not contribute towards the dynamics of the system.

However, we can remove this problem if we study the resonant condition $c_g = c_p$. With that condition, the only possible coupling between the left-hand side and the right-hand side of (2.22) is at order $O(\varepsilon^5)$. By doing some additional choice of coupling scale detailed in [59], the resulting equation from the computation at $O(\varepsilon^5)$ is

$$n_\tau = \alpha (|A|^2)_\xi \,, \tag{2.23}$$

where α is a constant. Note that (2.20) and (2.23) are again the Yajima-Oikawa equations as presented in (2.1) up to rescaling of the parameters.

Remarkably, the Yajima-Oikawa system can also be derived via multiscale analysis from other resonant conditions, for instance as a reduction of the Boussinesq equation [27, 28, 50], and it is of application in more branches of physics, such as water waves [77, 135].

2.2 Integrability properties and solutions

A very important property of the Yajima-Oikawa system that will be key for our purpose is that it is integrable (see Section 1.2). In their original paper [159], Yajima and Oikawa already showed that the system they had obtained as written in (2.2) and (2.5) is integrable by providing a Lax pair (in operator form), and applied the inverse scattering machinery [2, 78, 126] to obtain multi-soliton solutions.

The system written in the form (2.1) is also integrable. Ma found in 1978 its Lax pair in operator form [121] based on the results in [58], and applied the classical inverse scattering techniques to obtain soliton solutions (which can only go in one direction, as opposed to the ones proposed for the original equation by Yajima and Oikawa in [159]).

In [111], a Darboux transformation for the system based on Ma's Lax pair was introduced, and soliton solutions were also obtained by recursively applying the transformation upon the vaccuum solution. As for the ones introduced by Ma, these solitons can only move in one direction.

A Lax pair formulation in matrix form for the system is also known [33, 45]:

$$X(\lambda) = i\lambda\Sigma + Q\,, \quad T(\lambda) = (i\lambda)^2 T_2 + i\lambda T_1 + T_0\,, \tag{2.24}$$

where λ is the spectral parameter, Σ is the trace-less, diagonal matrix

$$\Sigma = \text{diag}\{1\,,\,0\,,\,-1\} \tag{2.25}$$

and the λ-independent matrices Q, A, B and C have the form

$$Q = \begin{pmatrix} 0 & S & iL \\ 0 & 0 & S^* \\ -i & 0 & 0 \end{pmatrix}, \qquad T_2 = \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.26a}$$

$$T_1 = \begin{pmatrix} 0 & iS & 0 \\ 0 & 0 & -iS^* \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_0 = \begin{pmatrix} 0 & iS_x & i|S|^2 \\ S^* & 0 & -iS_x^* \\ 0 & S & 0 \end{pmatrix}, \tag{2.26b}$$

where the asterisk denotes complex conjugation.

In [156], a Darboux-dressing approach was employed to obtain breather and rational solutions of the system via the introduction of an auxiliary version of the system featuring 4 complex potentials.

An explicit realisation of a rogue wave in the form of a rational solution for the system in the form

$$\begin{aligned} iS_t + \frac{1}{2}S_{xx} + LS = 0\,, \\ L_t = (|S|^2)_x\,, \end{aligned} \tag{2.27}$$

based on the results in [156], was presented in [13]:

$$\begin{aligned} S &= ae^{i(kx-\omega t)}\left[1 - \frac{it + \frac{ix}{2m-k} + \frac{1}{2(2m-k)(m-k)}}{(x-mt)^2 + n^2t^2 + \frac{1}{4n^2}}\right], \\ L &= b + 2\frac{n^2t^2 - (x-mt)^2 + \frac{1}{4n^2}}{\left[(x-mt)^2 + n^2t^2 + \frac{1}{4n^2}\right]^2}\,, \end{aligned} \tag{2.28}$$

where the real parameters a and b represent the amplitude of the background solution, the frequency ω satisfies the dispersion relation

$$\omega = \frac{k^2}{2} - b\,, \tag{2.29}$$

and the real parameters m and n are defined by

$$m = \frac{1}{6}\left[5k - \sqrt{3\left(k^2 + l + \frac{\nu}{l}\right)}\right], \tag{2.30a}$$

$$n = \pm\sqrt{(3m-k)(m-k)}\,, \tag{2.30b}$$

with $k \in \mathbb{R}$ and

$$\nu = \frac{1}{9}k^4 + 6ka^2\,, \tag{2.31a}$$

$$\begin{cases} l = -\left(\rho - \sqrt{\rho^2 - \nu^3}\right)^{1/3} & \text{for} \quad k \leqslant -3(2a^2)^{1/3}\,, \\ l = \left(-\rho + \sqrt{\rho^2 - \nu^3}\right)^{1/3} & \text{for} \quad -3(2a^2)^{1/3} < k \leqslant \frac{3}{2}(2a^2)^{1/3}\,, \end{cases} \tag{2.31b}$$

$$\rho = \frac{1}{2}k^6 - \frac{1}{54}(27a^2 + 5k^3)^2\,. \tag{2.31c}$$

System (2.27) can be transformed into the standard form (2.1) via different transformations, for instance

$$x \to -\frac{1}{\sqrt{2}}x\,, \tag{2.32a}$$

$$L \to -L\,, \tag{2.32b}$$

$$S \to 2^{1/4}\,S\,, \tag{2.32c}$$

or

$$t \to \sqrt[3]{2}\,t\,, \tag{2.33a}$$

$$x \to -\frac{1}{\sqrt[3]{2}}x\,, \tag{2.33b}$$

$$L \to -\frac{1}{\sqrt[3]{2}}L\,. \tag{2.33c}$$

In [35], Hirota bilinearisation techniques relying on reductions of the Kadomtsev-Petviashvili (KP) hierarchy (which will be detailed in Section 6.1) were applied in order to derive rogue wave solutions, which coincide with those introduced above in (2.28).

Hirota bilinearisation techniques were also employed in [45] to introduce rogue wave and breather solutions, but also so-called "double pole solutions", which can be interpreted as weakly bounded groups of solitons with very close wavenumbers.

The rogue waves introduced above were also employed in [33] to model internal behaviours in a stratified fluid under a resonant condition, which happened to follow Yajima-Oikawa-like equations.

In the recent paper [116], periodic-background solutions were obtained for the system by combining Darboux techniques with algebraic-geometric methods.

Auto-Bäcklund transformations for a still integrable generalised 3-variable version of the system were introduced in [157] and employed to construct periodic homoclinic connections of plane waves.

Finally, several integrable generalisations of the Yajima-Oikawa equations have been proposed throughout the years (most of them from a mathematical perspective, whereas their physical relevance is still to be investigated), including generalisations as vector systems [114], matrix systems [115], systems with additional or modified terms [79], systems with more than two equations and variables [156, 157] or systems in higher dimension [135, 62].

Chapter 3

Newell's Long Wave-Short Wave System

The other classical long wave-short wave system that we want to introduce is the one proposed by Newell in 1978 [128] as an integrable example of the family of long wave-short wave interaction systems introduced by Benney a year before [16]. We will refer to this system as the Newell system, and we will write it as

$$iS_t + S_{xx} + \left(iL_x + L^2 - 2\sigma|S|^2\right) S = 0\,, \tag{3.1a}$$

$$L_t = 2\sigma(|S|^2)_x\,, \qquad \sigma^2 = 1\,, \tag{3.1b}$$

where, as in Yajima-Oikawa, S is a complex variable representing the short wave and L is the real amplitude of the long wave. The extra parameter σ, which was not present in Yajima-Oikawa, is a sign splitting the equation into two different cases, that we can compare with the focusing and defocusing regimes of the nonlinear Schrödinger (NLS) equation. Note that in addition to the cross-interaction, different from the cross-interaction in Yajima-Oikawa, the system features a self-interaction term in the first equation, similar to the self-interaction in the NLS equation.

We will proceed as in the case before and introduce the system by reproducing its original computation, but in this case, contrary to Yajima-Oikawa, the derivation will be mainly mathematical as opposed to deriving it physically from first principles. Indeed, to the best of our knowledge, more than 40 years after its original derivation, the Newell system has never been derived from first principles in a physical context in fluid dynamics or nonlinear optics.

3.1 Mathematical derivation

Let us start reproducing the derivation of Benney's family of long-short interaction in [16].

Let us consider a triad of resonant waves whose wave numbers k_1, k_2 and k_3 and frequencies $\omega(k_1)$, $\omega(k_2)$ and $\omega(k_3)$ satisfy the identities

$$k_1 - k_2 = k_3 \,, \tag{3.2a}$$

$$\omega(k_1) - \omega(k_2) = \omega(k_3) \,. \tag{3.2b}$$

The first equation, (3.2a), is equivalent to

$$k_1 = k_s + \frac{1}{2}k_l \,, \qquad k_2 = k_s - \frac{1}{2}k_l \,, \qquad k_3 = k_l \,, \tag{3.3}$$

where k_s and k_l will represent the wave numbers of the short and long wave, respectively. Note that this setting is basically the same as the side-band modulations we introduced for the multiscale derivation of Yajima-Oikawa in Section 2.1. However, it is unclear whether there is a way to relate the multiscale approach and the derivation that follows and hence to give the Newell system a physical origin. This is indeed a relevant open question, on which we have plans to work in the future.

Now we can write (3.2b) as

$$\omega\left(k_s + \frac{1}{2}k_l\right) - \omega\left(k_s - \frac{1}{2}k_l\right) = \omega(k_l) \,. \tag{3.4}$$

We will now apply the long wave condition $k_l \ll k_s$, so that (3.4) gives us

$$k_l \cdot \nabla\omega(k_s) = \omega(k_l)\,, \tag{3.5}$$

With this setting in mind, Benney proposes a rather general partial differential equation of the form

$$u_t + \mathcal{L}(u) = \mathcal{N}(u)\,, \tag{3.6}$$

where $u(x,t)$ is a displacement taking into account the short and long waves, $\mathcal{L}$ is a linear operator involving only x-derivatives,

$$\mathcal{L} = \sum_n c_n \frac{\partial^n}{\partial x^n}\,, \tag{3.7}$$

and $\mathcal{N}$ is a nonlinear operator. In order to have a conservative system, we need $\mathcal{L}$ to only contain odd spatial derivatives. We also need $\mathcal{N}$ to satisfy that both $\mathcal{N}(u)$ and $u\mathcal{N}(u)$ be expressible in divergence form. Some simple forms for $\mathcal{N}$ that are acceptable are

$$\mathcal{N}(u) = uu_x\,, \tag{3.8a}$$

$$\mathcal{N}(u) = uu_{xxx} + 2u_x u_{xx}\,, \tag{3.8b}$$

$$\mathcal{N}(u) = u_x u_{xxxx} + 5u_{xx}u_{xxx}\,. \tag{3.8c}$$

For the case (3.8b), the breaking of large amplitude long waves is permitted, so that at larger amplitudes the long waves satisfy the Korteweg-de Vries (KdV) equation, while at small amplitudes they are essentially nondispersive. In the other two cases the breaking of the long waves is not permitted.

We want solutions of (3.6) made up of a long wave and a short one,

$$u(x,t) = \frac{\varepsilon_l}{\mu} u_l(X,T) + \varepsilon_s u_s(X,T)\,, \tag{3.9}$$

where u_l and u_s are long and short components, respectively, ε_l and ε_s are the slopes of the long wave and the short wave, μ is the ratio of the wave lengths, and X and T are slow variables.

The evolution equation coming from this computation will depend on the relative magnitudes of ε_l, ε_s and μ, so we will identify them with the triple (l, m, n), where

$$(\varepsilon_l, \varepsilon_s, \mu) = \left(\varepsilon^l, \varepsilon^m, \varepsilon^n\right), \qquad \varepsilon \ll 1\,, \tag{3.10}$$

with l, m and n natural numbers and ε a small constant.

In order to obtain the evolution equation, we will assume the short wave $u_s(X, T)$ to be periodic with constant wave numbers,

$$u_s(X,T) = \sum_n S_n(X,T) e^{in\theta}\,, \tag{3.11}$$

where $\theta = \theta(X, T)$ and, in order for u_s to be real, we set

$$S_{-n} = S_n^*\,. \tag{3.12}$$

The interactions are weak, so that the system will be governed by the first harmonic, which we will denote as $S = S_1(X, T)$, and the long wave $L = u_l(X, T)$.

With this, one can compute the evolution equations for different relative magnitudes of the parameters. For example, for the triple $(l, m, n) = (3, 1, 1)$, one obtains the equations

$$L_T + c_1 L_X = \alpha(|S|^2)_X\,, \tag{3.13a}$$

$$S_T + c_g S_X = \varepsilon\left(i\beta S_{XX} + i\gamma S^2 S^* + i\delta L S\right)\,, \tag{3.13b}$$

with $c_1, \alpha, \beta, \gamma$ and δ constants, and c_g the group velocity.

Now, the evolution equations that come from the different magnitudes are in general non-integrable. However, in [128] Newell found a system similar to the ones coming from Benney's computations that was indeed integrable.

He started with a system of the form

$$S_T - iK_2 S_{XX} = \left(-K_3 L_X + iK_4 L^2 - 2i\sigma K_5 |S|^2\right) S\,, \tag{3.14a}$$

$$L_T = 2\sigma K_1 (|S|^2)_X, \qquad \sigma = \pm 1\,, \tag{3.14b}$$

where K_1, K_2, K_3, K_4 and K_5 are constants, and, as before, $S(X,T)$ and $L(X,T)$ represent the amplitude of the long wave and the envelope of the short wave,

$$u(X,T) = u_l(X,T) + u_s(X,T)\,, \tag{3.15}$$

with

$$u_l(X,T) = L(X,T)\,, \qquad u_s(X,T) = S(X,T)e^{i\theta} + \text{c.c.} \tag{3.16}$$

where $\theta = kx - \omega t$.

Note that (3.14) is assuming that the triad interaction as explained at the beginning of the section is not direct, meaning that there are no $|L|^2$ or LS terms in (3.14b) (which, as for the non-resonant case in Yajima-Oikawa, would mean that the long wave follows the short wave simply by resonance).

The reasoning for the terms in the equation is that $|S|^2 S$ is a rather general self modal interaction while the term $L^2 S$ would describe a frequency adjustment of the short wave due to a mean current, and would arise from some $u^2 u_x$ term in (3.6).

Now, equations (3.14) are using the most general choice of variables. However, through scaling analysis one can normalise $K_1 = K_2 = K_5 = 1$ without loss of generality (as long as they are non-zero). It turns out in that case the system is integrable for $K_3 = K_4 = 1$ (in fact, for integrability it suffices to have the condition $K_1 K_3 = K_2 K_5$ in (3.14), however once applied that condition one can again rescale all the parameters to 1).

Also in [128], a gauge transformation was introduced providing a simplification of the system. Once applied the transformation

$$S = \tilde{S} e^{-i \int^X L(y,T)\, dy}\,, \tag{3.17}$$

equation (3.14a) becomes

$$\tilde{S}_T - 2L\tilde{S}_X = i\tilde{S}_{XX}\,. \tag{3.18}$$

The form (3.18) of the Newell system is still integrable.

3.2 Integrability properties and solutions

Newell himself proved in [128] that the system as presented in (3.1) is integrable by performing a classical inverse scattering analysis (see [2, 101, 129]). Through this analysis, he managed to obtain soliton solutions of the form

$$L = \frac{4\eta^2\sigma}{\xi D(\varphi)}\,, \tag{3.19a}$$

$$S = \frac{2\sigma\eta}{\sqrt{D(\varphi)}} e^{-i\chi(\varphi) - i\xi(x_0+y_0) - i\xi\varphi + i(\xi^2+\eta^2)T}\,, \tag{3.19b}$$

where ξ and η are, respectively, the real and imaginary part of the spectral parameter, $\lambda = \xi + i\eta$, x_0 and y_0 are arbitrary constants, and

$$\varphi = X - x_0 + 2\xi T\,, \tag{3.20a}$$

$$D(\varphi) = 2\left(1 + \frac{\eta^2}{\xi^2}\right)^{\frac{1}{2}} \left[\cosh(2\eta\varphi) - \frac{\sigma}{\left(1 + \frac{\eta^2}{\xi^2}\right)^{\frac{1}{2}}}\right]\,, \tag{3.20b}$$

$$\sin\big(\chi(\varphi)\big) = -\sigma\,\frac{\eta}{\xi}\left(1 + \frac{\eta^2}{\xi^2}\right)^{-\frac{1}{4}} \frac{e^{-\eta\varphi}}{\sqrt{D(\varphi)}}\,, \tag{3.20c}$$

$$\cos\big(\chi(\varphi)\big) = \frac{\left(1+\dfrac{\eta^2}{\xi^2}\right)^{\frac{1}{4}} e^{\eta\varphi} - \sigma\left(1+\dfrac{\eta^2}{\xi^2}\right)^{-\frac{1}{4}} e^{-\eta\varphi}}{\sqrt{D(\varphi)}}\,. \tag{3.20d}$$

Newell's system can also be presented through a matrix Lax pair formulation (which, to the best of our knowledge, was not introduced in the literature before we presented it as a particular choice in the Lax pair for the more general YON system in [30]) in the form

$$X(\lambda) = i\lambda\Sigma + Q\,, \quad T(\lambda) = (i\lambda)^2 T_2 + i\lambda T_1 + T_0\,, \tag{3.21}$$

where

$$\Sigma = \mathrm{diag}\{1\,,\,0\,,\,-1\}\,, \tag{3.22}$$

and

$$Q = \begin{pmatrix} 0 & S & i\sigma L \\ \sigma S^* & 0 & S^* \\ i\sigma L & \sigma S & 0 \end{pmatrix}, \quad T_2 = \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & iS & 0 \\ i\sigma S^* & 0 & -iS^* \\ 0 & -i\sigma S & 0 \end{pmatrix}, \tag{3.23a}$$

$$T_0 = \begin{pmatrix} -i\sigma|S|^2 & -LS + iS_x & i|S|^2 \\ -\sigma LS^* - i\sigma S_x^* & 2i\sigma|S|^2 & -LS^* - iS_x^* \\ i|S|^2 & -\sigma LS + i\sigma S_x & -i\sigma|S|^2 \end{pmatrix}. \tag{3.23b}$$

A Hirota bilinearisation approach (explained in Section 6.1) was applied to Newell's system in [34] to study the rogue waves of the system in the form (3.18). In the same paper, the modulational stability of the plane waves of the system was also studied, resulting in a full coincidence of the conditions for baseband modulational instability and for the existence of rogue waves (as also predicted in [13], see Chapter 5).

Chapter 4

Yajima-Oikawa-Newell: A More General System

Now that we have introduced the Yajima-Oikawa and Newell systems, their derivation and some properties, let us delve into the actual system we are going to study: the Yajima-Oikawa-Newell (YON) system, which unifies and generalises the two aforementioned systems, and which we introduced in [30].

One of the big advantages of using this new system is that one does not need to study the Yajima-Oikawa and Newell systems as separate systems. Instead, one can just obtain properties of the YON system, which then translate immediately as properties of the other two. Furthermore, in addition to having Yajima-Oikawa and Newell as subcases, YON system features an infinite family of integrable systems depending on two non-rescalable parameters, α and β, thus having the potential to better model physical phenomena while remaining integrable. Let us follow its derivation, as in [30].

4.1 Mathematical derivation and integrability

Let us recall, for handiness, a couple of formulae for Yajima-Oikawa and Newell systems. The form we take for Yajima-Oikawa is the system of PDEs

$$\begin{aligned} iS_t + S_{xx} - LS = 0\,, \\ L_t = 2(|S|^2)_x\,, \end{aligned} \tag{4.1}$$

while for Newell we use the form

$$iS_t + S_{xx} + \left(iL_x + L^2 - 2\sigma|S|^2\right) S = 0\,, \tag{4.2a}$$

$$L_t = 2\sigma(|S|^2)_x\,, \qquad \sigma^2 = 1\,, \tag{4.2b}$$

where the absolute value of S represents the amplitude of the short wave, L is the amplitude of the long wave, and σ is a sign.

Both systems feature a Lax pair of the form

$$X(\lambda) = i\lambda\Sigma + Q\,, \qquad T(\lambda) = (i\lambda)^2 T_2 + i\lambda T_1 + T_0\,, \tag{4.3}$$

where X and T are two complex 3-by-3 matrix-valued functions, $\lambda \in \mathbb{C}$ is the so-called spectral variable, Σ is the constant, traceless, diagonal matrix

$$\Sigma = \text{diag}\{1\,, 0\,, -1\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.4}$$

and $Q(x,t)$ is λ-independent and off-diagonal, that is, $Q_{jj} = 0$ for $j = 1, 2, 3$, and Q_{jk} with $j \neq k$ are complex-valued functions of x and t. T_2 is also a constant matrix, while T_1 and T_0 are also λ-independent and can be written as functions of Σ and Q.

In particular, as stated in [53] and explained later Section 5.1, for Lax pairs of the form (4.3), the only form allowed for T_2, T_1 and T_0 that avoids non-localities is

$$T_2 = -C_2\,, \tag{4.5a}$$

$$T_1 = -iC_1 - D_2(Q)\,, \tag{4.5b}$$

$$T_0 = C_0 - \frac{1}{2}[D_2(Q), \Gamma(Q)]^{(d)} - \Gamma\big(D_2(Q_x)\big) - \Gamma\Big([D_2(Q), Q]^{(o)}\Big) - iD_1(Q)\,, \tag{4.5c}$$

where the matrices C_j with $j = 0, 1, 2$ are constant and diagonal, the superindices (d) and (o) denote respectively the diagonal and off-diagonal part of the matrix, the linear invertible map Γ acts on off-diagonal matrices as

$$\big(\Gamma(\mathcal{M})\big)_{jk} = \frac{\mathcal{M}_{jk}}{s_j - s_k}\,, \tag{4.6}$$

where $\mathcal{M}$ is off-diagonal and s_j with $j = 1, 2, 3$ denotes the diagonal entry Σ_{jj}, so that

$$[\Sigma, \Gamma(\mathcal{M})] = \Gamma\big([\Sigma, \mathcal{M}]\big) = \mathcal{M}\,, \tag{4.7}$$

and the maps D_j with $j = 1, 2$ also act only on off-diagonal matrices in the following manner

$$D_j(\mathcal{M}) = [C_j, \Gamma(\mathcal{M})] = \Gamma\big([C_j, \mathcal{M}]\big)\,. \tag{4.8}$$

The proof for these formulae is provided in the Appendix A.

For Yajima-Oikawa, as presented in Section 2.2, the matrices above have the form

$$Q = \begin{pmatrix} 0 & S & iL \\ 0 & 0 & S^* \\ -i & 0 & 0 \end{pmatrix}, \qquad T_2 = \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.9a}$$

$$T_1 = \begin{pmatrix} 0 & iS & 0 \\ 0 & 0 & -iS^* \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_0 = \begin{pmatrix} 0 & iS_x & i|S|^2 \\ S^* & 0 & -iS_x^* \\ 0 & S & 0 \end{pmatrix}, \tag{4.9b}$$

while, for Newell, we derived the Lax pair as

$$Q = \begin{pmatrix} 0 & S & i\sigma L \\ \sigma S^* & 0 & S^* \\ i\sigma L & \sigma S & 0 \end{pmatrix}, \quad T_2 = \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & iS & 0 \\ i\sigma S^* & 0 & -iS^* \\ 0 & -i\sigma S & 0 \end{pmatrix}, \tag{4.10a}$$

$$T_0 = \begin{pmatrix} -i\sigma|S|^2 & -LS + iS_x & i|S|^2 \\ -\sigma LS^* - i\sigma S_x^* & 2i\sigma|S|^2 & -LS^* - iS_x^* \\ i|S|^2 & -\sigma LS + i\sigma S_x & -i\sigma|S|^2 \end{pmatrix}. \tag{4.10b}$$

As usual (see Section 1.2), the Lax pairs generate the original systems of PDEs through the compatibility condition

$$X_t - T_x + [X\,,\,T] = 0\,. \tag{4.11}$$

Now, we can try to unify the two systems by combining their Q matrices, given they have the same Σ. If we take

$$Q = \begin{pmatrix} 0 & S & iL \\ \alpha S^* & 0 & S^* \\ i\alpha^2 L - i\beta & \alpha S & 0 \end{pmatrix}, \tag{4.12}$$

where α and β are two arbitrary real constants, then using the relations (4.5) we can obtain the matrices

$$T_2 = \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & iS & 0 \\ i\alpha S^* & 0 & -iS^* \\ 0 & -i\alpha S & 0 \end{pmatrix}, \tag{4.13a}$$

$$T_0 = \begin{pmatrix} -i\alpha|S|^2 & -\alpha LS + iS_x & i|S|^2 \\ -\alpha^2 LS^* + \beta S^* - i\alpha S_x^* & 2i\alpha|S|^2 & -\alpha LS^* - iS_x^* \\ i\alpha^2|S|^2 & -\alpha^2 LS + \beta S + i\alpha S_x & -i\alpha|S|^2 \end{pmatrix}, \tag{4.13b}$$

where we have used the same choices for the matrices C_0, C_1 and C_2 as for the Lax pairs for Yajima-Oikawa and Newell. Then, through the compatibility condition (4.11), we get the new system

$$iS_t + S_{xx} + \left(i\alpha L_x + \alpha^2 L^2 - \beta L - 2\alpha|S|^2\right) S = 0\,, \tag{4.14a}$$

$$L_t = 2\left(|S|^2\right)_x\,, \tag{4.14b}$$

which we refer to as the Yajima-Oikawa-Newell (YON) system. It is integrable by construction, given that it comes from the compatibility condition of a Lax pair. It reduces to Yajima-Oikawa via the special choice of parameters $\alpha = 0$, $\beta = 1$, and to Newell via the choice $\alpha = \sigma$, $\beta = 0$ and the change of variable $L \to \sigma L$, where σ is the sign appearing in Newell's system, $\sigma = \pm 1$.

The YON system is invariant under the transformation

$$(x, t, S, L) \to \left(c^{-1}x,\, c^{-2}t,\, c\exp\left[i\frac{(c^2-1)\beta^2 t}{4\alpha^2}\right] S,\, cL - \frac{(c-1)\beta}{2\alpha^2}\right) \tag{4.15}$$

with $c \neq 0$ an arbitrary real parameter. For the Yajima-Oikawa limit $\alpha \to 0$, the limit of the transformation can be obtained by first applying the map $c \to \exp(-2\alpha^2 c)$, yielding

$$(x, t, S, L) \to \left(c^2 x,\, c^4 t,\, c^{-3} S,\, c^{-4} L\right)\,, \tag{4.16}$$

while for the Newell case $\beta = 0$ the transformation becomes

$$(x, t, S, L) \to \left(c^{-1}x,\, c^{-1}t,\, cS,\, cL\right)\,. \tag{4.17}$$

4.2 Miura transformation

The idea of combining the Yajima-Oikawa and Newell systems was fueled by the claim in the literature that both systems are related through a Miura transformation [119]. Similarly to the Gardner equation, which generalises the Korteweg-de Vries and modified Korteweg-de Vries systems [126], the existence of a Miura transformation is a good indicator of the potential existence of a more general integrable system encompassing them.

The claim is that the paper [117] provides a Miura transformation from the Yajima-Oikawa system into the Newell one. However, it is not claimed in the paper itself that the transformation provided is indeed a Miura transformation, but a Darboux transformation, and a preliminary analysis shows that there are some critical issues with assuming it is the former. The content of this section is based on an ongoing correspondence with Annalisa Calini [23].

Let us consider the Yajima-Oikawa system (4.1) with the Lax pair given by (4.9). We will try to relate it to the auxiliary system introduced in [117],

$$q_t = \frac{1}{2}(q_{xx} - uq)\,, \tag{4.18a}$$

$$r_t = \frac{1}{2}(-r_{xx} + ur)\,, \tag{4.18b}$$

$$u_t = (qr)_x\,, \tag{4.18c}$$

where q, r and u are complex variables. System (4.18) is also integrable, and it admits the Lax pair X_1, T_1, with

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ u+\lambda^2 & 0 & 1 \\ r & 0 & 0 \end{pmatrix}, \tag{4.19a}$$

$$T_1 = \begin{pmatrix} \frac{1}{3}\lambda^2 & 0 & q \\ qr & \frac{1}{3}\lambda^2 & q_x \\ -r_x & r & -\frac{2}{3}\lambda^2 \end{pmatrix}. \tag{4.19b}$$

It reduces to the Yajima-Oikawa system for

$$t = -2it'\,, \qquad q = S^*\,, \qquad r = iq^*\,, \qquad u = L\,, \tag{4.20}$$

where we have denoted the time variable of Yajima-Oikawa by t' (i. e. $S = S(x,t')$ and $L = L(x,t')$). The condition $r = iq^*$ is referred in [117] as the reality condition (since it allows for L to be real, and reduces the system to only two equations).

A gauge transformation is then introduced for (4.18),

$$\Psi = G\Phi\,, \tag{4.21}$$

where Φ is a solution of the Lax equations for the Lax pair (4.19), and G is the matrix

$$G = \begin{pmatrix} 1 & 0 & 0 \\ -\hat{u} & 1 & -\hat{q} \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.22}$$

Then the target system is defined by $\hat{\Psi} = \hat{X}\Psi$, where

$$\hat{X} = (G_x + GX_1)G^{-1}\,. \tag{4.23}$$

This gives us a relation between variables

$$\hat{u}_x + \hat{u} + \hat{q}\hat{r} = u\,, \tag{4.24a}$$

$$\hat{q}_x + \hat{q}\hat{u} = q\,, \tag{4.24b}$$

$$\hat{r} = r\,, \tag{4.24c}$$

which, according to [117], gives rise to the new system

$$\hat{q}_t = \frac{1}{2}(\hat{q}_{xx} + \hat{u}_x\hat{q} - \hat{u}^2\hat{q} - \hat{q}^2\hat{r})\,, \tag{4.25a}$$

$$\hat{r}_t = \frac{1}{2}(-\hat{r}_{xx} + \hat{u}_x\hat{r} + \hat{u}^2\hat{r} + \hat{q}\hat{r}^2)\,, \tag{4.25b}$$

$$\hat{u}_t = \frac{1}{2}(\hat{q}\hat{r})_x\,. \tag{4.25c}$$

One can readily show that, if $\hat{q}$, $\hat{r}$ and $\hat{u}$ satisfy (4.25), then the variables q, r and u given by (4.24) satisfy (4.18). However, it is unclear how to obtain the system (4.25) starting from (4.18) via (4.24) (which may mean that the transformation works from Newell to Yajima-Oikawa, and not the other way around as often claimed in the literature).

Furthermore, if one imposes the reality condition $\hat{r} = \hat{q}^*$ to (4.25), then the system reduces to the Newell system (4.2) with $\sigma = 1$. However, the reality condition to reduce (4.25) to the Newell system does not seem to entail the reality condition to reduce (4.18) to Yajima-Oikawa, which is a major issue to achieve a transformation between Yajima-Oikawa and Newell.

One can try and use a different approach to Miura transformations to construct one for our problem, for example [148]. However, when one does so, a similar problem occurs, and the condition for the reality of the long wave L in the Newell system seems to be incompatible with the reality of L in Yajima-Oikawa.

We are currently investigating whether one can overcome these issues and define a Miura transformation between the Yajima-Oikawa and Newell systems. At the time of the writing of the present thesis, this is still work in progress.

4.3 Periodic and travelling wave solutions

Being integrable, the YON system is fit for specific, elegant approaches in order to compute solutions, such as Hirota bilinearisation (which we will introduce later in the thesis) or inverse scattering machinery (introduced in Section 1.3).

However, some interesting solutions can also be obtained through a less sophisticated approach such as using an Ansatz. Let us recall our computations in [29] to obtain those solutions.

Let S and L have a travelling wave form, mimicking those of the NLS equation,

$$S(t,x) = s(z)e^{i(\phi(z)-\omega t)}, \qquad L(t,x) = \ell(z), \tag{4.26}$$

where $z = x - Vt$ with $V \neq 0$ and ω real constants which we will call the velocity and the frequency, respectively, and s, ϕ and ℓ real valued functions of z.

Introducing this Ansatz into (4.14b) we get the equation

$$-4s(z)s'(z) - V\ell(z) = 0\,, \tag{4.27}$$

where the apostrophe denotes derivation with respect to z. We can now integrate (4.27) with respect to z to obtain a formula for ℓ in terms of s:

$$\ell(z) = -\frac{2}{V}s(z)^2 + c_1, \tag{4.28}$$

where c_1 is an arbitrary integration constant. We can introduce the expression (4.28) into (4.14a) and separate it into real and imaginary parts to obtain the system of equations

$$s(z)\phi''(z) + 2s'(z)\phi'(z) - \left[V + \frac{4\alpha}{V}s(z)^2\right]s'(z) = 0\,, \tag{4.29a}$$

$$\begin{aligned} s''(z) - s(z)\phi'(z)^2 + V\,s(z)\phi'(z) + (\alpha^2 c_1^2 - \beta c_1 + \omega)\,s(z) \\ + \left(\frac{2\beta}{V} - \frac{4\alpha^2 c_1}{V} - 2\alpha\right)s(z)^3 + \frac{4\alpha^2}{V^2}s(z)^5 = 0\,. \end{aligned} \tag{4.29b}$$

for s and ϕ. Let us multiply (4.29a) by ϕ and integrate with respect to z to get a formula for the first derivative of ϕ as a function of s:

$$\phi'(z) = \frac{\alpha}{V}\,s(z)^2 + \frac{V}{2} + c_2\,s(z)^{-2}\,, \tag{4.30}$$

where c_2 is an integration constant. Now, we can use (4.30) to get rid of ϕ in (4.29b) and transform it into an ODE for s:

$$\begin{aligned} s''(z) + \frac{1}{4V}\left[V^3 + 4V\left(\alpha^2 c_1^2 - \beta c_1 + \omega\right) - 8\alpha c_2\right]\,s(z) \\ + \frac{2}{V}(\beta - \alpha V - 2\alpha^2 c_1)\,s(z)^3 + \frac{3\alpha^2}{V^2}\,s(z)^5 - c_2^2\,s(z)^{-3} = 0\,, \end{aligned} \tag{4.31}$$

which, again, we can integrate after multiplication by s', giving us a differential equation for $s'(z)^2$:

$$\begin{aligned} s'(z)^2 = -\frac{1}{4V}\left[V^3 + 4V\left(\alpha^2 c_1^2 - \beta c_1 + \omega\right) - 8\alpha c_2\right]s(z)^2 \\ + \frac{1}{V}(-\beta + \alpha V + 2\alpha^2 c_1)s(z)^4 - \frac{\alpha^2}{V^2}s(z)^6 + 2c_3 - c_2^2\,s(z)^{-2}\,, \end{aligned} \tag{4.32}$$

where c_3 is an integration constant. Note that for the Yajima-Oikawa case $\alpha = 0$, the coefficient of $s(z)^6$ becomes zero, making the equation lower order and leading to the Weierstrass elliptic function (as opposed to the Jacobi elliptic functions we will obtain in general). We will only consider the case $\alpha \neq 0$, leaving Yajima-Oikawa to the broad literature on the subject (see Section 2.2).

Let us then introduce the change of variable

$$\frac{V}{2\alpha}u(z) = s(z)^2 - u_0\,, \qquad u_0 = \frac{V(\alpha\,V + 2K_1\alpha^2 - \beta)}{4\alpha^2}\,, \qquad \alpha \neq 0\,, \tag{4.33}$$

so that equation (4.32) transforms into

$$u'(z)^2 = -u(z)^4 + \mu_2 u(z)^2 + \mu_1 u(z) + \mu_0 \tag{4.34}$$

where

$$\begin{aligned}\mu_0 = &-\frac{V}{2\alpha^3}c^3 + \frac{4\alpha^2\omega - 8\alpha^2\,V^2 - \beta^2}{4\alpha^4}c^2 + \frac{\alpha^2\mu_1 + 2\alpha\,V^2(\mu_2 + 6\omega) - \alpha^2\,V^3 - 3\beta^2\,V}{2\alpha^3}c \\ &+ \frac{1}{4}\left\{2\mu_1\,V + V^2\left(4\omega - \frac{\beta^2}{\alpha^2}\right) - \frac{\left[\beta^2 - \alpha^2(\mu_2 + 4\omega)\right]^2}{\alpha^4}\right\}\end{aligned} \tag{4.35}$$

and c, μ_1 and μ_2 are arbitrary real constants. Note that the number of arbitrary constants has not changed, as the old set of integration constants c_1, c_2 and c_3 can be expressed in terms of the new constants c, μ_1 and μ_2 as

$$c_1 = \frac{c + \beta}{2\alpha^2}, \tag{4.36a}$$

$$c_2 = -\frac{V\left\{c^2 + 6c\alpha\,V + 2\beta^2 + \alpha^2\left[V^2 - 2(\mu_2 + 4\omega)\right]\right\}}{16\alpha^3}\,. \tag{4.36b}$$

$$c_3 = \frac{V^3\left[-2\alpha^3\mu_1 + 2\alpha^2\mu_2(c + \alpha\,V) - (c + \alpha\,V)^3\right]}{32\alpha^4}\,. \tag{4.36c}$$

As for ϕ, the quadrature (4.30) becomes

$$\phi'(z) = \frac{\alpha\left[\alpha\,V^2 - c\,V + \alpha(\mu_2 + 4\omega)\right] - \beta^2 + 2\alpha u(z)[c + 2\alpha\,V + \alpha u(z)]}{2\alpha[c + \alpha\,V + 2\alpha u(z)]}\,. \tag{4.37}$$

In the case $\mu_1 = 0$, equation (4.34) admits periodic solutions in the form of Jacobi elliptic functions. Let us compute them.

