On feedback stabilization of nonlinear discrete-time state-delayed systems

Woihida Aggoune, Rupak Kharel* and K. Busawon*

***Abstract*— In this paper, we consider the problem of feedback stabilization of discrete-time systems with delays. The systems under consideration are nonlinear and nonaffine. By using the Lyapunov Razumikhin approach, we deduce general conditions for stabilizing the closed-loop system. Moreover stabilizing state feedback control laws are proposed.**

I. INTRODUCTION

During the last decades, the problem of stabilizability of control systems and the design of stabilizing feedback has been the subject of many papers, for both continuous and discrete time systems (see [1],[2], [3], [4], [5], [6], [7], [8], [9], [10], [11] and the references therein).

Very few works, however, have been performed to deal with the stabilization of nonlinear systems with delays. In the case of continuous systems, it is due to the difficulty induced by the infinite dimensionality of the state combined with the nonlinear structure (see [12],[13],[14]).

In [15] and [16], the problem of stabilization of continuous nonlinear systems with delays has been addressed. The approach developed is inspired by the classical result, well-known as the Jurdjevic and Quinn method [7], dedicated to the problem of stabilization of nonlinear systems. In [17], the problem of stabilization in the case of discrete time systems with delays was considered, but the study was restricted to the class of non linear affine systems. Moreover, the systems without the drift (i.e.,the input is equal to zero) was supposed to contain no delays.

Recently, in [18], we proposed results of stabilization for some nonlinear and nonaffine systems. However for a lot of cases of nonlinear systems, the results and the structure of control law that we proposed, cannot be applied.

In this paper, we propose a more general formulation of the approach proposed in [18]. We develop results on the state feedback stabilizability problem of equilibrium positions of discrete time nonlinear systems with delay. The system under consideration is nonaffine in control and the system without the drift still involve delays. By combining a suitable mathematical formalism and a Lyapunov Razumikhin approach (see [19],[20],[21]), sufficient conditions guaranteeing the stability of the closed loop system are developed and feedback controllers for these systems are proposed. The approach that is adopted in this paper allows for considering a large class of non linear systems.

W. Aggoune is with Equipe Commande des Systèmes (ECS), ENSEA, 6 avenue du Ponceau, 95014 Cergy-Pontoise Cedex, FRANCE aggoune@ensea.fr

* Rupak Kharel and K. Busawon are with University of Northumbria at Newcastle, School of Engineering, Ellison Building, Newcastle-upon-Tyne, NE1 8ST, UK.

In order to simplify the presentation, the single delay case is treated before considering the case of multiple delays.

The organization of the paper is as follows. In Section 2 the class of systems considered is presented and some basic notions are recalled . In Section 3, the main results are given and proved. Finally, Section 4 gives conclusions.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

The system under consideration is of the form :

$$\begin{cases} x(k+1) = f(x(k), x(k-n_d), u(k)) \\ \\ x(k) = x_0, \quad k = -n_d, 0. \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}$ is the input vector and $n_d \in \mathbb{N}$, is the delay. $\mathbb{N}$ is the set of all nonnegative integers.The vector x_0 represents the initial delay condition. f is a smooth vector field such that $f(0,0,0) = 0$.

Denote by f_0 the vector field defined by :

$$f_0(x(k), x(k-n_d)) = f(x(k), x(k-n_d), 0)$$

Suppose there exists a Lyapunov function V, i.e., a scalar function at least of class C^1 which is positive definite and proper, such that :

$$V(f_0(x(k), x(k-n_d))) - V(x(k)) \leq 0, \ \ \forall k \geq 0,$$

assuming that

$$V(x(k+i)) \leq V(x(k)), \ \ -n_d \leq i \leq -1, \ k \geq 0. \tag{2}$$

Resulting from the Lyapunov Razumikhin approach (see [21], recalled in the following), this means that the system without the drift is stable.

Now, we set the following notations, which will be used throughout the paper.
Let $(f^k)_{k\in\mathbb{N}}$ be the sequence of functions defined by :

$$f^0(x) = x, \quad \forall x \in \mathbb{R}^n$$

and recursively by

$$f^k(x) = f(f^{k-1}(x)), \quad \forall k \geq 1.$$

f^k is denoted the k-th multiple composition of the mapping f.