4.3.1 Jacobi elliptic sine solution

Let us assume $u(z)$ has the form

$$u(z) = \gamma_0 + \gamma_1 \operatorname{sn}\Big(a(z - z_0), m\Big)\,, \tag{4.38}$$

where $\operatorname{sn}(z)$ denotes the Jacobi elliptic sine of z, and γ_0, γ_1, a, z_0 and m are arbitrary real parameters with the constraint $0 \leqslant m \leqslant 1$.

Now, if we introduce the solution (4.38) into equation (4.34) and set $\mu_1 = 0$, after some manipulation we get the following polynomial in $\operatorname{sn}\Big(a(z - z_0), m\Big)$ equated to zero:

$$\begin{aligned} -\gamma_1^2(m^2a^2 + \gamma_1^2)\operatorname{sn}^4\Big(a(z-z_0),m\Big) &+ \gamma_1^2\big((1+m^2)a^2 + \mu_2\big)\operatorname{sn}^2\Big(a(z-z_0),m\Big) \\ &+ \gamma_1\mu_1 \operatorname{sn}\Big(a(z-z_0),m\Big) - \frac{1}{4}\Big\{2cV^3\alpha^3 + 4a^2\alpha^4\gamma_1^2 - 2c\alpha^3\mu_1 \\ &+ c^2(\beta^2 - 4\alpha^2\omega) + V^2\alpha^2(8c^2 + \beta^2 - 4\alpha^2\omega) + \big[\beta^2 - \alpha^2(\mu_2 + 4\omega)\big]^2 \\ &+ 2V\alpha\big[c^3 + 3c\beta^2 - \alpha^3\mu_1 - 2c\alpha^2(\mu_2 + 6\omega)\big]\Big\} = 0\,, \end{aligned} \tag{4.39}$$

where for simplicity we have already set $\gamma_0 = 0$ since it results from the system of equations explained in the paragraph below.

Since this equality must hold for every value of z, the only possibility is that all the coefficients are identically zero. By doing so, we obtain a set of algebraic equations for the parameters ω, m, a, γ_0, γ_1, μ_2, and c. In particular, from the independent term of (4.39) one can get an expression for ω in terms of the rest of parameters. However, this expression turns out to always be non-real for any choice of parameter, which contradicts our initial assumption that ω is a real constant. Hence, we can conclude that no solution of the form (4.38) exists starting from an Ansatz of the form (4.26).

4.3.2 Jacobi elliptic cosine solution

Let us now assume a solution with the elliptic cosine $\mathrm{cn}(z)$ replacing $\mathrm{sn}(z)$ in (4.38). Then, going through the same computation as (4.39) and solving the system of equations for the parameters, we obtain that the solution has the form

$$u(z) = ma\,\mathrm{cn}\Big(a(z-z_0), m\Big)\,, \tag{4.40}$$

and that

$$\mu_2 = (2m^2-1)a^2\,, \qquad c = b - \alpha V\,, \tag{4.41a}$$

and

$$\begin{aligned}\omega = \frac{1}{8\alpha^2}\Big\{&2\beta^2 + 2\alpha^2\left[a^2(1-2m^2) - 2V^2\right] + 4\alpha V b + b^2 \\ &\pm\sqrt{(b-2am\alpha)\,(b+2am\alpha)\,\left[b^2+4a^2(1-m^2)\alpha^2\right]}\Big\}\,,\end{aligned} \tag{4.41b}$$

where m, z_0, $a \neq 0$ and b are real parameters with $0 \leqslant m \leqslant 1$. Introducing the solution (4.40) back into the Ansatz (4.26) via the change of variable (4.33), we get the following solution for the YON system:

$$S(x,t) = \frac{1}{2}e^{i(\phi(z-\omega t)}\sqrt{\frac{V\left[b + 2m\alpha a\,\mathrm{cn}\Big(a(z-z_0),m\Big)\right]}{\alpha^2}}\,, \tag{4.42a}$$

$$L(x,t) = \frac{\beta - \alpha V - 2m\alpha a\,\mathrm{cn}\Big(a(z-z_0),m\Big)}{2\alpha^2}\,, \qquad z = x - Vt\,, \tag{4.42b}$$

where $\phi(z)$ satisfies the quadrature

$$\phi'(z) = \frac{1}{4\alpha}\Bigg\{ b + 2\alpha\left[V + ma\,\mathrm{cn}\Big(a(z-z_0), m\Big)\right] \\ \pm \frac{\sqrt{(b-2am\alpha)\,(b+2am\alpha)\,\left[b^2+4a^2(1-m^2)\alpha^2\right]}}{2m\alpha a\,\mathrm{cn}\Big(a(z-z_0), m\Big)+b}\Bigg\}, \tag{4.42c}$$

where the sign in front of the square root must be the same sign chosen for ω.

Let us note that, in addition to the arbitrary coupling parameters α and β coming from the YON system (4.14), the solution (4.42) features five additional real parameters –namely a, b, m, V, and z_0,– with a sixth one coming from the integration of (4.42c).

We now need to do one last check. In order for our computations to be true, we need our initial assumption that s, ϕ and ω are real to be true. However, our formulae feature square roots that, depending on the values of the parameters, may become imaginary. Hence, we will need to check the sign inside each of the square roots and impose constraints on the parameters to make sure they are all positive. These constraints turn out to be

$$\left|\frac{b}{\alpha a}\right| \geqslant 2m\,, \qquad Vb \geqslant 0\,. \tag{4.43}$$

Additionally, in the special case $m = 1$, the value $b = 0$ is allowed as long as $\alpha V a > 0$. We will treat that special case in Section 4.3.4.

Let us now obtain a few properties of the cnoidal solution. Its short wave $|S|$ oscillates between the values

$$\frac{1}{2}\sqrt{\frac{V(b+2m\alpha a)}{\alpha^2}} \qquad \text{and} \qquad \frac{1}{2}\sqrt{\frac{V(b-2m\alpha a)}{\alpha^2}}\,, \tag{4.44}$$

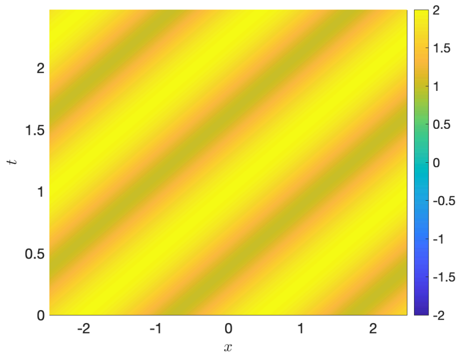
that is to say, $|S|^2$ oscillates with an amplitude $|Vma/\alpha|$, whereas the long wave L oscillates between the values

$$\frac{\beta-\alpha\,V-2m\alpha a}{2\alpha^2} \qquad \text{and} \qquad \frac{\beta-\alpha\,V+2m\alpha a}{2\alpha^2}\,, \tag{4.45}$$

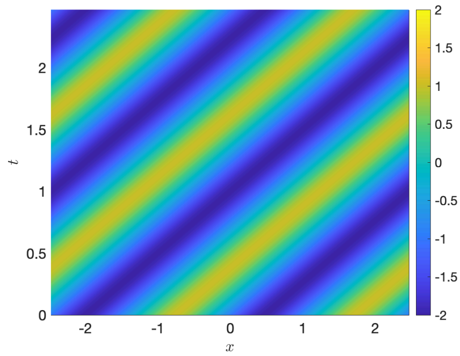
that is to say, with an amplitude $|2ma/\alpha|$. Both S and L are periodic in x with period $|4K(m)/a|$

and in t with period $|4K(m)/(aV)|$, where $K(m)$ is the complete elliptic integral of the first kind of m,

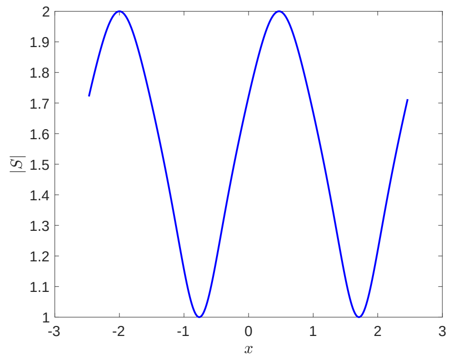
$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} \,. \tag{4.46}$$



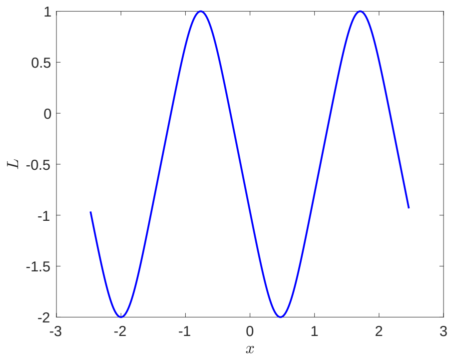
(a) Short wave $|S|$.



(b) Long wave L.



(c) Short wave profile at $t = 0$.



(d) Long wave profile at $t = 0$.

Figure 1: Elliptic cosine solution with $\alpha = 1$, $\beta = 2$, $V = 1.2$, $b = 5$, $a = 1.3$, $z_0 = 0$, $m = 0.5$.

Finally, let us note that two special solutions appear when taking the particular choices $m = 0$ and $m = 1$. When $m = 0$, the elliptic cosine reduces to the trigonometric cosine, so the solution becomes a plane wave. For the choice $m = 1$, the elliptic cosine reduces to the hyperbolic secant, hence leading to a localised solution, which we will cover in Section 4.3.4.

4.3.3 Jacobi delta amplitude solution

We will now proceed as above but using the Jacobi delta amplitude $\mathrm{dn}(z)$ instead $\mathrm{sn}(z)$ or $\mathrm{cn}(z)$ in (4.38), thus obtaining the following solution to (4.34):

$$u(z) = \frac{a}{m}\,\mathrm{dn}\left(a(z-z_0), m\right), \tag{4.47}$$

with

$$\mu_2 = \left(\frac{2}{m^2}-1\right)a^2\,, \quad c = b - \alpha V\,, \tag{4.48a}$$

and

$$\omega = \frac{1}{8m^2\alpha^2}\Bigg\{2\alpha^2\big[-2V^2m^2+(m^2-2)a^2\big]+4Vm^2\alpha b+m^2(2\beta^2+b^2) \\ \pm\sqrt{(mb-2a\alpha)\,(mb+2a\alpha)\,[m^2b^2-4a^2(1-m^2)\alpha^2]}\Bigg\}\,, \tag{4.48b}$$

where, as in the previous case, m, z_0, $a \neq 0$, and b are real parameters with $0 \leqslant m \leqslant 1$.

Again, we can undo the change of variable (4.33) to get a solution of the original system,

$$S(x,t) = \frac{1}{2}e^{i(\phi(z)-\omega t)}\sqrt{\frac{V\left[mb+2\alpha a\,\mathrm{dn}\left(a(z-z_0),m\right)\right]}{m\alpha^2}}\,, \tag{4.49a}$$

$$L(x,t) = \frac{m(\beta-\alpha V)-2\alpha a\,\mathrm{dn}\left(a(z-z_0),m\right)}{2m\alpha^2}\,, \qquad z = x - Vt \tag{4.49b}$$

where ϕ satisfies the quadrature

$$\phi'(z) = \frac{1}{4m\alpha}\Bigg[2Vm\alpha+mb+2\alpha a\,\mathrm{dn}\left(a(z-z_0),m\right) \\ \pm\frac{\sqrt{(mb-2a\alpha)\,(mb+2a\alpha)\,[m^2b^2-4a^2(1-m^2)\alpha^2]}}{2\alpha a\,\mathrm{dn}\left(a(z-z_0),m\right)+mb}\Bigg]\,, \tag{4.49c}$$

where the sign in front of the square root must coincide with the sign chosen for ω. Similarly to the cnoidal solutions, the dnoidal solutions feature five arbitrary real parameters –namely a, b,

m, V, and z_0,– with a sixth one coming from the integration of (4.49c).

As with the cnoidal solutions, we need to check the sign inside the square roots involved in the formulae above to ensure that all the variables are real. In this case, the constraints on the parameters turn out to be

$$2\sqrt{\frac{1-m}{m^2}} \leqslant \frac{b}{\alpha a} \leqslant 2\sqrt{\frac{1-m^2}{m^2}}\,, \qquad \alpha V a > 0\,, \qquad 0 < m \leqslant 1\,. \tag{4.50}$$

Additionally, it admits the special values $\frac{b}{\alpha a} = -2\sqrt{\frac{1-m}{m^2}}$ and $\frac{b}{\alpha a} = \frac{2}{m}$, for $0 < m \leqslant 1$. We will treat the special case $m = 1$ in Section 4.3.4.

Let us also introduce some properties of the solution. It is periodic in x for both L and $|S|$ with period $|2K(m)/a|$, and in t with period $|2K(m)/(aV)|$, where $K(m)$ is the complete elliptic integral of the first kind of m as written in (4.46). The phase of S is also periodic, with period $|4K(m)/a|$.

As for the amplitudes, the short wave $|S|$ oscillates between the values

$$\frac{1}{2}\sqrt{\frac{V[2\alpha a(1-m)^{1/2}+mb]}{m\alpha^2}} \qquad \text{and} \qquad \frac{1}{2}\sqrt{\frac{V(2\alpha a+mb)}{m\alpha^2}}\,, \tag{4.51}$$

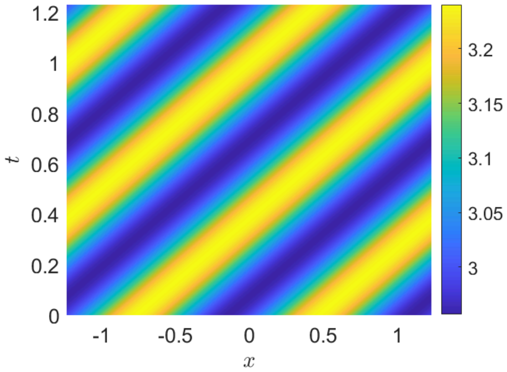
that is to say, the oscillations in $|S|^2$ have an amplitude $\left|Va(1-\sqrt{1-m})/(2\alpha)\right|$, whereas the long wave L oscillates between the values

$$\frac{m(\beta-\alpha V)-2\alpha a}{2m\alpha^2} \qquad \text{and} \qquad \frac{m(\beta-\alpha V)-2\alpha a\sqrt{1-m}}{2m\alpha^2}\,, \tag{4.52}$$

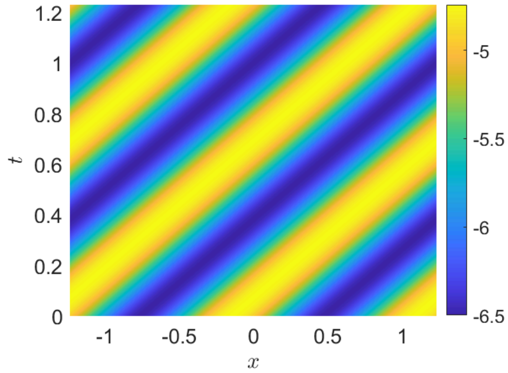
that is to say, with an amplitude $\left|a(1-\sqrt{1-m})/\alpha\right|$.

4.3.4 Traveling waves: Solitons

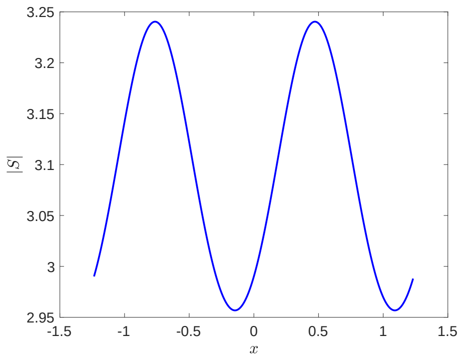
Let us now study the special choice $m = 1$ in (4.40). In that case, the period of the elliptic cosine diverges, and hence the solution becomes localised. The resulting object is of solitonic nature, and can be either bright or dark.



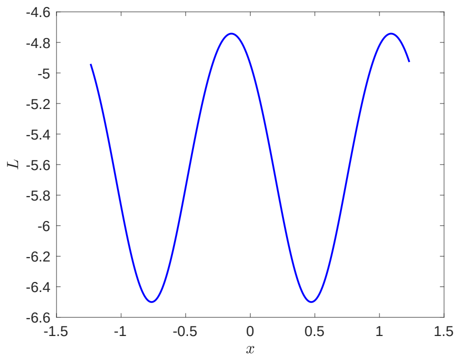
(a) Short wave $|S|$.



(b) Long wave L.



(c) Short wave profile at $t = 0$.



(d) Long wave profile at $t = 0$.

Figure 2: Delta amplitude solution with $\alpha = 1$, $\beta = 1$, $V = 2$, $b = 9$, $a = 3$, $z_0 = -2$, $m = 0.5$.

The generic solution coming from the cnoidal one when $m = 1$ corresponds to a dark soliton of the form

$$S(x,t) = \frac{1}{2} e^{i(\phi(z) - \omega t)} \sqrt{\frac{V\left[b + 2\alpha a \,\mathrm{sech}\left(a(z - z_0)\right)\right]}{\alpha^2}}\,, \tag{4.53a}$$

$$L(x,t) = \frac{\beta - \alpha V - 2\alpha a \,\mathrm{sech}\left(a(z - z_0)\right)}{2\alpha^2}\,, \qquad z = x - Vt\,, \tag{4.53b}$$

$$\omega = \frac{1}{8\alpha^2}\left[2\beta^2 - 2\alpha^2(2V^2 + a^2) + 4V\alpha b + b^2 \pm \sqrt{b^4 - 4\alpha^2 a^2 b^2}\right], \tag{4.53c}$$

with

$$\begin{aligned}\phi(z) =& \frac{z - z_0}{4\alpha}\left(2\alpha V + b \pm \mathrm{sgn}(b)\sqrt{b^2 - 4\alpha^2 a^2}\right) + \arctan\left(\tanh\left(\frac{a}{2}(z - z_0)\right)\right) \\ & \mp \mathrm{sgn}(b)\, \arctan\left(\frac{(b - 2\alpha a)\tanh\left(\frac{a}{2}(z - z_0)\right)}{\sqrt{b^2 - 4\alpha^2 a^2}}\right) + \phi_0\,,\end{aligned} \tag{4.53d}$$

where the phase ϕ_0 is an arbitrary integration constant, and the sign function satisfies $\mathrm{sgn}(0) = 0$.

As a direct consequence of (4.43), and as it can be observed from the formulae above, the constraint on the parameters that ensures the validity of the soliton solution is, when $b \neq 0$,

$$\left| \frac{b}{\alpha a} \right| \geqslant 2 \, . \tag{4.54}$$

As mentioned before, when $m = 1$ the special choice $b = 0$ is also allowed by the system. In that case, the phase becomes

$$\phi(z) = \frac{V\,(z - z_0)}{2} + \arctan\Big(\tanh\Big(\frac{a}{2}(z - z_0) \Big) \Big) + \phi_0 \, , \tag{4.55}$$

and the soliton in S becomes a bright one.

As for the amplitudes, the square of the short wave, $|S|^2$, has an amplitude $\frac{1}{2}\left|\frac{Va}{\alpha}\right|$ over the background $\left|\frac{Vb}{4\alpha}\right|$, whereas the long wave L has an amplitude $-\frac{a}{\alpha}$ over the background $\frac{\beta - \alpha V}{2\alpha^2}$. Note that both amplitudes and the background of S do not depend on β at all, while they all depend inversely on α.

By construction, both $S(x, 0)$ and $L(x, 0)$ are centred at $x = z_0$, whereas $S(x, t_0)$ and $L(x, t_0)$ for a given t_0 are both centred at $x = z_0 + V t_0$.

Whenever $b = 0$, the short wave S has zero background (and hence is of bright nature), whereas for $V = \beta/\alpha$, the long wave L has zero background. When both equalities are true one has a bright soliton solution, and when both are false the solution is dark. When only one of the equalities is true the solution is mixed bright-dark one.

The resulting formulae for the fully bright case is

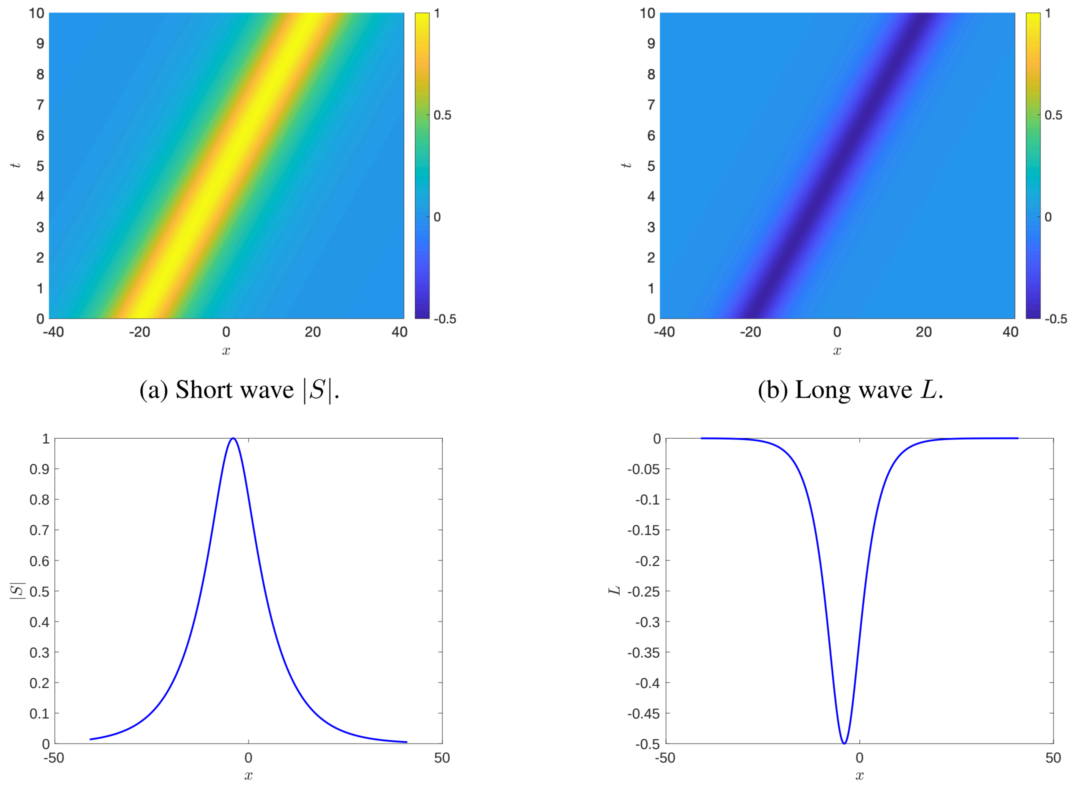
$$S(x,t) = \frac{\sqrt{2}}{2} e^{i(\phi(z)-\omega t)} \sqrt{\frac{\beta a \operatorname{sech}\Big(a(z-z_0)\Big)}{\alpha^2}}\,, \tag{4.56a}$$

$$L(x,t) = -\frac{a \operatorname{sech}\Big(a(z-z_0)\Big)}{\alpha}\,, \qquad z = x - \frac{\beta}{\alpha} t\,, \tag{4.56b}$$

$$\omega = -\frac{\beta^2 + \alpha^2 a^2}{4\alpha^2}\,, \tag{4.56c}$$

$$\phi(z) = \frac{\beta(z-z_0)}{2\alpha} + \arctan\left(\tanh\left(\frac{a}{2}(z-z_0)\right)\right) + \phi_0\,. \tag{4.56d}$$

A similar computation can be carried for the dnoidal solution (4.47) by taking the limit $m = 1$. However, the resulting formulae are the exact same soliton formulae that resulted from the cnoidal case.



(a) Short wave $|S|$.

(b) Long wave L.

(c) Short wave profile at $t = 4$.

(d) Long wave profile at $t = 4$.

Figure 3: Bright soliton solution, with $\alpha = 0.5$, $\beta = 2$, $a = 0.25$, $b = 0$, $V = 4$, $z_0 = -20$.

4.3.5 Traveling waves: Rational solutions

We can obtain further types of solution by making different choices of parameters in our original differential equations. Let us get back to (4.32) and make the choice $c_1 = c_2 = c_3 = 0$. Then, the equation becomes

$$s'(z)^2 = -\frac{\alpha^2}{V^2}s(z)^2\left[s(z)^4 - \frac{V}{\alpha^2}(\alpha V-\beta) + \frac{V^2}{3\alpha^4}(\alpha V-\beta)^2\right], \tag{4.57}$$

It can be solved through an integration process,

$$\int_s^0 \frac{3\sqrt{3}\,\alpha^2|\alpha|}{(V(\alpha V-\beta)-3\alpha^2\zeta^2)^{3/2}}d\zeta = \pm\sqrt{\frac{\alpha^2}{V^2}}z\,. \tag{4.58}$$

By computing the integral, solving with respect to $s(z)$ and introducing it into the formulae for $\ell(z)$ and $\phi(z)$, we get the solution

$$S(x,t) = \frac{z}{\sqrt{3}\,\alpha}\,e^{i(\phi(z)-\omega t)}\sqrt{\frac{V\,(\alpha V-\beta)^3}{9\alpha^2+(\alpha V-\beta)^2z^2}}\,, \tag{4.59a}$$

$$L(x,t) = -\frac{2(\alpha V-\beta)^3z^2}{3\alpha^2[9\alpha^2+(\alpha V-\beta)^2z^2]}\,, \qquad z = x - Vt\,, \tag{4.59b}$$

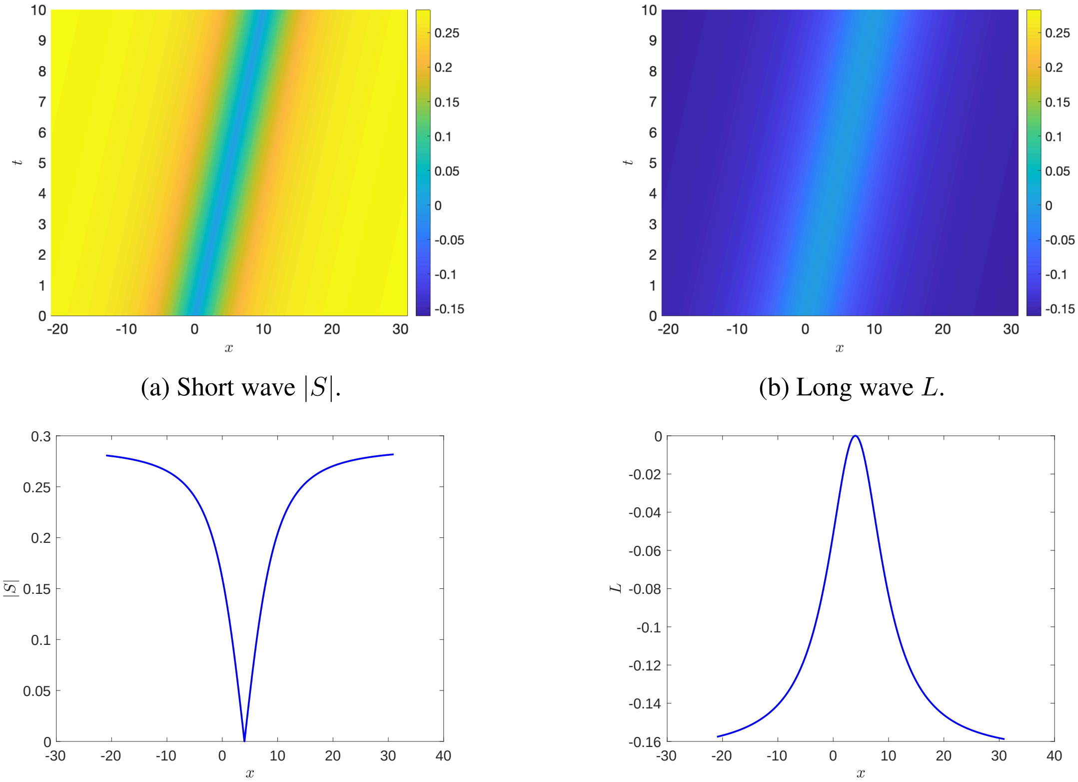
$$\omega = \frac{\alpha^2V^2 - 8\alpha\beta V + 4\beta^2}{12\alpha^2}\,, \tag{4.59c}$$

$$\phi(z) = \arctan\left(\frac{3\alpha}{(\alpha V-\beta)z}\right) + \left(\frac{V}{2}+\frac{\alpha V-\beta}{3\alpha}\right)z + \phi_0\,, \tag{4.59d}$$

where the phase ϕ_0 is an arbitrary integration constant. As in the cases before, one needs to check the inside of the square root involved in S to ensure it is a positive quantity. That entails the following constraint for the parameters:

$$V(\alpha V-\beta) \geqslant 0\,. \tag{4.60}$$

The short wave $|S|$ turns out to be a dark rational solution with an amplitude depression of $\sqrt{V(\alpha V-\beta)/(3\alpha^2)}$ propagating over a non-vanishing background $\sqrt{V(\alpha V-\beta)/(3\alpha^2)}$ (that is, the minimum is at zero), whereas the long wave L has an amplitude $2(\alpha V-\beta)/(3\alpha^2)$ over the asymptotic background $-2(\alpha V - \beta)/(3\alpha^2)$ (again meaning that the peak is at zero, either growing from a negative background or decreasing from a positive one). Also note that the profile of the short wave is that of a peakon, produced by the non-differentiability of the $\arctan$ function inside the phase, while the profile of the long wave is akin to a soliton.



(a) Short wave $|S|$.

(b) Long wave L.

(c) Short wave profile at $t = 4$.

(d) Long wave profile at $t = 4$.

Figure 4: Rational solution with $\alpha = 2$, $\beta = 1$, $V = 1$.

In the special case $\beta = 0$, that is, going back to the Newell system (3.1), the rational solution becomes

$$S(x,t) = \frac{z}{\sqrt{3}\,\alpha}\, e^{i(\phi(z)-\omega t)} \sqrt{\frac{\alpha\, V^4}{9 + V^2 z^2}}\,, \tag{4.61a}$$

$$L(x,t) = -\frac{2V^3 z^2}{3\alpha[9 + V^2 z^2]}\,, \qquad z = x - Vt\,, \tag{4.61b}$$

$$\omega = \frac{V^2}{12}\,, \tag{4.61c}$$

$$\phi(z) = \arctan\left(\frac{3}{V\,z}\right) + \left(\frac{V}{2} + \frac{\alpha V}{3\alpha}\right) z + \phi_0\,, \tag{4.61d}$$

and the constraint on the parameters (4.60) becomes

$$\alpha > 0\,. \tag{4.62}$$

Note that taking $\alpha > 0$ is equivalent to making the choice of sign $\sigma = 1$ in the Newell system, showing that the two choices of sign are most likely intrinsically different.

To the best of our knowledge, this was the first time such a rational solution was derived for the Newell system (3.1).

Another special solution can be obtained by taking the extra choice of parameters $\omega = -c^2/4$, in addition to $c_1 = c_2 = c_3 = 0$. The quadrature for $s(z)$, (4.32), then becomes

$$s'(z)^2 = -\frac{\alpha^2}{V^2}s(z)^4\left[s(z)^2 - \frac{V}{\alpha^2}(\alpha V - \beta)\right]\,, \tag{4.63}$$

and an integration process similar to the one before leads to the solution

$$S(x,t) = e^{i(\phi(z)-\omega t)}\sqrt{\frac{V(\alpha V-\beta)}{\alpha^2+(\alpha V-\beta)^2 z^2}}\,, \tag{4.64a}$$

$$L(t,x) = -\frac{2(\alpha V-\beta)}{\alpha^2+(\alpha V-\beta)^2 z^2}\,, \qquad z = x - Vt\,, \tag{4.64b}$$

$$\omega = -c^2/4\,, \tag{4.64c}$$

$$\phi(z) = \frac{V}{2}z + \arctan\left(\frac{(\alpha V-\beta)z}{\alpha}\right) + \phi_0\,. \tag{4.64d}$$

As before, the constraint after checking the square root is

$$V(\alpha V - \beta) \geqslant 0\,. \tag{4.65}$$

This is a bright rational solution, where the short wave $|S|$ and the long wave L have amplitudes $\sqrt{V(\alpha V-\beta)/(\alpha^2)}$ and $-2(\alpha V-\beta)/(\alpha^2)$, respectively, on a zero background.

Again, to the best of our knowledge, it is also a novel solution of the Newell system once taken

$\beta = 0$, and again it is only valid for $\alpha > 0$ (that is, for $\sigma = 1$).

4.4 Symmetries and conservation laws

As the last part of this chapter, we will obtain some additional properties of the YON system related to symmetries and conserved quantities. Namely, we will first study the Lie point symmetries of the system, which are continuous transformation groups that map every solution of the system into another solution.

Afterwards, we will proceed to study the conservation laws of the system. Since it is integrable, it allows infinitely many conservation laws. However, we will just derive a few explicit ones by studying multipliers.

4.4.1 Lie point symmetries

Let us start with the Lie point symmetries. The computations that follow were done as part of the collaboration for [29] although they did not make the final version of the paper. In order to perform the computation we will need all the quantities and variables involved to be real. Hence, since S is complex, we will split it into real and imaginary part, $S(x,t) = S_1(x,t) + iS_2(x,t)$, introduce it into (4.14a) and also split the equation into real and imaginary part. By doing so, we are able to rewrite the YON system as a system of 3 real PDEs, which we will denote

$$\mathcal{F}_1 = 0\,, \qquad \mathcal{F}_2 = 0\,, \qquad \mathcal{F}_3 = 0\,, \tag{4.66a}$$

where

$$\mathcal{F}_1 = -S_{2,t} + S_{1,xx} + \alpha^2 L^2 S_1 - \beta L S_1 - 2\alpha S_1 S_2^2 - 2\alpha S_1^3 - \alpha L_x S_2\,, \tag{4.66b}$$

$$\mathcal{F}_2 = S_{1,t} + S_{2,xx} + \alpha L_x S_1 - \beta L S_2 - 2\alpha S_2^3 - 2\alpha S_1^2 S_2 + \alpha^2 L^2 S_2\,, \tag{4.66c}$$

$$\mathcal{F}_3 = L_t - 2(S_1^2 + S_2^2)_x\,. \tag{4.66d}$$

We say that the 1-parameter group of transformations

$$\bar{t} = \bar{t}(x,t,S_1,S_2,L,\varepsilon), \quad \bar{x} = \bar{x}(x,t,S_1,S_2,L,\varepsilon),$$
$$\bar{S}_1 = \bar{S}_1(x,t,S_1,S_2,L,\varepsilon), \quad \bar{S}_2 = \bar{S}_2(x,t,S_1,S_2,L,\varepsilon), \quad \bar{L} = \bar{L}(x,t,S_1,S_2,L,\varepsilon),$$

acting on a subset of $\mathbb{R}^3$ and depending on a real continuous parameter ε, is a Lie point symmetry of (4.66a) if its infinitesimal generator

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial S_1} + \eta_2 \frac{\partial}{\partial S_2} + \eta_3 \frac{\partial}{\partial L}\,, \tag{4.67}$$

where

$$\begin{aligned} &\xi \equiv \xi(x,t,S_1,S_2,L,\varepsilon) = \left.\frac{\partial \bar{x}}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \tau \equiv \tau(x,t,S_1,S_2,L,\varepsilon) = \left.\frac{\partial \bar{t}}{\partial \varepsilon}\right|_{\varepsilon=0}, \\ &\eta_1 \equiv \eta_1(x,t,S_1,S_2,L,\varepsilon) = \left.\frac{\partial \bar{S}_1}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \eta_2 \equiv \eta_2(x,t,S_1,S_2,L,\varepsilon) = \left.\frac{\partial \bar{S}_2}{\partial \varepsilon}\right|_{\varepsilon=0}, \\ &\eta_3 \equiv \eta_3(x,t,S_1,S_2,L,\varepsilon) = \left.\frac{\partial \bar{L}}{\partial \varepsilon}\right|_{\varepsilon=0}, \end{aligned} \tag{4.68}$$

satisfies the invariance condition

$$X^{(2)}\mathcal{F}_i = 0 \quad \text{whenever} \quad \mathcal{F}_i = 0, \qquad i = 1,\,2,\,3\,, \tag{4.69}$$

where the second prolongation $X^{(2)}$ is given by

$$\begin{aligned} X^{(2)} = X &+ \eta_1^{(x)} \frac{\partial}{\partial S_{1,x}} + \eta_1^{(t)} \frac{\partial}{\partial S_{1,t}} + \eta_2^{(x)} \frac{\partial}{\partial S_{2,x}} + \eta_2^{(t)} \frac{\partial}{\partial S_{2,t}} \\ &+ \eta_3^{(x)} \frac{\partial}{\partial L_x} + \eta_3^{(t)} \frac{\partial}{\partial L_t} + \eta_1^{(xx)} \frac{\partial}{\partial S_{1,xx}} + \eta_2^{(xx)} \frac{\partial}{\partial S_{2,xx}}\,, \end{aligned} \tag{4.70}$$

where the extended infinitesimals are found through the formulae

$$\eta_j^{(x)} = D_x \eta_j - (D_x \xi) S_{j,x} - (D_x \tau) S_{j,t}\,, \tag{4.71a}$$

$$\eta_j^{(t)} = D_t \eta_j - (D_t \xi) S_{j,x} - (D_t \tau) S_{j,t}\,, \tag{4.71b}$$

$$\eta_j^{(xx)} = D_x \eta_{j(x)} - (D_x \xi) S_{j,xx} - (D_x \tau) S_{j,xt}\,, \tag{4.71c}$$

for $j = 1, 2$, and

$$\eta_3^{(x)} = D_x \eta_3 - (D_x \xi) L_x - (D_x \tau) L_t\,, \tag{4.71d}$$

$$\eta_3^{(t)} = D_t \eta_3 - (D_t \xi) L_x - (D_t \tau) L_t\,, \tag{4.71e}$$

with D_x and D_t denoting the total derivative with respect to x or t, respectively.

The invariance condition (4.69) leads to an overdetermined system of linear equations for the infinitesimals ξ, τ and η_j and its solution leads to a unique determination of Lie point symmetries, see [136]. By employing a computational approach as described in [40, 41, 42, 43, 44], we can obtain, for every choice of the parameters α and β in (4.14), the vector fields

$$X_1 = \frac{\partial}{\partial x}\,, \qquad X_2 = \frac{\partial}{\partial t}\,, \qquad X_3 = S_2 \frac{\partial}{\partial S_1} - S_1 \frac{\partial}{\partial S_2}\,, \tag{4.72}$$

which correspond, by making use of the exponential map, see [136], to translations in x,

$$(\bar{x}, \bar{t}, \bar{S}_1, \bar{S}_2, \bar{L}) = (x + \varepsilon, t, S_1, S_2, L)\,, \tag{4.73}$$

translations in t,

$$(\bar{x}, \bar{t}, \bar{S}_1, \bar{S}_2, \bar{L}) = (x, t + \varepsilon, S_1, S_2, L)\,, \tag{4.74}$$

and rotations around the origin in the (S_1, S_2)-plane,

$$(\bar{x}, \bar{t}, \bar{S}_1, \bar{S}_2, \bar{L}) = (x, t, S_1 \cos\varepsilon + S_2 \sin\varepsilon, -S_1 \sin\varepsilon + S_2 \cos\varepsilon, L)\,, \tag{4.75}$$

respectively. One can obtain additional Lie point symmetries by taking special choices for the parameters. If $\alpha \neq 0$, then we have the additional generator

$$X_4 = 4t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} - \frac{2\alpha^2 S_1 - \beta^2 t S_2}{\alpha^2} \frac{\partial}{\partial S_1} - \frac{\beta^2 t S_1 + 2\alpha^2 S_2}{\alpha^2} \frac{\partial}{\partial S_2} - \frac{2\alpha^2 L - \beta}{\alpha^2} \frac{\partial}{\partial L}\,, \tag{4.76}$$

which corresponds to the transformation group

$$
\begin{aligned}
(\bar{x},\bar{t},\bar{S}_1,\bar{S}_2,\bar{L}) = \Bigg(& e^{2\varepsilon}x, e^{4\varepsilon}t, \\
& S_1 e^{-2\varepsilon}\cos\left(\frac{\beta^2}{\alpha^2}\varepsilon\right) + S_2 e^{-2\varepsilon}\sin\left(\frac{\beta^2}{\alpha^2}\varepsilon\right), \\
& -S_1 e^{-2\varepsilon}\sin\left(\frac{\beta^2}{\alpha^2}\varepsilon\right) + S_2 e^{-2\varepsilon}\cos\left(\frac{\beta^2}{\alpha^2}\varepsilon\right) \\
& \frac{\beta}{\alpha^2}\left(1-\frac{1}{2}e^{-2\varepsilon}\right) + Le^{-2\varepsilon}\Bigg).
\end{aligned}
\tag{4.77}
$$

$[\cdot,\cdot]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$
$\mathbf{X_1}$	0	0	0	$2X_1$
$\mathbf{X_2}$	0	0	0	$4X_2 - \frac{\beta^2}{\alpha^2}X_3$
$\mathbf{X_3}$	0	0	0	0
$\mathbf{X_4}$	$-2X_1$	$-4X_2 + \frac{\beta^2}{\alpha^2}X_3$	0	0

Table 1: Commutator table for $\alpha \neq 0$.

If $\alpha = 0$ and $\beta \neq 0$, then we also have the generators

$$X_5 = 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} - 3S_1\frac{\partial}{\partial S_1} - 3S_2\frac{\partial}{\partial S_2} - 4L\frac{\partial}{\partial L}\,, \tag{4.78a}$$

$$X_6 = \beta t S_2\frac{\partial}{\partial S_1} - \beta t S_1\frac{\partial}{\partial S_2} + \frac{\partial}{\partial L}\,. \tag{4.78b}$$

The generator X_5 is associated to a transformation group corresponding to a scaling transformation,

$$(\bar{x},\bar{t},\bar{S}_1,\bar{S}_2,\bar{L}) = (e^{2\varepsilon}x, e^{4\varepsilon}t, e^{-3\varepsilon}S_1, e^{-3\varepsilon}S_2, e^{-4\varepsilon}L)\,, \tag{4.79}$$

which was already introduced in (4.16), whereas the transformation group associated to X_6 is given by

$$(\bar{x},\bar{t},\bar{S}_1,\bar{S}_2,\bar{L}) = (x, t, S_1\cos(\beta t\varepsilon)+S_2\sin(\beta t\varepsilon), -S_1\sin(\beta t\varepsilon)+S_2\cos(\beta t\varepsilon), L+\varepsilon)\,, \tag{4.80}$$

and it corresponds to a precession with frequency $\beta\varepsilon$ around the origin in the (S_1, S_2)-plane.

$[\cdot,\cdot]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_5}$	$\mathbf{X_6}$
$\mathbf{X_1}$	0	0	0	$2X_1$	0
$\mathbf{X_2}$	0	0	0	$4X_2$	βX_3
$\mathbf{X_3}$	0	0	0	0	0
$\mathbf{X_5}$	$-2X_1$	$-4X_2$	0	0	$4X_6$
$\mathbf{X_6}$	0	$-\beta X_3$	0	$-4X_6$	0

Table 2: Commutator table for $\alpha = 0$, $\beta \neq 0$.

4.4.2 Conservation laws

As stated before, one of the key properties of integrable systems is that they possess an infinite number of linearly independent conserved quantities. Although there exists specific machinery to compute them for the special case of integrable systems, such as using Lenard chains, which allow one to obtain the conserved quantities via recursions (see [122]), as for the rest of this chapter we will ignore the integrability of the system and apply a standard method in order to obtain a few explicit conservation laws following [29]. A conservation law is a relation of the form

$$\rho_t + f_x = 0\,, \tag{4.81}$$

where $\rho \equiv \rho(S, L, S_x, L_x, \ldots)$ is the density and $f \equiv f(S, L, S_x, L_x, \ldots)$ the corresponding flux, respectively. The second equation of the YON system, (4.14b), is trivially a conservation law of the system, with

$$\rho_0 = L\,, \tag{4.82a}$$

$$f_0 = -2|S|^2\,. \tag{4.82b}$$

As stated in [7, 8, 9], conservation laws correspond to symmetries of the equation, but in many cases those symmetries may turn out to be non-classical or even non-local. When Noether's theorem can be applied, the correspondence between conservation laws and symmetries are made explicit; however, in order to apply it one needs to write the system in a variational formulation, which we were unable to find. Because of that, we will employ the multiplier method (see [7, 8, 9, 136]), which consists on finding a vector $g = (g_1, g_2, g_3)$, called multiplier, depending on S, L and their derivatives up to a fixed but arbitrary order, such that

$$\frac{\delta g_1 \mathcal{E}_1}{\delta S} = 0\,, \qquad \frac{\delta g_2 \mathcal{E}_2}{\delta S^*} = 0\,, \qquad \frac{\delta g_3 \mathcal{E}_3}{\delta L} = 0\,, \tag{4.83}$$

where

$$\mathcal{E}_1 = iS_t + S_{xx} + \left(i\alpha L_x + \alpha^2 L^2 - \beta L - 2\alpha|S|^2\right) S\,, \qquad \mathcal{E}_2 = \mathcal{F}_1^*\,, \qquad \mathcal{E}_3 = L_t - 2(|S|^2)_x \tag{4.84}$$

are the equations in YON system (and its complex conjugate for the complex equation), and where $\delta/\delta u$ denotes the variational derivative with respect to the variable u. If a multiplier g like that can be found, then it ensures the existence of some ρ and f such that

$$\rho_t + f_x = g_1\mathcal{E}_1 + g_2\mathcal{E}_2 + g_3\mathcal{E}_3 = 0\,. \tag{4.85}$$

By using the specially designed GeM software package for symmetries and conservation laws (see [40, 42, 41, 43, 44]), one can, for any choice of α and β, compute pairs of conserved densities and fluxes depending on derivatives up to second order, namely

$$\rho_1 = \frac{\alpha}{2}L^2 - |S|^2\,, \tag{4.86a}$$

$$f_1 = -2\alpha L|S|^2 - 2\ \mathrm{Im}(S^* S_x)\,; \tag{4.86b}$$

$$\rho_2 = 2\alpha^2 L|S|^2 - \beta|S|^2 + 2\alpha\,(S^* S_x)\,, \tag{4.87a}$$

$$f_2 = 2\alpha|S_x|^2 - 2\alpha\ \mathrm{Re}(S^* S_{xx}) + 4\alpha^2 L\ \mathrm{Im}(S^* S_x) - 2\beta\ \mathrm{Im}(S^* S_x)\,; \tag{4.87b}$$

$$\rho_3 = 2\alpha^2 tL|S|^2 + \frac{\alpha^2}{2}xL^2 - \frac{\beta}{2}xL - \beta t|S|^2 - \alpha x|S|^2 + 2\alpha t \,\mathrm{Im}(S^* S_x)\,, \tag{4.88a}$$

$$\begin{aligned} f_3 =& -2\alpha^2 xL|S|^2 + \beta x|S|^2 + 4\alpha^2 tL \,\mathrm{Im}(S^* S_x) + 2\alpha t|S_x|^2 - 2\alpha x \,\mathrm{Im}(S^* S_x) \\ & - 2\beta t \,\mathrm{Im}(S^* S_x) - 2\alpha t \,\mathrm{Re}(S^* S_{xx}); \end{aligned} \tag{4.88b}$$

$$\begin{aligned} \rho_4 =& \frac{1}{2}\alpha^5 L^4 - 2\alpha^4 L^2|S|^2 - \alpha^3\beta L^3 + 2\alpha^3|S|^4 + 4\alpha^3 L \,\mathrm{Im}(S^* S_x) \\ & - \frac{1}{2}\alpha^3 L_x^2 + 2\beta^2|S|^2 - 4\alpha\beta \,\mathrm{Im}(S^* S_x) + 4\alpha^2|S_x|^2\,, \end{aligned} \tag{4.89a}$$

$$\begin{aligned} f_4 =& 6\alpha^3\beta L^2|S|^2 + 8\alpha^4 L|S|^4 - 4\alpha^2\beta|S|^4 - 4\alpha\beta|S_x|^2 + 4\alpha^3 L|S_x|^2 \\ & - 4\alpha^5 L^3|S|^2 + 8\alpha^3|S|^2 \,\mathrm{Im}(S^* S_x) + 4\beta^2 \,\mathrm{Im}(S^* S_x) - 4\alpha^4 L^2 \,\mathrm{Im}(S^* S_x) \\ & + 8\alpha^3 L_x \,\mathrm{Re}(S^* S_x) + 4\alpha\beta \,\mathrm{Re}(S^* S_{xx}) - 4\alpha^3 L \,\mathrm{Re}(S^* S_{xx}) + 8\alpha^2 \,\mathrm{Im}(S_x^* S_{xx}). \end{aligned} \tag{4.89b}$$

Chapter 5

Stability of Plane Waves for the YON System

Once we have studied some basic properties of the YON system in the previous chapter ignoring the fact that it is integrable, now we will devote to the study of the stability of its solutions (see Section 1.4). In this case we will make use of its integrability properties by following the stability method introduced in [53], which makes use of the Lax pair of the system (4.3) to systematically construct local perturbations of solutions (namely, of plane waves, though we plan to extend the method to broader kinds of solutions) and study their stability in an algebro-geometric framework through the so-called stability spectrum. The theory on this method will be presented in Section 5.1.

This algebro-geometric method had already been applied to the study and completely characterise the instabilities of plane waves in the vector nonlinear Schrödinger (vNLS) equation (see [53, 54, 55]), and the 3-wave resonant interaction model (see [120, 141]), and we succesfully applied it to the YON system too (see [30]). Let us present how the method works to then show its application to the YON system.

5.1 The stability spectrum

We will follow the original introduction of the method in [53]. Let us consider an integrable system of PDEs as introduced in Section 1.2, with a *Lax pair* of the form

$$\Psi_x = X\Psi\,, \qquad \Psi_t = T\Psi\,, \tag{5.1}$$

where Ψ, X, and T are $N \times N$ matrix-valued complex functions of x, t, the unknown variables of the system of PDEs at hand, and their derivatives. X and T will also depend on an additional complex parameter called the *spectral parameter*, which we will denote by λ. For our purpose, we will consider this dependence to be polynomial up to linear order for X, and to quadratic order for T, that is,

$$X(\lambda) = i\lambda\Sigma + Q\,, \qquad T(\lambda) = \lambda^2 T_2 + \lambda T_1 + T_0\,. \tag{5.2}$$

The system of PDEs arises from this framework as the *compatibility condition* of the Lax pair,

$$X_t - T_x + [X,\, T] = 0\,, \tag{5.3}$$

where $[X,\, T]$ denotes the commutator of X and T, $[X,\, T] = XT - TX$. That means that the equality (5.3) will hold if and only if the original system of PDEs is satisfied.

Given that both X and T are polynomial in λ, the left-hand side of the compatibility condition will also be polynomial. In particular, since X is linear and T is quadratic, the left-hand side of the compatibility condition will be cubic, and, since it must hold for every choice of λ, it yields 4 equations for the matrix coefficients Σ, Q, T_0, T_1 and T_2.

Let us now introduce a small perturbation. Given a pair X and T solving the compatibility condition (5.3), let us consider a new solution $X + \delta X$ and $T + \delta T$ differing by a small change of the original X and T. Then, the perturbations δX and δT must satisfy, at first order, the *linearised equation*

$$(\delta X)_t - (\delta T)_x + [\delta X,\, T] + [X,\, \delta T] = 0\,. \tag{5.4}$$

That means that the task at hand is finding a solution $A(x, t, \lambda)$, $B(x, t, \lambda)$ of the linearised equation

$$A_t - B_x + [A,\, T] + [X,\, B] = 0\,, \tag{5.5}$$

where we have denoted $A = \delta X$ and $B = \delta T$ to simplify the notation. Let us note, however, that both A and B are related to the fundamental solution of the Lax equations (5.1), $\Psi(x, t, \lambda)$. Namely, if we define a new matrix-valued function $\Phi(x, t, \lambda)$ by

$$\Phi(x, t, \lambda) = \Psi(x, t, \lambda) M(\lambda) \Psi^{-1}(x, t, \lambda)\,, \tag{5.6}$$

where $M(\lambda)$ is an arbitrary constant matrix depending only on λ, then it satisfies the pair of linear ODEs

$$\Phi_x = [X,\, \Phi]\,, \qquad \Phi_t = [T,\, \Phi]\,. \tag{5.7}$$

These relations are a direct consequence of the Lax equations (5.1):

$$\begin{aligned}\Phi_x &= (\Psi M \Psi^{-1})_x \\ &= \Psi_x M \Psi^{-1} + \Psi M \Psi^{-1}_x \\ &= X \Psi M \Psi^{-1} - \Psi M \Psi^{-1} X \\ &= [X,\, \Psi M \Psi^{-1}] = [X,\, \Phi]\,,\end{aligned}$$

where we have used the matrix relation

$$\mathcal{M}^{-1}_x = -\mathcal{M}^{-1} \mathcal{M}_x \mathcal{M}^{-1} \tag{5.8}$$

for any invertible matrix $\mathcal{M}$, so that

$$\Psi^{-1}_x = -\Psi^{-1} \Psi_x \Psi^{-1} = -\Psi^{-1} X \Psi \Psi^{-1} = -\Psi^{-1} X\,,$$

where we have again applied the Lax equations (5.1). The proof for Φ_t is completely analogous. The compatibility of equations (5.7), $\Phi_{xt} = \Phi_{tx}$, coincides with the compatibility condition for the Lax pair. With this, we can now prove the following result, previously introduced in [53].

Proposition 5.1.1 *Let A and B be a pair of matrix-valued functions solving the linearised equation* (5.5)*. Then, also the pair*

$$F = [A,\, \Phi]\,, \qquad G = [B,\, \Phi] \tag{5.9}$$

is a solution of the same linearised equation, that is,

$$F_t - G_x + [F,\, T] + [X,\, G] = 0\,. \tag{5.10}$$

Proof: *To prove the result we will only need the relations* (5.7) *and the Jacobi identity.*

$$F_t - G_x + [F,\, T] + [X,\, G] = ([A,\, \Phi])_t - ([B,\, \Phi])_x + [[A,\, \Phi], T] + [X, [B,\, \Phi]]\,.$$

We have that

$$\begin{aligned}
[A,\, \Phi]_t &= [A_t,\, \Phi] + [A,\, \Phi_t] \\
&= [A_t,\, \Phi] + [A, [T,\, \Phi]] \\
&= [A_t,\, \Phi] - [\Phi,\, [A,\, T]] - [T,\, [\Phi,\, A]] \\
&= [A_t,\, \Phi] + [[A,\, T],\, \Phi] - [[A,\, \Phi],\, T]\,,
\end{aligned}$$

and, similarly,

$$[B,\, \Phi]_x = [B_x,\, \Phi] - [[X,\, B],\, \Phi] + [X,\, [B,\, \Phi]]\,.$$

Putting all together we have that

$$F_t - G_x + [F,\, T] + [X,\, G] = \big[A_t - B_x + [A,\, T] + [X,\, B],\, \Phi\big]\,,$$

but the first entry of the commutator is the linearised equation for A and B, so it is equal to 0, *which proves the proposition.*

With this proposition, we can also get the following corollary:

Corollary 5.1.2 *The matrix-valued functions*

$$F = \left[\frac{\partial X}{\partial \lambda}, \Phi\right], \qquad G = \left[\frac{\partial T}{\partial \lambda}, \Phi\right] \tag{5.11}$$

are a solution of the linearised equation (5.10).

It follows from Proposition 5.1.1 and the fact that

$$A = \frac{\partial X}{\partial \lambda}, \qquad B = \frac{\partial T}{\partial \lambda} \tag{5.12}$$

trivially satisfy the linearised equation (5.5).

Let us now go back to our system through the Lax pair (5.2). We will need to impose a few additional properties on the matrix coefficients. First of all, we will impose that Σ be constant and Hermitian, and hence, without loss of generality, we will consider it diagonal and real, in block-diagonal notation

$$\Sigma = \mathrm{diag}\{\alpha_1 \mathbb{1}_1, \ldots, \alpha_L \mathbb{1}_L\}, \qquad 2 \leqslant L \leqslant N, \tag{5.13}$$

where the eigenvalues $\alpha_1, \ldots, \alpha_L$ are real and distinct, $\mathbb{1}_j$ is the $n_j \times n_j$ identity matrix where n_j is the multiplicity of the eigenvalue α_j, and N is the order of the matrix Σ.

Then, the structure of Σ allows us to split the set of $N \times N$ matrices into two subspaces: the subspace of block-diagonal matrices and the subspace of block-off-diagonal matrices, where the size of the blocks is defined by the size of the blocks with the same eigenvalue in Σ. We can then, given any $N \times N$ matrix $\mathcal{M}$, adopt the notation

$$\mathcal{M} = \mathcal{M}^{(d)} + \mathcal{M}^{(o)}, \tag{5.14}$$

where $\mathcal{M}^{(d)}$ denotes the block-diagonal part of $\mathcal{M}$ and $\mathcal{M}^{(o)}$ denotes its block-off-diagonal part.

Consistently with this notation, we will define the entries $\mathcal{M}_{jk}$ as matrices themselves of dimension $n_j \times n_k$ demarcated by the blocks above. Note that the matrix entries $\mathcal{M}_{jk}$ do not necessarily commute with each other; in fact, they are in general rectangular matrices.

Also note that, given two arbitrary matrices $\mathcal{M}$ and $\mathcal{N}$, we have that $\mathcal{M}^{(d)}\mathcal{N}^{(d)}$ is block-diagonal, $\mathcal{M}^{(d)}\mathcal{N}^{(o)}$ is block-off-diagonal, whereas, for $N > 2$, $\mathcal{M}^{(o)}\mathcal{N}^{(o)}$ is neither necessarily block-diagonal nor block-off-diagonal.

Furthermore, we will impose $Q(x,t)$ in (5.2) to be block-off-diagonal, and differentiable up to sufficiently high order so that all the derivatives involved in the computation are properly defined.

After this consideration, we can express the matrix coefficients of T as functions of Σ and Q through the compatibility condition of the Lax pair (5.3):

$$T_2 = C_2\,, \tag{5.15a}$$

$$T_1 = C_1 - iI_1 - iD_2(Q)\,, \tag{5.15b}$$

$$\begin{aligned} T_0 = C_0 + I_0 - \frac{1}{2}[D_2(Q),\Gamma(Q)]^{(d)} - \Gamma\big(D_2(Q_x)\big) \\ - \Gamma\Big([D_2(Q),Q]^{(o)}\Big) - iD_1(Q) - [I_1,\Gamma(Q)]\,, \end{aligned} \tag{5.15c}$$

where the matrices C_j with $j = 0,1,2$ are constant and diagonal (we will set $C_0 = 0$ since its value is irrelevant for our purpose), the linear invertible map Γ acts on block-off-diagonal matrices as

$$\big(\Gamma(\mathcal{M})\big)_{jk} = \frac{\mathcal{M}_{jk}}{\alpha_j - \alpha_k}\,, \tag{5.16}$$

where $\mathcal{M}$ is block-off-diagonal, so that

$$[\Sigma,\Gamma(\mathcal{M})] = \Gamma\big([\Sigma,\mathcal{M}]\big) = \mathcal{M}\,, \tag{5.17}$$

and the maps D_j with $j = 1, 2$ also act only on block-off-diagonal matrices as

$$D_j(\mathcal{M}) = [C_j, \Gamma(\mathcal{M})] = \Gamma\big([C_j, \mathcal{M}]\big)\,. \tag{5.18}$$

Finally, the non-local matrices I_1 and I_0 are block-diagonal and are defined by

$$I_1(x,t) = \int^x dy \Big[Q(y,t),\, D_2\big(Q(y,t)\big)\Big]^{(d)} \tag{5.19a}$$

$$\begin{aligned} I_0(x,t) = \int^x dy \Bigg\{ &-\frac{1}{2}\Big[C_2,\, \big[\Gamma(Q_y(y,t)),\, \Gamma(Q(y,t))\big]^{(d)} \\ &- \Big[Q(y,t),\, \Gamma\big([D_2(Q(y,t)),\, Q(y,t)]^{(o)}\big)\Big]^{(d)} \\ &- i\Big[Q(y,t),\, D_1\big(Q(y,t)\big)\Big]^{(d)} - \Big[Q(y,t),\, \big[I_1(t),\, \Gamma(Q(y,t))\big]\Big]^{(d)} \Bigg\}\,. \end{aligned} \tag{5.19b}$$

The detailed proof for these formulae is provided in the Appendix A.

The nonlocality generated by I_1 and I_0, whose consequence is that $Q(x,t)$ will satisfy an integro-differential equation through the compatibility condition, rather than a partial differential equation, is problematic, since it is non-physical and it is unclear whether the system can be treated via spectral methods in case they are non-zero (see [51]). Because of that, we will only consider the local case, and will therefore give the condition for I_1 and I_0 to vanish.

Proposition 5.1.3 *The matrices I_1 and I_0 vanish if and only if the blocks of C_1 and C_2 are proportional to the identity matrix, that is,*

$$C_1 = \mathrm{diag}\{\beta_1 \mathbb{1}_1, \ldots, \beta_L \mathbb{1}_L\}\,, \qquad C_2 = \mathrm{diag}\{\gamma_1 \mathbb{1}_1, \ldots, \gamma_L \mathbb{1}_L\}\,. \tag{5.20}$$

The proof of this proposition is also provided in the Appendix A.

We will keep these conditions for locality from now on, so that the evolution equation for the matrix Q reads

$$\begin{aligned} Q_t = -\Gamma\big(D_2(Q_{xx})\big) - \Big[\Gamma\big(D_2(Q_x)\big),\, Q\Big]^{(o)} - \Gamma\Big(\big[D_2(Q),\, Q\big]_x^{(o)}\Big) - \Big[\big(D_2(Q)\Gamma(Q)\big)^{(d)},\, Q\Big] \\ - \Big[\Gamma\Big(\big[D_2(Q),\, Q\big]^{(o)}\Big),\, Q\Big]^{(o)} - iD_1(Q_x) - i\big[D_1(Q),\, Q\big]^{(o)}\,. \end{aligned} \tag{5.21}$$

With this, we can introduce a local perturbation by substituting the given solution Q with $Q+\delta Q$, and then linearise the equation (5.21) by neglecting the nonlinear terms in δQ. That way, we obtain the linear PDE

$$\begin{aligned} \delta Q_t = & -\Gamma\big(D_2(\delta Q_{xx})\big) - \Big[\Gamma\big(D_2(\delta Q_x)\big),\, Q\Big]^{(o)} - \Gamma\Big(\big[D_2(Q_x),\, \delta Q\big]^{(o)}\Big) \\ & - \Gamma\Big(\big[D_2(\delta Q),\, Q\big]^{(o)} + \big[D_2(Q),\, \delta Q\big]^{(o)}\Big)_x - \Big[\big(D_2(Q)\Gamma(Q)\big)^{(d)},\, \delta Q\Big] \\ & - \Big[\big(D_2(\delta Q)\Gamma(Q)\big)^{(d)},\, Q\Big] - \Big[\big(D_2(Q)\Gamma(\delta Q)\big)^{(d)},\, Q\Big] \\ & - \Big[\Gamma\Big(\big[D_2(Q),\, Q\big]^{(o)}\Big),\, \delta Q\Big]^{(o)} - \Big[\Gamma\Big(\big[D_2(\delta Q),\, Q\big]^{(o)}\Big),\, Q\Big]^{(o)} \\ & - \Big[\Gamma\Big(\big[D_2(Q),\, \delta Q\big]^{(o)}\Big),\, Q\Big]^{(o)} - iD_1(\delta Q_x) \\ & - i\big[D_1(Q),\, \delta Q\big]^{(o)} - i\big[D_1(\delta Q),\, Q\big]^{(o)}\,. \end{aligned} \tag{5.22}$$

This leads us to the main result of this section:

Proposition 5.1.4 *The matrix*

$$F = [\Sigma,\, \Phi]\,, \tag{5.23}$$

defined as in the Corollary 5.1.2 by taking the Lax pair (5.2)*, satisfies the same linear PDE* (5.22) *as δQ if and only if the matrices C_1 and C_2 satisfy the locality condition* (5.20)*.*

The proof of the proposition is rather long but straightforward. Roughly, the idea of the proof is to combine the ODEs for Φ, (5.7), to obtain a λ-independent PDE for the matrix F. In order to do so, one can split the first of the ODEs in (5.7) into block diagonal $\Phi^{(d)}$ and block-off-diagonal $\Phi^{(o)} = \Gamma(F)$ parts,

$$\lambda F = \Gamma(F_x) - \big[Q,\, \Phi^{(d)}\big] - [Q,\, \Gamma(F)]^{(o)}\,, \qquad \Phi_x^{(d)} = [Q,\, \Gamma(F)]^{(d)}\,. \tag{5.24}$$

With this one can write F_t from (5.23) together with the second ODE in (5.7), that is,

$$F_t = \big[\Sigma,\, [T,\, \Phi]\big]\,, \tag{5.25}$$

expand T with its expression in (5.2) along with (5.15) and replace all the terms λF in the resulting expression using the formula (5.24).

Once that is done, all the terms containing $\Phi^{(d)}$ cancel out, provided the condition (5.20), and the remaining terms in F_t can be rearranged in the same form as (5.22) purely algebraically.

With this we have arrived to an important conclusion. The matrix F that we defined in (5.23) has the same block-off-diagonal structure as the perturbation δQ and is a λ-dependent solution of the linearised equation (5.22), with a λ-dependence originating from both the arbitrary matrix $M(\lambda)$ and from the fundamental solution of the Lax equations Ψ, both of which make up Φ. Its role is analogous to that of the exponential solutions of linear equations with constant coefficients, that is, by "moving" the spectral parameter λ over what we call the stability spectrum (which will be introduced later on in this section), F provides the Fourier-like modes of the system of PDEs.

Note that throughout the whole chapter we have not made any reference to the boundary conditions of $Q(x,t)$ at $x = \pm\infty$. In fact, F does not depend at all on the boundary conditions and hence it is fit to treat problems with either vanishing or non-vanishing boundary conditions, periodic solutions, and even other different kinds of problems which may be of physical interest.

Now, the results in Proposition 5.1.4 imply that any sum (or, for our interest, integral) of $F(x,t,\lambda)$ over the spectral variable λ is also a solution of the linearised equation (5.22) for δQ, and, as such, a proper perturbation of the solution. Henceforth, we will simplify the notation by simply calling that object δQ,

$$\delta Q(x,t) = \int_{\mathbb{S}_x} d\lambda\, F(x,t,\lambda)\,. \tag{5.26}$$

The choice of integrand and path of integration must be performed in such a way that δQ is bounded in the x variable at any fixed t and in the proper function space (e. g. a localised perturbation). The construction of perturbations via a series,

$$\delta Q(x,t) = \sum_{\lambda \in I \subset \mathbb{S}_x} F(x,t,\lambda)\,, \tag{5.27}$$

which is potentially a good framework to represent periodic perturbations.

Both in the integral and in the series cases, the boundedness of the solution implies that $F(x,t,\lambda)$ itself must be bounded, which leads to the construction of what we called the *stability spectrum* $\mathbb{S}_x$, which consists on the subset of the complex λ-plane for which the perturbation $F(x,t,\lambda)$ is bounded in the x variable.

Whether a given λ belongs or not in the stability spectrum depends on the asymptotic behaviour of $Q(x,t)$ for large $|x|$. When $Q(x,t)$ vanishes sufficiently fast for $|x| \to \infty$, for example when its entries are in L^1, then $\mathbb{S}_x$ coincides with the so-called Lax spectrum, that is, the spectrum of the X operator that acts on the fundamental solution through the Lax equation $\Psi_x = X\Psi$. That is the case when the matrices X, T in the Lax pair are 2×2 matrices. In that cases, the procedure introduced in the present chapter is equivalent to the method of squared eigenfunctions (see [100, 160]), making it unnecessary to use this more convoluted method.

However, when dealing with $N \times N$ matrices, this is not true anymore, as $Q(x,t)$ tends in general to some finite, nonzero value when $|x| \to \infty$. In that case the Lax spectrum and the stability spectrum do not coincide, and the squared eigenfunctions method cannot be applied anymore [110]. In fact, the action that should be studied to consider stability is not the direct action but the action through the commutator, $\Phi_x = [X, \Phi]$. The corresponding stability spectrum $\mathbb{S}_x$ will consist on a piecewise continuous curve, potentially along with a finite number of isolated points.

Before continuing, note that the study of the completeness of the $F(x,t,\lambda)$ in (5.26) and (5.27) is out of the scope of this work, and hence we will only describe solutions of the linearised equation for δQ (5.22) belonging to an adequate function space and that can be obtained through the integral or series representations introduced above.

We will claim that a given solution $Q(x,t)$ is linearly stable (with respect to t) if any small change $\delta Q(x,t_0)$ compatible with our representation remains small as time grows.

Let us now show how we can represent the matrix $F(x,t,\lambda)$ in the case of 3×3 matrices. We can consider the fundamental matrix solution Ψ of the Lax equations (5.1), and denote its three column vectors by $\psi^{(1)}$, $\psi^{(2)}$ and $\psi^{(3)}$,

$$\Psi = \begin{pmatrix} \psi^{(1)} & \psi^{(2)} & \psi^{(3)} \end{pmatrix} . \tag{5.28}$$

We can consider without loss of generality that Ψ has unit determinant, so that

$$\det \Psi = \psi^{(1)} \cdot \psi^{(2)} \wedge \psi^{(3)} . \tag{5.29}$$

Then, the expression for $F(x,t,\lambda)$ (5.23) can be rewritten as

$$F(x,t,\lambda) = \left[\Sigma,\ \Psi(x,t,\lambda) M(\lambda) \Psi^A(x,t,\lambda)\right] , \tag{5.30}$$

where Ψ^A denotes the adjugate matrix of Ψ. We can express it through its rows,

$$\Psi^A = \begin{pmatrix} {\psi^{A(1)}}^T \\ {\psi^{A(2)}}^T \\ {\psi^{A(3)}}^T \end{pmatrix} , \tag{5.31}$$

where the superscript T denotes transposition, transforming column vectors into rows. In this framework, the vectors $\psi^{A(j)}$ are given by the expression

$$\psi^{A(j)} = \psi^{(m)} \wedge \psi^{(n)} , \tag{5.32}$$

where $\{j,m,n\}$ is a cyclic permutation of $\{1,2,3\}$.

Since for a generic non-reduced $Q(x,t)$ and a given value of λ we need six eigenfunctions, we can represent the perturbation using the basis of matrices $M^{(jm)}$ given by

$$M_{ab}^{(jm)} = \delta_{ja}\delta_{mb}\,, \qquad j, m = 1, 2, 3, \qquad j \neq m\,, \tag{5.33}$$

where the deltas denote the Kronecker delta. Then we will denote

$$F^{(jm)} = \left[\Sigma,\, \Psi M^{(jm)} \Psi^A\right]. \tag{5.34}$$

The strength of this choice relies on the fact that

$$\Psi(x,t,\lambda) M^{(jm)} \Psi^A(x,t,\lambda) = \psi^{(j)} \left(\psi^{(j)} \wedge \psi^{(n)}\right)^T \epsilon_{jnm}\,, \tag{5.35}$$

where ϵ_{jnm} denotes the parity of the permutation $\{j, n, m\}$ of $\{1, 2, 3\}$ (that is, $\epsilon_{jnm} = 1$ if it is a cyclic permutation and $\epsilon_{jnm} = -1$ otherwise). We can then write $F^{(jm)}$ as

$$F^{(jm)}(x,t,\lambda) = \left[\Sigma,\, \psi^{(j)} \left(\psi^{(j)} \wedge \psi^{(n)}\right)^T\right] \epsilon_{jnm}\,. \tag{5.36}$$

With this representation, we can write the perturbation as

$$F(x,t,\lambda) = \sum_{j,m} \mu^{(jm)}(\lambda) F^{(jm)}(x,t,\lambda)\,, \tag{5.37}$$

where the six functions $\mu^{(jm)}(\lambda)$ play the role of a Fourier-like transform.