Let δ be the delay operator given for any function $a(.)$ by :

$$\delta a(k) = a(k - n_d). \tag{3}$$

For any function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, define :

$$h_\delta(x(k)) = h(x(k), x(k - n_d)), \quad \forall k \in \mathbb{N},$$

$$h_\delta^0(x) = x, \quad \forall x \in \mathbb{R}^n$$

and by induction,

$$h_\delta^k(x) = h_\delta(h_\delta^{k-1}(x)), \quad \forall k \geq 1.$$

Before proceeding further, we recall some preliminary results (see [20],[21]).
Consider the nonlinear discrete-time delay systems of the general form :

$$x(k+1) = f(k, x_k) \tag{4}$$

where $f : \mathbb{N} \times \mathcal{C}_H \mapsto \mathbb{R}^n$ is continuous with respect to the first argument, lipschitzian with respect to the second and satisfy $f(k, 0) = 0$ for all $k \in \mathbb{N}$. $\mathcal{C}$ is the set of function

$$\mathcal{C} = \{\phi : \{-r, -r+1, \ldots, -1, 0\} \to \mathbb{R}^n\}$$

with r for some integer $r \geq 0$. For $\phi \in \mathcal{C}$, define the norm of ϕ by $\|\phi\| = \max_{s \in J} |\phi(s)|$. The Euclidean norm of $\phi(s) \in \mathbb{R}^n$ is denoted by $|\phi(s)|$ and $J = \{-r, -r+1, \ldots, -1, 0\}$. For all $\alpha \geq 0$, we denote by $\mathcal{C}_\alpha$, the set defined by

$$\mathcal{C}_\alpha = \{\phi \in \mathcal{C} : \|\phi\| < \alpha\}.$$

For $n_0 \in \mathbb{N}$, $x(n_0, \phi)(k)$ will represent the solution of (4) at time k with initial data ϕ, specified at time n_0, i.e., $x(n_0, \phi)(n_0 + s) = \phi(s)$, for $s \in J$.
For $s \in J$,

$$x_k(s) = x(k+s)$$

and represents the state of the delay system.

DEFINITION 1: The equilibrium solution, $x \equiv 0$ of the delay difference system (4) is said to be :

1) stable, if for any $n_0 \in \mathbb{N}$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, n_0)$ such that $\phi \in \mathcal{C}_\delta$ implies $x_k(n_0, \phi) \in \mathcal{C}_\varepsilon$ for $k \geq n_0$.
2) uniformly stable if the number δ in the definition is independent of n_0.

DEFINITION 2: The equilibrium solution, $x \equiv 0$ of the delay difference system (4) is said to be asymptotically stable, if it is stable and there exists $b_0 = b_0(n_0) > 0$ such that $\phi \in \mathcal{C}_{b_0}$ implies $x_k(n_0, \phi) \to 0$ as $k \to \infty$.

To prove the stability of the zero solution of the system, the following result can be used.

THEOREM 1:
Suppose $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, nondecreasing functions, $u(x), v(x)$ positive for $x > 0$, $u(0) = v(0) = 0$, v strictly increasing. If there is a continous function $V : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$u(|x|) \leq V(k, x) \leq v(|x|)$$

and

$$\begin{aligned} \Delta V(k, x(.)) &\equiv V(k+1, x(k+1)) - V(k, x(k)) \\ &\equiv V(k+1, f(k, x_k)) - V(k, x(k)) \\ &\leq -w(|x(k)|) \end{aligned}$$

if $V(k+s, x(k+s)) \leq V(k+1, x(k+1))$ for $s \in J$

then the zero solution of (4) is uniformly stable.

Now, the main results can be stated and proved.

III. MAIN RESULTS

Consider the system (1) where the vector field f is rewritten as :

$$f(x, \delta x, u) = f_0(x, \delta x) + g(x, \delta x, u),$$

with :

$$g(x, \delta x, u) = f(x, \delta x, u) - f_0(x, \delta x).$$

For simplicity, k is dropped. The vector field f_0 has been introduced in the previous section. It corresponds to the case where the control $u = 0$.

With the notations introduced previously, the vector field $f_{0,\delta}$ correspond to $f_{0,\delta}(x) = f_0(x, \delta x)$.

Let F_δ and $\tilde{V}$ be the functions defined by :

$$F_\delta(x, u) = f(x, \delta x, u)$$

and

$$\tilde{V}(x, u) = V(F_\delta(x, u)),$$

We suppose that the integer $\alpha \in \mathbb{N}^\star$ defined by :

$$\alpha = \inf\{k \in \mathbb{N} / \frac{\partial^k \tilde{V}}{\partial u^k}(x, 0) \neq 0\} \tag{5}$$

is odd.

Then we have the following result :

THEOREM 2:
If the intersection of the sets

$$W_1 = \{x \in \mathbb{R}^n / V(f_{0,\delta}^{k+1}(x)) - V(f_{0,\delta}^k(x)) = 0; \quad \forall k \in \mathbb{N}\}$$

and (6)

$$W_2 = \{x \in \mathbb{R}^n / \frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(f_{0,\delta}^k(x), 0) = 0; \forall k \in \mathbb{N}\}$$

is reduced to the origin, then the system (1) can be made globally asymptotically stable at the origin.