Let us illustrate how the method works by applying it to the plane wave solutions of the Yajima-Oikawa-Newell system introduced in Chapter 4.

5.2 The YON case

Let us recall the YON system introduced in Chapter 4,

$$iS_t + S_{xx} + \left(i\alpha L_x + \alpha^2 L^2 - \beta L - 2\alpha |S|^2\right) S = 0\,, \tag{5.38a}$$

$$L_t = 2\left(|S|^2\right)_x\,, \tag{5.38b}$$

which has a Lax pair

$$X(\lambda) = i\lambda\Sigma + Q\,, \qquad T(\lambda) = \lambda^2 T_2 + \lambda T_1 + T_0\,, \tag{5.39}$$

with

$$\Sigma = \mathrm{diag}\{1\,,\,0\,,\,-1\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5.40}$$

$$Q = \begin{pmatrix} 0 & S & iL \\ \alpha S^* & 0 & S^* \\ i\alpha^2 L - i\beta & \alpha S & 0 \end{pmatrix}, \tag{5.41}$$

which, using formulae (5.15), give rise to

$$T_2 = \frac{i}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & -S & 0 \\ -\alpha S^* & 0 & S^* \\ 0 & \alpha S & 0 \end{pmatrix}, \tag{5.42a}$$

$$T_0 = \begin{pmatrix} -i\alpha|S|^2 & -\alpha LS + iS_x & i|S|^2 \\ -\alpha^2 LS^* + \beta S^* - i\alpha S^*_x & 2i\alpha|S|^2 & -\alpha LS^* - iS^*_x \\ i\alpha^2|S|^2 & -\alpha^2 LS + \beta S + i\alpha S_x & -i\alpha|S|^2 \end{pmatrix}. \tag{5.42b}$$

We will study the plane waves of the system following the computations in [29], which have the general form

$$S(x,t) = ae^{i\theta} \quad L(x,t) = b\,, \quad \theta = qx - \nu t\,, \quad \nu = q^2 - \alpha^2 b^2 + \beta b + 2\alpha a^2\,, \tag{5.43}$$

where we have introduced three new arbitrary real parameters, namely, the two amplitudes a and b and the wavenumber of the short wave, q. The last formula of (5.43) shows the dispersion relation for the plane wave.

The first step to apply on the solution (5.43) the stability method introduced before is finding a corresponding fundamental solution $\hat{\Psi}(x,t,\lambda)$ of the Lax equations (5.1). One can find such solution with the form

$$\hat{\Psi}(x,t,\lambda) = e^{i\rho(\lambda)t} R(x,t) e^{i(xW(\lambda) - tW^2(\lambda))}\,, \qquad R(x,t) = \text{diag}\{1, e^{-i\theta}, 1\}\,, \tag{5.44}$$

where

$$\rho(\lambda) = \frac{2}{3}\lambda^2 + \alpha^2 b^2 - 2\alpha a^2 - \beta b, \tag{5.45}$$

and the x,t-independent matrix $W(\lambda)$ takes the form

$$W(\lambda) = \begin{pmatrix} \lambda & -ia & b \\ -i\alpha a & q & -ia \\ \alpha^2 b - \beta & -i\alpha a & -\lambda \end{pmatrix}. \tag{5.46}$$

Since $W(\lambda)$ has the trace properties

$$\text{tr}(W) = q\,, \qquad \text{tr}(W^2) = \nu + 3\rho\,, \tag{5.47}$$

then the solution $\hat{\Psi}(x,t,\lambda)$ in (5.44) has unit determinant.

For our purpose, and according to the formalism in the previous section, it is convenient to choose the alternative solution $\Psi(x,t,\lambda)$ whose column vectors $\psi^{(j)}$ are defined by

$$\psi^{(j)} = \hat{\Psi}(x,t,\lambda)\, f^{(j)}(\lambda)\,, \qquad j = 1,2,3, \tag{5.48}$$

where the three vectors $f^{(j)}(\lambda)$ are the eigenvalues of $W(\lambda)$,

$$W f^{(j)} = w_j f^{(j)}\,, \tag{5.49}$$

where w_j denotes the corresponding eigenvalue of $W(\lambda)$. We will consider them normalised so that

$$f^{(1)} \cdot \left(f^{(2)} \wedge f^{(3)}\right) = 1\,, \tag{5.50}$$

which entails the condition $\det\left(\Psi(x,t,\lambda)\right) = 1$. Proceeding this way, we can compute the column vectors $\psi^{(j)}$, which turn out to be

$$\psi^{(j)} = e^{i(\eta_j + \rho t)} R f^{(j)}\,, \tag{5.51}$$

with

$$\eta_j = w_j x - w_j^2 t\,, \qquad j = 1\,,2\,,3\,. \tag{5.52}$$

With this, we can use formula (5.36) to compute the eigenfunctions $F^{(jm)}(x,t,\lambda)$ corresponding to the plane wave solution at hand. With further simplifications coming from the fact that

$$\eta_1 + \eta_2 + \eta_3 = \theta - 3\rho t\,, \tag{5.53}$$

and the matrix relation (for any pair of arbitrary vectors u and v)

$$e^{i\theta}(Ru)(Ru \wedge Rv)^T = R\left[u(u \wedge v)^T\right] R^{-1}\,, \qquad R(x,t) = \mathrm{diag}\left\{1, e^{-i\theta}, 1\right\}\,, \tag{5.54}$$

we get the expression

$$F^{(jm)}(x,t,\lambda) = e^{i(\eta_j - \eta_m)}\, R\left[\Sigma\,,\, f^{(j)}(f^{(j)} \wedge f^{(n)})^T\right] R^{-1}\, \epsilon_{jnm}\,, \qquad j \neq m\,, \tag{5.55}$$

where, as before, ϵ_{jnm} denotes the parity of the permutation $\{j, n, m\}$.

This shows that, apart from the phase θ, which is independent of λ, the eigenfunctions $F^{(jm)}(x, t, \lambda)$ depend on x and t only through the exponentials $e^{i(\eta_j - \eta_m)}$, which, thanks to the expression (5.52), one can write in the form

$$e^{\pm i(k_n x - \omega_n t)}\,, \quad k_n = w_{n+1} - w_{n+2}\,, \quad \omega_n = w_{n+1}^2 - w_{n+2}^2\,, \quad n = 1, 2, 3 \mod 3\,, \tag{5.56}$$

where the wave numbers $k_j(\lambda)$ and the corresponding frequencies $\omega_j(\lambda)$ are defined in terms of the eigenvalues w_j of the matrix W, that is, the roots of the characteristic polynomial

$$\begin{aligned} P(w, \lambda) = \det[w\,\mathbb{1} - W(\lambda)] &= (w - w_1)(w - w_2)(w - w_3) \\ &= (w - q)(w^2 - \lambda^2 + p) + r\,, \end{aligned} \tag{5.57}$$

where the parameters p and r are defined as

$$p = 2\alpha a^2 - \alpha^2 b^2 + \beta b = \nu - q^2\,, \qquad r = a^2[2\alpha(q + \alpha b) - \beta]\,. \tag{5.58}$$

Since we require our solutions to be bounded functions of x, we can conclude that the set of values of λ that allows us to construct valid perturbations is the subset of the complex λ-plane where at least one of the wave numbers $k_j(\lambda)$ is real. That leads us to the following definition.

Definition 5.2.1 *The stability spectrum $\mathbb{S}_x$ is defined as the set of complex values of λ for which at least one of the three wave number functions $k_1(\lambda)$, $k_2(\lambda)$, and $k_3(\lambda)$ is real.*

The dispersion relation between the wave number k_j and the corresponding frequency ω_j is parametrically defined by the functions $k_j(\lambda)$ and $\omega_j(\lambda)$ by varying the spectral parameter λ over the stability spectrum $\mathbb{S}_x$. The corresponding plane wave will be linearly stable if $\omega_j(\lambda)$ is real for the values of j such that $k_j(\lambda)$ is real for every λ in the spectrum $\mathbb{S}_x$. Otherwise, if for some j and $\lambda \in \mathbb{S}_x$, $\omega_j(\lambda)$ is non-real, then the plane wave solution is linearly unstable. In that case, the relevant physical information of the instability is given by the gain function,

$$\Gamma_j(\lambda) = |\mathrm{Im}(\omega_j(\lambda))| \,, \qquad \lambda \in \mathbb{S}_x \,, \qquad \mathrm{Im}(k_j(\lambda)) = 0 \,. \tag{5.59}$$

We can then proceed to classify the different stability spectra for each choice of parameters and the corresponding gain functions. Note that the characteristic polynomial $P(w, \lambda)$ in (5.57) depends only on the wave number q of the short wave solution, the parameters r and p defined in (5.58), and the spectral parameter λ only via the combination $\lambda^2 - p$ (which, in turn, is also the only appearance of p). Because of that, it is sufficient to fix the values of q and r in the characteristic polynomial and redefine the stability spectrum as a curve in the complex plane of $\lambda^2 - p$. By doing so, our parameter space reduces to the (q, r)-plane. We will then introduce the more convenient, complex variable Λ as

$$\Lambda = \lambda^2 - p \,, \tag{5.60}$$

so that the parameter p becomes irrelevant for the classification of spectra. In a minor abuse of notation, we will refer to the characteristic polynomial as a function of Λ as

$$\begin{aligned} P(w, \Lambda) = \det[w\,\mathbf{1} - W] &= (w - w_1)(w - w_2)(w - w_3) \\ &= (w - q)(w^2 - \Lambda) + r \,. \end{aligned} \tag{5.61}$$

To make the change of variable explicit, we will denote by $\mathbb{S}^\Lambda$ the stability spectrum in the complex Λ-plane through the following definition.

Definition 5.2.2 *The stability spectrum $\mathbb{S}^\Lambda$ is defined as the set of complex values of Λ for which at least one of the three wave number functions $k_1(\Lambda)$, $k_2(\Lambda)$, and $k_3(\Lambda)$ is real*

Let us also note that the parameter q, if non-zero, can be rescaled to $q = 1$ by rescaling w by q, Λ by q^2, and r by q^3, that is, through the change of variables

$$w \to qw \,, \qquad \Lambda \to q^2\Lambda \,, \qquad r \to q^3 r \,. \tag{5.62}$$

We will however keep q in the formulae so that we can set $q = 1$ numerically when desired, and study separately the case $q = 0$.

Let us consider first the part of the spectrum $\mathbb{S}^\Lambda$ that lies on the real axis, $\operatorname{Im}\Lambda = 0$. In that case, all the coefficients of the characteristic polynomial $P(w, \Lambda)$ are real, and hence either the three zeros $w_1(\Lambda)$, $w_2(\Lambda)$, and $w_3(\Lambda)$ are real, or one is real and two are complex conjugate. In the first case, the three wave numbers $k_j(\Lambda)$ are real, while in the second one none of them is, leading to the following.

Proposition 5.2.3 *If Λ is real then it belongs to the spectrum $\mathbb{S}^\Lambda$ if and only if the w-discriminant of the polynomial (5.61) is non-negative, that is, if $\Delta_w P(w, \Lambda) \geqslant 0$, where*

$$\Delta_w P(w, \Lambda) = k_1^2 k_2^2 k_3^2 = 4\Lambda^3 - 8q^2\Lambda^2 + 4q(q^3 - 9r)\Lambda - 27r^2 + 4rq^3 . \tag{5.63}$$

As shown by the expression (5.63), the large and positive real values of Λ do always belong to the spectrum $\mathbb{S}^\Lambda$, while the large and negative real values of Λ do not. We can study the asymptotics of the equation $P(w, \Lambda) = 0$ around the point at infinity of the complex Λ-plane to obtain the behaviours of the roots w_j:

$$\begin{aligned} w_1(\Lambda) &= \sqrt{\Lambda} - \tfrac{r}{2\Lambda} + O(1/\Lambda^{3/2}) \\ w_2(\Lambda) &= -\sqrt{\Lambda} - \tfrac{r}{2\Lambda} + O(1/\Lambda^{3/2}) \\ w_3(\Lambda) &= q + \tfrac{r}{\Lambda} + O(1/\Lambda^2) , \end{aligned} \tag{5.64}$$

where the labels $1, 2, 3$ are arbitrary. It is clear from (5.64) that if Λ is real, large and positive, then also the wave numbers $k_j(\Lambda)$ are real and large. Conversely, if Λ is real, large and negative, then none of the wave numbers $k_j(\Lambda)$ are real and hence Λ does not belong to the spectrum.

Let us turn our attention to the w-discriminant (5.63). It is a cubic polynomial of Λ with real coefficients, so it can either have one or three real zeros. If it only has one zero, say Λ_+, then the discriminant $\Delta_w P(w, \Lambda)$ must be negative for all values $\Lambda < \Lambda_+$ and positive for $\Lambda > \Lambda_+$. That means that the real part of the spectrum $\mathbb{S}^\Lambda$ is the semi-axis $\Lambda_+ \leqslant \Lambda < \infty$. If, on the contrary, $\Delta_w P(w,\Lambda)$ has three real roots, say $\Lambda_0 < \Lambda_- < \Lambda_+$, then the discriminant will be negative both for $\Lambda < \Lambda_0$ and for $\Lambda_- < \Lambda < \Lambda_+$, meaning that the real part of the spectrum $\mathbb{S}^\Lambda$ consists on the finite interval $\Lambda_0 \leqslant \Lambda \leqslant \Lambda_-$ and the semi-axis $\Lambda_+ \leqslant \Lambda < \infty$. This finite gap between the different pieces of the real part of the spectrum will be a distinctive feature of the topology of the spectra in our classification. Its existence is determined by the Λ-discriminant of $\Delta_w P(w,\Lambda)$, that is $\Delta_\Lambda \Delta_w P(w,\Lambda)$, which depends only on the two parameters q and r, and can be summarised in the following result.

Proposition 5.2.4 *Let $\Delta_\Lambda \Delta_w P(w,\Lambda)$ be the Λ-discriminant of the discriminant* (5.63)*, that is,*

$$\Delta_\Lambda \Delta_w P(w,\Lambda) = 16r(8q^3 - 27r)^3 \,. \tag{5.65}$$

The spectrum $\mathbb{S}^\Lambda$ has one, and only one, finite gap (G) on the real axis if and only if $\Delta_\Lambda \Delta_w P(w,\Lambda) > 0$, namely, if and only if $r(8q^3 - 27r) > 0$, and it has no gap if $r(8q^3 - 27r) < 0$. The gap opening and closing threshold values of the parameters are $r = 0$ and $r = (8/27)\,q^3$.

Now that we have completed the study on the real part of the spectrum, we can switch our attention to the non-real values of Σ that yet belong to the spectrum, $\Sigma \in \mathbb{S}^\Lambda$ with $\mathrm{Im}(\Lambda) \neq 0$. For this we will introduce what we call the *polynomial of the squares of the differences*,

$$\mathcal{P}(\zeta,\Lambda) = (\zeta - k_1^2)(\zeta - k_2^2)(\zeta - k_3^2) = \zeta^3 + \gamma_2\,\zeta^2 + \gamma_1\,\zeta + \gamma_0\,, \tag{5.66a}$$

whose the roots $\zeta_j(\Lambda),\ j = 1,2,3$, are the squares of the differences of the Λ-dependent roots $w_j(\Lambda)$ of the characteristic polynomial $P(w,\Lambda)$,

$$\zeta_j(\Lambda) = k_j^2(\Lambda) = (w_{j+1} - w_{j+2})^2\,, \quad j = 1,\,2,\,3 \mod 3\,. \tag{5.66b}$$

The coefficients of the polynomial $\mathcal{P}(\zeta,\Lambda)$ (5.66a) can be computed explicitly in terms of the coefficients of the polynomial $P(w,\Lambda)$ (5.61) using symmetry properties (see [54]). Let us consider a generic monic polynomial

$$f(x) = x^N + f_{N-1}x^{N-1} + \ldots + f_0\,, \tag{5.67}$$

and try to construct a new polynomial $g(y)$ whose roots are the squared differences of the N roots x_j of $f(x)$, that is,

$$g(y) = \prod_{j<m}\Big[y - (x_j - x_m)^2\Big]\,. \tag{5.68}$$

Since the roots of $g(y)$ are, by construction, quadratic symmetric functions of the roots x_j of $f(x)$, the coefficients of $g(y)$ are necessarily polynomials of the coefficients of $f(x)$. In particular, for the case $N = 3$, $g(y)$ is also a cubic polynomial,

$$g(y) = y^3 + g_2y^2 + g_1y + g_0\,, \tag{5.69}$$

and its coefficients are given by Vieta's formulae,

$$\begin{aligned} g_2 &= 2\big(3f_1 - f_2^2\big)\,,\\ g_1 &= \big(3f_1 - f_2^2\big)^2\,,\\ g_0 &= 27f_0^2 + 4f_1^3 - 18f_0f_1f_2 + 4f_0f_2^3 - f_1^2f_2^2\,. \end{aligned} \tag{5.70}$$

Note that these formulae give $g(y)$ the non-generic form $g(y) = y(y + g_2/2)^2 + g_0$. Also note that, when $y = 0$ in (5.68), we have that

$$-g(0) = \Delta_x(f) = \prod_{j<m}(x_j - x_m)^2\,. \tag{5.71}$$

By using the formulae (5.70) with the coefficients of $P(w,\Lambda)$ in (5.61), one can compute the coefficients γ_j of the polynomial of the squares of the differences (5.66a), which turn out to be

$$\begin{aligned} \gamma_2 &= -2(3\Lambda + q^2)\,,\\ \gamma_1 &= (3\Lambda + q^2)^2\,,\\ \gamma_0 &= -4\Lambda^3 + 8q^2\Lambda^2 - 4q(q^3 - 9r)\Lambda + 27r^2 - 4rq^3\,. \end{aligned} \tag{5.72}$$

Also the property (5.71) tells us that the w-discriminant (5.63) is related to this new polynomial by

$$\Delta_w P(w, \Lambda) = -\mathcal{P}(0, \Lambda)\,. \tag{5.73}$$

Relation (5.73) allows us to rewrite the Proposition 5.2.4 for characterisation of gaps in terms of the new polynomial $\mathcal{P}$.

We now get to the main point of the method, which is that we can also redefine our stability spectrum $\mathbb{S}^\Lambda$ by changing the point of view on $\mathcal{P}(\zeta, \Lambda)$, and instead of seeing it as a function of ζ with coefficients on Λ we can see it as a function of Λ with coefficients in ζ, or rather as an implicit equation of a curve, similar to an algebraic curve. By doing that, we can express the stability spectrum as the locus of the Λ-zeros of $\mathcal{P}(\zeta, \Lambda)$ corresponding to a real, non-negative value of the variable $\zeta \geqslant 0$. We can then construct the stability spectrum by moving ζ in the interval $[0, \infty)$ and, for each value of ζ, computing the three Λ-roots of $\mathcal{P}(\zeta, \Lambda)$. Let us note that the polynomial of the squares of the differences has the expression as a polynomial in Λ

$$\begin{aligned}\mathcal{P}(\zeta, \Lambda) &= -4\,[\Lambda - \Lambda_1(\zeta)]\,[\Lambda - \Lambda_2(\zeta)]\,[\Lambda - \Lambda_3(\zeta)] \\ &= -4\Lambda^3 + \Lambda^2(9\zeta + 8q^2) - 2\Lambda(3\zeta^2 - 3q^2\zeta + 2q^4 - 18qr) \\ &\quad + \zeta^3 - 2q^2\zeta^2 + q^4\zeta + 27r^2 - 4q^3 r\,,\end{aligned} \tag{5.74}$$

which, assuming ζ is real, has real coefficients, and hence the following.

Proposition 5.2.5 *The stability spectrum $\mathbb{S}^\Lambda$ is symmetric with respect to the real axis.*

That means that the stability spectrum is a symmetric piecewise smooth curve in the complex Λ-plane. Switching from a triplet of real Λ-roots of $\mathcal{P}(\zeta, \Lambda)$ to a pair of complex conjugate roots and one real root (or vice versa) when moving ζ from 0 to ∞ comes from a collision of two real (or two complex conjugate) Λ-roots. That happens at a zero ζ_j of the discriminant

$$Q(\zeta) = \Delta_\Lambda \mathcal{P}(\zeta, \Lambda)\,, \tag{5.75}$$

that is, when $Q(\zeta_j) = 0$, provided the discriminant changes sign. The polynomial $Q(\zeta)$ turns out to be of fifth degree. It factorises as

$$
\begin{aligned}
Q(\zeta) &= 4\,Q_1^2(\zeta)\,Q_2(\zeta) \\
Q_1(\zeta) &= 18q\zeta + 27r - 8q^3, \\
Q_2(\zeta) &= \zeta^3 - 8q^2\zeta^2 + 8q(2q^3 - 9r)\zeta + 4r(8q^3 - 27r)\,.
\end{aligned}
\tag{5.76}
$$

After a collision where the discriminant $Q(\zeta)$ changes sign, two real Λ-roots scatter off the real axis, or two complex conjugate Λ-roots scatter into the real axis. This makes the factor $Q_1^2(\zeta)$ irrelevant for our analysis since it does not encode a change of sign. It is worth noting, however, that, for more convoluted systems where $\mathcal{P}(\zeta, \Lambda)$ is of a higher degree, components that do not encompass changes of sign may be relevant to characterise meaningful collisions of roots taking place off the real axis, or even several simultaneous collisions in the real axis.

Be that as it may, for the case at hand we only need to focus on $Q_2(\zeta)$. It is a cubic polynomial with real coefficients, so its three roots ζ_1, ζ_2, and ζ_3 can either be all real if the discriminant of $Q_2(\zeta)$ is positive,

$$
\Delta_\zeta Q_2(\zeta) = 16r\big(16q^3 - 27r\big)^3 > 0\,, \tag{5.77}
$$

or two complex conjugate and one real if the discriminant is negative.

We can also write our condition for the existence of a gap in terms of $Q_2(\zeta)$, reformulating it as $Q_2(0) = -\zeta_1\zeta_2\zeta_3 = 4r(8q^3 - 27r) > 0$. It is more convenience for the study of the condition to consider the sign of qr instead of the sign of r and q separately. In particular, we can use the condition $q^2 Q_2(0) > 0$, which translates into

$$
(qr)\big[8q^4 - 27(qr)\big] > 0\,, \tag{5.78}
$$

which is never satisfied if $qr < 0$ and, in fact, is only satisfied if $0 < qr < (8/27)q^4$.

We can do something similar with condition (5.77) on the discriminant of $Q_2(\zeta)$ to transform it into the condition

$$
(qr)\big[16q^4 - 27(qr)\big] > 0\,, \tag{5.79}
$$

which in turn gives the condition $0 < qr < (16/27)q^4$. If that is the case, then for large enough values of ζ, namely $\zeta_3 < \zeta < \infty$, as well as for $\zeta_1 < \zeta < \zeta_2$, $Q_2(\zeta)$ is positive, while for $\zeta_2 < \zeta < \zeta_3$ and for $-\infty < \zeta < \zeta_1$, $Q_2(\zeta)$ is negative. We can now analyse all the possible cases according to the sign of the ζ-roots of $Q_2(\zeta)$.

(i) $\zeta_1 < \zeta_2 < \zeta_3 < 0$: this case (all three roots are negative) is not allowed due to Vieta's relation $\zeta_1 + \zeta_2 + \zeta_3 = 8q^2$.

(ii) $\zeta_1 < \zeta_2 < 0 < \zeta_3$: Vieta's relation $\zeta_1\zeta_2\zeta_3 = 4r(27r - 8q^3)$, implies $(qr)\left[27(qr) - 8q^4\right] > 0$, while the positive discriminant condition (5.79) does not allow qr to be negative. That means that only the interval $(8/27)q^4 < qr < (16/27)q^4$ is allowed. In that case, two Λ-roots collide for $\zeta = \zeta_3$ and, for $0 \leqslant \zeta < \zeta_3$, the spectrum exhibits two complex conjugate curves in the Λ-plane. We refer to this complex part of the spectrum as branch (B), since the two end-points at $\zeta = 0$ of the two Λ-roots trajectories do not generically coincide with each other (see Figure 5a below). Gaps are not allowed in this case, which we denote as 0G 1B 0L, or B-type.

(iii) $\zeta_1 < 0 < \zeta_2 < \zeta_3$: the same Vieta's relation above requires $0 < qr < (8/27)q^4$, which is also compatible with the positive discriminant condition (5.79). Two Λ-roots collide for $\zeta = \zeta_3$, get off the real axis and collide again for $\zeta = \zeta_2$ on the real Λ-axis thereby forming one complex closed curve, which we term loop (L) (see Figure 6a below). In this case there exist no branches. However, a gap does exist since its existence condition (5.78) is satisfied. Thus, we denote this spectrum type 1G 0B 1L, or LG-type.

(iv) $0 < \zeta_1 < \zeta_2 < \zeta_3$: this case is not allowed by two Vieta relations, which lead to the inequalities $6q^4 - 27(qr) > 0$ and $(qr)\left[27(qr) - 8q^4\right] > 0$, and by the positive discriminant condition (5.79).

Let us now consider the spectra $\mathbb{S}^\Lambda$ whose parameters q, r satisfy the negative discriminant inequality, namely $(qr)\left[16q^4 - 27(qr)\right] < 0$. In that case only one ζ-root, say ζ_3, is real, while the other two are complex conjugate roots. Again we will study the possible cases according to the sign of ζ_3.

(i) $\zeta_3 < 0$: not possible since Vieta's relations for the polynomial $Q_2(\zeta)$, see (5.76), and the sign of the discriminant $\Delta_\zeta Q_2(\zeta)$ are never compatible.

(ii) $\zeta_3 > 0$: again, combining Vieta's relations with the negative discriminant condition shows that this is possible only for $qr < 0$ or $qr > (16/27)q^4$. The resulting spectrum is of type 0G 1B 0L.

With all the observations above we can finalise our full characterisation of the possible spectra, which is shown in Table 3. As explained before, we have completed the computation assuming

Table 3: Stability spectra

$qr < 0$	$0 < qr < \frac{8}{27}q^4$	$\frac{8}{27}q^4 < qr$
0G 1B 0L	1G 0B 1L	0G 1B 0L

$q \neq 0$ (so that, for the sake of plotting, we can just take $q = 1$ and it will be equivalent to any other non-zero value). For completeness, we can study the non-general choice $q = 0$ separately, assuming $r \neq 0$. In that case the polynomial of the squares of the differences $\mathcal{P}(\zeta, \Lambda)$ becomes

$$\mathcal{P}(\zeta, \Lambda) = -\left(4\Lambda - \zeta\right)\left(\Lambda - \zeta\right)^2 + 27r^2\,, \tag{5.80}$$

and hence $\mathcal{P}(\zeta, \Lambda) = -4\Lambda^3 + 27r^2$. By means of the Proposition 5.2.3, the real part of the spectrum is the semi-axis $\Lambda_0 \leqslant \Lambda < \infty$ with $\Lambda_0 = \left[(27/4)r^2\right]^{1/3}$. For the complex part of the spectrum, let us note that $Q_2(\zeta) = \zeta^3 - 108r^2$, which means that the only real zero of $Q_2(\zeta)$ is the positive number $\zeta_3 = \left(108r^2\right)^{1/3}$. For this value, the polynomial $\mathcal{P}(\zeta_3, \Lambda)$ has a double real Λ-root Λ_B, which can be found to be $\Lambda_B = (1/2)\zeta_3 = 2^{1/3}\Lambda_0$. As ζ is moved back from ∞, two Λ-roots collide at Λ_B when $\zeta = \zeta_3$ and then scatter off the real axis to produce a branch, whose end-points for $\zeta = 0$ are located at the complex conjugate points $\Lambda_0 e^{2i\pi/3}$ and $\Lambda_0 e^{-2i\pi/3}$. That means that for $q = 0$, $r \neq 0$, the spectrum is of type 0G 1B 0L.

The only additional choices left to study are the non-generic thresholds between regions, in particular $qr = 0$ and $qr = (8/27)q^4$. We have already seen what happens when $q = 0$ and $r \neq 0$, we can now look at the spectrum when $r = 0$. In that case, the characteristic polynomial

(5.61) becomes

$$P(w,\Lambda) = (w-q)(w^2-\Lambda)\,, \tag{5.81}$$

making the roots explicit. The corresponding spectrum is entirely real, which we will denote as 0G 0B 0L. The wave number functions have the expression

$$k_1(\Lambda) = -q-\sqrt{\Lambda}\,,\quad k_2(\Lambda) = q-\sqrt{\Lambda}\,,\quad k_3(\Lambda) = 2\sqrt{\Lambda}\,,\quad 0 \leqslant \Lambda < +\infty\,. \tag{5.82}$$

The last choice of parameters we have to check is $27r - 8q^3 = 0$, where a gap disappears. In that case the discriminant $Q(0)$ goes to zero, meaning that the polynomial of the squares of the differences,

$$\mathcal{P}(0,\Lambda) = -4\left[\Lambda + \frac{1}{3}q^2\right]^2\left[\Lambda - \frac{8}{3}q^2\right]\,, \tag{5.83}$$

has a double Λ-root, $\Lambda_D = -(1/3)q^2$, where a branch closes up and becomes a loop. The corresponding spectrum can be classified as 1G 0B 1L, where the real segment before the loop is reduced to only a point.

Now, once we have completed the study of the stability spectrum, regarding the space variable x, we can switch our attention to the time variable t to look at the stability of the perturbations. That means we have to investigate whether the frequencies $\omega_1(\Lambda)$, $\omega_2(\Lambda)$, and $\omega_3(\Lambda)$ (see (5.56)) are real numbers for $\Lambda \in \mathbb{S}^\Lambda$. We can write their relation with the wave numbers $k_j(\Lambda)$ as

$$\omega_j = \frac{1}{3}k_j\left(2q + k_j + 2k_{j+1}\right)\,,\qquad j = 1,2,3 \bmod 3\,, \tag{5.84}$$

which can be derived from their definition (5.56) by first inverting the map $\{w_j\} \to \{k_j\}$, that is

$$w_j = \frac{1}{3}\left(q + k_j + 2k_{j+2}\right)\,,\qquad j = 1,2,3 \bmod 3\,. \tag{5.85}$$

However, this is not a dispersion relation, since every frequency ω_j is written in terms of two different wave numbers, k_j and k_{j+1}. We can obtain a proper dispersion relation by eliminating the variable Λ among the algebraic relations $P(w,\Lambda) = 0$, $\mathcal{P}(\zeta,\Lambda)$, and $\omega_j = k_j(q - w_j)$. In order to do that, we also need to introduce the polynomial of the squares of the differences for W^2, which we will denote by $\mathcal{R}(\xi, \Lambda)$,

$$\mathcal{R}(\xi, \Lambda) = (\xi - \omega_1^2)(\xi - \omega_2^2)(\xi - \omega_3)^2 = \xi^3 + \delta_2\xi^2 + \delta_1\xi + \delta_0\,. \tag{5.86}$$

where the coefficients can be computed using Vieta's relations (5.70):

$$\begin{aligned}
\delta_2 &= -2\left(q^4 - 6qr - 2q^2\Lambda + \Lambda^2\right),\\
\delta_1 &= \left(q^4 - 6qr - 2q^2\Lambda + \Lambda^2\right)^2,\\
\delta_0 &= r^2\left(-4q^3r + 27r^2 - 4q^4\Lambda + 36qr\Lambda + 8q^2\Lambda^2 - 4\Lambda^3\right)\,,
\end{aligned} \tag{5.87}$$

The problem then reduces to finding a Gröbner basis for our set of polynomials, which can be obtained via standard methods, e. g. Buchberger's algorithm or Faugère's algorithms [22, 64, 65]. With that, we can compute the three-branch dispersion relation $H(\pm k_j, \pm\omega_j) = 0$, where $H(k, \omega)$ has the expression

$$H(k, \omega) = \omega^3 - 4qk\omega^2 + k^2\left(4q^2 - k^2\right)\omega - 4rk^3\,. \tag{5.88}$$

Note here that the dispersion relation (5.88) can also be obtained via a stardard Fourier approach to the linearised equations. However, the Fourier expansion approach can only be used for plane wave, whereas the approach shown in this chapter has the potential to be applied to broader classes of solutions, which will be a topic of further research. Further reasons justifying the convenience to use this approach are explained in Section 5.3.

Now that we have the dispersion relation, let us study the stability for generic values of q and r. We have seen that for any generic value of q and r, the stability spectrum exhibits both a real component and a complex, non-real one. For each point Λ of the real part of the spectrum, all three wave numbers $k_1(\Lambda)$, $k_2(\Lambda)$, and $k_3(\Lambda)$ are real. That means, by using the expression (5.84), that the corresponding frequencies $\omega_j(\Lambda)$ are also real, so the perturbations constructed using only the real part of the spectrum are bounded in t and thus do not cause instabilities.

On the other hand, for points Λ on the non-real part of the spectrum (that is, either on a branch or on a loop), only one of the three wave numbers, say $k_3(\Lambda)$, is real, whereas the other two are non-real. Again by using the expression (5.84), the corresponding frequency $\omega_3(\Lambda)$ is then non-real, since its real part is

$$\mathrm{Re}(\omega_3) = \frac{1}{3}k_3\,(2q + k_3) + \frac{2}{3}k_3\,\mathrm{Re}(k_1)\,. \tag{5.89}$$

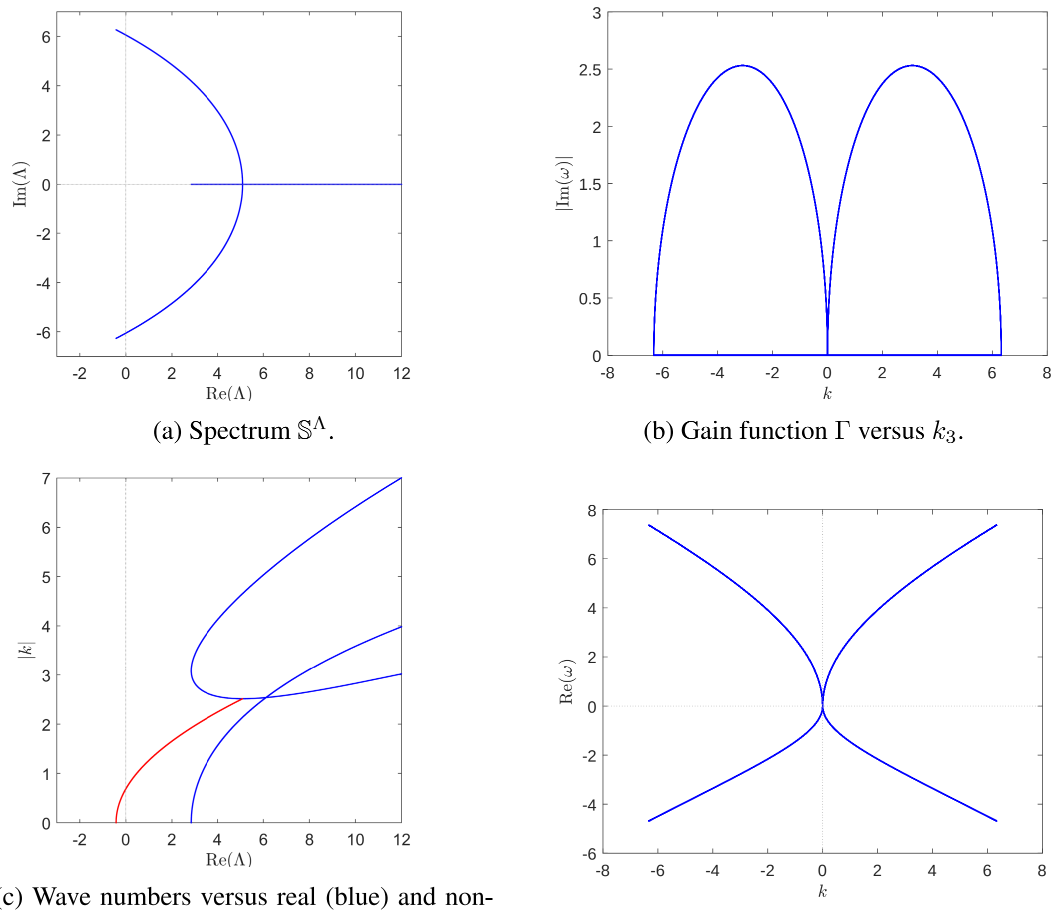
while its imaginary part, which produces the instability, has the form

$$\Gamma_3(\lambda) = |\mathrm{Im}(\omega_3)| = \frac{2}{3}\,|k_3\,\mathrm{Im}(k_1)|\;, \tag{5.90}$$

which is non-zero as long as Λ is off the real axis. We can conclude the analysis of stability with the following result.