Proof
Consider the difference of the Lyapunov function V along trajectories of (1) :

$$\Delta V(x) = V(f(x, \delta x, u)) - V(x).$$

With the notations introduced previously, this turns out to write that :

$$\Delta V(x) = \tilde{V}(x, u) - V(x).$$

According to assumption (5), we obtain the following Taylor expansion :

$$\tilde{V}(x, u) = \tilde{V}(x, 0) + \frac{u^\alpha}{\alpha!}\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0) + u^{\alpha+1}h(x, u) \quad (7)$$

where h is a function of $C^\infty(\mathbb{R}^n, \mathbb{R})$.

Note that :

$$\tilde{V}(x, 0) = V(f_{0,\delta}(x)).$$

Let $\theta \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n;]0, \infty))$, a function satisfying :

$$\theta(x, \delta x) \leq \frac{1}{\sup_{|u|\leq 1} |\alpha!\ h(x, u)|^2 + |\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0)|^2 + 2}. \quad (8)$$

We define the control law u by :

$$u(x, \delta x) = -\theta(x, \delta x)\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0) \quad (9)$$

It is easy to check that, with condition (8), the control law verify the following inequalities :

$$\begin{aligned} & u(x, \delta x)\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0) \leq 0, \\ & |u(x, \delta x)| \leq \frac{1}{2}, \quad (10) \\ & \text{and} \quad |\alpha!\ \theta(x, \delta x)\, h(x, u)| \leq \frac{1}{2}, \quad \forall x \in \mathbb{R}. \end{aligned}$$

By using (7), we find that :

$$\begin{aligned} \Delta V(x) &= V(f(x, \delta x, u)) - V(x) \\ &= \big(\tilde{V}(x, 0) - V(x)\big) + \frac{u^\alpha}{\alpha!}\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0) \\ &\quad + u^{\alpha+1}h(x, u) \end{aligned}$$

By substituting the control u by its expression (9) we get :

$$\begin{aligned} \Delta V(x) &= \big(V(f_{0,\delta}(x)) - V(x)\big) \\ &\quad -\theta_\delta(x)^\alpha\{\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0)\}^{\alpha+1}. \\ &\qquad .\{\frac{1}{\alpha!} - \theta_\delta(x)h(x, u)\} \end{aligned}$$

By taking into account the fact that the control u satisfy (10) and by using and the assumption (2) (related to the Lyapunov-Razumikhin approach), we obtain :

$$\begin{aligned} \Delta V(x) &\leq \big(V(f_{0,\delta}(x)) - V(x)\big) \\ &\quad -\frac{\theta_\delta(x)^\alpha}{2}\{\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0)\}^{\alpha+1}, \\ &\leq 0. \end{aligned}$$

Then the closed-loop system formed by (1) and (9) is stable.

By the LaSalle's invariance principle for difference system (see [22]), all solution of the closed-loop system converges to the largest invariant set I contained in

$$\Omega = \{x \in \mathbb{R}^n /\ \ \Delta V(x) = 0\}.$$

Since α is odd, we can remark that if $x \in \Omega$ then:

$$V(f_{0,\delta}(x)) - V(x) = 0$$

and

$$\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(x, 0) = 0.$$

Let $x(k)$ be a solution of the closed-loop system with $x(0) = x \in I$. By invariance of I :

$$x(k) \in I \qquad \forall k \geq 0.$$

Since $u = 0$ on the set I, it follows that :

$$x(k) = f_{0,\delta}^k(x).$$

Then, the invariance of I implies that :

$$V(f_{0,\delta}^{k+1}(x)) - V(f_{0,\delta}^k(x)) = 0$$

and

$$\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(f_{0,\delta}^k(x), 0) = 0.$$

Note that

$$\frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(f_{0,\delta}^k(x), 0) = \frac{\partial V}{\partial x}(f_{0,\delta}^{k+1}(x))\frac{\partial^\alpha g_\delta}{\partial u^\alpha}(f_{0,\delta}^k(x), 0)$$

if $g_\delta(x, 0) = g(x, \delta x, 0)$.

Then x is an element of $W_1 \cap W_2$ defined by (6). Since by assumption $W_1 \cap W_2 = \{0\}$, we can conclude that the attractivity of the origin is proved.
This finishes the proof of the Theorem.

Let us consider now the nonlinear system with delay of the general form :

$$\begin{cases} x(k+1) = f_0(x(k), x(k-n_d)) + u^\mu f_1(x(k), x(k-n_d)) \\ \qquad\qquad + u^{\mu+1} f_2(x(k), x(k-n_d), u(k)) \\ x(k) = x_0, \quad k = -n_d, 0. \end{cases} \quad (11)$$

where $\mu > 1$ and μ is odd.
For a smooth vector field f, this correspond to a Taylor expansion of order μ for which the first terms of the expansion (except the first one), are equal to zero .
Then we can derive the following result :

COROLLARY 1:
If the set

$$\begin{aligned} W = \{x \in \mathbb{R}^n \ / \quad & V(f_{0,\delta}^{k+1}(x)) - V(f_{0,\delta}^{k}(x)) = \\ & \frac{\partial V}{\partial x}(f_{0,\delta}^{k+1}(x)) f_{1,\delta}(f_{0,\delta}^{k}(x)) = 0; \\ & \forall k \in \mathbb{N}\} \end{aligned} \tag{12}$$

is reduced to the origin, then the system (1) is globally asymptotically stabilizable.