Proposition 5.2.6 *All stability spectra $\mathbb{S}^\Lambda$ are classified with respect to the parameters q and $r \neq 0$ in two types: the B-type, containing a real part and one branch, and the LG-type, containing a real part with one finite gap, and one loop. Only for $r = 0$ the spectrum is totally real with no complex part. The plane wave solutions are then linearly stable if and only if $r = 0$, with $\omega_1 = k_1^2 + 2qk_1$, $\omega_2 = -k_2^2 + 2qk_2$, and $\omega_3 = 0$, and is otherwise unstable for every value of q and every non-vanishing value of r.*

Examples of these two types of spectra have been numerically computed. In Figure 5, B-type spectrum is presented (Figure 5a); with the corresponding gain function Γ on the branch (Figure 5b), which proves that this instability is of baseband type, that is, waves are unstable around $|k| = 0$; the functions $k_j(\Lambda)$, for $j = 1, 2, 3$, if Λ is real, together with the function $k_3(\mathrm{Re}(\Lambda))$ on the branch (Figure 5c); the real frequency on the branch as function of $k = k_3$ (Figure 5d). The same functions are plotted in Figures 6b, 6c and 6d, for an LG-type spectrum, which is shown in Figure 6a. In particular, Figure 6b shows that in this case the instability is of passband type, namely, waves are stable for sufficiently small values of $|k|$.



(a) Spectrum $\mathbb{S}^{\Lambda}$.

(b) Gain function Γ versus k_3.

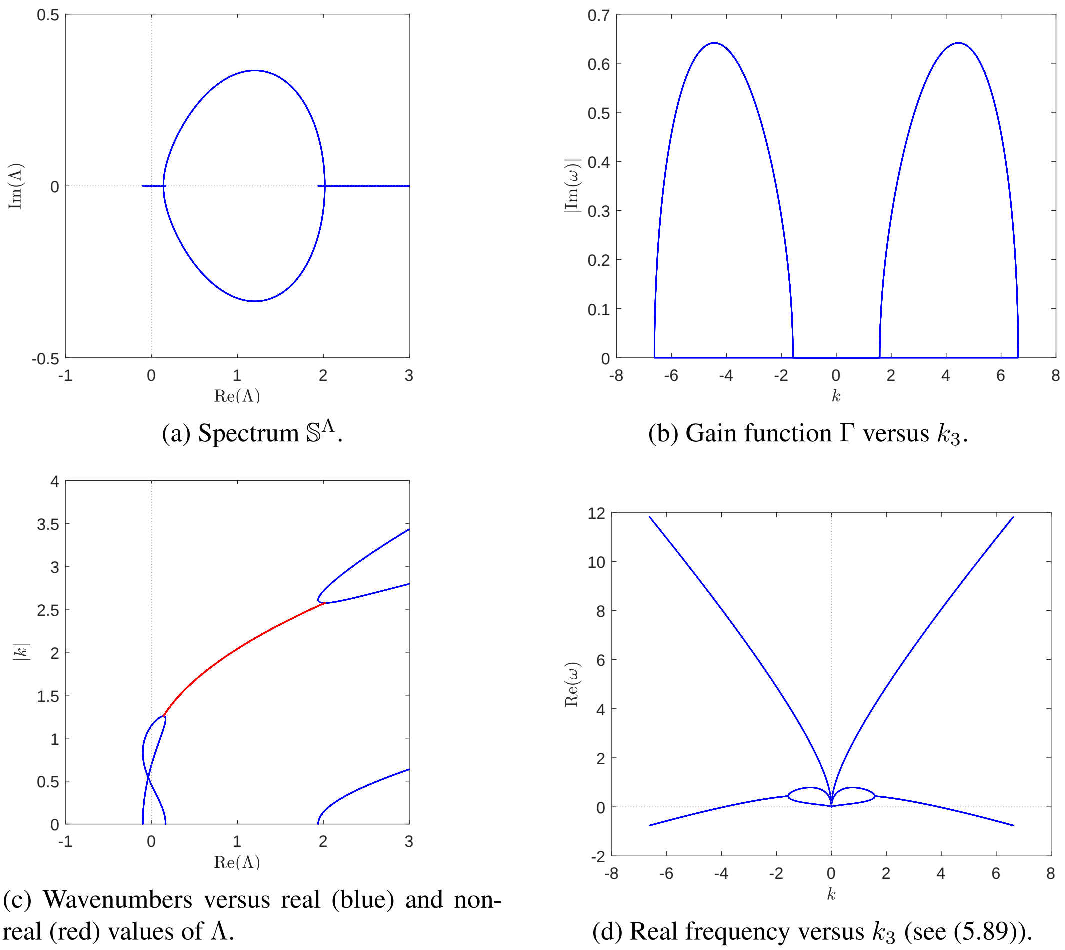
(c) Wave numbers versus real (blue) and non-real (red) values of Λ.

(d) Real frequency versus k_3 (see (5.89)).

Figure 5: B-type spectrum, $q = 1$, $r = -4$.

5.3 Stability and rogue waves

One relevant application of the method is that the classification of the spectra in having either passband or baseband instability gives a straightforward prediction on the existence of rogue waves understood as rational solitons. As shown in the literature, baseband modulation instability is one of the proposed mechanisms for the origin of rogue waves [13, 14, 152]. Using the stability spectra, the existence of baseband instability is associated to the existence of branches in the spectrum, which enables to relate the choices of parameters that allow for rogue waves with the topological properties of the stability spectrum. In fact, we propose the following conjecture, which is yet to be proven rigorously but holds true for all the systems studied and has some geometrical intuition.



(a) Spectrum $\mathbb{S}^{\Lambda}$.

(b) Gain function Γ versus k_3.

(c) Wavenumbers versus real (blue) and non-real (red) values of Λ.

(d) Real frequency versus k_3 (see (5.89)).

Figure 6: LG-type spectrum, $q = 1$, $r = 0.1$.

Conjecture 5.3.1 *The existence of branches in a stability spectrum for a plane wave is equivalent to the existence of rational solitons constructed using that same plane wave as background. The value of the spectral parameter λ at the end of the branches coincides exactly with the value of λ that allows one to turn breathers into rational solitons in the Darboux-dressing framework.*

This conjecture, if true, tells us that not only can we predict the existence of rogue waves from the shape of the spectrum, but it also gives the exact value of the spectral parameter we have to use to construct them via spectral methods. That would be a strong point remarking the convenience of using the approach in this chapter instead of simpler methods like a Fourier expansion. We also have an addition conjecture, still not proven rigorously, regarding the existence of gaps.

Conjecture 5.3.2 *The existence of gaps or, otherwise, parts of the real axis not included in* *the stability spectrum is related to the existence of dark solitons having that plane wave as background. The values of the spectral parameter λ in the gap can be used to construct the dark soliton solution via spectral methods.*

Note that in this case we do not propose an equivalence, given that for some systems there exist dark solitons that use values of the spectral parameter out of the gaps for their construction. For example, for the YON system both the finite gaps and infinite gaps in the Λ-spectra, which may be used to construct dark solitons, can move to the imaginary axis for big enough values of p when going back to the original spectral parameter λ, which serves as a counterexample.

A rigorous proof and, if true, a geometric analysis for these conjectures are yet to be produced, and are a topic of further research.

Summarising, some strenghts of the method introduced in this section in comparison with Fourier-based methods are:

- It is better suited to deal with multicomponent systems, which are often hard to treat or untreatable with other methods.
- It provides additional information in the form of the stability spectrum, which allows to predict the types of instability that the system presents without the need to derive the gain function.
- The stability spectrum can also potentially provide predictions for the existence of special types of solution, such as rational solutions or dark solitons.
- Although further research is needed, the links between its topology and the behaviour of the system may provide deeper insights on the nature of integrability from the geometric point of view.

Chapter 6

Hirota Bilinear Method for the YON System

The last part of the thesis will be devoted to the application of Hirota bilinearisation techniques (see [88, 89, 90, 91, 93]) to obtain general classes of soliton and rational soliton solutions for the YON system. Multiple techniques have been developed throughout the years to construct solutions in closed form for nonlinear equations of integrable type. Some of the most notable methods include inverse scattering techniques [1, 2, 3, 6, 78, 126, 131], Bäcklund transformations [85, 140], and Darboux methods [111, 156]. Nevertheless, the Hirota bilinear method has two very important points that distinguish it from the methods before: it is not an analytical method but an algebraic one, and it is not a spectral method, meaning that it does not rely on the existence of a Lax pair to work. In fact, some authors claim it can even be applied to non-integrable systems [113, 137]; however, other results seem to point otherwise, finding evidence that the existence of multisoliton solutions via Hirota bilinearisation may indeed be a characterisation of integrability (see [87]). Studying this characterisation and the connections between the method and the classical definitions of integrability is currently an open field of research. The construction of a Hirota bilinearisation of a system is also useful for constructing Bäcklund transformations [92].

The specific approach we are going to take, known as the method of τ-functions, makes use of the Kadomtsev-Petviashvili (KP) equation (see [98]) and the discrete KP (dKP) equation, also known as Hirota-Miwa (HM) equation (see [93, 127]) to write the corresponding bilinear forms as elements of the KP hierarchy. It has been succesfully applied to several integrable systems including the massive Thirring model [38], the semidiscrete Camassa-Holm equation [134, 143], the complex short pulse equation [72], the Sasa-Satsuma equation [73, 158], the Yajima-Oikawa system [35], the Newell system [34, 39], the vector nonlinear Schrödinger (NLS) equation, also known as Manakov system [66], the discrete NLS equation [132], the generalised derivative NLS equation [37], the nonlocal NLS equation, also known as Ablowitz-Musslimani equation [68], the multicomponent coupled Ito equation [36], the Degasperis-Procesi equation [70, 71], or the reduced Ostrovsky equation [69], among others.

6.1 The Hirota bilinear method

A very nice introduction to the Hirota bilinear method is presented in [86]. We will follow results from that paper along with workshop notes [67] that Baofeng Feng kindly provided to give some insights on the topic.

The first step in the method consists on bilinearising the equation, which means rewriting the original equations with Hirota's bilinear operator D,

$$D_x^n f \cdot g = (\partial_x - \partial_y)^n f(x)g(y)\big|_{x=y}\,, \tag{6.1}$$

instead of the regular derivatives. Let us illustrate how we can perform this bilinearisation by performing it on the Korteweg-de Vries (KdV) equation,

$$u_{xxx} + 6uu_x + u_t = 0\,, \tag{6.2}$$

following the computations in [86]. A way to find it is to change variables so that the leading derivative has the same number of derivatives than the nonlinear term, that is, to define a new variable w such that the first two terms in (6.2) have the same number of derivatives. If we establish each x-derivative having degree 1, then the leading derivative u_{xxx} has degree 3, so, in order to balance the term uu_x, u needs to have degree 2, so we can introduce the change of variable

$$u = \partial_x^2 w \,, \tag{6.3}$$

so that (6.2) transforms into

$$w_{xxxxx} + 6w_{xx}w_{xxx} + w_{xxt} = 0\,. \tag{6.4}$$

Now we can integrate (6.4) with respect to x to get

$$w_{xxxx} + 3w_{xx}^2 + w_{xt} = 0\,. \tag{6.5}$$

This integration should introduce an integration constant, but given that w is just defined in (6.3) up to

$$w \to w + xC_1(t) + C_0(t)\,, \tag{6.6}$$

we can absorb the integration constant by making an appropriate choice for $C_0(t)$ and $C_1(t)$.

Now that we have an equation with balanced derivatives, namely (6.4), we can bilinearise it by introducing a new variable with natural degree (as explained above) equal to zero, that is, either $\log(F)$ or f/g. For this case, the former works, so let us define

$$w = \gamma \log(F)\,, \tag{6.7}$$

with γ a free parameter. By introducing it into (6.4), we get an equation of fourth degree in F, with a structure

$$F^2 \times \text{(quadratic term)} + 3\gamma(2-\gamma)(2FF'' - F'^2)F'^2 = 0\,. \tag{6.8}$$

We want the equation to be quadratic, so we will choose $\gamma = 2$, so that the equation becomes

$$F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 + F_{xt}F - F_xF_t = 0\,. \tag{6.9}$$

Once we have the equation in a quadratic form, we can try and replace the derivatives with Hirota's bilinear operator (6.1). D operates on a product of two functions similarly to the Leibniz rule, but with an important sign change,

$$D_x f \cdot g = f_x g - f g_x\,, \tag{6.10a}$$

$$D_x^2 f \cdot g = f g_{xx} - 2 f_x g_x + f_{xx} g\,, \tag{6.10b}$$

$$D_x D_t f \cdot g = f_{xt} g - f_x g_t - f_t g_x + f g_{xt}\,. \tag{6.10c}$$

In particular, we can write

$$D_x D_t F \cdot F = 2(F_{xt}F - F_xF_t)\,, \tag{6.11a}$$

$$D_x^4 F \cdot F = 2(F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2)\,, \tag{6.11b}$$

so that we can rewrite (6.9) as

$$\left(D_x^4 + D_x D_t\right) F \cdot F = 0\,, \tag{6.12}$$

which we will call the Hirota bilinear form of the KdV equation. Overall, to obtain the Hirota bilinear form we have applied the transformation

$$u = 2\partial_x^2 \log(F) \tag{6.13}$$

and integrated the equation once, to then transform the derivatives into Hirota derivatives using the D-operator. This is, however, not an algorithmic process, and the exact transformation and the number of new variables needed to get to a bilinear form may vary from one system to another. In fact, for some systems a Hirota trilinear formalism has been introduced, due to the impossibility of finding a bilinear form for the equation (see [80]).

Now that we have the bilinear form (6.12) we can use it to obtain solutions of the KdV equation. Let us give a few useful propertes of the D-operator before. Let P be a polynomial. Then,

$$P(D)1 \cdot f = P(-\partial)f\,, \qquad P(D)f \cdot 1 = P(\partial)f\,, \tag{6.14a}$$

$$P(D)e^{px} \cdot e^{qx} = P(p-q)e^{(p+q)x}\,, \tag{6.14b}$$

$$\partial_x^2 \log(f) = (D_x^2 f \cdot f)/(2f^2)\,, \tag{6.14c}$$

$$\partial_x^4 \log(f) = (D_x^4 f \cdot f)/(2f^2) - 3(D_x^2 f \cdot f)^2/(2f^4)\,. \tag{6.14d}$$

Let us start obtaining the soliton solutions. The bilinear form of KdV, like for many other systems, is of the polynomial form

$$P(D_x, D_y, \ldots)F \cdot F = 0\,. \tag{6.15}$$

We can do the further assumption that P is an even polynomial, since all the odd terms cancel out due to the antisymmetry of the D-operator.

The most basic solution we can look for is the vacuum solution, corresponding to $u = 0$ in (6.2). By means of (6.13), the corresponding F-solution is of the form

$$F = e^{C_1(t)x + C_0(t)}\,, \tag{6.16}$$

but, as we said, we have the freedom to choose $C_1(t)$ and $C_0(t)$ however we want, so we can just define the vacuum solution as $F = 1$. It is a solution of the system as long as

$$P(0, 0, \ldots) = 0\,, \tag{6.17}$$

that is, as long as the independent term of the polynomial is zero. The soliton solutions are obtained as finite perturbation expansions around the vacuum solution,

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots \tag{6.18}$$

where ε is a formal expansion parameter. Each extra term we take in the finite expansion will contribute an additional soliton to the solution. Let us start with the 1-soliton solution, which only uses one perturbation term,

$$F = 1 + \varepsilon f_1 \,. \tag{6.19}$$

If we substitute it into (6.15), we get

$$P(D_x, D_t, \ldots)[1 \cdot 1 + \varepsilon 1 \cdot f_1 + \varepsilon f_1 \cdot 1 + \varepsilon^2 f_1 \cdot f_1] \,. \tag{6.20}$$

The order ε^0 vanishes due to (6.17). For the order ε^1, both terms are actually the same due to relation (6.14a) and P being an even polynomial. It gives the equation

$$P(\partial_x, \partial_t, \ldots) f_1 = 0 \,. \tag{6.21}$$

The soliton solutions will be associated to the exponential solutions of this equation. For the 1-soliton solution, we can take

$$f_1 = e^\eta \,, \qquad \eta = px + qt + \ldots + \text{const} \,, \tag{6.22}$$

so the equation (6.21) provides a dispersion relation for the parameters $p, q, \ldots$ in the exponential,

$$P(p, q, \ldots) = 0 \,. \tag{6.23}$$

Once that we have established f_1 is an exponential solution, the term of order ε^2 also vanishes due to property (6.14b),

$$P(D_x, D_t, \ldots) e^\eta \cdot e^\eta = P(p - p, q - q, \ldots) e^{2\eta} = P(0, 0, \ldots) e^{2\eta} = 0 \,. \tag{6.24}$$

For the particular case of KdV, with

$$F = 1 + e^{px+qt+\phi_0} \tag{6.25}$$

the equation (6.23) provides the well known dispersion relation $q^3 = p$.

For the 2-soliton solution, one has to add one more term to the vacuum solution, $F = 1+f_1+f_2$, and, in order to combine the solitons, we want that

$$f_1 = e^{\eta_1} + e^{\eta_2}\,, \tag{6.26}$$

with the notation

$$\eta_i = p_i x + q_i t + \ldots + \text{const}\,, \tag{6.27}$$

with $p_i, q_i, \ldots$ complex parameters related through the dispersion relation (6.23).

A similar analysis to the one performed for the 1-soliton solution tells us that the combination we are looking for is

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}\,, \tag{6.28}$$

where A_{12} has the form

$$A_{12} = -\frac{P(p_1-p_2, q_1-q_2, \ldots)}{P(p_1+p_2, q_1+q_2, \ldots)}\,. \tag{6.29}$$

For the 3-soliton solution, the corresponding form would be

$$F = 1+e^{\eta_1}+e^{\eta_2}+e^{\eta_3}+A_{12}e^{\eta_1+\eta_2}+A_{13}e^{\eta_1+\eta_3}+A_{23}e^{\eta_2+\eta_3}+A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}\,, \tag{6.30}$$

where the A_{ij} are defined as in (6.29). The computation is based on imposing that the solution reduces to the 2-soliton solution when the third soliton goes to infinity (when $\eta_3 \to \pm\infty$). The general N-soliton solution can be obtained in a similar way (see [80]) and has the form

$$F = \sum_{\substack{\mu_i=0,1\\ 1\leqslant i\leqslant N}} \exp\left(\sum_{1\leqslant i<j\leqslant N} \varphi(i,j)\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\eta_i \right)\,, \tag{6.31}$$

where we have defined $A_{ij} = e^{\varphi(i,j)}$. Note that going from the 2-soliton solution to the N-soliton solution does not add additional conditions or constraints on the individual solitons or on the equation, and is completely fixed.

The existence of these multisoliton solutions without additional constraints on them is a distinctive feature of integrable systems. Indeed, one can define a system as Hirota integrable if such a solution exist, and, for all the systems studied up to our knowledge, this definition of integrability is equivalent to more conventional definitions of integrability, like Lax integrability (see [86]).

Now that we have an idea of how the general method works, let us introduce the method we will employ, the method of τ-functions, following [67]. We will make use of the discrete Kadomtsev-Petviashvili (dKP) equation, also known as Hirota-Miwa equation (see [93, 127]), which is also integrable. It is an equation for a complex-valued τ-function defined on a three-dimensional lattice with lattice constants a_1, a_2 and a_3, so that the coordinates of the vertices are $(k_1 a_1, k_2 a_2, k_3 a_3)$ with $k_1, k_2, k_3 \in \mathbb{Z}$. In a slight abuse of notation we will denote these coordinates as (k_1, k_2, k_3), so that we will denote $\tau(k_1, k_2, k_3) = \tau(k_1 a_1, k_2 a_2, k_3 a_3)$. The standard way of writing the dKP equation in bilinear form is

$$\begin{aligned} &a_1 (a_2 - a_3) \tau (k_1 + 1, k_2, k_3) \tau (k_1, k_2 + 1, k_3 + 1) \\ &\quad + a_2 (a_3 - a_1) \tau (k_1, k_2 + 1, k_3) \tau (k_1 + 1, k_2, k_3 + 1) \\ &\quad + a_3 (a_1 - a_2) \tau (k_1, k_2, k_3 + 1) \tau (k_1 + 1, k_2 + 1, k_3) = 0 \,. \end{aligned} \tag{6.32}$$

The dKP equation appears naturally when generalising certain geometric settings to a nonlinear stage (see [60]). By defining

$$x = \sum_{i=1}^{3} a_i k_i \,, \qquad y = \frac{1}{2} \sum_{i=1}^{3} a_i^2 k_i \,, \qquad t = \frac{1}{3} \sum_{i=1}^{3} a_i^3 k_i \,, \tag{6.33}$$

and taking the continuous limit $a_i \to 0$, we get the bilinear equation [67]

$$\left(D_x^4 - 4 D_x D_t + 3 D_y^2\right) \tau(x, y, t) \cdot \tau(x, y, t) = 0 \,, \tag{6.34}$$

which, by means of the change of variable $u = 2(\log\tau)_{xx}$, transforms into the original KP equation [98],

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0\,. \tag{6.35}$$

By using the more compact notation

$$\tau\left(k_1+1,k_2,k_3\right) = \tau_1\,, \qquad \tau\left(k_1,k_2+1,k_3+1\right) = \tau_{23} = \widehat{\tau}_1\,, \tag{6.36}$$

we can write the dKP equation as

$$a_1\left(a_2-a_3\right)\tau_1\widehat{\tau}_1 + a_2\left(a_3-a_1\right)\tau_2\widehat{\tau}_2 + a_3\left(a_1-a_2\right)\tau_3\widehat{\tau}_3 = 0\,. \tag{6.37}$$

Note that it can also be rewritten as a determinant,

$$\begin{vmatrix} 1 & a_1 & a_1\tau_1\widehat{\tau}_1 \\ 1 & a_2 & a_2\tau_2\widehat{\tau}_2 \\ 1 & a_3 & a_3\tau_3\widehat{\tau}_3 \end{vmatrix} = 0\,. \tag{6.38}$$

The dKP hierarchy is then defined as the set of dKP equations for lattices (k_i,k_j,k_m) taken from the bigger lattice $(k_1,k_2,k_3,\ldots)$,

$$\left(a_j^{-1}-a_m^{-1}\right)\tau_i\tau_{jm} + \left(a_m^{-1}-a_i^{-1}\right)\tau_j\tau_{mi} + \left(a_i^{-1}-a_j^{-1}\right)\tau_m\tau_{ij} = 0\,. \tag{6.39}$$

We can further rewrite the dKP equation by using the gauge transformation

$$\tau \to \left(a_2^{-1}-a_3^{-1}\right)^{k_2k_3}\left(a_1^{-1}-a_3^{-1}\right)^{k_1k_3}\left(a_1^{-1}-a_2^{-1}\right)^{k_1k_2}\tau\,, \tag{6.40}$$

so that it becomes

$$\tau_1\tau_{23} - \tau_2\tau_{31} + \tau_3\tau_{12} = 0\,. \tag{6.41}$$

Via a similar transformation, the equations in the dKP hierarchy can also be transformed into

$$\tau_m \tau_{ij} - \tau_j \tau_{im} + \tau_i \tau_{jm} = 0\,. \tag{6.42}$$

This is integrable, and its Lax pair is known and has been employed to study its Darboux transformation (see [130]). It also admits solutions in terms of the following discrete Gram-type determinant [133]

$$\tau(k_1, k_2, k_3) = |m_{ij}(k_1, k_2, k_3)|_{1\leqslant i,j\leqslant N} \tag{6.43}$$

with

$$m_{ij}(k_1, k_2, k_3) = c_{ij} + \frac{1}{p_i + q_j} \prod_{l=1}^{3} \phi(a_l, k_l)\, \bar{\phi}(a_l, k_l)\ , \tag{6.44}$$

where

$$\phi(a_l, k_l) = (1 - a_l p_i)^{-k_l}\ , \qquad \bar{\phi}(a_l, k_l) = (1 + a_l q_j)^{k_3}\ . \tag{6.45}$$

That means it can also be explicitly expressed as

$$\tau(k_1, k_2, k_3) = \left| c_{ij} + \frac{1}{p_i + q_j} \left(-\frac{1 - a_1 p_i}{1 + a_1 q_j} \right)^{-k_1} \left(-\frac{1 - a_2 p_i}{1 + a_2 q_j} \right)^{-k_2} \left(\frac{1 - a_3 p_i}{1 + a_3 q_j} \right)^{-k_3} \right| . \tag{6.46}$$

The dKP equation also admits the Casorati-type solution [133]

$$\tau(k_1, k_2, k_3) = \begin{vmatrix} \varphi_1^{(0)} & \varphi_1^{(1)} & \cdots & \varphi_1^{(N-1)} \\ \varphi_2^{(0)} & \varphi_2^{(1)} & \cdots & \varphi_2^{(N-1)} \\ \vdots & \vdots & \vdots & \\ \varphi_N^{(0)} & \varphi_N^{(1)} & \cdots & \varphi_N^{(N-1)} \end{vmatrix}, \tag{6.47}$$

with

$$\varphi_i^{(n)}(k_1, k_2, k_3) = \alpha_i p_i^n \prod_{j=1}^{3} (1 - a_j p_i)^{-k_j} + \beta_i q_i^n \prod_{j=1}^{3} (1 - a_j q_i)^{-k_j}\ . \tag{6.48}$$

One can also define higher order equations in the dKP hierarchy via the determinant [133]

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} & a_1^{n-2}\tau_1\widehat{\tau}_1 \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} & a_2^{n-2}\tau_2\widehat{\tau}_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-2} & a_n^{n-2}\tau_n\widehat{\tau}_n \end{vmatrix} = 0\,, \tag{6.49}$$

so that we can get, for example, the equation

$$\begin{aligned} &a_1^2\,(a_2-a_3)\,(a_2-a_4)\,(a_3-a_4)\,\tau_1\widehat{\tau}_1 - a_2^2\,(a_1-a_3)\,(a_1-a_4)\,(a_3-a_4)\,\tau_2\widehat{\tau}_2 \\ &\quad + a_3^2\,(a_1-a_2)\,(a_1-a_4)\,(a_2-a_4)\,\tau_3\widehat{\tau}_3 - a_4^2\,(a_1-a_2)\,(a_1-a_3)\,(a_2-a_3)\,\tau_4\widehat{\tau}_4 = 0\,. \end{aligned} \tag{6.50}$$

Together with the following equations written using (6.39)

$$\left(a_2^{-1}-a_4^{-1}\right)\tau_1\tau_{24} + \left(a_4^{-1}-a_1^{-1}\right)\tau_2\tau_{41} + \left(a_1^{-1}-a_2^{-1}\right)\tau_4\tau_{12} = 0\,, \tag{6.51a}$$

$$\left(a_3^{-1}-a_4^{-1}\right)\tau_1\tau_{34} + \left(a_1^{-1}-a_3^{-1}\right)\tau_4\tau_{13} + \left(a_4^{-1}-a_1^{-1}\right)\tau_3\tau_{41} = 0\,, \tag{6.51b}$$

$$\left(a_2^{-1}-a_3^{-1}\right)\tau_4\tau_{23} + \left(a_3^{-1}-a_4^{-1}\right)\tau_2\tau_{34} + \left(a_4^{-1}-a_2^{-1}\right)\tau_3\tau_{42} = 0\,, \tag{6.51c}$$

and the original dKP equation (6.37) we can write the following equation of hydrodynamic type

$$\begin{pmatrix} 0 & (a_1-a_2)\,\tau_{12} & (a_3-a_1)\,\tau_{13} & (a_2-a_3)\,\tau_{23} \\ (a_2-a_1)\,\tau_{12} & 0 & (a_1-a_4)\,\tau_{14} & (a_4-a_2)\,\tau_{24} \\ (a_1-a_3)\,\tau_{13} & (a_4-a_1)\,\tau_{41} & 0 & (a_3-a_4)\,\tau_{34} \\ (a_3-a_2)\,\tau_{23} & (a_2-a_4)\,\tau_{24} & (a_4-a_3)\,\tau_{34} & 0 \end{pmatrix} \begin{pmatrix} a_4\tau_4 \\ a_3\tau_3 \\ a_2\tau_2 \\ a_1\tau_1 \end{pmatrix} = 0\,. \tag{6.52}$$

In order to have a non-trivial solution we require that the determinant of the coefficient matrix be equal to zero,

$$(a_1 - a_2)(a_3 - a_4)\,\tau_{12}\tau_{34} - (a_1 - a_3)(a_2 - a_4)\,\tau_{13}\tau_{24} + (a_1 - a_4)(a_2 - a_3)\,\tau_{14}\tau_{23} = 0\,. \tag{6.53}$$

Now that we have introduced this language of τ-functions we can see how other systems can be derived in this same language following similar definitions to (6.46). Let us illustrate it by seeing how it enables to construct and study the continuous two-dimensional Toda hierarchy. The Toda lattice is an integrable model describing a chain with a nearest neighbour interaction under a certain special potential (see [147]). It can be written in the so-called Flaschka's variables as

$$\begin{cases} \dfrac{da_n}{dt} & = a_n(b_n - b_{n+1})\,, \\ \dfrac{db_n}{dt} & = 2(a_{n-1}^2 - a_n^2)\,, \end{cases} \tag{6.54}$$

where physically a_n and b_n are functions of the generalised coordinate q_n and generalised momentum p_n of the n-th particle in the chain.

The two-dimensional Toda lattice is a still integrable extension of the one-dimensional one obtained by adding a linear coupling in the vertical direction (see [112, 124]), so that the equation of motion for the (n, m)-th particle is

$$\frac{d^2 y_{n,m}}{dt^2} = e^{-(y_{n,m} - y_{n-1,m})} - e^{-(y_{n+1} - y_{n,m})} + \kappa(y_{n,m+1} + 2y_{n,m} + y_{n,m-1})\,, \tag{6.55}$$

where $y_{n,m}$ is the position of the (n, m)-th particle and κ is a coupling constant. We will consider a continuous limit in the vertical direction, that is, $y_{n,m} \to y_n(x, t)$. Expanding $y_{n,m\pm1}$ as

$$y_{n,m\pm1} = y_n \pm h\frac{\partial y_n}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 y_n}{\partial x^2} + \mathcal{O}(h^3)\,, \tag{6.56}$$

we end up getting

$$\frac{\partial^2 y_n}{\partial t^2} = e^{-(y_n - y_{n-1})} - e^{-(y_{n+1} - y_{n,m})} + \kappa h^2\frac{\partial^2 y_n}{\partial x^2} + \mathcal{O}(h^4)\,. \tag{6.57}$$

If we define $r_n = y_n - y_{n-1}$, after normalising the constants and neglecting the terms of higher order in h, we get

$$\frac{\partial^2 r_n}{\partial t^2} - \frac{\partial^2 r_n}{\partial x^2} = 2e^{-r_n} - e^{-r_{n+1}} - e^{r_{n-1}} \,. \tag{6.58}$$

By defining the new variables $z = \frac{1}{2}(t+x)$ and $s = \frac{1}{2}(x-t)$, so that

$$\partial_t = \frac{1}{2}(\partial_y - \partial_s)\,, \qquad \partial_x = \frac{1}{2}(\partial_y + \partial_s)\,, \tag{6.59}$$

then (6.58) becomes

$$\frac{\partial^2 r_n}{\partial z \partial s} = 2e^{-r_n} - e^{-r_{n+1}} - e^{r_{n-1}}\,, \tag{6.60}$$

the two-dimensional Toda equation in light-cone coordinates [67]. We can now introduce a change of variable to linearise the equation,

$$e^{-r_n} - 1 = \frac{\partial^2}{\partial z \partial s}\log(\tau_n)\,, \tag{6.61}$$

so that (6.60) adopts the Hirota bilinear form

$$(D_y D_s - 2)\tau_n \cdot \tau_n = -\tau_{n+1}\tau_{n-1}\,. \tag{6.62}$$

Let us get back to the dKP equation (6.32). If we introduce the change of variable

$$1 - a_1 p_i = -\tilde{p}_i\,, \qquad 1 + a_1 q_j = \tilde{q}_j\,, \tag{6.63a}$$

$$\frac{1 - a_2 p_i}{1 + a_2 q_j} = \frac{1 - a\tilde{p}_i}{1 + a\tilde{q}_j}\,, \qquad a = \frac{a_2}{a_1 - a_2}\,, \tag{6.63b}$$

$$\frac{1 - a_3 p_i}{1 + a_3 q_j} = \frac{1 - b^{-1}\tilde{p}_i}{1 + b^{-1}\tilde{q}_j}\,, \qquad b^{-1} = \frac{a_3}{a_1 - a_3}\,, \tag{6.63c}$$

then through the indices transformation

$$n = -(k_1 + k_3)\,, \tag{6.64a}$$

$$k = k_2\,, \tag{6.64b}$$

$$l = k_3\,, \tag{6.64c}$$

the dKP equation becomes

$$\begin{aligned}&(a - b^{-1})\tau_n(k,l)\tau_n(k+1,l+1)\\&\quad - a\tau_{n-1}(k+1,l)\tau_{n+1}(k,l+1) + b^{-1}\tau_n(k,l+1)\tau_n(k+1,l) = 0\,,\end{aligned} \tag{6.65}$$

which we can rewrite as

$$\begin{aligned}&ab\Big(\tau_n(k,l)\tau_n(k+1,l+1) - \tau_{n-1}(k+1,l)\tau_{n+1}(k,l+1)\Big)\\&\quad = \tau_n(k,l)\tau_n(k+1,l+1) - \tau_n(k,l+1)\tau_n(k+1,l)\,,\end{aligned} \tag{6.66}$$

namely, the discrete analogue of the two-dimensional Toda lattice (6.62). That means it admits a Gram-type solution similar to (6.43),

$$\tau_n(k,l) = \left| c_{ij} + \frac{1}{p_i + q_j}\left(-\frac{p_i}{q_j}\right)^n \left(\frac{1 - ap_i}{1 + aq_j}\right)^{-k} \left(\frac{1 - bp_i^{-1}}{1 + bq_j^{-1}}\right)^{-l} e^{\xi_i + \bar{\xi}_j} \right|\,, \tag{6.67}$$

where

$$\xi_i = p_i^{-1}x_{-1} + p_i x_1 + \dots\,, \qquad \bar{\xi}_j = q_j^{-1}x_{-1} + q_j x_1 + \dots \tag{6.68}$$

6.2 Hirota bilinearisation for the YON system

Let us apply the theory introduced above to study the Yajima-Oikawa-Newell (YON) system

$$iS_t + S_{xx} + (i\alpha L_x + \alpha^2 L^2 - \beta L - 2\alpha|S|^2)S = 0\,, \tag{6.69a}$$

$$L_t = 2(|S|^2)_x\,, \tag{6.69b}$$

following results from our paper in preparation [31] By replacing L with L/α and $\beta = 2\alpha\delta$, we can rewrite it as

$$iS_t + S_{xx} + (iL_x + L^2 - 2\delta L - 2\alpha|S|^2)S = 0\,, \tag{6.70a}$$

$$L_t = 2\alpha(|S|^2)_x\,. \tag{6.70b}$$

We will bilinearise (6.70) by using the change of variables

$$L = i\left(\log\frac{f^*}{f}\right)_x\,, \qquad S = \frac{g}{f}\,, \tag{6.71}$$

where f and g are complex functions and f^* denotes the complex conjugate of f. The rationale for the form of L is to ensure the resulting quantity is real, plus balancing the derivatives as explained in the previous section.

The application of (6.71) into (6.70b) entails

$$i\left(\log\frac{f^*}{f}\right)_{tx} = 2\alpha\left(\frac{gg^*}{ff^*}\right)_x\,. \tag{6.72}$$

By integrating with respect to x, we get

$$i\left(\log\frac{f^*}{f}\right)_t = 2\alpha\frac{gg^*}{ff^*} + C_1 \tag{6.73a}$$

$$\Longrightarrow i\frac{D_t f^*\cdot f}{ff^*} = 2\alpha\frac{gg^*}{ff^*} + C_1 \tag{6.73b}$$

$$\Longrightarrow iD_t f\cdot f^* = -2\alpha gg^* - C_1 ff^*\,, \tag{6.73c}$$

where we have used antisymmetry and the property

$$\frac{\partial}{\partial x}\log\frac{a}{b} = \frac{D_x a\cdot b}{ab} \tag{6.74}$$

for arbitrary functions a and b. By setting $C_1 = 0$ in (6.73c), we get the first of our bilinear equations,

$$iD_t f\cdot f^* = -2\alpha|g|^2\,. \tag{6.75}$$

Introducing (6.71) into (6.70a) is a little more convoluted,

$$i\left(\frac{g}{f}\right)_t + \left(\frac{g}{f}\right)_{xx} - \left\{\left(-\log\frac{f^*}{f}\right)_{xx} + \left[i\left(\log\frac{f^*}{f}\right)_x\right]^2 - 2i\delta\left(\log\frac{f^*}{f}\right)_x - 2\alpha\frac{gg^*}{ff^*}\right\}\frac{g}{f} = 0\,, \tag{6.76a}$$

$$\begin{aligned}\Longrightarrow \frac{iD_t g\cdot f}{f^2} + \frac{D_x^2 g\cdot f}{f^2} + \Bigg[&-\frac{D_x^2 f\cdot f}{f^2} - \left(\frac{D_x^2 f^*\cdot f^*}{2f^{*2}} - \frac{D_x^2 f\cdot f}{2f^2}\right)\\ &+\left(i\frac{D_x f^*\cdot f}{ff^*}\right)^2 - 2i\delta\frac{D_x f^*\cdot f}{ff^*} - 2\alpha\frac{gg^*}{ff^*}\Bigg]\frac{g}{f} = 0\,,\end{aligned} \tag{6.76b}$$

$$\begin{aligned}\Longrightarrow \frac{(iD_t + D_x^2)g\cdot f}{f^2} - \frac{g}{f}\Bigg(&\frac{D_x^2 f^*\cdot f^*}{2f^{*2}} + \frac{D_x^2 f\cdot f}{2f^2} + \frac{(D_x f^*\cdot f)^2}{f^2 f^{*2}}\\ &+ \frac{2i\delta D_x f^*\cdot f + 2\alpha gg^*}{ff^*}\Bigg) = 0\,,\end{aligned} \tag{6.76c}$$

$$\begin{aligned}\Longrightarrow \frac{(iD_t + D_x^2)g\cdot f}{f^2} - \frac{g}{f}\Bigg(&\frac{f^2(2f^*_{xx}f^* - 2f^{*2}_x)}{2f^2f^{*2}} + \frac{f^{*2}(2f_{xx}f - 2f_x^2)}{2f^2f^{*2}}\\ &+ \frac{(D_x f^*\cdot f)^2}{f^2 f^{*2}} + \frac{2i\delta D_x f^*\cdot f + 2\alpha gg^*}{ff^*}\Bigg) = 0\,,\end{aligned} \tag{6.76d}$$

$$\Longrightarrow \frac{(iD_t + D_x^2)g \cdot f}{f^2} - \frac{g}{f}\left(\frac{f^*_{xx}}{f^*} + \frac{f^{*2}_{\ x}}{f^{*2}} + \frac{f_{xx}}{f} - \frac{f_x^2}{f^2} + \frac{(f^*_x f - f^* f_x)^2}{f^2 f^{*2}}\right.$$
$$\left. + \frac{2i\delta D_x f^* \cdot f + 2\alpha g g^*}{f f^*}\right) = 0\,, \tag{6.76e}$$

$$\Longrightarrow \frac{(iD_t + D_x^2)g \cdot f}{f^2} - \frac{g}{f}\left(\frac{f^*_{xx}}{f} - \frac{f_{xx}}{f} - \frac{2f^*_x f_x}{f f^*} + \frac{2i\delta D_x f^* \cdot f + 2\alpha g g^*}{f f^*}\right) = 0\,, \tag{6.76f}$$

$$\Longrightarrow \frac{(iD_t + D_x^2)g \cdot f}{f^2} - \frac{g}{f}\left(\frac{f^*_{xx} f - 2f^*_x f_x + f^* f_{xx}}{f f^*} + \frac{2i\delta D_x f^* \cdot f + 2\alpha g g^*}{f f^*}\right) = 0\,, \tag{6.76g}$$

$$\Longrightarrow \frac{(iD_t + D_x^2)g \cdot f}{f^2} - \frac{g}{f}\frac{(D_x^2 - 2i\delta D_x) f \cdot f^* + 2\alpha g g^*}{f f^*} = 0\,. \tag{6.76h}$$

By decoupling (6.76h), we obtain two additional bilinear equations,

$$(iD_t + D_x^2)g \cdot f = 0\,, \tag{6.77}$$

$$(D_x^2 - 2i\delta D_x) f \cdot f^* + 2\alpha |g|^2 = 0\,. \tag{6.78}$$

We can use the previous bilinear equation (6.75) to rewrite (6.78) as

$$iD_t f \cdot f^* = (D_x^2 - 2i\delta D_x) f \cdot f^*\,. \tag{6.79}$$

So, summarising, the YON system transforms into a system of three Hirota bilinear equations:

$$(iD_t + D_x^2)g \cdot f = 0\,, \tag{6.80a}$$

$$iD_t f \cdot f^* = (D_x^2 - 2i\delta D_x) f \cdot f^*\,, \tag{6.80b}$$

$$iD_t f \cdot f^* = -2\alpha |g|^2\,. \tag{6.80c}$$

6.3 Bright soliton solutions

Now, to produce bright soliton solutions (understood as soliton solutions on a zero background), we can take our variables f and g with the form

$$f = 1 + \epsilon f_2 + \epsilon f_4 + \dots\,, \qquad g = \epsilon g_1 + \epsilon^3 g_3 + \epsilon^5 g_5 + \dots \tag{6.81}$$

and introduce them into our bilinear equations (6.80). In order to obtain the one-bright-soliton solutions we can assume

$$f_4 = f_6 = \dots = 0\,, \qquad g_3 = g_5 = \dots = 0\,. \tag{6.82}$$

After this choices, to the lowest order in ϵ our bilinear equations (6.80) yields

$$(iD_t + D_x^2) g_1 \cdot 1 = 0\,, \tag{6.83a}$$

$$(D_x^2 - 2i\delta D_x) f_2 \cdot 1 + (D_x^2 - 2i\delta D_x) 1 \cdot f_2^* = -2\alpha g_1 g_1^*\,, \tag{6.83b}$$

$$iD_t f_2 \cdot 1 + iD_t 1 \cdot f_2^* = -2\alpha g_1 g_1^*\,. \tag{6.83c}$$

From (6.83a) we get

$$i\frac{\partial g_1}{\partial t} + \frac{\partial^2 g_1}{\partial x^2} = 0\,, \tag{6.84}$$

which gives us a solution

$$g_1 = \gamma_1 e^{\eta_1}\,, \qquad \eta_1 = k_1 x + i k_1^2 t + \phi_0\,, \tag{6.85}$$

with k_1 the corresponding wave number and ϕ_0 an arbitrary initial phase.

From (6.83b) and (6.83c) we get

$$\frac{\partial^2 f_2}{\partial x^2} - 2i\delta\frac{\partial f_2}{\partial x} + \frac{\partial^2 f_2^*}{\partial x^2} + 2i\delta\frac{\partial f_2^*}{\partial x} = -2\alpha g_1 g_1^*\,, \tag{6.86a}$$

$$i\frac{\partial f_2}{\partial t} - i\frac{\partial f_2^*}{\partial t} = -2\alpha g_1 g_1^*\,. \tag{6.86b}$$

We can plug the expression for g_1 obtained above into (6.86) to get

$$\frac{\partial^2 f_2}{\partial x^2} - 2i\delta\frac{\partial f_2}{\partial x} + \frac{\partial^2 f_2^*}{\partial x^2} + 2i\delta\frac{\partial f_2^*}{\partial x} = -2\alpha|\gamma_1|^2 e^{\eta_1+\eta_1^*}\,, \tag{6.87a}$$

$$i\frac{\partial f_2}{\partial t} - i\frac{\partial f_2^*}{\partial t} = -2\alpha|\gamma_1|^2 e^{\eta_1+\eta_1^*}\,. \tag{6.87b}$$

For these equations we can try an Ansatz

$$f_2 = A_2 e^{\eta_1+\eta_1^*}\,, \qquad f_2^* = A_2^* e^{\eta_1+\eta_1^*}\,, \tag{6.88}$$

and after substituting we get that

$$A_2 = \frac{2\alpha|\gamma_1|^2(i\delta + k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)}\,. \tag{6.89}$$

Putting everything together, we end up with the solution

$$f = 1 + \frac{2\alpha|\gamma_1|^2(i\delta + k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\eta_1+\eta_1^*} = \begin{vmatrix} \frac{i\delta+k_1^*}{k_1+k_1^*}e^{\eta_1+\eta_1^*} & 1 \\ -1 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} \end{vmatrix}\,, \tag{6.90a}$$

$$f^* = 1 + \frac{2\alpha|\gamma_1|^2(i\delta - k_1)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\eta_1+\eta_1^*} = \begin{vmatrix} \frac{i\delta-k_1}{k_1+k_1^*}e^{\eta_1+\eta_1^*} & 1 \\ -1 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} \end{vmatrix}\,, \tag{6.90b}$$

$$g = \gamma_1 e^{\eta_1} = \begin{vmatrix} \frac{i\delta+k_1^*}{k_1+k_1^*}e^{\eta_1+\eta_1^*} & 1 & e^{\eta_1} \\ -1 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & 0 \\ 0 & -\gamma_1 & 0 \end{vmatrix}\,, \qquad \eta_1 = k_1 x + ik_1^2 t + \phi_0\,. \tag{6.90c}$$

Upon getting back to the S and L variables through the change (6.71), the formulae above provide the one-soliton solution. For example, for the short wave we have

$$S = \frac{g}{f} = \frac{\gamma_1 e^{\eta_1}}{1 + \frac{2\alpha|\gamma_1|^2(i\delta+k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\eta_1+\eta_1^*}} \tag{6.91}$$

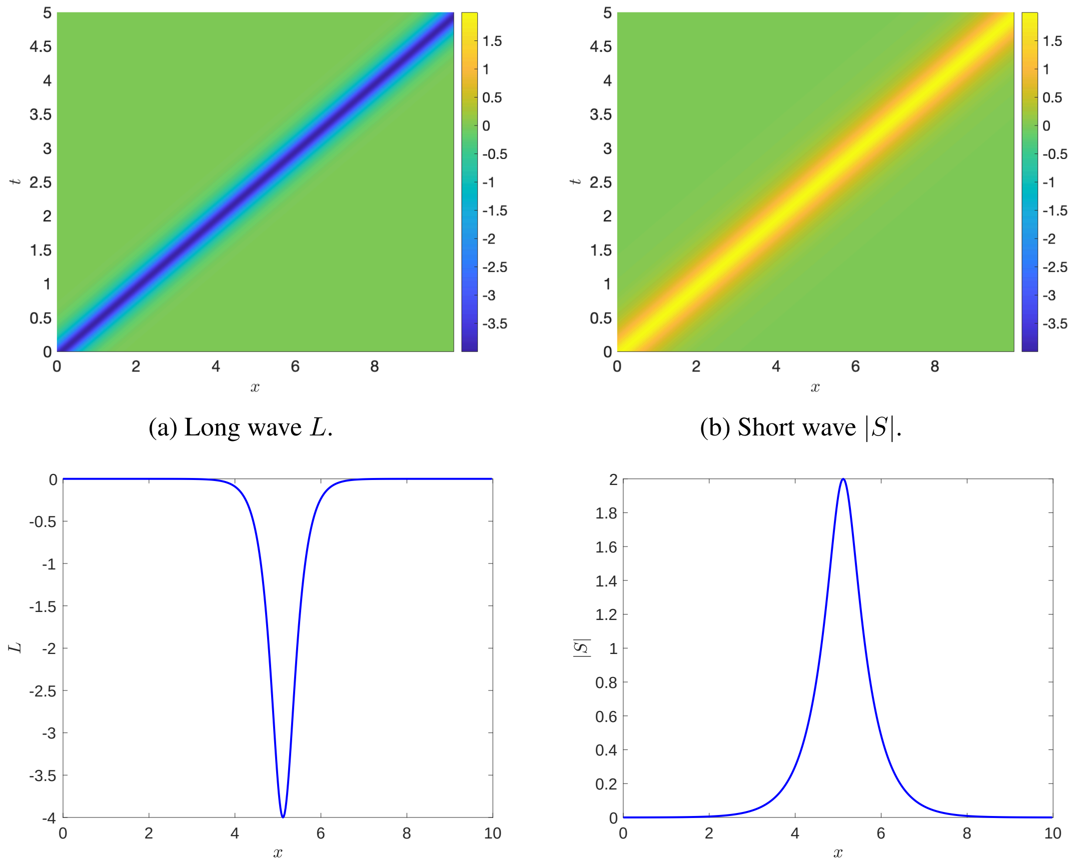
and

$$|S|^2 = \frac{gg^*}{ff^*} = \frac{|\gamma|^2 e^{\eta_1+\eta_1^*}}{\left(1+\frac{2\alpha|\gamma_1|^2(i\delta+k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\eta_1+\eta_1^*}\right)\left(1+\frac{2\alpha|\gamma_1|^2(i\delta-k_1)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\eta_1+\eta_1^*}\right)}. \tag{6.92}$$

We can denote $k_1 = k_r + ik_i$ and get back to the original variables in (6.69) to write explicitly the general form of the one-bright-soliton solution as

$$S(x,t) = \frac{8e^{i(k^{*2}t+k^*x)}k_i k_r^2\gamma}{8e^{4k_ik_rt}k_ik_r^2 + |\gamma|^2 e^{2k_rx}[\beta - 2(k_i+ik_r)\alpha]}, \tag{6.93a}$$

$$L(x,t) = -\frac{64e^{2k_r(2k_it+x)}k_ik_r^4|\gamma|^2}{64e^{8k_ik_rt}k_i^2k_r^4 - 16e^{2k_r(2k_it+x)}k_ik_r^2|\gamma|^2(2k_i\alpha-\beta) + |\gamma|^4e^{4k_rx}(4\alpha^2|k|^2 - 4\alpha\beta k_i + \beta^2)}. \tag{6.93b}$$



(a) Long wave L. (b) Short wave $|S|$.

(c) Long wave profile at $t = 2.5$. (d) Short wave profile at $t = 2.5$.

Figure 7: 1-bright-soliton solution with $\alpha = 1$, $\beta = 2$, $k_1 = 2+i$, $\gamma_1 = 2+i$.

Both S and L are solitons moving with velocity

$$V = 2k_i\,, \tag{6.94}$$

so that $L(x,0) = L(x+Vt,t)$ for every x and t (and the same applies for $|S(x,t)|$).

When $t=0$, both $|S|$ and $|L|$ have their maximum at

$$x_{\max} = \frac{1}{4k_r}\log\left(\frac{64k_i^2k_r^4}{|\gamma|^4(4\alpha^2|k|^2 - 4\alpha\beta k_i + \beta^2)}\right)\,, \tag{6.95}$$

with the soliton in L having an amplitude

$$A_L = \frac{4k_r^2}{-\mathrm{sgn}(k_i)\sqrt{4\alpha^2|k|^2 - 4\alpha\beta k_i + \beta^2} + (2\alpha k_i - \beta)}\,, \tag{6.96}$$

where $\mathrm{sgn}(k_i)$ denotes the sign of k_i, while the amplitude of $|S|$ satisfies

$$A_S^2 = -k_i A_L \tag{6.97}$$

as a direct consequence of the property

$$\frac{L(x,t)}{|S(x,t)|^2} = -\frac{1}{k_i}\,. \tag{6.98}$$

Note that all the formulae above reduce nicely to the Newell case $\beta = 0$ for every value of the parameters. However, for the Yajima-Oikawa case $\alpha = 0$, the amplitude A_L in (6.96) diverges whenever $k_i < 0$. That, together with the expression for the velocity (6.94) indicates that Yajima-Oikawa admits only bright solitons traveling to the right direction.

Once we have computed the formulae we can check that the solitons obtained in Section 4.3.4 via an Ansatz approach are indeed subcases of the ones we obtained via Hirota. In particular, the bright soliton solution coincides with the Hirota soliton for the special case $k_i = \beta/2\alpha$, so that the velocity of the soliton is exactly $V = \beta/\alpha$. This reduction does not work well for the Yajima-Oikawa case $\alpha = 0$, which is not surprising considering our Ansatz computation was only valid for $\alpha \neq 0$.

Through a similar process, we can compute the two-bright-soliton solutions by assuming k_4 and g_3 are also nonzero in (6.81). By doing that, one ends up with the expressions

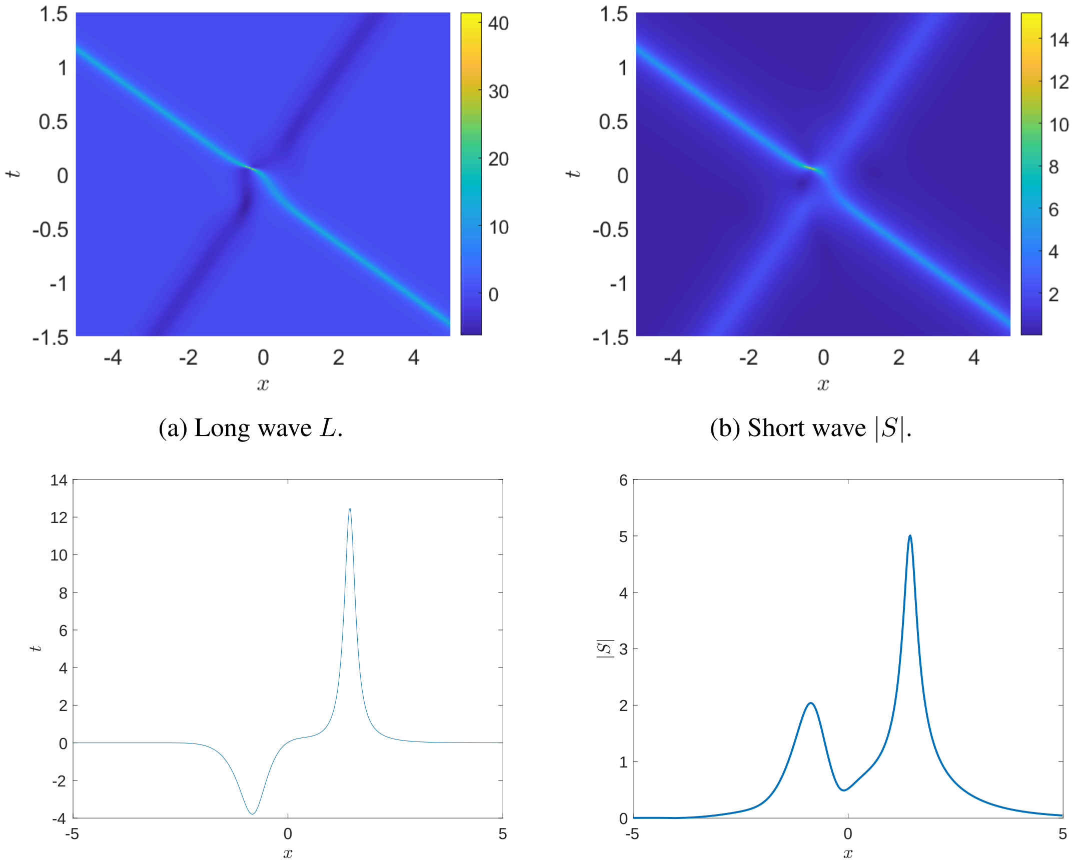
$$f = \begin{vmatrix} \frac{i\delta+k_1^*}{k_1+k_1^*}e^{\eta_1+\eta_1^*} & \frac{i\delta+k_2^*}{k_1+k_2^*}e^{\eta_1+\eta_2^*} & 1 & 0 \\ \frac{i\delta+k_1^*}{k_2+k_1^*}e^{\eta_2+\eta_1^*} & \frac{i\delta+k_2^*}{k_2+k_2^*}e^{\eta_2+\eta_2^*} & 0 & 1 \\ -1 & 0 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & -\frac{2\alpha\gamma_1^*\gamma_2}{k_1^{*2}-k_2^2} \\ 0 & -1 & -\frac{2\alpha\gamma_2^*\gamma_1}{k_2^{*2}-k_1^2} & -\frac{2\alpha|\gamma_2|^2}{k_2^{*2}-k_2^2} \end{vmatrix}, \tag{6.99a}$$

$$f^* = \begin{vmatrix} \frac{i\delta-k_1}{k_1+k_1^*}e^{\eta_1+\eta_1^*} & \frac{i\delta-k_2}{k_1+k_2^*}e^{\eta_1+\eta_2^*} & 1 & 0 \\ \frac{i\delta-k_1}{k_2+k_1^*}e^{\eta_2+\eta_1^*} & \frac{i\delta-k_2}{k_2+k_2^*}e^{\eta_2+\eta_2^*} & 0 & 1 \\ -1 & 0 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & -\frac{2\alpha\gamma_1^*\gamma_2}{k_1^{*2}-k_2^2} \\ 0 & -1 & -\frac{2\alpha\gamma_2^*\gamma_1}{k_2^{*2}-k_1^2} & -\frac{2\alpha|\gamma_2|^2}{k_2^{*2}-k_2^2} \end{vmatrix}, \tag{6.99b}$$

$$g = \begin{vmatrix} \frac{i\delta+k_1^*}{k_1+k_1^*}e^{\eta_1+\eta_1^*} & \frac{i\delta+k_2^*}{k_1+k_2^*}e^{\eta_1+\eta_2^*} & 1 & 0 & e^{\eta_1} \\ \frac{i\delta+k_1^*}{k_2+k_1^*}e^{\eta_2+\eta_1^*} & \frac{i\delta+k_2^*}{k_2+k_2^*}e^{\eta_2+\eta_2^*} & 0 & 1 & e^{\eta_2} \\ -1 & 0 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & -\frac{2\alpha\gamma_1^*\gamma_2}{k_1^{*2}-k_2^2} & 0 \\ 0 & -1 & -\frac{2\alpha\gamma_2^*\gamma_1}{k_2^{*2}-k_1^2} & -\frac{2\alpha|\gamma_2|^2}{k_2^{*2}-k_2^2} & 0 \\ 0 & 0 & -\gamma_1 & -\gamma_2 & 0 \end{vmatrix}. \tag{6.99c}$$

One can study the phase shift via the changes of variable $X = x+2k_{1i}$ and $X = x+2k_{2i}$, where k_{1i} and k_{2i} are the imaginary parts of k_1 and k_2, in order to make "stationary" the first and second soliton, respectively. That way one can make t go to $\pm\infty$ to collapse to a one-soliton solution, and then study the difference in phase between the two asymptotic one-soliton solutions. The phase shift for the first soliton in the 2-soliton solution then takes the form

$$\phi_0 = \sigma\frac{1}{2k_{1r}}\log\left[\frac{(k_{1i}^2+k_{1r}^2+k_{2i}^2+k_{2r}^2-2k_{1i}k_{2i}-2k_{1r}k_{2r})^2(k_{1i}^2+k_{1r}^2+k_{2i}^2+k_{2r}^2+2k_{1i}k_{2i}+2k_{1r}k_{2r})}{(k_{1i}^2+k_{1r}^2+k_{2i}^2+k_{2r}^2-2k_{1i}k_{2i}+2k_{1r}k_{2r})^2(k_{1i}^2+k_{1r}^2+k_{2i}^2+k_{2r}^2+2k_{1i}k_{2i}-2k_{1r}k_{2r})}\right], \tag{6.100}$$



(a) Long wave L.

(b) Short wave $|S|$.

(c) Long wave profile at $t = -0.5$.

(d) Short wave profile at $t = -0.5$.

Figure 8: 2-bright-soliton solution with $\alpha = 1$, $\beta = 2$, $k_1 = 2 + i$, $k_2 = 1 - 2i$, $\gamma_1 = 2 + i$, $\gamma_2 = 1 + 2i$.

where $k_1 = k_{1r} + ik_{1i}$, $k_2 = k_{2r} + ik_{2r}$, and the sign σ is

$$\sigma = \operatorname{sgn}[k_{2r}(k_{2i} - k_{1i})] \, . \tag{6.101}$$

The corresponding shift for the second soliton can be obtained by simply swapping the subindices 1 and 2 in the formulae above.

Further solitons can be added to the solution by taking more nonzero terms in f and g. The resulting f and g have similar forms as above, by extending the size of the determinant in 2 for each soliton added, so the N-soliton solution has the form

$$f = \begin{vmatrix} A & \mathbb{1}_N \\ -\mathbb{1}_N & B \end{vmatrix} , \tag{6.102a}$$

$$g = \begin{vmatrix} A & \mathbb{1}_N & c_\eta \\ -\mathbb{1}_N & B & 0_{N\times 1} \\ 0_{1\times N} & r_\gamma & 0 \end{vmatrix} , \tag{6.102b}$$

where the $N \times N$ matrices A and B are defined by

$$A_{ij} = \frac{i\delta + k_j^*}{k_i + k_j^*} e^{\eta_i + \eta_j^*} , \qquad B_{ij} = -\frac{2\alpha\gamma_i^*\gamma_j}{k_i^{*2} - k_j^2} , \tag{6.103a}$$

the column vector c_η satisfies $(c_\eta)_i = e^{\eta_i}$, $i = 1, \ldots, N$, and the row vector r_γ satisfies $(r_\gamma)_i = -\gamma_i$.

As one would expect, the soliton solutions above coincide with the ones obtained for the Newell system in [34] when setting $\beta = 0$ and $\alpha = 1$, and with the solutions for Yajima-Oikawa obtained in [35] upon setting $\alpha = 0$ and $\beta = 1$.

6.4 Dark soliton solutions

In the previous section we computed the bright soliton solutions. Now we will also use the Hirota techniques to produce dark soliton solutions (by which we mean soliton solutions on a constant non-zero background), also following our paper in preparation [31]. In order to do so, we will have to modify the change of variables we used for the bilinearisation. In the bright case, we just wrote a change of variable on a zero background, however now we need to explicitly account for the plane wave

$$S = \rho e^{i[qx-(q^2+2\rho^2)t]} , \qquad L = \nu \, . \tag{6.104}$$

in our change of variable. To do so, we can modify our previous change of variable (6.71) into

$$S = \rho \frac{g}{f} e^{i[qx-(q^2+2\rho^2)t]}\,, \qquad L = \nu + i\left(\log\frac{f^*}{f}\right)_x\,. \tag{6.105}$$

From (6.70b) we get

$$i\left(\log\frac{f^*}{f}\right)_{xt} = 2\alpha\rho^2\left(\frac{gg^*}{ff^*}\right)_x\,. \tag{6.106}$$

By integrating it with respect to x we get

$$\begin{aligned} i\left(\log\frac{f^*}{f}\right)_t &= 2\alpha\rho^2\frac{gg^*}{ff^*} + C_1 & (6.107a)\\ \Longrightarrow i\frac{D_t f^*\cdot f}{ff^*} &= 2\alpha\rho\frac{gg^*}{ff^*} + C_1 & (6.107b)\\ \Longrightarrow iD_t f\cdot f^* &= -2\alpha\rho^2 gg^* - C_1 ff^*\,. & (6.107c)\end{aligned}$$

In this case we will set $C_1 = -2\alpha\rho^2$ to obtain

$$iD_t f\cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,, \tag{6.108}$$

which will be one of our bilinear equations. To introduce the change of variable in (6.70b), the computation will again be a longer one,

$$\begin{aligned} & i\left(\frac{g}{f}\right)_t + (q^2+2\alpha\rho^2-\nu^2+2\delta\nu)\frac{g}{f} + 2iq\left(\frac{g}{f}\right)_x + \left(\frac{g}{f}\right)_{xx} - \frac{g}{f}\left(\log\frac{f^*}{f}\right)_{xx}\\ &\quad + \frac{g}{f}\left[\nu + i\left(\log\frac{f^*}{f}\right)_x\right]^2 - 2i\delta\frac{g}{f}\left[\nu + i\left(\log\frac{f^*}{f}\right)_x\right] - 2\alpha\rho^2\frac{gg^*}{ff^*}\frac{g}{f} = 0\,, & (6.109a)\\ \Longrightarrow\ & \frac{iD_t g\cdot f}{f^2} + 2\alpha\rho^2\frac{g}{f} + 2iq\frac{D_x g\cdot f}{f^2} + \frac{D_x^2 g\cdot f}{f^2} - \frac{g}{f}\frac{D_x^2 f\cdot f}{f^2}\\ &\quad - \frac{g}{f}\left(\frac{D_x^2 f^*\cdot f^*}{2f^{*2}} - \frac{D_x^2 f\cdot f}{2f^2}\right) + \frac{g}{f}\left[2i\nu\frac{D_x f^*\cdot f}{ff^*} + \left(i\frac{D_x f^*\cdot f}{ff^*}\right)^2\right] & (6.109b)\\ &\quad - 2i\delta\frac{g}{f}\frac{D_x f^*\cdot f}{ff^*} - 2\alpha\rho^2\frac{gg^*}{ff^*}\frac{g}{f} = 0\,,\\ \Longrightarrow\ & \frac{(iD_t + 2iD_x + D_x^2 + 2\alpha\rho^2)g\cdot f}{f^2} & (6.109c)\\ &\quad - \frac{g}{f}\left(\frac{D_x^2 f^*\cdot f^*}{2f^{*2}} + \frac{D_x^2 f\cdot f}{2f^2} + \frac{(D_x f^*\cdot f)^2}{f^2 f^{*2}} + \frac{2i(\delta-\nu)D_x f^*\cdot f + 2\alpha\rho gg^*}{ff^*}\right) = 0\,, \end{aligned}$$

$$\Longrightarrow \frac{(iD_t + 2iqD_x + D_x^2 + 2\alpha\rho^2)g \cdot f}{f^2} - \frac{g}{f}\left(\frac{f^2(2f_{xx}^* - 2f_x^{*2})}{2f^2 f^{*2}} + \frac{f^{*2}(2f_{xx}f - 2f_x^2)}{2f^2 f^{*2}}\right.$$
$$\left. + \frac{(D_x f^* \cdot f)^2}{f^2 f^{*2}} + \frac{2i(\delta-\nu)D_x f^* \cdot f + 2\alpha\rho^2 g g^*}{ff^*}\right) = 0\,, \tag{6.109d}$$
$$\Longrightarrow \frac{(iD_t + 2iqD_x + D_x^2 + 2\alpha\rho^2)g \cdot f}{f^2} - \frac{g}{f}\left(\frac{f_{xx}^*}{f^*} - \frac{f_x^{*2}}{f^{*2}} + \frac{f_{xx}}{f} - \frac{f_x^2}{f^2}\right.$$
$$\left. - \frac{f_x^2}{f^2} + \frac{(f_x^* f - f^* f_x)^2}{f^2 f^{*2}} + \frac{2i(\delta-\nu)D_x f^* \cdot f + 2\alpha\rho^2 g g^*}{ff^*}\right) = 0\,, \tag{6.109e}$$
$$\Longrightarrow \frac{(iD_t + 2iqD_x + D_x^2 + 2\alpha\rho^2)g \cdot f}{f^2}$$
$$- \frac{g}{f}\left(\frac{f_{xx}^*}{f^*} + \frac{f_{xx}}{f} - \frac{2f_x f_x^*}{ff^*} + \frac{2i(\delta-\nu)D_x f^* \cdot f + 2\alpha\rho^2 g g^*}{ff^*}\right) = 0\,, \tag{6.109f}$$
$$\Longrightarrow \frac{(iD_t + 2iqD_x + D_x^2 + 2\alpha\rho^2)g \cdot f}{f^2}$$
$$- \frac{g}{f}\left(\frac{f_{xx}^* f - 2f_x f_x^* + f^* f_{xx}}{ff^*} + \frac{2i(\delta-\nu)D_x f^* \cdot f + 2\alpha\rho^2 g g^*}{ff^*}\right) = 0\,, \tag{6.109g}$$
$$\Longrightarrow \frac{(iD_t + 2iqD_x + D_x^2 + 2\alpha\rho^2)g \cdot f}{f^2}$$
$$- \frac{g}{f}\frac{(D_x^2 - 2i(\delta-\nu)D_x)f \cdot f^* + 2\alpha\rho^2 g g^*}{ff^*} = 0\,. \tag{6.109h}$$

By decoupling (6.109h), we obtain the bilinear equations

$$(iD_t + 2iqD_x + D_x^2)g \cdot f = 0\,, \tag{6.110}$$
$$(D_x^2 - 2i(\delta-\nu)D_x)f \cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,. \tag{6.111}$$

We can use the previous bilinear equation (6.108) to rewrite the latter as

$$iD_t f \cdot f^* = [D_x^2 - 2i(\delta-\nu)D_x]f \cdot f^*\,. \tag{6.112}$$

To summarise, by using the change of variable (6.105) we were able to rewrite the YON system as

$$(iD_t + 2iqD_x + D_x^2)g \cdot f = 0\,, \tag{6.113a}$$

$$iD_t f \cdot f^* = [D_x^2 - 2i(\delta - \nu)D_x]f \cdot f^*\,, \tag{6.113b}$$

$$iD_t f \cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,. \tag{6.113c}$$

One can compare these with the following equations in the extended KP hierarchy

$$(D_{x_2} - 2aD_{x_1} - D_{x_1}^2)\tau_{n,k+1} \cdot \tau_{n,k} = 0\,, \tag{6.114a}$$

$$(D_{x_2} - 2bD_{x_1} + D_{x_1}^2)\tau_{n,k} \cdot \tau_{n+1,k} = 0\,, \tag{6.114b}$$

$$[(a-b)D_{x_{-1}} + 1]\tau_{n,k} \cdot \tau_{n+1,k} = \tau_{n,k+1}\tau_{n+1,k-1}\,. \tag{6.114c}$$

As members of the KP hierarchy, they have a Gram-type solution

$$\tau_{n,k} = |m_{ij}^{n,k}|_{1\leqslant i,j\leqslant N}\,, \tag{6.115}$$

with

$$m_{ij}^{n,k} = \delta_{ij} + \frac{ik_j - b}{k_i + k_j^*}\left(-\frac{k_i - b}{k_j + b}\right)^n\left(-\frac{k_i - a}{k_j^* + a}\right)^k e^{\xi_i + \xi_j^*}\,, \tag{6.116}$$

where

$$\xi_i = \frac{1}{k_i - a}x_{-1} + k_i x_1 + k_i^2 x_2 + \xi_{i0}\,, \quad \xi_i^* = \frac{1}{k_i^* - a}x_{-1} + k_i^* x_1 + k_i^{*2} x_2 + \xi_{i0}^*\,. \tag{6.117}$$

If we add the constraint condition

$$\frac{1}{k_i - a} + \frac{1}{k_i^* + a} = \frac{1}{2\alpha(a-b)\rho^2}(k_i^2 - k_i^{2*})\,, \tag{6.118}$$

which we can rewrite as

$$\frac{2\alpha(a-b)\rho^2}{(k_i - a)(k_i^* + a)} = k_i - k_i^*\,, \tag{6.119}$$

to the N-soliton solutions, then the τ-functions satisfy

$$(a-b)\partial_{x_{-1}}\tau_{n,k} = \frac{1}{2\alpha\rho^2}\partial_{x_2}\tau_{n,k}\,, \tag{6.120}$$

so that (6.114c) becomes

$$(D_{x_2} + 2\alpha\rho^2)\tau_{n,k}\cdot\tau_{n+1,k} = 2\alpha\rho^2\tau_{n,k+1}\tau_{n+1,k-1}\,. \tag{6.121}$$

After adding this constraint, we can set $n=-1$ and $k=0$ in the three KP bilinear equations to get

$$(D_{x_2} - 2aD_{x_1} - D_{x_1}^2)\tau_{-1,1}\cdot\tau_{-1,0} = 0\,, \tag{6.122a}$$

$$(D_{x_2} - 2bD_{x_1} + D_{x_1}^2)\tau_{-1,0}\cdot\tau_{0,0} = 0\,, \tag{6.122b}$$

$$(D_{x_2} + 2\alpha\rho^2)\tau_{-1,0}\cdot\tau_{0,0} = 2\alpha\rho^2\tau_{-1,1}\tau_{0,-1}\,. \tag{6.122c}$$

We can then introduce

$$f = \tau_{-1,0}\,, \qquad g = \tau_{-1,1}\,, \qquad f^* = \tau_{0,0}\,, \qquad g^* = \tau_{0,-1}\,, \tag{6.123}$$

so the bilinear equations above become

$$(D_{x_2} - 2aD_{x_1} - D_{x_1}^2)g\cdot f = 0\,, \tag{6.124a}$$

$$(D_{x_2} - 2bD_{x_1} + D_{x_1}^2)f\cdot f^* = 0\,, \tag{6.124b}$$

$$(D_{x_2} + 2\alpha\rho^2)f\cdot f^* = 2\alpha\rho^2 gg^*\,. \tag{6.124c}$$

Finally, by taking $x_2 = it$, $a = iq$ and $b = i(\delta - \nu)$, they become

$$(iD_t + 2iqD_x + D_x^2)g \cdot f = 0\,, \tag{6.125a}$$

$$iD_t f \cdot f^* = [D_x^2 - 2i(\delta - \nu)D_x]f \cdot f^*\,, \tag{6.125b}$$

$$iD_t f \cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,, \tag{6.125c}$$

which are exactly the equations (6.113) that we obtained for the YON system.