This result is a consequence of Theorem 2 applied to system (11). We can check easily that

$$\frac{\partial^p \tilde{V}}{\partial u^p}(f_{0,\delta}^{k}(x), 0) = 0, \quad for \quad 1 \leq p < \mu,$$

and

$$\frac{\partial^\mu \tilde{V}}{\partial u^\mu}(f_{0,\delta}^{k}(x), 0) = \mu! \, \frac{\partial V}{\partial x}(f_{0,\delta}(x)) f_{1,\delta}(x)$$

Then, the system is stabilizable by means of a feedback of the form :

$$u(x, \delta x) = -\Psi(x, \delta x) \frac{\partial V}{\partial x}(f_{0,\delta}(x)) \; f_{1,\delta}(x) \, . \tag{13}$$

where $\Psi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n;]0, \infty))$ is an appropriate function.

REMARK 1: In the particular case where $\mu = 1$, this result is still valid and we can then recover a result establish in [18].

REMARK 2: We can note that Theorem 2, can be extended to the case of multiple commensurate delays.

Consider the system of the form :

$$\begin{cases} x(k+1) = f(x(k), x(k-n_{d1}), ..., x(k-n_{dm}), u(k)) \\ \\ x(k) = x_0, \quad k = -n_{d1}, \dots, -n_{dm}, 0. \end{cases} \tag{14}$$

where f is a smooth vector field satisfying $f(0, \dots, 0) = 0$.

Let us now introduce the delay operators, δ_i, $(i \in \mathbb{N})$ defined by :

$$\delta_i x(k) = x(k - n_{di})$$

We can define the delay operator τ_p by:

$$\tau_p = (\delta_1, .., \delta_p).$$

Let F be a function mapping $\mathbb{R}^{n \times (p+1)}$ into $\mathbb{R}^n$ and $F_{\tau_p}(x(t))$ the function defined by

$$\begin{aligned} F_{\tau_p}(x(k)) &= F(x(k), x(k-n_{d1}), \dots, x(k-n_{dp})) \\ &= F(x(k), \tau_p(x(k))). \end{aligned}$$

We define the sequence of function $F_{\tau_p}^k$ for $k \in \mathbb{N}$ as follows:

$$F_{\tau_p}^0(x) = x, \quad \forall x \in \mathbb{R}^n,$$

and by recurrence

$$F_{\tau_p}^k(x) = F_{\tau_p}(F_{\tau_p}^{k-1}(x)), \quad \forall k \geq 1.$$

By setting $f_0(x(k), x(k - n_{d1}), ..., x(k - n_{dm})) = f(x(k), x(k-n_{d1}), ..., x(k-n_{dm}), 0)$ and thus

$$f_{0,\tau_m}(x(k)) = f_0(x(k), x(k-n_{d1}), ..., x(k-n_{dm})),$$

we can state the following result:

THEOREM 3:
If the intersection of the sets

$$\begin{aligned} \widetilde{W_1} = \{x \in \mathbb{R}^n / \, & V(f_{0,\tau_m}^{k+1}(x)) - V(f_{0,\tau_m}^{k}(x)) = 0; \\ & \forall k \in \mathbb{N}\} \end{aligned}$$

and

$$\begin{aligned} \widetilde{W_2} = \{x \in \mathbb{R}^n / \, & \frac{\partial^\alpha \tilde{V}}{\partial u^\alpha}(f_{0,\tau_m}^{k}(x), 0) = 0; \\ & \forall k \in \mathbb{N}\} \end{aligned} \tag{15}$$

with $\tau_m = (\delta_1, .., \delta_m)$, is reduced to the origin, then the system with multiple delays (14) is globally asymptotically stabilizable at the origin.

Proof :
The proof of this result is similar to the proof of Theorem 2. We replace the delay operator δ given in (3) by the delay operator $\tau = (\delta_1, .., \delta_m)$.

IV. CONCLUSIONS

In this paper, we have presented results on stabilization of discrete time nonlinear systems with time delay. More precisely, we have used the Invariance Principle of LaSalle dedicated to difference systems combined with a Lyapunov Razumikhin type approach. We obtained sufficient conditions, for guaranteeing the asymptotic stability of the closed-loop system and stabilizing state feedback control laws have been derived.

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