That means that we can adapt the solutions (6.115) and (6.116) as solutions of the YON system. The 1-dark-soliton solution is given by

$$f = 1 - \frac{ik_1^* - \delta + \nu}{k_1 + k_1^*} e^{\xi_1 + \xi_1^*}\,, \tag{6.126a}$$

$$f^* = 1 + \frac{ik_1 + \delta - \nu}{k_1 + k_1^*} e^{\xi_1 + \xi_1^*}\,, \tag{6.126b}$$

$$g = 1 + \frac{ik_1^* - \delta + \nu}{k_1 + k_1^*} \frac{k_1 - iq}{k_1^* + iq} e^{\xi_1 + \xi_1^*}\,, \tag{6.126c}$$

$$g^* = 1 - \frac{ik_1 + \delta - \nu}{k_1 + k_1^*} \frac{k_1^* + iq}{k_1 - iq} e^{\xi_1 + \xi_1^*}\,, \tag{6.126d}$$

$$\xi_1 = k_1 x + ik_1^2 t + \xi_{10}\,, \quad \xi_1^* = k_1^* x - ik_1^{*2} t + \xi_{10}^*\,, \tag{6.126e}$$

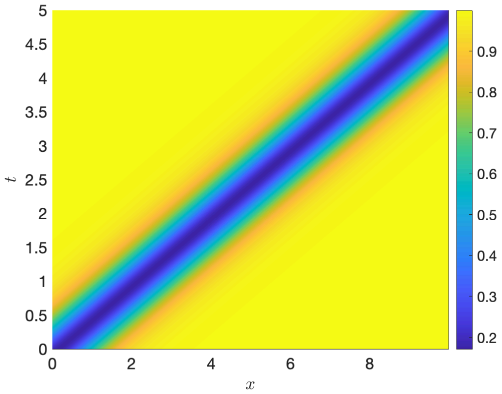
where the parameters must satisfy the constraint condition

$$\frac{2i\alpha(q - \delta + \nu)\rho^2}{|k_1 - iq|^2} = k_1 - k_1^*\,. \tag{6.127}$$

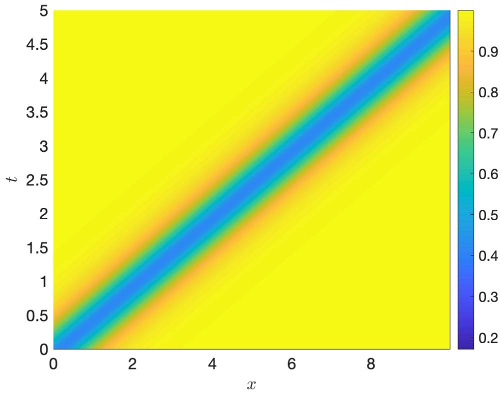
By taking $k_1 = k_{1r} + ik_{1i}$, we can rewrite the constraint condition as

$$k_{1r} = \pm\left(\frac{\alpha(q - \delta + \nu)\rho^2}{k_{1i}} - (k_{1i} - q)^2\right)^{\frac{1}{2}}\,. \tag{6.128}$$

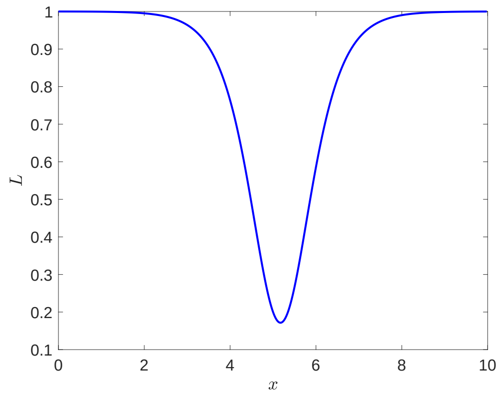
As with the bright case, the dark solitons move with a velocity $V = 2k_i$, that is, they satisfy $L(x, t) = L(x + 2k_i t, 0)$ and $|S(x, t)| = |S(x + 2k_i t, 0)|$.



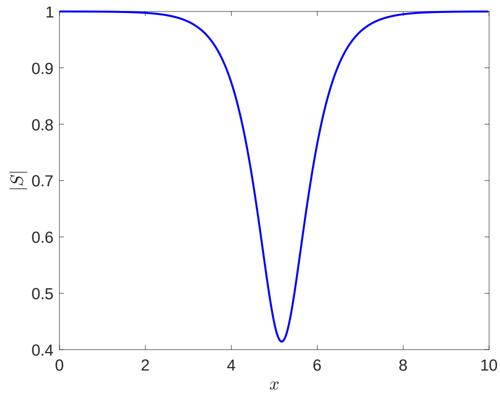
(a) Long wave L.



(b) Short wave $|S|$.



(c) Long wave profile at $t = 2.5$.



(d) Short wave profile at $t = 2.5$.

Figure 9: 1-dark-soliton solution with $\alpha = -1$, $\delta = 3$, $k_1 = 1 + i$, $\rho = 1$, $\nu = 1$, $q = 1$.

The N-dark-soliton solution is, as in the bright case, given in determinant form,

$$f = \left|\delta_{ij} - \frac{ik_j^* - \delta + \nu}{k_i + k_j^*} e^{\xi_i + \xi_j^*}\right|_{N\times N}, \tag{6.129a}$$

$$f^* = \left|\delta_{ij} + \frac{ik_j + \delta - \nu}{k_i^* + k_j} e^{\xi_i^* + \xi_j}\right|_{N\times N}, \tag{6.129b}$$

$$g = \left|\delta_{ij} + \frac{ik_j^* - \delta + \nu}{k_i + k_j^*} \frac{k_i - iq}{k_i^* + iq} e^{\xi_i + \xi_j^*}\right|_{N\times N}, \tag{6.129c}$$

$$g^* = \left|\delta_{ij} - \frac{ik_j + \delta - \nu}{k_i^* + k_j} \frac{k_i^* + iq}{k_i - iq} e^{\xi_i^* + \xi_j}\right|_{N\times N}, \tag{6.129d}$$

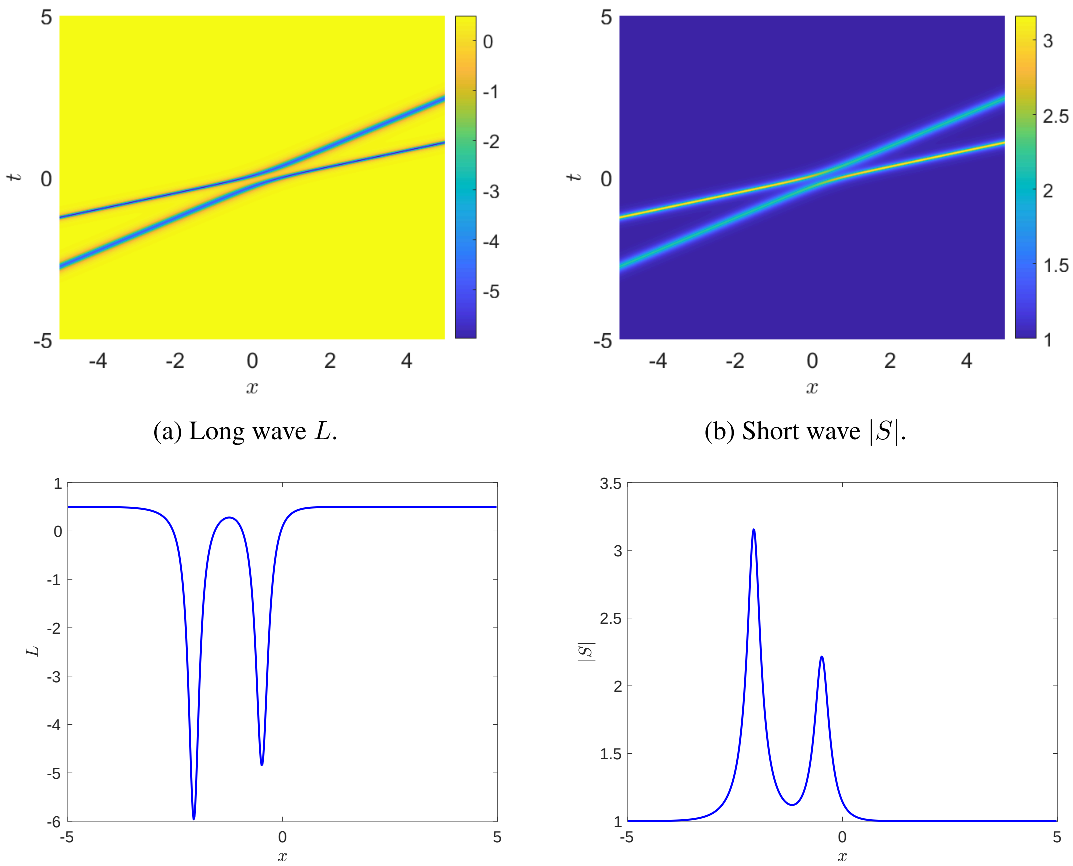
$$\xi_i = k_i x + ik_i^2 t + \xi_{i0}\,, \quad \xi_i^* = k_i^* x - ik_i^{*2} t + \xi_{i0}^*\,, \tag{6.129e}$$

where the parameters are subject to the constraint

$$\frac{2i\alpha(q-\delta+\nu)\rho^2}{|k_i - iq|^2} = k_i - k_i^* \,. \tag{6.130}$$

As with the 1-dark-soliton solution, we can set $k_i = k_{ir} + ik_{ii}$ to write the constraint as

$$k_{ir} = \pm\left(\frac{\alpha(q-\delta+\nu)\rho^2}{k_{ii}} - (k_{ii}-q)^2\right)^{\frac{1}{2}} \,. \tag{6.131}$$



(a) Long wave L.

(b) Short wave $|S|$.

(c) Long wave profile at $t = -0.5$.

(d) Short wave profile at $t = -0.5$.

Figure 10: 2-dark-soliton solution with $\alpha = 2$, $\delta = -3$, $k_1 = \sqrt{2} + 2i$, $k_2 = \sqrt{6} + i$, $\rho = 1$, $\nu = 1$, $q = 1$.

The phase shift for the 2-dark-soliton solution can be written explicitly by denoting $k_1 = k_{1r} + ik_{1i}$ and $k_2 = k_{2r} + ik_{2i}$ and proceeding as in the bright case, that is, moving with the velocity of one of the solitons to make it stationary, so that for $t \to \pm\infty$ the solution collapses into a

1-soliton solution. For the first soliton, the phase shift reads

$$\phi_0 = \frac{\log\left(\dfrac{k_{1r}^2 + k_{1i}^2 + k_{2r}^2 + k_{2i}^2 + 2k_{1r}k_{2r} - 2k_{1i}k_{2i}}{k_{1r}^2 + k_{1i}^2 + k_{2r}^2 + k_{2i}^2 - 2k_{1r}k_{2r} - 2k_{1i}k_{2i}}\right)}{2k_{1r}} . \tag{6.132}$$

The formula for the second soliton can be also written by simply exchanging indices 1 and 2 in the formula above.

6.5 Breathers and rogue waves

To derive the breather solutions, we can use the same change of variable (6.105) we used for the dark solitons in the previous sections. For the breather case, we will employ the breather solutions for the KP hierarchy (6.114)

$$\tau_{n,k} = |m_{ij}^{n,k}|_{1\leqslant i,j\leqslant N} , \tag{6.133}$$

where

$$m_{ij}^{n,k} = \sum_{m,r=1}^{2} \frac{a_{im}b_{jr}(k_{im}-b)}{k_{im}+k_{jr}^*}\left(-\frac{k_{im}-b}{k_{jr}^*+b}\right)^n\left(-\frac{k_{im}-a}{k_{jr}^*+a}\right)^k e^{\xi_{im}+\xi_{jr}^*} , \tag{6.134}$$

with

$$\xi_{im} = \frac{1}{k_{im}-a}x_{-1} + k_{im}x_1 + k_{im}^2x_2 + \xi_{im,0} , \tag{6.135a}$$

$$\xi_{jr}^* = \frac{1}{k_{jr}^*+a}x_{-1} + k_{jr}^*x_1 + {k_{jr}^*}^2x_2 + \xi_{jr,0}^* , \tag{6.135b}$$

and k_{im}, k_{jr}, a_{im}, a_{jr}, $\xi_{im,0}$, $\xi_{jr,0}$ arbitrary complex parameters.

If we add the constraint condition

$$\frac{1}{k_{i1}-a} + \frac{1}{k_{i2}-a} = -\frac{1}{2\alpha(a-b)\rho^2}(k_{i1}+k_{i2}) , \tag{6.136}$$

which we can rewrite as

$$\frac{2\alpha(a-b)\rho^2}{(k_{i1}-a)(k_{i2}-a)} = k_{i1} + k_{i2}\,, \tag{6.137}$$

then the τ-functions satisfy

$$(a-b)\partial_{x_{-1}}\tau_{n,k} = \frac{1}{2\alpha\rho^2}\partial_{x_2}\tau_{n,k}\,, \tag{6.138}$$

so that (6.114c) becomes

$$(D_{x_2} + 2\alpha\rho^2)\tau_{n,k}\cdot\tau_{n+1,k} = 2\alpha\rho^2\tau_{n,k+1}\tau_{n+1,k-1}\,. \tag{6.139}$$

By performing the same transformation from the KP bilinear system to the one coming from the YON system, we can write the N-breather solution as

$$f = \left|\sum_{m,r=1}^{2}\frac{a_{im}b_{jr}[k_{jr}^* + i(\delta-\nu)]}{k_{im}+k_{jr}^*}e^{\xi_{im}+\xi_{jr}^*}\right|_{N\times N}\,, \tag{6.140a}$$

$$f^* = \left|\sum_{m,r=1}^{2}\frac{a_{im}^*b_{jr}^*[k_{jr} - i(\delta-\nu)]}{k_{im}^*+k_{jr}}e^{\xi_{im}^*+\xi_{jr}}\right|_{N\times N}\,, \tag{6.140b}$$

$$g = \left|\sum_{m,r=1}^{2}\frac{a_{im}b_{jr}[k_{jr}^* + i(\delta-\nu)]}{k_{im}+k_{jr}^*}\frac{k_{im}-iq}{k_{jr}+iq}e^{\xi_{im}+\xi_{jr}^*}\right|_{N\times N}\,, \tag{6.140c}$$

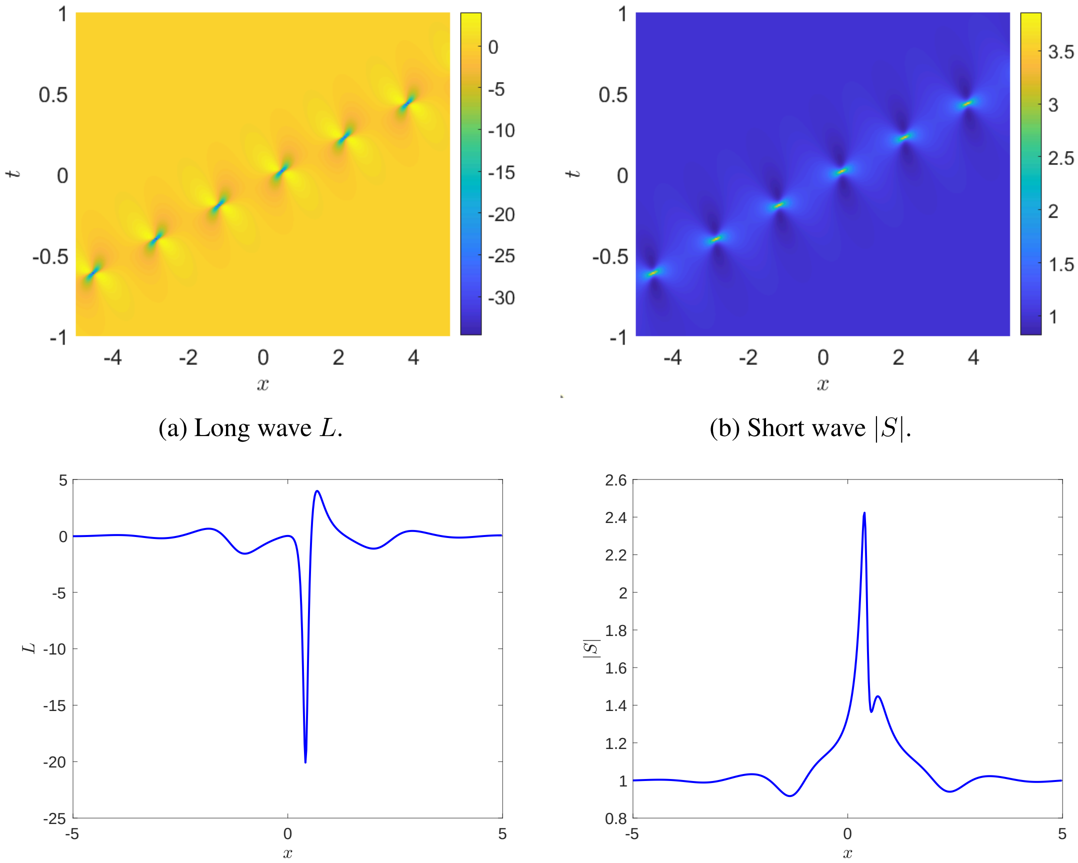
$$g^* = \left|\sum_{m,r=1}^{2}\frac{a_{im}^*b_{jr}^*[k_{jr} - i(\delta-\nu)]}{k_{im}^*+k_{jr}}\frac{k_{im}^*+iq}{k_{jr}^*-iq}e^{\xi_{im}^*+\xi_{jr}}\right|_{N\times N}\,. \tag{6.140d}$$

The constraint condition becomes

$$\frac{2\alpha(q-\delta+\nu)\rho^2}{(k_{i1}-iq)(k_{i2}-iq)} = k_{i1} + k_{i2}\,. \tag{6.141}$$

We can also study the fundamental rogue wave solution (understood as a rational solution) using the same bilinear equations in the KP hierarchy. Let us consider the solution

$$\tau_{n,k} = ABm^{n,k}\,, \tag{6.142}$$



(a) Long wave L.

(b) Short wave $|S|$.

(c) Long wave profile at $t = 0$.

(d) Short wave profile at $t = 0$.

Figure 11: Breather solution with $\delta = 1$, $k_1 = 2 + i$, $k_2 = 1 - 2i$, $a_1 = 1 + 2i$, $b_1 = 1 - 2i$, $a_2 = 2 - i$, $b_2 = 2 + 1$, $\rho = 1$, $\nu = 1$, $q = 1$.

where

$$m^{n,k} = \frac{-k_1 + b}{k_1 + k_1^*} \left(-\frac{k_1 - b}{k_1^* + b}\right)^n \left(-\frac{k_1 - a}{k_1^* + a}\right)^k e^{\xi_i + \xi_j^*} , \tag{6.143}$$

with

$$\xi_i = \frac{1}{k_1 - a} x_{-1} + k_1 x_1 + k_1^2 x_2 , \qquad \xi_j^* = \frac{1}{k_1^* + a} x_{-1} + k_1^* x_1 + k_1^{*2} x_2 , \tag{6.144}$$

and A and B are the differential operators

$$A = a_0(k_1 \partial_{k_1}) + a_1 , \qquad B = b_0(k_1^* \partial_{k_1^*}) + b_1 , \tag{6.145}$$

with a_0, a_1, b_0 and b_1 arbitrary constants. In this case, the constraint condition turns out to be

$$\frac{2\alpha(a-b)\rho^2}{(k_1-a)^2} = 2k_1^2\,, \tag{6.146}$$

which coincides with the constraint for breathers (6.137) for $k_{i1} = k_{i2} = k_1$. By taking the choice of parameters leading to the YON system, we get our fundamental rogue wave solution,

$$f = AB\left[\frac{k_1^* + i(\delta-\nu)}{k_1+k_1^*}e^{\xi_i+\xi_j^*}\right], \tag{6.147a}$$

$$g = -AB\left[\frac{k_1^* + i(\delta-\nu)}{k_1+k_1^*}\frac{k_1-iq}{k_1^*+iq}e^{\xi_i+\xi_j^*}\right]. \tag{6.147b}$$

The constraint condition then becomes

$$\frac{2i\alpha(\delta-\nu-q)\rho^2}{(k_1-iq)^2} = 2k_1^2\,, \tag{6.148}$$

which we can rewrite using $iz = k_1$ as

$$2z(z-q)^2 + 2\alpha(\delta-\nu-q)\rho^2 = 0\,. \tag{6.149}$$

Now, since we have $k_1 + k_1^*$ in the denominators, we need k_1 to have a non-zero real part. That means that (6.149) needs to feature complex conjugate roots, since it is a cubic equation with real coefficients. We can write it in terms of the original α and β instead of using δ as

$$2z(z-q)^2 + (\beta - 2\alpha\nu - 2\alpha q)\rho^2\,. \tag{6.150}$$

The condition to have two complex conjugate roots in a cubic polynomial is a condition on the discriminant, namely

$$\Delta = 4\rho^2[2\alpha(q+\nu)-\beta]\Big[8q^3 + 27\rho^2[2\alpha(q+\nu)-\beta]\Big] < 0\,, \tag{6.151}$$

which coincides exactly with the condition for branches in the stability analysis (see (5.79), with r given in (5.58)) after identifying $\rho = a$ and $\nu = \alpha b$, hence supporting the conjecture in Section 5.3. Note that, when taking $\alpha = 0$, hence reducing the YON system to the Yajima-Oikawa system, the condition for the existence of rogue waves coincides with the one derived using Darboux-dressing in [156].

Chapter 7

Conclusions and Outlook

To close out the thesis, let us give a brief tally of the results presented in the thesis and an outlook of prospective lines of research branching from them.

In this work we have presented a new long wave-short wave interaction system, unifying and generalising the Yajima-Oikawa and Newell systems into a what we called the Yajima-Oikawa-Newell system. It has the remarkable property of being Lax integrable for any choice of the two arbitrary, non-rescalable parameters it features. We have derived its bright and dark multi-soliton solutions, as well as the rational solutions by means of a Hirota bilinear approach. We have studied the stability spectra associated to its plane wave solutions, showing that the condition for the existence of the topological feature referred to as branch in the stability spectrum of a plane wave corresponds to the existence condition of the corresponding rogue wave coming from the Hirota bilinear method.

In Chapter 1 we gave an introduction on several techniques and pieces of theory on integrable systems, including multiscale method, Lax pairs and inverse scattering. We showed how the standard inverse scattering techniques are not of application to certain kinds of systems where the Lax pair features a singular matrix (and, later on, we verified that the Yajima-Oikawa-Newell system is an example of those). Trying to derive new machinery for this kind of systems is one of our current lines of research, jointly with Cornelis van der Mee. We finished the introductory chapter giving a review of different approaches to the stability of solutions of nonlinear systems, and their advantages and disadvantages in terms of applicability and additional information they provide. Another of our lines of research focuses on how these different techniques relate to each other, especially to the one we later employ for Chapter 5.

We then went on to reproduce the historical introduction of two very well-known integrable systems to model the interaction between long and short waves, namely the Yajima-Oikawa (Chapter 2) and Newell (Chapter 3) systems, after which we provided a review of the different results on them available in the literature.

After having presented those two classical systems, in Chapter 4 we introduced a new system generalising them into a single integrable system, which we called Yajima-Oikawa-Newell, via working with their Lax pairs. In a first survey of the system, we employed an Ansatz approach to compute periodic solutions in the form of elliptic functions, soliton solutions (by making the periodic of the elliptic solutions go to infinity), and rational solutions. We are currently working on finding connections and applications of the Yajima-Oikawa-Newell system for physical settings, jointly with Antonio Degasperis and others. In this chapter we also present existing literature about links between the Yajima-Oikawa system and the Newell system, and how, despite some authors claiming it provides a Miura transformation, the arguments provided fell short of a proof of this fact. We are currently trying to derive a Miura transformation for these systems jointly with Annalisa Calini.

In Chapter 5, we present a method to investigate the stability of solutions of integrable systems by transforming the problem of stability into the problem of studying the geometric and topologic features of a certain curve in the spectral parameter space, which we call the stability spectrum. We applied the method to study the stability of the plane wave solutions of the Yajima-Oikawa-Newell system, and concluded that they are unstable for almost any choice of parameters. We also introduced a few conjectures relating the topology of the stability spectrum to the existence of various types of solutions, namely rational solutions and solitons. A rigorous proof for these conjectures is not available at this point of the research, and is currently work in progress. Another of our current lines of research deals with extending the applicability of the stability method, either to broader classes of Lax pairs (which could potentially lead to the study of some systems of great physical relevance, like the massive Thirring model), or to broader classes of solutions, particularly to periodic solutions and soliton solutions of multicomponent systems (both of which have been barely studied). Finally, jointly with Sara Lombardo and Priscila Leal da Silva, we are investigating the relation between our stability spectra and the ones studied by Deconinck and collaborators.

Finally, in Chapter 6 we provided an overview of the general theory of Hirota bilinearisation and, more in particular, of the theory of τ-functions, relating the bilinear form of integrable systems to that of members of the Kadomtsev-Petviashvili hierarchy. We then applied these Hirota techniques to obtain various kinds of solutions of the Yajima-Oikawa-Newell system, namely bright and dark solitons, breathers, and rational solutions. These results form part of a paper currently in preparation, written jointly with Baofeng Feng and Kenichi Maruno. In particular, the condition derived for the existence of rational solutions for the Yajima-Oikawa-Newell equation coincides with our condition for the existence of branches in the stability spectrum in Chapter 5, hence supporting our conjectures in Section 5.3. As a final note, we want to mention that a further line of research related to the material in Chapter 6 focuses on the connection between the Hirota techniques and spectral techniques, such as the Darboux-dressing method. This is also work in progress.

Appendix A

Proof for the Lax Pair Formulae

In this appendix we will provide the proofs for formulae (5.15) to (5.19) and Propositions 5.1.3 and 5.1.4.

Let us consider a Lax pair of the form

$$X = i\lambda\Sigma + Q\,, \qquad T = \lambda^2 T_2 + \lambda T_1 + T_0\,, \tag{A.1}$$

where Σ is a constant diagonal matrix, and Q, T_0, T_1 and T_0 are matrices depending only on x and t, but not on λ. For the sake of simplicity, we will assume that all eigenvalues of Σ have multiplicity 1 and Q is off-diagonal (although the proof will be also valid for a matrix Σ with repeated eigenvalues and Q a block-off-diagonal as introduced in Chapter 5). The compatibility condition of the Lax pair reads

$$X_t - T_x + [X, T] = 0\,, \tag{A.2}$$

which, for the choices of X and T in (A.1), translates as

$$Q_t - \Big(\lambda^2 (T_2)_x + \lambda (T_1)_x + (T_0)_x\Big) + \Big[i\lambda\Sigma + Q,\, \lambda^2 T_2 + \lambda T_1 + T_0\Big] = 0\,. \tag{A.3}$$

We will assume the matrices Σ and Q are known and try to find an expression for T_2, T_1 and T_0 in terms of them. Since the relation (A.3) has to hold for every choice of λ, the coefficient of every power of λ has to be identically zero, so we will study each of the coefficients to obtain relations among the matrices. Henceforth, for any given matrix M we will denote its diagonal part as $M^{(d)}$

and its off-diagonal part as $M^{(o)}$. From the coefficient of λ^3, we get the expression

$$i[\Sigma, T_2] = 0 \quad \Longrightarrow \quad T_2^{(o)} = 0\,, \tag{A.4}$$

since Σ is a diagonal matrix with distinct eigenvalues. Now, the coefficient of λ^2 gives us the expression

$$-(T_2)_x + i[\Sigma, T_1] + [Q, T_2] = 0\,. \tag{A.5}$$

Note that both Σ and T_2 are diagonal matrices, which imply that the result of both commutators in (A.5) is off-diagonal. In order to study it, we will split the equation in diagonal and off-diagonal part. For the diagonal part we have

$$\left(T_2^{(d)}\right)_x = 0\,, \tag{A.6}$$

which, along with (A.4) tells us that T_2 is a constant, diagonal matrix. We will denote it as $T_2 = C_2$. From the off-diagonal part of (A.5) we also get the expression

$$i\big[\Sigma, T_1^{(o)}\big] + [Q, T_2] = 0\,, \tag{A.7}$$

where we have used that, since Σ is diagonal, $[\Sigma, T_1] = \big[\Sigma, T_1^{(o)}\big]$. To obtain a formula for $T_1^{(o)}$, let us take a look at the expression

$$\big[\Sigma, T_1^{(o)}\big] = i[Q, C_2]\,. \tag{A.8}$$

Note that the linear map $[\Sigma, \bullet]$ is actually an automorphism of the subalgebra of off-diagonal matrices, and hence it is invertible when restricted to that class. Let us denote its inverse by Γ. Given any off-diagonal matrix M, one can explicitly write $\Gamma(M)$ as

$$\big(\Gamma(M)\big)_{ij} = \frac{M_{ij}}{\alpha_i - \alpha_j}\,, \quad i \neq j\,, \qquad \big(\Gamma(M)\big)_{ii} = 0\,, \tag{A.9}$$

where $\alpha_j = \Sigma_{jj}$ denotes the j-th diagonal entry of Σ. It is easy to check that, as expected,

$$\Gamma\big([\Sigma, M]\big) = \big[\Sigma, \Gamma(M)\big] = M\,. \tag{A.10}$$

With this definition of Γ, we can use the equation (A.8) to get an explicit expression

$$T_1^{(o)} = i\Gamma\big([Q, C_2]\big) = -iD_2(Q)\,, \tag{A.11}$$

where we have arbitrarily defined

$$D_j(M) = \Gamma\big([C_j, M]\big) \tag{A.12}$$

for any off-diagonal matrix M. Now, from the coefficient of λ, we have the relation

$$-(T_1)_x + i[\Sigma, T_0] + [Q, T_1] = 0\,. \tag{A.13}$$

We will proceed as before and split it into diagonal and off-diagonal part. Note that, as opposed to T_2, now T_1 has a non-zero off-diagonal part. From the diagonal part of (A.13), we get

$$-\big(T_1^{(d)}\big)_x + \big[Q, T_1^{(o)}\big]^{(d)} = 0\,, \tag{A.14}$$

so that

$$T_1^{(d)} = \int \big[Q, T_1^{(o)}\big]^{(d)} = i\int \Big[Q, \Gamma\big([Q, C_2]\big)\Big]^{(d)}\,. \tag{A.15}$$

We have that, since C_2 is diagonal,

$$[Q, C_2]_{ij} = Q_{ij}(C_2)_{jj} - (C_2)_{ii}Q_{ij} = Q_{ij}(\gamma_j - \gamma_i)\,, \tag{A.16}$$

where we have denoted by $\gamma_j = (C_2)_{jj}$ the diagonal entries of C_2. Then, from the definition of Γ in (A.9), we have

$$\Gamma\big([Q, C_2]\big)_{ij} = Q_{ij}\frac{\gamma_j - \gamma_i}{\alpha_i - \alpha_j}\,, \tag{A.17}$$

so that

$$\begin{aligned} \Big[Q, \Gamma\big([Q, C_2]\big)\Big]_{ij} &= \sum_k \left(Q_{ik}Q_{kj}\frac{\gamma_j - \gamma_k}{\alpha_k - \alpha_j} - Q_{ik}Q_{kj}\frac{\gamma_k - \gamma_i}{\alpha_i - \alpha_k}\right) \\ &= \sum_k Q_{ik}Q_{kj}\left(\frac{\gamma_j - \gamma_k}{\alpha_k - \alpha_j} - \frac{\gamma_k - \gamma_i}{\alpha_i - \alpha_k}\right)\,. \end{aligned} \tag{A.18}$$

Now, when $i = j$, the constants inside the bracket cancel out, so that

$$\Big[Q, \Gamma\big([Q, C_2]\big)\Big]_{ii} = 0\,, \tag{A.19}$$

and hence from (A.15) we get that

$$T_1^{(d)} = \int 0_{N\times N} = C_1\,, \tag{A.20}$$

where C_1 denotes an arbitrary, constant, diagonal matrix. Furthermore, from the off-diagonal part of (A.13) we get the relation

$$-\big(T_1^{(o)}\big)_x + i\big[\Sigma, T_0^{(o)}\big] + [Q, T_1]^{(o)} = 0\,, \tag{A.21}$$

so that, proceeding as before,

$$\begin{aligned} T_0^{(o)} &= i\Gamma\big([Q, T_1]^{(o)}\big) - i\Gamma\Big(\big(T_1^{(o)}\big)_x\Big) \\ &= i\Gamma\Big(\big[Q, C_1 - iD_2(Q)\big]^{(o)}\Big) + \Gamma\Big(\Gamma\big([Q_x, C_2]\big)\Big) \\ &= -iD_1(Q) + \Gamma\Big(\big[Q, D_2(Q)\big]^{(o)}\Big) - \Gamma\big(D_2(Q_x)\big)\,. \end{aligned} \tag{A.22}$$

Finally, the independent term gives

$$Q_t - (T_0)_x + [Q, T_0] = 0\,. \tag{A.23}$$

Its diagonal part gives the relation

$$-\big(T_0^{(d)}\big)_x + \big[Q, T_0^{(o)}\big]^{(d)} = 0\,, \tag{A.24}$$

so that

$$\begin{aligned} T_0^{(d)} = \int \big[Q, T_0^{(o)}\big]^{(d)} &= i\int \Big[Q, \Gamma\big([Q, C_1]\big)\Big]^{(d)} \\ &+ \int \Big[Q,\, \Gamma\Big(\big[Q, D_2(Q)\big]\Big)\Big]^{(d)} - \int \Big[Q,\, \Gamma\big(D_2(Q_x)\big)\Big]^{(d)}\,. \end{aligned} \tag{A.25}$$

Let us study each of those integrals individually. First, note that

$$\int \Big[Q, \Gamma\big([Q, C_1]\big)\Big]^{(d)} = \int 0_{N\times N} \tag{A.26}$$

by the same reasoning as (A.15)-(A.20). For the next one, let us recall that

$$D_2(Q)_{ij} = Q_{ij}\frac{\gamma_i - \gamma_j}{\alpha_i - \alpha_j}\,, \tag{A.27}$$

so that

$$\begin{aligned}\big[Q, D_2(Q)\big]_{ij} &= \sum_k \left(Q_{ik}Q_{kj}\frac{\gamma_k - \gamma_j}{\alpha_k - \alpha_j} - Q_{ik}Q_{kj}\frac{\gamma_i - \gamma_k}{\alpha_i - \alpha_k}\right)\\ &= \sum_k Q_{ik}Q_{kj}\left(\frac{\gamma_k - \gamma_j}{\alpha_k - \alpha_j} - \frac{\gamma_i - \gamma_k}{\alpha_i - \alpha_k}\right)\,,\end{aligned} \tag{A.28}$$

and then

$$\Gamma\Big(\big[Q, D_2(Q)\big]\Big)_{ij} = \sum_k \frac{Q_{ik}Q_{kj}}{\alpha_i - \alpha_j}\left(\frac{\gamma_k - \gamma_j}{\alpha_k - \alpha_j} - \frac{\gamma_i - \gamma_k}{\alpha_i - \alpha_k}\right)\,. \tag{A.29}$$

Now, we have

$$\begin{aligned}\Big[Q,\, \Gamma\Big(\big[Q, D_2(Q)\big]\Big)\Big]_{ij} &= \sum_{k,\,l} \frac{Q_{il}Q_{lk}Q_{kj}}{\alpha_l - \alpha_j}\left(\frac{\gamma_k - \gamma_j}{\alpha_k - \alpha_j} - \frac{\gamma_l - \gamma_k}{\alpha_l - \alpha_k}\right)\\ &\quad - \sum_{k,\,l} \frac{Q_{ik}Q_{kl}Q_{kj}}{\alpha_i - \alpha_l}\left(\frac{\gamma_k - \gamma_l}{\alpha_k - \alpha_l} - \frac{\gamma_i - \gamma_k}{\alpha_i - \alpha_k}\right)\,.\end{aligned} \tag{A.30}$$

Swapping the indices k and l in the second sum, we have that

$$\begin{aligned}&\Big[Q,\, \Gamma\Big(\big[Q, D_2(Q)\big]\Big)\Big]_{ij} =\\ &= \sum_{k,\,l} Q_{il}Q_{lk}Q_{kj}\left[\frac{1}{\alpha_l - \alpha_j}\left(\frac{\gamma_k - \gamma_j}{\alpha_k - \alpha_j} - \frac{\gamma_l - \gamma_k}{\alpha_l - \alpha_k}\right) - \frac{1}{\alpha_i - \alpha_k}\left(\frac{\gamma_l - \gamma_k}{\alpha_l - \alpha_k} - \frac{\gamma_i - \gamma_l}{\alpha_i - \alpha_l}\right)\right]\,,\end{aligned} \tag{A.31}$$

but then when $i = j$, all the constants in the bracket cancel out, so that

$$\Big[Q,\, \Gamma\Big(\big[Q, D_2(Q)\big]\Big)\Big]_{ii} = 0 \tag{A.32}$$

and

$$\int \Big[Q,\, \Gamma\Big(\big[Q, D_2(Q)\big]\Big)\Big]^{(d)} = \int 0_{N\times N}\,. \tag{A.33}$$

We now have only one integral left in (A.25),

$$\int \Big[Q,\, \Gamma\big(D_2(Q_x)\big)\Big]^{(d)}. \tag{A.34}$$

Let us proceed with it. First, we have that

$$D_2(Q_x)_{ij} = Q_{ij}\frac{\gamma_i - \gamma_j}{\alpha_i - \alpha_j}\,, \tag{A.35}$$

so that

$$\Gamma\big(D_2(Q_x)\big)_{ij} = Q_{ij}\frac{\gamma_i - \gamma_j}{(\alpha_i - \alpha_j)^2}\,, \tag{A.36}$$

and hence

$$\Big[Q,\, \Gamma\big(D_2(Q_x)\big)\Big]_{ij} = \sum_k \left[Q_{ik}(Q_{kj})_x\frac{\gamma_k - \gamma_j}{(\alpha_k - \alpha_j)^2} - (Q_{ik})_x Q_{kj}\frac{\gamma_i - \gamma_k}{(\alpha_i - \alpha_k)^2}\right]. \tag{A.37}$$

Now, when $i = j$, we have

$$\Big[Q,\, \Gamma\big(D_2(Q_x)\big)\Big]_{ii} = \sum_k \frac{\gamma_k - \gamma_i}{(\alpha_k - \alpha_i)^2}\Big(Q_{ik}(Q_{ki})_x + (Q_{ik})_x Q_{ki}\Big)\,. \tag{A.38}$$

Now the key point is that we can distribute the constant to construct entries of $\Gamma\big(D_2(Q)\big)$ or $\Gamma\big(D_2(Q_x)\big)$ in any way we want, so that we get the relation

$$\Big[Q,\, \Gamma\big(D_2(Q_x)\big)\Big]_{ii} = \Big[Q,\, \Gamma\big(D_2(Q_x)\big)\Big]_{ii} = -\Big[\Gamma\big(D_2(Q)\big),\, Q_x\Big]_{ii}\,. \tag{A.39}$$

Let us call $R = \Gamma\big(D_2(Q)\big)$ and $R_x = \Gamma\big(D_2(Q_x)\big)$ to simplify the notation. Then we have

$$\begin{aligned}
[Q, R_x] &= \frac{1}{2}\Big([Q, R_x] - [R, Q_x]\Big)\\
&= \frac{1}{2}\Big(QR_x - R_xQ - RQ_x + Q_xR\Big)\\
&= \frac{1}{2}\Big(\big(QR\big)_x - \big(RQ\big)_x\Big)\\
&= \frac{1}{2}\Big(QR - RQ\Big)_x = \frac{1}{2}\big[Q, R\big]_x\,.
\end{aligned} \tag{A.40}$$

That means that

$$\Big[Q,\,\Gamma\big(D_2(Q_x)\big)\Big]_{ii} = \Big[Q,\,\Gamma\big(D_2(Q_x)\big)\Big]_{ii} = \frac{1}{2}\Big(\Big[Q,\Gamma\big(D_2(Q_x)\big)\Big]_x\Big)_{ii} \tag{A.41}$$

and hence

$$\Big[Q,\,\Gamma\big(D_2(Q_x)\big)\Big]^{(d)} = \frac{1}{2}\Big[Q,\Gamma\big(D_2(Q_x)\big)\Big]_x^{(d)}\,. \tag{A.42}$$

Putting all these results back into (A.25), we have that

$$T_0^{(d)} = -\frac{1}{2}\int \Big[Q,\Gamma\big(D_2(Q_x)\big)\Big]_x^{(d)} = C_0 - \frac{1}{2}\Big[Q,\Gamma\big(D_2(Q_x)\big)\Big]^{(d)}\,, \tag{A.43}$$

where C_0 is a constant, diagonal matrix.

We can now put all the results together to give the full expression for T_2, T_1 and T_0:

$$T_2 = T_2^{(d)} + T_2^{(o)} = C_2\,, \tag{A.44a}$$

$$T_1 = T_1^{(d)} + T_1^{(o)} = C_1 - iD_2(Q)\,, \tag{A.44b}$$

$$\begin{aligned} T_0 &= T_0^{(d)} + T_0^{(o)} \\ &= C_0 - \frac{1}{2}\Big[Q,\Gamma\big(D_2(Q_x)\big)\Big]^{(d)} - iD_1(Q) + \Gamma\Big(\big[Q,D_2(Q)\big]^{(o)}\Big) - \Gamma\big(D_2(Q_x)\big)\,, \end{aligned} \tag{A.44c}$$

which coincide with the formulae given in (5.15) after setting $I_1 = I_0 = 0$. We will also set $C_0 = 0$ since it does not matter for our purpose.

The off-diagonal part of (A.23) gives us the formula for Q_t, which coincides exactly with the one presented in (5.21) after substituting the formulae in (A.44).

I_0 and I_1 being zero comes from the fact that we have assumed that all the diagonal matrices involved have distinct eigenvalues. To understand how it works and prove the Proposition 5.1.3 we can look at how the proof adapts to having a matrix Σ with repeated eigenvalues, so that (d) and (o) represent respectively the block-diagonal and block-off-diagonal part of the matrix, as explained in Chapter 5. In that framework, M_{ij} represents the i,j-th block of the matrix, instead of a single matrix, so that we have to be careful not to assume commutativity. The only point of the proof where we used commutativity was in (A.16) and the analogous computation for C_1. That relation is only true if we impose that $(C_2)_{jj} = \gamma_j \mathbb{1}_j$ and $(C_1)_{jj} = \beta_j \mathbb{1}_j$, which means that C_2

and C_1 have the same diagonal structure as Σ and is exactly the condition (5.20) of the Proposition 5.1.3.

References

[1] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. London Mathematical Society Lecture Note Series, vol. 149. Cambridge University Press, Cambridge, UK, 1991.

[2] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Physica D*, 18:455–463, 1974.

[3] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. *Discrete and Continuous Nonlinear Schrödinger Systems*. Cambridge University Press, Cambridge, UK, 2004.

[4] M. J. Ablowitz and C. M. Schober. Chaotic dynamics of modulational instability in optical fibers. *AIP Conf. Proc.*, 755(40), 2005.

[5] M. J. Ablowitz and H. Segur. *Solitons and the Inverse Scattering Transform*. Studies in Applied and Numerical Mathematics. SIAM, Philadelphia, PA, US, 1981.

[6] T. Aktosun, F. Demontis, and C. van der Mee. Exact solutions to the focusing nonlinear Schrödinger equation. *Inverse Probl.*, 23(5):2171, 2007.

[7] S. C. Anco and G. W. Bluman. Direct construction of conservation laws from field equations. *Phys. Rev. Lett.*, 78:2869–2873, 1997.

[8] S. C. Anco and G. W. Bluman. Direct construction method for conservation laws of partial differential equations. I: Examples of conservation law classifications. *Eur. J. Appl. Math.*, 13:545–566, 2002.

[9] S. C. Anco and G. W. Bluman. Direct construction method for conservation laws of partial differential equations. II: General treatment. *Eur. J. Appl. Math.*, 13:567–585, 2002.

[10] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics, vol. 60. Springer, New York, NY, US, 1989.

[11] O. Babelon, D. Bernard, and M. Talon. *Introduction to Classical Integrable Systems*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, UK, 2003.

[12] D. Baldwin and W. Hereman. Symbolic software for the Painlevé test of nonlinear ordinary and partial differential equations. *J. Nonlinear Math. Phys.*, 13:90–111, 2006.

[13] F. Baronio, S. Chen, P. Grelu, S. Wabnitz, and M. Conforti. Baseband modulation instability as the origin of rogue waves. *Phys. Rev. A*, 91:033804, 2015.

[14] F. Baronio, M. Conforti, A. Degasperis, S. Lombardo, M. Onorato, and S. Wabnitz. Vector rogue waves and baseband modulational instability in the defocusing regime. *Phys. Rev. Lett.*, 113:034101, 2014.

[15] T. B. Benjamin and J. E. Feir. The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.*, 27:417–430, 1967.

[16] D. J. Benney. A general theory for interactions between short and long waves. *Stud. Appl. Math.*, 56:81–94, 1977.

[17] D. J. Benney and A. C. Newell. The propagation of nonlinear wave envelopes. *J. Math. Phys.*, 46:133–139, 1967.

[18] R. Bhatia. *Matrix Analysis*. Graduate Texts in Mathematics, vol. 169. Springer, New York, NY, US, 1997.

[19] G. Biondini, S. Li, D. Mantzavinos, and S. Trillo. Universal behavior of modulationally unstable media. *SIAM Rev.*, 60(4):888–908, 2018.

[20] J. D. Bjorken and S. D. Drell. *Relativistic Quantum Fields*. McGraw-Hill, New York, NY, US, 1965.

[21] H. Borluk and S. Erbay. Stability of solitary waves for three-coupled long wave-short wave interaction equations. *IMA J. Appl. Math.*, 76:582–598, 2011.

[22] B. Buchberger. Theoretical basis for the reduction of polynomials to canonical forms. *ACM SIGSAM Bull.*, 10(3):19–29, 1976.

[23] A. Calini. Personal communication.

[24] A. Calini, C. M. Schober, and M. Strawn. Linear instability of the Peregrine breather: Numerical and analytical investigations. *Appl. Numer. Math.*, 141:36–43, 2018.

[25] F. Calogero. Why are certain nonlinear PDEs both widely applicable and integrable? In V. E. Zakharov, editor, *What Is Integrability?* Springer, Berlin/Heidelberg, Germany, 1991.

[26] F. Calogero and A. Degasperis. *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations*. North-Holland, Amsterdam, The Netherlands, 1982.

[27] F. Calogero, A. Degasperis, and J. Xiaoda. Nonlinear Schrödinger-type equations from multiscale reduction of PDEs. I. Systematic derivation. *J. Math. Phys.*, 41:6399–6443, 2000.

[28] F. Calogero, A. Degasperis, and J. Xiaoda. Nonlinear Schrödinger-type equations from multiscale reduction of PDEs. II. Necessary conditions of integrability for real PDEs. *J. Math. Phys.*, 42:2635–2652, 2001.

[29] M. Caso-Huerta, A. Degasperis, P. Leal da Silva, S. Lombardo, and M. Sommacal. Periodic and solitary wave solutions of the long wave–short wave Yajima–Oikawa–Newell model. *Fluids*, 7(227), 2022.

[30] M. Caso-Huerta, A. Degasperis, S. Lombardo, and M. Sommacal. A new integrable model of long wave-short wave interaction and linear stability spectra. *Proc. R. Soc. A*, 477:20210408, 2021.

[31] M. Caso-Huerta, B.-F. Feng, S. Lombardo, K. Maruno, and M. Sommacal. Solitary and rational solutions of the Yajima-Oikawa-Newell long wave–short wave model via Hirota bilinearization. Paper in preparation.

[32] M. Caso-Huerta, S. Lombardo, M. Sommacal, and C. van der Mee. The scattering problem for the Yajima-Oikawa-Newell equations. Paper in preparation.

[33] H. N. Chan, R. H. J. Grimshaw, and K. W. Chow. Modeling internal rogue waves in a long wave-short wave resonance framework. *Phys. Rev. Fluids*, 3:124801, 2018.

[34] J. Chen, L. Chen, B.-F. Feng, and K. Maruno. High-order rogue waves of a long-wave–short-wave model of Newell type. *Phys. Rev. E*, 100:052216, 2019.

[35] J. Chen, Y. Chen, B.-F. Feng, K. Maruno, and Y. Ohta. General high-order rogue waves of the (1+1)-dimensional Yajima-Oikawa system. *J. Phys. Soc. Jpn.*, 87:094007, 2018.

[36] J. Chen, Y. Chen, B.-F. Feng, and H.-M. Zhu. Pfaffian-type soliton solution to a multi-component coupled Ito equation. *Chin. Phys. Lett.*, 31(11):110502, 2014.

[37] J. Chen and B.-F. Feng. A note on the bilinearization of the generalized derivative nonlinear Schrödinger equation. *J. Phys. Soc. Jpn.*, 90(2):023001, 2021.

[38] J. Chen and B.-F. Feng. Tau-function formulation for bright, dark soliton and breather solutions to the massive Thirring model. *Stud. Appl. Math.*, 150(1):35–68, 2022.

[39] J. Chen, B.-F. Feng, K. Maruno, and Y. Ohta. The derivative Yajima–Oikawa system: Bright, dark soliton and breather solutions. *Stud. Appl. Math.*, 141(2):145–185, 2017.

[40] A. F. Cheviakov. GeM software package for computation of symmetries and conservation laws of differential equations. *Comput. Phys. Commun.*, 176:48–61, 2007.

[41] A. F. Cheviakov. Computation of fluxes of conservation laws. *J. Eng. Math.*, 66:153–173, 2010.

[42] A. F. Cheviakov. Symbolic computation of local symmetries of nonlinear and linear partial and ordinary differential equations. *Math. Comput. Sci.*, 4:203–222, 2010.

[43] A. F. Cheviakov. Symbolic computation of nonlocal symmetries and nonlocal conservation laws of partial differential equations using the GeM package for Maple. In J.-F. Ganghoffer and I. Mladenov, editors, *Similarity and Symmetry Methods*, Lecture Notes in Applied and Computational Mechanics, pages 165–184. Springer, Cham, Switzerland, 2014.

[44] A. F. Cheviakov. Symbolic computation of equivalence transformations and parameter reduction for nonlinear physical models. *Comput. Phys. Commun.*, 220:56–73, 2017.

[45] K. W. Chow, H. N. Chan, D. J. Kedziora, and R. H. J. Grimshaw. Rogue wave modes for the long wave-short wave resonance model. *J. Phys. Soc. Jpn.*, 82:074001, 2013.

[46] A. Das. *Integrable Models*. World Scientific Lecture Notes in Physics, vol. 30. World Scientific, Singapore, 1989.

[47] B. Deconinck, P. McGill, and B. L. Segal. The stability spectrum for elliptic solutions to the sine-Gordon equation. *Physica D*, 360:17–35, 2017.

[48] B. Deconinck and B. L. Segal. The stability spectrum for elliptic solutions to the focusing NLS equation. *Physica D*, 346:1–19, 2017.

[49] B. Deconinck and J. Upsal. The orbital stability of elliptic solutions of the focusing nonlinear Schrödinger equation. *SIAM J. Math. Anal.*, 52(1):1–41, 2020.

[50] A. Degasperis. Multiscale expansion and integrability of dispersive wave equations. In A. V. Mikhailov, editor, *Integrability*, Lecture Notes in Physics, vol. 767, pages 215–244. Springer, Berlin/Heidelberg, Germany, 2009.

[51] A. Degasperis. Integrable nonlocal wave interaction models. *J. Phys. A*, 44(5):052002, 2011.

[52] A. Degasperis and S. Lombardo. Integrability in action: Solitons, instability and rogue waves. In M. Onorato, S. Resitorio, and F. Baronio, editors, *Rogue and Shock Waves in Nonlinear Dispersive Media*, Lecture Notes in Physics, vol. 926, pages 23–53. Springer, Berlin/Heidelberg, Germany, 2016.

[53] A. Degasperis, S. Lombardo, and M. Sommacal. Integrability and linear stability of nonlinear waves. *J. Nonlinear Sci.*, 28:1251–1291, 2018.

[54] A. Degasperis, S. Lombardo, and M. Sommacal. Coupled nonlinear Schrödinger equations: Spectra and instabilities of plane waves. In N. Euler and M. C. Nucci, editors, *Nonlinear Systems and Their Remarkable Mathematical Structures*, volume 2. CRC Press, Boca Raton, FL, US, 2019.

[55] A. Degasperis, S. Lombardo, and M. Sommacal. Rogue wave type solutions and spectra of coupled nonlinear Schrödinger equations. *Fluids*, 4(1):57, 2019.

[56] F. Demontis, S. Lombardo, M. Sommacal, C. van der Mee, and F. Vargiu. Effective generation of closed-form soliton solutions of the continuous classical Heisenberg ferromagnet equation. *Commun. Nonlinear Sci. Numer. Simul.*, 64:35–65, 2018.

[57] F. Demontis, B. Prinari, C. van der Mee, and F. Vitale. The inverse scattering transform for the focusing nonlinear Schrödinger equation with asymmetric boundary conditions. *J. Math. Phys.*, 55(10):101505, 2018.

[58] V. D. Djordjevic and L. G. Redekopp. On two-dimensional packets of capillary-gravity waves. *J. Fluid Mech.*, 79:703–714, 1977.

[59] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris. *Solitons and Nonlinear Wave Equations*. Academic Press, New York, NY, US, 1982.

[60] A. Doliwa. Desargues maps and the Hirota–Miwa equation. *Proc. R. Soc. A*, 466:1177–1200, 2010.

[61] P. G. Drazin and R. S. Johnson. *Solitons: An Introduction*. Cambridge University Press, Cambridge, UK, 1972.

[62] H. A. Erbay and S. Erbay. Transverse linear instability of solitary waves for coupled long-wave short-wave interaction equations. *Appl. Math. Lett.*, 25:2402–2406, 2012.

[63] L. D. Faddeev and L. A. Takhtajan. *Hamiltonian Methods in the Theory of Solitons*. Springer, Berlin, Germany, 1987.

[64] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases (F_4). *J. Pure Appl. Algebra*, 139(1):61–88, 1999.

[65] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases without reduction to zero (F_5). In *Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation (ISSAC)*, pages 75–83. ACM Press, New York, NY, US, 2002.

[66] B.-F. Feng. General N-soliton solution to a vector nonlinear Schrödinger equation. *J. Phys. A Math.*, 47(35):355203, 2014.

[67] B.-F. Feng. Discrete Kadomtsev-Petviashvili (KP) equation and its applications, March 2022. Workshop notes for the 12th IMACS International Conference on Nonlinear Evolution Equations and Wave Phenomena.

[68] B.-F. Feng, X. Luo, M. J. Ablowitz, and Z. H. Musslimani. General soliton solution to a nonlinear nonlinear Schrödinger equation with zero and nonzero boundary conditions. *Nonlinearity*, 31(12):5385, 2017.

[69] B.-F. Feng, K. Maruno, and Y. Ohta. On the τ-functions of the reduced Ostrovsky equation and the $A_2^{(2)}$ two-dimensional Toda system. *J. Phys. A Math.*, 45(35):355203, 2012.

[70] B.-F. Feng, K. Maruno, and Y. Ohta. On the τ-functions of the Degasperis–Procesi equation. *J. Phys. A Math.*, 46(4):045205, 2013.

[71] B.-F. Feng, K. Maruno, and Y. Ohta. An integrable semi-discrete Degasperis-Procesi equation. *Nonlinearity*, 30(6):2246, 2015.

[72] B.-F. Feng, K. Maruno, and Y. Ohta. Geometric formulation and multi-dark soliton solution to the defocusing complex short pulse equation. *Stud. Appl. Math.*, 138:215–232, 2016.

[73] B.-F. Feng, C. Shi, G. Zhang, and C. Wu. Higher-order rogue wave solutions of the Sasa-Satsuma equation. *J. Phys. A Math.*, 55:235701, 2022.

[74] H. Flaschka, M. G. Forest, and D. W. McLaughlin. Multiphase averaging and the inverse spectral solution of the Korteweg–de Vries equation. *Commun. Pure Appl. Math.*, 33:739–784, 1980.

[75] M. G. Forest, D. W. McLaughlin, D. J. Muraki, and O. C. Wright. Nonfocusing instabilities in coupled, integrable nonlinear Schrödinger PDEs. *J. Nonlinear Sci.*, 10(3):291–331, 2000.

[76] M. G. Forest and O. C. Wright. An integrable model for stable:unstable wave coupling phenomena. *Physica D*, 178(3–4):173–189, 2003.

[77] M. Funakoshi and M. Oikawa. The resonant interaction between a long internal gravity wave and a surface gravity wave packet. *J. Phys. Soc. Jpn.*, 52(6):1982–1995, 1983.

[78] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett.*, 19(19):1095–1097, 1967.

[79] X. Geng and R. Li. On a vector modified Yajima Oikawa long-wave short-wave equation. *Mathematics*, 7(10):958, 2019.

[80] B. Grammaticos, A. Ramani, and J. Hietarinta. Multilinear operators: The natural extension of Hirota's bilinear formalism. *Phys. Lett. A*, 190(1):65–70, 1994.

[81] S. D. Griffiths, R. H. J. Grimshaw, and K. R. Khusnutdinova. Modulational instability of two pairs of counter-propagating waves and energy exchange in a two-component system. *Physica D*, 214(1):1–24, 2006.

[82] R. H. J. Grimshaw. The modulation of an internal gravity-wave packet, and the resonance with the mean motion. *Stud. Appl. Math.*, 56:241–266, 1977.

[83] P. G. Grinevich and P. M. Santini. The exact rogue wave recurrence in the NLS periodic setting via matched asymptotic expansions, for 1 and 2 unstable modes. *Phys. Lett. A*, 382(14):973–979, 2018.

[84] P. G. Grinevich and P. M. Santini. The finite gap method and the analytic description of the exact rogue wave recurrence in the periodic NLS Cauchy problem. *Nonlinearity*, 31(11):5258, 2018.

[85] R. Hermann. *The Geometry of Nonlinear Differential Equations, Bäcklund Transformations, and Solitons*. Mathematics Sciences Press, Berkeley, CA, US, 1976.

[86] J. Hietarinta. Introduction to the Hirota bilinear method. In Y. Kosmann-Schwarzbach, B. Grammaticos, and K. M. Tamizhmani, editors, *Integrability of Nonlinear Systems*, Lecture Notes in Physics, vol. 495, pages 95–103. Springer, Cham, Switzerland, 1997.

[87] J. Hietarinta. Hirota's bilinear method and its connection with integrability. In A. V. Mikhailov, editor, *Integrability*, Lecture Notes in Physics, vol. 767, pages 279–314. Springer, Berlin/Heidelberg, Germany, 2008.

[88] R. Hirota. Exact solutions of the Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.*, 27:1192–1194, 1971.

[89] R. Hirota. *Direct Method in Soliton Theory*. Cambridge University Press, Cambridge, UK, 1972.

[90] R. Hirota. Exact solution of the modified Korteweg-de Vries equation for multiple collisions of solitons. *J. Phys. Soc. Jpn.*, 33:1456–1458, 1972.

[91] R. Hirota. Exact solution of the sine-Gordon equation for multiple collisions of solitons. *J. Phys. Soc. Jpn.*, 33:1459–1463, 1972.

[92] R. Hirota. A new form of Bäcklund transformations and its relation to the inverse scattering problem. *Prog. Theor. Phys.*, 52(5):1498–1512, 1974.

[93] R. Hirota. Discrete analogue of a generalized Toda equation. *J. Phys. Soc. Jpn.*, 50:3785–3791, 1981.

[94] N. J. Hitchin, G. B. Segal, and R. S. Ward. *Integrable Systems: Twistors, Loop Groups, and Riemann Surfaces*. Oxford Graduate Texts in Mathematics, vol. 4. Oxford University Press, Oxford, UK, 2013.

[95] E. L. Ince. *Ordinary Differential Equations*. Courier Corporation, North Chelmsford, MA, US, 1956.

[96] L. Jetto and V. Orsini. Some remarks on the Schur stability of interval polynomials. In *16th Mediterranean Conference on Control and Automation*. IEEE, New York, NY, US, 2008.

[97] M. Jimbo and T. Miwa. Solitons and infinite dimensional Lie algebras. *Publ. Res. Inst. Math. Sci.*, 19:943–1001, 1983.

[98] B. B. Kadomtsev and V. I. Petviashvili. On the stability of solitary waves in weakly dispersive media. *Sov. Phys. Dokl.*, 15:539–541, 1970.

[99] V. I. Karpman. On the dynamics of sonic-Langmuir solitons. *Phys. Scr.*, 11(5):263–265, 1975.

[100] D. J. Kaup. Closure of the squared Zakharov-Shabat eigenstates. *J. Math. Anal. Appl.*, 54:849–864, 1976.

[101] D. J. Kaup. The three wave interaction: A nondispersive phenomenon. *Stud. Appl. Math.*, 55(1):9–44, 1976.

[102] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-Gonzalez. *Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment*. Springer Series on Atomic, Optical, and Plasma Physics. Springer, Berlin, Germany, 2007.

[103] G. N. Koutsokostas, T. P. Horikis, D. J. Frantzeskakis, B. Prinari, and G. Biondini. Multiscale expansions avector solitons of a two-dimensional nonlocal nonlinear Schrödinger system. *Stud. Appl. Math.*, 145:739–764, 2020.

[104] C. S. Kueny and P. J. Morrison. Nonlinear instability and chaos in plasma wave-wave interactions. II. Numerical methods and results. *Phys. Plasmas*, 2(11):4149–4160, 1995.

[105] E. A. Kuznetsov. Solitons in a parametrically unstable plasma. *Dokl. Akad. Nauk SSSR*, 236(9):575–577, 1977.

[106] L. D. Landau and E. M. Lifshitz. *Mechanics*. Course of Theoretical Physics, vol. 1. Butterworth-Heinemann, Oxford, UK, 1976.

[107] H. B. Lawson. *The Quantitative Theory of Foliations*. CBMS Regional Conference Series in Mathematics, vol. 27. American Mathematical Society, Providence, RI, US, 1977.

[108] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Math.*, 21(5):467–490, 1968.

[109] P. D. Lax and R. S. Phillips. Scattering theory for automorphic functions. *Bull. Am. Math. Soc.*, 2(2):261–295, 1980.

[110] P. Leal da Silva, M. Sommacal, and C. Van der Mee. Personal communication.

[111] S. B. Leble and N. V. Ustinov. Darboux transforms, deep reductions and solitons. *J. Phys. A: Math. Gen.*, 26:5007–5016, 1993.

[112] A. V. Leznov and M. V. Saveliev. Representation of zero curvature for the system of nonlinear partial differential equations $x_{\alpha,z\bar{z}} = \exp(kx)_\alpha$ and its integrability. *Lett. Math. Phys.*, 3:489–494, 1979.

[113] L. Li, C. Duan, and F. Yu. An improved Hirota bilinear method and new application for a nonlocal integrable complex modified Korteweg-de Vries (MKdV) equation. *Phys. Lett. A*, 383(14):1578–1582, 2019.

[114] R. Li and X. Geng. On a vector long wave-short wave-type model. *Stud. Appl. Math.*, 144:164–184, 2019.

[115] R. Li and X. Geng. A matrix Yajima Oikawa long-wave-short-wave resonance equation, Darboux transformations and rogue wave solutions. *Commun. Nonlinear Sci.*, 90:105408, 2020.

[116] R. Li and X. Geng. Periodic-background solutions for the Yajima-Oikawa long wave–short wave equation. *Nonlinear Dyn.*, 94:1053–1067, 2022.

[117] L. Ling and Q. P. Liu. A long waves-short waves model: Darboux transformation and soliton solutions. *J. Math. Phys.*, 52:053513, 2011.

[118] J. Liouville. Note sur l'intégration des équations différentielles de la dynamique, présentée au Bureau des Longitudes le 29 juin 1853. *J. Math. Pures Appl.*, pages 137–138, 1855.

[119] Q. P. Liu. Modifications of k-constrained KP hierarchy. *Phys. Lett. A*, 187(5–6):373–381, 1994.

[120] S. Lombardo, M. Sommacal, and M. Romano. The 3-wave resonant interaction model: Spectra and instabilities of plane waves. Submitted for publication, 2022.

[121] Y.-C. Ma. The complete solution of the long-wave short-wave resonance equations. *Stud. Appl. Math.*, 59:201–221, 1978.

[122] F. Magri. Lenard chains for classical integrable systems. *Theor. Math. Phys.*, 137(3):1716–1722, 2003.

[123] V. A. Marchenko. *Sturm-Liouville Operators and Applications*. Operator Theory: Advances and Applications, vol. 22. Springer, New York, NY, US, 1986.

[124] A. V. Mikhailov. Integrability of a two-dimensional generalization of the Toda chain. *JETP Lett.*, 30(7):414–418, 1979.

[125] P. Miller. Lax pairs and other integrable equations. Lecture notes: https://dept.math.lsa.umich.edu/~millerpd/docs/651_Winter18/Topic03-651-W18.pdf. Department of Mathematics, University of Michigan. [Online; accessed 19 October 2022].

[126] R. M. Miura, C. S. Gardner, and M. D. Kruskal. Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *J. Math. Phys.*, 9:1204–1209, 1968.

[127] T. Miwa. On Hirota's difference equations. *Proc. Jpn. Acad. A: Math.*, 58(1):9–12, 1982.

[128] A. C. Newell. Long waves-short waves: A solvable model. *SIAM J. Appl. Math.*, 35:650–664, 1978.

[129] A. C. Newell. The general structure of integrable evolution equations. *Proc. R. Soc. A*, 365:1722, 1979.

[130] J. J. C. Nimmo. Darboux transformations and the discrete KP equation. *J. Phys. A: Math. Gen.*, 30(24):8693–8704, 1997.

[131] S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov. *Theory of Solitons. The Inverse Scattering Method*. Plenum Press, New York, NY, US, 1984.

[132] Y. Ohta and B.-F. Feng. General rogue wave solution to the discrete nonlinear Schrödinger equation. *Physica D*, 439:133400, 2022.

[133] Y. Ohta, R. Hirota, S. Tsujimoto, and T. Imai. Casorati and discrete Gram determinant representations of solutions to the discrete KP hierarchy. *J. Phys. Soc. Jpn.*, 62:1872–1886, 1993.

[134] Y. Ohta, K. Maruno, and B.-F. Feng. An integrable semi-discretization of the Camassa-Holm equation and its determinant solution. *J. Phys. A Math.*, 41:355205, 2008.

[135] M. Oikawa, M. Okamura, and M. Funakoshi. Two-dimensional resonant interaction between long and short waves. *J. Phys. Soc. Jpn.*, 58(12):4416–4430, 1989.

[136] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer, New York, NY, US, 1993.

[137] A. Pekcan. Solutions of non-integrable equations by the Hirota direct method. *arXiv:nlin/0603072v1*, 2006.

[138] D. E. Pelinovsky. Instability of double-periodic waves in the nonlinear Schrödinger equation. *Front. Phys.*, 9(6):599146, 2020.

[139] B. Prinari, M. J. Ablowitz, and G. Biondini. Inverse scattering transform for the vector nonlinear Schrödinger equation with nonvanishing boundary conditions. *J. Math. Phys.*, 47:063508, 2006.

[140] C. Rogers and W. F. Shadwick. *Bäcklund Transformations and Their Applications*. Academic Press, Cambridge, MA, US, 1982.

[141] M. Romano. *The 3-Wave Resonant Interaction Model: Spectra and Instabilities of Plane Waves*. PhD thesis, Northumbria University, 2019.

[142] C. M. Schober and A. Islas. The routes to stability of spatially periodic solutions of the linearly damped NLS equation. *Eur. Phys. J. Plus*, 135(8):642, 2020.

[143] H. Sheng, G.-F. Yu, and B.-F. Feng. An integrable semidiscretization of the modified Camassa–Holm equation with linear dispersion term. *Stud. Appl. Math.*, 149(1):230–265, 2022.

[144] L. A. Takhtajan, A. Y. Alekseev, I. Y. Afareva, M. A. Semenov-Tian-Shansky, E. K. Sklyanin, F. A. Smirnov, and S. L. Shatashvili. Scientific heritage of L. D. Faddeev. Survey of papers. *Russ. Math. Surv.*, 72(6):977–1081, 2017.

[145] V. I. Talanov. Self focusing of wave beams in nonlinear media. *JETP Lett.*, 2:138, 1965.

[146] T. Taniuti and N. Yajima. Perturbation method for a nonlinear wave modulation. I. *J. Math. Phys.*, 10(8):1369–1372, 1969.

[147] M. Toda. Vibration of a chain with a non-linear interaction. *J. Phys. Soc. Jpn.*, 22:431–436, 1967.

[148] T. Tsuchida. Systematic method of generating new integrable systems via inverse Miura maps. *J. Math. Phys.*, 52(5):053503, 2018.

[149] J. Upsal and B. Deconinck. Real Lax spectrum implies spectral stability. *Stud. Appl. Math.*, 145(4):765–790, 2020.

[150] C. Van der Mee. *Exponentially Dichotomous Operators and Applications*. Birkhauser, Basel, Switzerland, 2009.

[151] A. Varga. A Schur method for pole assignment. *IEEE Trans. Autom. Control*, 26(2):517–519, 1981.

[152] X. Wang, J. Wei, L. Wang, and J. Zhang. Baseband modulation instability, rogue waves and state transitions in a deformed Fokas–Lenells equation. *Nonlinear Dyn.*, 97:343–353, 2019.

[153] J. Weiss, M. Tabor, and G. Carnevale. The Painlevé property for partial differential equations. *J. Math. Phys.*, 24(3):522–526, 1983.

[154] G. B. Whitham. Nonlinear dispersion of water waves. *J. Fluid Mech.*, 27:399–412, 1966.

[155] G. B. Whitham. *Linear and Nonlinear Waves*. Wiley-Interscience, New York, NY, US, 1974.

[156] O. C. Wright. Homoclinic connections of unstable plane waves of the long-wave–short wave equations. *Stud. Appl. Math.*, 117:71–93, 2006.

[157] O. C. Wright. On a homoclinic manifold of a coupled long-wave–short-wave system. *Commun. Nonlinear Sci.*, 15:2066–2072, 2010.

[158] C. Wu, B. Wei, C. Shi, and B.-F. Feng. Multi-breather solutions to the Sasa-Satsuma equation. *Proc. R. Soc A*, 478:20210711, 2022.

[159] N. Yajima and M. Oikawa. Formation and interaction of sonic-Langmuir solitons: Inverse scattering method. *Prog. Theor. Phys.*, 56(6):1719–1739, 1976.

[160] J. Yang and D. J. Kaup. Squared eigenfunctions for the Sasa-Satsuma equation. *J. Math. Phys.*, 50(2):023504, 2009.

[161] H. C. Yuen and B. M. Lake. Instabilities of waves on deep water. *Annu. Rev. Fluid Mech.*, 12(1):303–334, 1980.

[162] V. E. Zakharov. The collapse of Langmuir waves. *Sov. Phys. JETP*, 35:908–914, 1972.

[163] V. E. Zakharov and E. A. Kuznetsov. Multi-scale expansions in the theory of systems integrable by the inverse scattering transform. *Physica D*, 18:455–463, 1986.

[164] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP*, 34:62–69, 1972.